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PORTFOLIOS OPTIMIZATION UNDER CONSTRAINT IN INCOMPLETE MARKETS BASED UPON RECURSIVE UTILITIES

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Abstract. We study the problem of recursive utility maximization in the presence of nonlinear constraint on the wealth for a model driven by Lévy processes. We extend the notion of W-divergence to vector valued functions and then reduce the problem to the classical problem of recursive utility maximization problem under the W-projection. Using BSDE technics, we derive a first order condition which gives a necessary and sufficient condition of optimality under the W-projection, which generalizes the characterization of optimal solution obtained in [6] in the case of continuous–time, and also the result obtained in [9] in the case of standard utility.

1. Introduction

Due to the emergence of the theories of backward stochastic differential equations (BSDE) and risk measure in the last decades, attention of researchers has been devoted to the applications of theses theories in some optimization problems in mathematical finance. (e.g, utility maximization, hedging, pricing etc...) In particular, the notion of utility and more precisely expected utility was initiated to modeled preference of an investor. Motivated by the desire to disentangle risk aversion and willingness to substitute inter-temporally, the notion of stochastic differential utility (SDU) or recursive utility was introduced to generalized the standard utility (see Duffie and Epstein [3]).

In this paper, we consider the problem of SDU maximization from terminal wealth and an intermediate consumption under budget constraint, when the wealth is supposed to satisfy non–linear equation. Standard utility maximization under a budget constraint is a fundamental problem in finance. In the case of linear wealth, this problem has been largely studied in both complete and incomplete market using Markovian methods via HJB equation or martingale and duality methods. (see for e.g. Merton [19], Cox and

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Huang [2], Karatzal et al. [13], Duffie et al. [4], Kramkov and Schachermayer [15], Göll and Rüschendorf [9], etc...)

El Karoui et al. [6] have considered the continuous–time portfolio–consumption problem of an agent with recursive utility in the presence of nonlinear constraints on the wealth in a complete market. They used BSDE to derive a dynamic maximum principle which generalized the optimal policies obtained by Duffie and Skiadas [5] in the case of a linear wealth. They obtained a characterization of optimal wealth and utility processes as the unique solution of a forward–backward system. Øksendal and Sulem in [21] solved this problem for more general utility functions under partial information. A malliavin calculus approach is also explore in the later.

The first motivation of this paper is to generalized the result of El Karoui et al. [6] to discontinuous process and then to study the maximization problem. However by doing so, the market becomes incomplete. Therefore the budget constraint which can be formalized in a complete market in term of an expectation under the single pricing measure is not valid. In fact, in an incomplete market, the set of equivalent martingale measures is infinite, and the analysis of the budget constraint needs more attentions.

The second motivation is the paper by Göll and Rüschendorf [9]. They gave a characterization of minimal distance martingale measures with respect to W–divergence distance in a general semimartingale model. They also show that the minimal distance martingale measures are equivalent to minimax martingale measure with respect to related utility functions and that optimal portfolio can be characterized by them.

Using techniques of optimization by vector space method, we can extend the result of Göll and Rüschendorf [9] in the framework of recursive utility. This extension enables to use the notion of W–divergence (where W is a vector value function) over the set of equivalent martingales measures to give the optimal solution of the recursive utility maximization problem under budget constraint in incomplete market.

We also derive a stochastic maximum principle which gives necessary and sufficient condition of optimality, and which generalized the result of El Karoui et al.[6] in the case of processes with jumps and for quasi–strong generators.

The paper is organized as follows: In Section 2, we present our model of recursive utility problem under nonlinear wealth constraint. In Section 3, fixing an equivalent martingale measure, we derive a maximum principle which gives a necessary and sufficient conditions of optimality. In Section 4, we introduce the notion of W–divergences and recall some importance results about W–projection. This section also contains our main results.

2. The constrained maximization problem

2.1. **The model.**

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ be a filtered complete probability space satisfying usual conditions. Suppose that the filtration is generated by the following two mutually independent processes:

- $W = \{W_t, t \in [0,T]\}$ is a m–Brownian motion,

- $\widetilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is a m–independent compensated Poisson random measures where ν_i is measure of Lévy process with jump measure N_i, $i = 1, ..., m$. We assume that

$$\int_{\mathbb{R}_0} (1 \wedge |z|^2)\nu(dz) < +\infty, \text{ with } \mathbb{R}_0 = \mathbb{R} \setminus \{0\}.$$

We deal with a financial market consisting of $n+1$ traded financial assets on a fixed time horizon $0 \le T < \infty$. The first asset is a bond whose price $S_0(t)$ is modeled by the following equation

$$dS^0(t) = S^0(t)r(t)dt,\ 0 \le t \le T, \qquad S^0(0) = s_0. \tag{2.1}$$

The remaining n assets are risky with the price process S^i of the i^{th} stock satisfying the following SDE

$$\begin{cases} dS^i(t) &= S^i(t)\Big(a^i(t)dt + \sum_{j=1}^m \sigma^{i,j}(t)dW_t^j \\ & \qquad + \sum_{j=1}^m \int_{\mathbb{R}_0} \gamma^{i,j}(t,z)\widetilde{N}(dt,dz)\Big), \\ S^i(0) &= s_i, \qquad 1 \le i \le n. \end{cases} \tag{2.2}$$

Here $r(t)$ is a deterministic function, $a(t)$, $\sigma(t)$ and $\gamma(t,z)$ are given $\mathcal{F}_t$-predictable, uniformly bounded functions satisfying the following integrability condition:

$$E\Big[\int_0^T \Big\{ \mid r(s) \mid + \mid a(s) \mid + \frac{1}{2}\sigma(s)^2 \\ + \int_{\mathbb{R}} \mid \log(1+\gamma(s,z)) - \gamma(s,z) \mid \nu(dz)\Big\} ds\Big] < \infty,$$

where T is fixed. We assume that

$$\gamma(t,z) \ge -1 \qquad \text{for a.a. } t,z \in [0,T] \times \mathbb{R}_0.$$

Let consider now an economic agent with initial wealth $v \ge 0$ who can invest in the $(n+1)$ assets. Let us denote by $\pi_i = (\pi_i(t))_{0 \le t \le T}$ the amount of money invested in the stock at time t by the trader. A portfolio process $\pi = (\pi_i(t))_{1 \le i \le n}$ is an $\mathcal{F}_t$–predictable process on $(\Omega, \mathcal{F}, \mathbb{P})$ for which

$$\sum_{i=1}^n \int_0^T \pi_i^2(t)dt < \infty \text{ a.s.} \tag{2.3}$$

A consumption process $c = \{c(t), 0 \le t \le T\}$ is a measurable adapted process with value in $[0,\infty)$ such that

$$\int_0^T c(t)dt < \infty \text{ a.s.} \tag{2.4}$$

If we assume that the pair (π, c) is self–financing, then the wealth process of the investor can be expressed in term of the portfolio strategy $\pi = (\pi_i)_{1\leq i\leq n}$ as

$$-dV^{(v,\pi,c)}(t) = b(t, c(t), V^{(v,\pi,c)}(t), \pi(t))dt - \pi'(t)\sigma(t)dW(t) - \int_{\mathbb{R}_0} \pi'(t)\gamma(t,z)\tilde{N}(dt,dz),\ V^{(v,\pi,c)}(0) = v > 0, \tag{2.5}$$

where

1) c is the consumption plan,

2) $b(t, c(t), V^{(v,\pi,c)}(t), \pi(t)) = -[V^{(v,\pi,c)}(t)r(t) - \sum_{i=1}^n \pi_i(t)(r(t) - a^i(t) - c(t))]$,

3) $\pi'(t)\sigma(t)dW(t) = \sum_{i=1}^n \pi_i(t) \sum_{j=1}^n \sigma^{i,j}(t)dW^j(t)$,

4) $\pi'(t)\gamma(t,z)\widetilde{N}(dt,dz) = \sum_{i=1}^n \pi_i(t) \sum_{j=1}^l \gamma^{i,j}(t,z)\widetilde{N}^j(dt,dz)$.

We will assume in general that the drift coefficient b of the wealth process satisfies the following assumptions.

Assumption A1.

(i) b is Lipschitz with respect to π, v and uniformly with respect to t, c, v,

(ii) there exists a positive constant k such that for all $c \in \mathbb{R}_+$

$$|b(t, c, 0, 0)| \leq k\, c \text{ a.s. } ,$$

(iii) $\forall c \in \mathbb{R}_+$, $b(t, c, 0, 0) \geq 0$ a.s.,

(iv) the function b is non–decreasing with respect to c and convex with respect to c, x, π.

Definition 2.1. The consumption–portfolio pair (π, c) satisfying (2.3) and (2.4) is *admissible* for initial wealth $v \geq 0$ if

$$V^{(v,\pi,c)}(t) \geq 0, \text{ for } 0 \leq t \leq T \text{ a.s.} \tag{2.6}$$

The set $\mathcal{V}(v)$ is the set of admissible pair (π, c) for initial wealth v.

In order to give a characterization of admissible pairs, we first recall the following definition.

Definition 2.2. A probability Q which is absolutely continuous with respect to P is called an *absolutely continuous martingale measure* if S is a local martingale measure under Q. The family of these measures is denoted by $\mathcal{P}$. Any $Q \in \mathcal{P}$ which is equivalent to P is called the *equivalent local martingale measure*. The family of theses measures will be denoted by $\mathcal{P}_e$.

Since the market is governed by a Lévy process, there is no unique equivalent local martingale measure. Let $\Gamma = (\Gamma(t))_{0\leq t\leq T}$ the process defined by

$$dQ(\omega) = \Gamma(T)dP(\omega) \qquad \text{on } \mathcal{F}_T, \text{ for } Q \in \mathcal{P}.$$

Fixing initial wealth of $v > 0$, a contingent claim $H \geq 0$ is *affordable* if there is a self–financing portfolio $V \in \mathcal{V}(v)$ such that $V(T) \geq H$ P–a.s. As shown by Jeanblanc and Pontier [12] this notion of affordability is equivalent to

$$\sup_{Q\in\mathcal{P}_e} \mathbb{E}_Q\Big[HR_T + \int_0^T R(s)c(s)ds\Big] \leq v,$$

where $R(t) = \exp(-\int_0^t r(s)ds)$ is a discounted rate. The following proposition gives necessary and sufficient condition of admissibility.

Proposition 2.3. *(i) Let H be a non-negative, $\mathcal{F}_T$–measurable random variable such that*

$$\mathbb{E}_Q[HR(T) + \int_0^T R(s)c(s)ds] \leq v. \tag{2.7}$$

Then there exists a portfolio process π such that the pair (π, c) is admissible for the initial endowment v and such that the terminal wealth is at least H.

(ii) If (π, c) is an admissible strategy, then we have

$$\mathbb{E}_Q[V(T)R(T) + \int_0^T R(s)c(s)ds] \leq v. \tag{2.8}$$

Proof. The same as in Jeanblanc and Pontier [12]. □

2.2. **The Recursive utility problem.**

In this section, $(\Omega, \mathcal{F}, P)$ represent again a probability space. We consider a small agent endowed with an initial wealth v, who can consume between time 0 and time T. Let $\pi(t)$ be the stock portfolios and $c(t)$ be the positive consumption rate at time t. We suppose that there exists a terminal reward X at time T. In this setting, the utility at time t depends on the consumption rate $c(t)$ and on the future utility. More precisely, the recursive utility at time t is defined by

$$X(t) = \mathbb{E}\Big[X + \int_t^T f(s, c(s), X(s), Y(s), K(s,\cdot))ds\Big|\mathcal{F}_t\Big], \tag{2.9}$$

where f is called a standard drive. As has shown in El Karoui et al. [7] for continuous case, and in Situ [24] and Halle [11] for jumps case, such utility process can be regarded as the solution of the following BSDE,

$$\left\{\begin{array}{ll} -dX(t) & = f(t, X(t), Y(t), K(t,\cdot))dt - Y(t)dW(t) - \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz), \\ X(T) & = X. \end{array}\right. \tag{2.10}$$

We denote by $\{\mathcal{F}_t\}_{0\leq t\leq T}$ the natural filtration generated by B and $\widetilde{N}$ where $\mathcal{F}_0$ contains all P null sets of $\mathcal{F}$. It is a complete right continuous filtration. From now on, we fix the final time $T > 0$.

In the following, we define some space of processes.

Definition 2.4. For $\beta \geq 0$,

- $L^2_{T,\beta}(\mathbb{R}^n)$ is the space of all $\mathcal{F}_T$–measurable random variables $X : \Omega \to \mathbb{R}^n$ such that
$$\mathbb{E}^p[e^{\beta T}\|X\|^2] < \infty,$$
- $S^2_{T,\beta}(\mathbb{R}^n)$ is the space of all càdlàg, adapted processes $X : \Omega \times [0,T] \to \mathbb{R}^n$ such that
$$\mathbb{E}[e^{\beta T} \sup_{0\le t\le T} \|X\|^2] < \infty,$$
- $H^2_{T,\beta}(\mathbb{R}^n)$ is the space of all predictable processes $X : \Omega \times [0,T] \to \mathbb{R}^n$ such that
$$\mathbb{E}\Big[\int_0^T e^{\beta T}\|X\|^2 dt\Big] < \infty,$$
- $\hat{H}^2_{T,\beta}(\mathbb{R}^n)$ is the space of all mappings $U : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}^n$ which are $\mathcal{G}\otimes\mathcal{B}$–measurable and satisfy
$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}_0} e^{\beta T}\|U(t,z)\|^2 \nu(dz)dt\Big] < \infty,$$
where $\mathcal{G}$ denote the σ–field of $\mathcal{F}_t$–predictable subsets of $\Omega \times [0,T]$,
- $\hat{L}^2_T(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu, \mathbb{R}^n)$ is the space of all $\mathcal{B}(\mathbb{R}_0)$–measurable mappings $U : \mathbb{R}_0 \to \mathbb{R}^n$ such that
$$\int_{\mathbb{R}_0} \|U(t,z)\|^2 \nu(dz) < \infty,$$
- we define $\mathcal{V}_\beta = S^2_{T,\beta}(\mathbb{R}^n) \times H^2_{T,\beta}(\mathbb{R}^n) \times \hat{H}^2_{T,\beta}(\mathbb{R}^n)$.

For notational simplicity, we will write $L^2_{T,\beta}$, $S^2_{T,\beta}$, $H^2_{T,\beta}$, $\hat{H}^2_{T,\beta}$ instead of $L^2_{T,\beta}(\mathbb{R}^n)$, $S^2_{T,\beta}(\mathbb{R}^n)$, $H^2_{T,\beta}(\mathbb{R}^n)$, $\hat{H}^2_{T,\beta}(\mathbb{R}^n)$, respectively.

In order to ensure that equation (2.10) has a unique solution, we make some classical assumptions.

Assumption A2. $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \hat{L}^2_T(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu, \mathbb{R}^n) \to \mathbb{R}$ is $\mathcal{G}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(\hat{H}^2_{T,\beta}(\mathbb{R}^n))$–measurable such that $f(t,0,0,0,0) \in H^2_T$ and it is uniformly Lipschitz, i.e., there exists K such that, for every $(x_1, y_1, k_1(y))$ and $(x_2, y_2, k_2(y))$,
$$\begin{aligned}&|f(\omega,t,c,x_1,y_1,k_1(\cdot)) - f(\omega,t,c,x_2,y_2,k_2(\cdot))|\\ &\le K(|x_1-x_2| + |y_1-y_2| + |k_1-k_2|_{\nu(\cdot)}).\end{aligned}$$

Assumption A3. We suppose that there exists some constants M_1 and M_2 such that, for all $c \in \mathbb{R}_+$,
$$|f(t,c,0,0,0)| \le M_1 + M_2\frac{c^p}{p} \qquad \text{a.s,}$$
with $0 \le p < 1$ and $p \neq 0$.

For each $c \in H_T^2$ and each terminal reward $X \in L_T^2$, Assumption A2 and Assumption A3 guaranty that the BSDE (2.10) has a unique solution $(X, Y, K) \in H_{T,\beta}^2 \times H_{T,\beta}^2 \times \hat{H}_{T,\beta}^2$.

As means to have "nice" linear BSDE for our stochastic gradient[a] and to apply a comparison theorem as in Duffie and Skiadas [5], we need the following assumptions.

Assumption A4. The generator f is of the kind

$$f(\omega, t, c, x, y, k) = h\Big(\omega, t, c, x, y, \int_{\mathbb{R}_0} k(z)\varphi(t,z)\nu(dz)\Big) \tag{2.11}$$

for $(\omega, t, c, x, y, k) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times L^2(\mathbb{R}_0, \mathcal{B}(\mathbb{R}_0), \nu, \mathbb{R})$ where $\varphi : \Omega \times [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ is $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}_0)$–measurable and satisfies

$$0 \le \varphi(t,z) \le c(1 \wedge |z|), \ \forall z \in \mathbb{R}_0,$$

and $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{R})$–measurable and satisfies

(i) $\mathbb{E}\Big[\int_0^T |h(t,0,0,0,0)|^2 dt\Big] < +\infty$,
(ii) h is uniformly Lipschitz with respect to x, y, k,
(iii) $k \mapsto f(\omega, t, c, x, y, k)$ is non–decreasing for all (t, c, x, y, k).

From now on, we assume that our generator is of the form (2.11), and when we write $f(\omega, t, c, x, y, k)$, we mean $h\Big(\omega, t, c, x, y, \int_{\mathbb{R}_0} k(z)\varphi(t,z)\nu(dz)\Big)$.

To guarantee that our recursive utility satisfies properties of utility functions, we need the following assumptions.

Assumption A5. f is strictly concave with respect to c, x, y, k and f is strictly non–decreasing function with respect to c. Furthermore we assume that f satisfies the Inada condition.

In general, the terminal value X will measure the utility of terminal wealth, that is, $X(\omega) = U(V(T,\omega), \omega)$, where $V(T)$ is the value of the agent's wealth at terminal time T and U satisfies the following assumptions.

Assumption A6. $U : \Omega \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{F}_T \times \mathcal{B}(\mathbb{R})$–measurable. U is strictly concave and non-decreasing with respect to v and satisfies, for $\forall v \in \mathbb{R}_+$,

$$|U(v)| \le k_1 + k_2 \frac{v^p}{p} \text{ a.s. with } 0 < p < 1, v \in \mathbb{R}_+.$$

We also assume that U satisfies the Inada condition.

This assumption ensures that for $V(T) \in L_T^2$, $U(V(T)) \in L_T^2$ and the recursive utility (2.9) associated with the terminal reward X is non–decreasing and concave with respect to the terminal wealth $V(T)$. We will also impose some restriction on the set $\mathcal{P}_e$.

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[a] See Duffie and Skiadas [5] and El Karoui et. al. for the definition

Assumption A7. We assume that $\mathcal{P}_e$ is convex and the set

$$\mathcal{K}_{\mathcal{P}_e} = \Big\{ \frac{dQ}{dP}; Q \in \mathcal{P}_e \Big\},$$

is convex and weakly compact.

2.3. **The problem.**

Let us fix an initial wealth $v > 0$. As already pointed, we are interested in a small investor with initial wealth v, and who decides at each time t his stock portfolio $\pi(t)$ and his consumption $c(t)$. He wishes to select a self–financing portfolio–consumption strategy in $\mathcal{V}(v)$ that maximizes the recursive utility of consumption and terminal wealth $V^{v,\pi,c}(T)$. The optimization problem can then be written as follows.

Problem A.

$$\sup_{(\pi,c)\in\mathcal{V}(v)} X^{v,\pi,c}(0) = \sup_{(\pi,c)\in\mathcal{V}(v)} \mathbb{E}\Big[\int_0^T f(s, c(s), X(s), Y(s), K(s,\cdot))ds + U(V^{(v,\pi,c)}(T))\Big]. \tag{2.12}$$

As stated in Proposition 2.3, this path constraint can be changed into an expectation constraint with respect to martingale measure. Using the notion of affordability, Proposition 2.3, the optimal decomposition theorem in Kramkov [14] and Föllmer and Kabanov [8], the path constraint is equivalent to

$$\sup_{Q\in\mathcal{P}_e} \mathbb{E}_Q[HR(T) + \int_0^T R(s)c(s)ds] \le v.$$

We can now pose the SDU maximization under budget constraint that we want to solve. We denote the set of all terminal financial positions and consumption with well defined utility and prices by

$$\mathcal{I}_1 = \{H \ge 0, H \in L^2_{T,\beta} : HR(T) \in \text{Ł}^1(Q) \text{ for all } Q \in \mathcal{P}_e \text{ and } U(H)^- \in L^1(P)\}.$$

and

$$\begin{aligned}\mathcal{I}_2 =&\{c = (c(t))_{t\ge 0} : c \in H^2_{T,\beta} : c(t)R(t) \in \text{Ł}^1(\lambda \times Q) \\ &\text{for all } Q \in \mathcal{P}_e \text{ and } f(t,\cdot,y,z,k(\cdot)) \in L^1(\lambda\times P)\},\end{aligned}$$

where λ denotes the Lebesgue measure.

For $v > 0$, the optimization problem to be solved under budget constraint is the following

Problem B.

$$\begin{aligned}&\max\nolimits_{(H,c)\in\mathcal{I}_1\times\mathcal{I}_2} \mathbb{E}\Big[\textstyle\int_0^T f(t, c(t), X(t), Y(t), K(t,\cdot))dt + U(H)\Big] \\ &\text{subject to} \\ &\qquad \sup\nolimits_{Q\in\mathcal{P}_e} \mathbb{E}_Q[HR(T) + \textstyle\int_0^T R(s)c(s)ds] \le v.\end{aligned} \tag{2.13}$$

Denote by $\mathcal{X}(x)$ the set of financial positions in $\mathcal{I}_1 \times \mathcal{I}_2$ that satisfy the budget constraint, i.e.,

$$\mathcal{X}(x) = \{(H, c) \in \mathcal{I}_1 \times \mathcal{I}_2; \sup_{Q \in \mathcal{P}_e} \mathbb{E}_Q[HR(T) + \int_0^T R(s)c(s)ds] \leq v\}. \tag{2.14}$$

In the following theorem, we show that if the stock price S is locally bounded, then the solution to the Problem A is equivalent to the solution of the Problem B.

Theorem 2.5. *The optimization Problem A admits a solution if and only if the optimization Problem B admits a solution.*

If $(H^*, c^*) \in \mathcal{X}(v)$ is a solution to the Problem B, then there exists a solution (π^*, c^*) to the Problem A with $V^{(\pi^*,c^*)}(T) \geq H^*$ P–almost surely. Moreover if the solution is unique P–almost surely, then $(V^{(\pi^*,c^*)}(T), c^*) = (H^*, c^*)$ P–almost surely.

Conversely, if (π^*, c^*) is solution to the Problem A, then $(V^{(\pi^*,c^*)}(T), c^*) \in \mathcal{X}(v)$ is solution to the Problem B.

Proof. The theorem is proven in the same way as in [10, Theorem 2.7].

Assume that the Problem B admits a solution (H^*, c^*). Let Z be a right–continuous version of

$$Z_t = ess \sup_{Q \in \mathcal{P}_e} \mathbb{E}_Q\Big[H^* R(T) + \int_0^T R(s)c^*(s)ds|\mathcal{F}_t\Big]. \tag{2.15}$$

Since $\sup_{Q \in \mathcal{P}_e} \mathbb{E}_Q[H^* R(T) + \int_0^T R(s)c^*(s)ds] < \infty$, then Z is a supermartingale for every $Q \in \mathcal{P}_e$ (see [14, Proposition 4.2]). By [14, Theorem 2.1], there exists a predictable process π^0 and an adapted increasing process c^0 such that

$$R(T)V^0(T) = v + \int_0^T \pi^0(t)d\tilde{S}(t) - \tilde{c}(T) \geq H^* R(T) + \int_0^T R(s)c^*(s)ds \geq 0 \tag{2.16}$$

where $\tilde{c}(T) = \int_0^T R(s)c^0(s)ds$ and $V^0(T) = (V^0(t))_{0 \leq t \leq T} = V^{(0,\pi^0,c^0)}(t)$ is consumption portfolios and the capital of a wealth. Define the process M by

$$M(t) = R(t)V^0(t) + \int_0^t R(s)c^0(s)ds. \tag{2.17}$$

Then, under all $Q \in \mathcal{P}_e$, M is a supermartingale as a sigma–martingale which is bounded from below. Thus

$$\mathbb{E}_Q\Big[R(T)V^0(T) + \int_0^T R(s)c^0(s)ds\Big] \leq v, \ \forall Q \in \mathcal{P}_e, \tag{2.18}$$

which implies that

$$\sup_{Q \in \mathcal{P}_e} \mathbb{E}_Q\Big[R(T)V^0(T) + \int_0^T R(s)c^*(s)ds\Big] \leq v, \ \forall Q \in \mathcal{P}_e, \tag{2.19}$$

We have $R(T)V^0(T)+\int_0^T R(s)c^0(s)ds \geq H^*R(T)+\int_0^T R(s)c^*(s)ds$ and $V^0(T) \geq H^*$ (from 2.16), i.e., $U(V^0(T)) \geq U(H^*)$. It follows from the comparison theorem of BSDE with jump, see Situ [24], that

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T f(t,c^0(t),X^0(t),Y^0(t),K^0(t,\cdot))dt+U(V^0(T))\Big]\\&\geq \mathbb{E}\Big[\int_0^T f(t,c^*(t),X^*(t),Y^*(t),K^*(t,\cdot))dt+U(H^*)\Big].\end{aligned} \tag{2.20}$$

$(V^0(T),c^0) \in \mathcal{X}(v)$ and since (H^*,c^*) is solution to the Problem B, we obtain fron 2.20

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T f(t,c^0(t),X^0(t),Y^0(t),K^0(t,\cdot))dt+U(V^0(T))\Big]\\&= \mathbb{E}\Big[\int_0^T f(t,c^*(t),X^*(t),Y^*(t),K^*(t,\cdot))dt+U(H^*)\Big].\end{aligned} \tag{2.21}$$

It remains to show that (π^0,c^0) is a solution to the Problem A. Let $(\pi,c)\in\mathcal{V}(v)$. Using similar argument as above, we can show that $(V(T),c)=(V^{(\pi,c)}(T),c)\in\mathcal{X}(v)$. Thus

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T f(t,c(t),X(t),Y(t),K(t,\cdot))dt+U(V(T))\Big]\\&\leq \mathbb{E}\Big[\int_0^T f(t,c^*(t),X^*(t),Y^*(t),K^*(t,\cdot))dt+U(H^*)\Big]\\&= \mathbb{E}\Big[\int_0^T f(t,c^0(t),X^0(t),Y^0(t),K^0(t,\cdot))dt+U(V^0(T))\Big]\end{aligned} \tag{2.22}$$

Then (π^0,c^0) is a solution to the Problem A. If (H^*,c^*) is P–almost surely unique, then $V^0(T)=H^*$ and $c^0=c^*$ P–almost surely since $(V^0(T),c^0)$ is a solution to the Problem A.

Conversely, assume that $(V^*,c^*)=(V^{(\pi^*,c^*)},c^*)\in\mathcal{V}(v)$ is solution to the Problem A. Let $M(T)=R(T)V^*(T)+\int_0^T R(s)c^*ds = x+\int_0^T \pi^*(s)d\tilde{S}(s)ds$. Then M is supermartingale. Thus,

$$\mathbb{E}_Q[R(T)V^*(T)+\int_0^T R(s)c^*(s)ds]\leq v,\ \forall Q\in\mathcal{P}_e, \tag{2.23}$$

i.e.,

$$\sup_{Q\in\mathcal{P}_e}\mathbb{E}_Q[R(T)V^*(T)+\int_0^T R(s)c^*(s)ds]\leq v. \tag{2.24}$$

This implies that $(V^*(T),c^*)\in\mathcal{X}(v)$.

Assume that there exists $(H^0, c^0) \in \mathcal{X}(v)$ such that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T f(t, c^0(t), X^0(t), Y^0(t), K^0(t,\cdot))dt + U(H^0)\Big] \\
&> \mathbb{E}\Big[\int_0^T f(t, c^*(t), X^*(t), Y^*(t), K^*(t,\cdot))dt + U(V^*(T))\Big]. \qquad (2.25)
\end{aligned}$$

Then using the same argument as above, we can find $(V^0(T), c^0) = (V^{(\pi^0,c^0)}(T), c^0)$ such that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T f(t, c^0(t), X^0(t), Y^0(t), K^0(t,\cdot))dt + U(V^0(T))\Big] \\
&= \mathbb{E}\Big[\int_0^T f(t, c^0(t), X^0(t), Y^0(t), K^0(t,\cdot))dt + U(H^0)\Big] \\
&> \mathbb{E}\Big[\int_0^T f(t, c^*(t), X^*(t), Y^*(t), K^*(t,\cdot))dt + U(V^*(T))\Big]. \qquad (2.26)
\end{aligned}$$

This contradict the fact that $(V^*(T), c^*) = (V^{(\pi^*,c^*)}(T), c^*)$ is optimal to the Problem A. It follows that $(V^*(T), c^*)$ is a solution to the Problem B. □

3. The solution of the optimization problem under one probability measure

In this section, we fix an equivalent martingale measure $Q^* \in \mathcal{P}_e$. Let $\mathcal{I}_1^{Q^*}$ denote the set of terminal financial position,

$$\mathcal{I}_1^{Q^*} = \{H \geq 0 : HR(T) \in \text{Ł}^1(Q^*) \text{ and } U(H)^- \in L^1(P)\},$$

and $\mathcal{I}_2^{Q^*}$ the set of consumption,

$$\mathcal{I}_2^{Q^*} = \{c = (c(t))_{t\geq 0} : c(t)R(t) \in \text{Ł}^1(\lambda \times Q^*) \text{ and } f(t,\cdot,y,z,k(\cdot)) \in L^1(\lambda \times P)\}.$$

Put $\mathcal{I}^{Q^*} = \mathcal{I}_1^{Q^*} \times \mathcal{I}_2^{Q^*}$. Let $v > 0$ be an initial wealth. We study now an auxiliary maximization problem under budget constraint:

$$\begin{array}{ll}
\max_{(H,c)\in\mathcal{I}^{Q^*}} \mathbb{E}\Big[\int_0^T f(t, c(t), X(t), Y(t), K(t,\cdot))dt + U(H)\Big] & \\
\text{subject to} \qquad \mathbb{E}_{Q^*}[HR(T) + \int_0^T R(s)c(s)ds] \leq v. &
\end{array} \qquad (3.1)$$

We shall show that the constrained optimization problem (3.1) has a unique solution.

First note that problem (3.1) can be reduce to the following

$$\max_{(H,c)} X(0) \qquad \text{over all } (H,c) \in \mathcal{I}^{Q^*},$$

$$\text{subject to} \qquad \mathbb{E}_{Q^*}[HR(T) + \int_0^T R(s)c(s)ds] \leq v.$$

Recall from standard result on BSDEs that X solves the following adapted BSDE

$$\begin{aligned} dX(t) &= -f(t, c(t), X(t), Y(t), K(t,\cdot))dt + Y(t)^* dW(t) \\ &\quad + \int_{\mathbb{R}_0} K(t,z)\tilde{N}(dt,dz), \\ X(T) &= U(H) = U(V^{(\pi,c)}(T)). \end{aligned}$$

Define, for a fixed $\lambda > 0$, the map

$$L(0,H,c) = X(0) - \lambda\Big(\mathbb{E}_{Q^*}\Big[HR(T) + \int_0^T R(s)c(s)ds\Big] - v\Big).$$

Using classical result in convex analysis for characterization of optimal solution (see for e.g. [6, Theorem 4.1]) we study now the following unconstrained optimization problem

$$\sup_{(H,c)\in\mathcal{I}^{Q^*}} L(0,H,c), \tag{3.2}$$

which optimal solution is equivalent to that of (3.1).

In order to solve the optimization problem, we use the stochastic gradient approach introduced by Duffie and Skiadas [5] and El Karoui et al. [6]. Define $\Delta c(t) = c(t) - c^*(t)$ and $\Delta X^{(\pi,c)}(t) = X^{(\pi,c)}(t) - X^{(\pi^*,c^*)}(t)$. Denote the stochastic gradient of (X,V) by $(\partial_h X, \partial_h V)$. Then $(\partial_h X, \partial_h V)$ are solution of the following linear backward–forward SDE's

$$\begin{cases} d\partial_h X(t) &= -\Big[\bigtriangledown_c f(t)\Delta c(t) + \bigtriangledown_x f(t)\partial_h X(t) + \bigtriangledown_y f(t)\partial_h Y(t) \\ &\quad + \int_{\mathbb{R}_0} \bigtriangledown_k f(t)\partial_h K(t,z)\nu(dz)\Big]dt - \partial_h Y(t)dW(t) - \int_{\mathbb{R}_0} \partial_h K(t,z)\widetilde{N}(dt,dz) \\ \partial_h X(T) &= \bigtriangledown_v U(V^{(\pi,c)}(T))\partial_h V(T), \end{cases} \tag{3.3}$$

and

$$\begin{cases} d\partial_h V(t) &= -[\bigtriangledown_c b(t)\Delta c(t) + \bigtriangledown_v b(t)\partial_h V(t)]dt \\ \partial_h V(0) &= 0. \end{cases} \tag{3.4}$$

The linear BSDE (3.3) of $\partial_h X(t)$ can be solved explicitly by introducing the following adjoint process

$$\begin{cases} dG(t) &= G(t^-)[\bigtriangledown_x f(t)dt + \bigtriangledown_y f(t)dW(t) + \int_{\mathbb{R}_0} \bigtriangledown_k f(t)\widetilde{N}(dt,dz)] \\ G(0) &= 1. \end{cases} \tag{3.5}$$

The stochastic representation of $\partial_h X(t)$ is given by

$$\partial_h X(t) = \mathbb{E}_t\Big[G(T)\partial_h X(T) + \int_t^T G(s)\bigtriangledown_c f(s)\Delta c(s)ds\Big|\mathcal{F}_t\Big] \tag{3.6}$$

With this expression, we have that

$$\partial_h L(t,H,c) = \mathbb{E}_t\Big[\{G(T)\bigtriangledown_x U(V(T)) - \lambda R(T)\Gamma^*(T)\}\partial_h V(T) \\ + \int_t^T G(s)\bigtriangledown_c f(s) - \lambda R(s)\Gamma^*(s)\Delta c(s)ds\Big] \tag{3.7}$$

and therefore

$$\partial_h L(0,H,c) = \mathbb{E}\Big[\{G(T)\bigtriangledown_v U(V(T)) - \lambda R(T)\Gamma^*(T)\}\partial_h V(T) \\ + \int_0^T G(s)\bigtriangledown_c f(s) - R(s)\Gamma^*(s)\Delta c(s)ds\Big] \tag{3.8}$$

We can then give the first Theorem giving necessary condition of optimality:

Theorem 3.1. *Suppose the Assumptions A1– A6 are satisfied. If $(H^*,c^*) \in \mathcal{I}^{Q^*}$ is an optimal solution for the optimization problem (3.2), then the following conditions hold:*

$$G(s)\bigtriangledown_c f(s) - \lambda R(s)\Gamma^*(s) = 0, \tag{3.9}$$

$$G(T)\bigtriangledown_v U(V(T)) - \lambda R(T)\Gamma^*(T) = 0. \tag{3.10}$$

Proof. Let (H^*,c^*) be an optimal solution of the optimization problem (3.2) and let (H,c) be another plan. For $\epsilon \in (0,1)$, we have

$$L(0,H^*+\epsilon(H-H^*),c^*+\epsilon(c-c^*)) \le L(0,H^*,c^*).$$

Then

$$\frac{1}{\epsilon}L(0,H^*+\epsilon(H-H^*),c^*+\epsilon(c-c^*)) - L(0,H^*,c^*) \le 0.$$

Taking the limit as ϵ tends to 0, we obtain $\partial_h L(0) \le 0$. The result then follows from [6, Theorem 4.2]. □

We can then derive an explicit expression of an optimal solution for the Problem B if it exists.

Corollary 3.2. *The optimal solution is given by*

$$c^*(t) = (\bigtriangledown_c f)^{-1}\Big(\frac{R(t)\Gamma^*(t)}{G(t)}\Big), \tag{3.11}$$

$$V^*(T) = (\bigtriangledown_v U)^{-1}\Big(\frac{\lambda^* R(T)\Gamma^*(T)}{G(T)}\Big). \tag{3.12}$$

Proof. We note that $\lambda > 0$. We have that

$$\frac{\partial}{\partial x}[G(T)\bigtriangledown_v U(V(T)) - \lambda R(T)\Gamma^*(T)] = G(T)\bigtriangledown_v^2 U(V(T)) \le 0, \tag{3.13}$$

where $\bigtriangledown_v U(v) = U''(x)$ denotes the second partial derivative. If we have a strict inequality, then there exists an optimal wealth V^* such that the value $V^* = V^*(T)$ is the solution to the (3.10).

We can in similar way check for the optimal consumption in (3.9).

The Lagrange multiplier satisfies

$$\mathbb{E}\Big[\int_0^T R(t)\Gamma(t)(\bigtriangledown_c f)^{-1}\Big(\frac{R(t)\Gamma^*(t)}{G(t)}\Big)dt + R(T)\Gamma(T)(\bigtriangledown_v U)^{-1}\Big(\frac{xR(T)\Gamma^*(T)}{G(T)}\Big)\Big] = v. \tag{3.14}$$

□

Let now check that the necessary conditions are also sufficient. Assume that there exists (V^*, c^*) that satisfies the necessary conditions of Theorem 3.1. Let ΔX be the variation of the utility associated with an arbitrary admissible control (V^*, c^*). Put

$$\begin{aligned}
&\Delta X(t) = X(t) - X^*(t);\ \Delta Y(t) = Y(t) - Y^*(t);\\
&\Delta K(t,z) = K(t,z) - K^*(t,z);\ \Delta V(t) = V(t) - V^*(t);\\
&\Delta c(t) = c(t) - c^*(t).
\end{aligned}$$

We have the following proposition.

Proposition 3.3. *We have the following inequalities*

$$\Delta X(t) \leq \partial_h X^*(t), \tag{3.15}$$

$$\Delta V(t) = \partial_h V^*(t), \tag{3.16}$$

P–a.s. for $0 \leq t \leq T$.

Proof. The triplet $(\Delta X, \Delta Y, \Delta K)$ satisfies the following BSDE

$$\begin{cases} d\Delta X(t) &= -\Delta f_B(t, \Delta X(t), \Delta Y(t), \Delta K(t,z))dt + \Delta Y(t)dW(t)\\ &\quad + \int_{\mathbb{R}_0} \Delta K(t,z)\widetilde{N}(dt,dz),\\ \Delta X(T) &= U(V(T)) - U(V^*(T)), \end{cases} \tag{3.17}$$

where

$$\begin{aligned}\Delta f_B(t,x,y,k) =& f(t, c(t), X^*(t) + x, Y^*(t) + y, K^*(t,z) + k)\\ &- f(t, c^*(t), X^*(t), Y^*(t), K^*(t,z)).\end{aligned}$$

Recall also that the triple $(\partial_h X^*, \partial_h Y^*, \partial_h K^*)$ satisfies the following BSDE

$$\begin{cases} d\partial_h X^*(t) &= -\partial_h f(t, \partial_h X^*(t), \partial_h Y^*(t), \partial_h K^*(t,z))dt + \partial_h Y^*(t)dW(t)\\ &\quad + \int_{\mathbb{R}_0} \partial_h K(t,z)\widetilde{N}(dt,dz),\\ \partial_h X^*(T) &= \bigtriangledown_v U(V^*(T)) \cdot \Delta V(T), \end{cases} \tag{3.18}$$

where

$$\partial_h f(t,x,y,k) = \bigtriangledown_c f(t)\Delta c(t) + \bigtriangledown_x f(t)x + \bigtriangledown_y f(t)y + \int_{\mathbb{R}_0} \bigtriangledown_k f(t)k\nu(dz),$$

and $\Delta V(T) = V(T) - V^*(T)$. By concavity argument, we have:

$$U(V(T)) - U(V^*(T)) \leq \bigtriangledown_v U(V^*(T))\Delta V(T). \tag{3.19}$$

$$
\begin{aligned}
&\Delta f_B \\
&= f(t, c(t), X^*(t) + x, Y^*(t) + y, K^*(t,z) + k) - f(t, c(t), X^*(t), Y^*(t) + y, K^*(t,z) + k) \\
&+ f(t, c(t), X^*(t), Y^*(t) + y, K^*(t,z) + k) - f(t, c(t), X^*(t), Y^*(t), K^*(t,z) + k) \\
&+ f(t, c(t), X^*(t), Y^*(t), K^*(t,z) + k) - f(t, c(t), X^*(t), Y^*(t), K^*(t,z)) \\
&+ f(t, c(t), X^*(t), Y^*(t), K^*(t,z)) - f(t, c^*(t), X^*(t), Y^*(t), K^*(t,z)) \\
&\le \nabla_x f(t) \cdot x + \nabla_y f(t) \cdot y + \nabla_c f(t) \cdot \Delta c + \partial_h f(t) \int_{\mathbb{R}_0} K(t,z) \cdot k\nu(dz) \\
&= \partial_h f(t, c, x, y, k).
\end{aligned}
\tag{3.20}
$$

By comparison theorem of BSDE with jumps, see [23, Theorem 2.6] or [24, Theorem 252], we have $\Delta X(t) \le \partial_h X^*(t)$, $\forall t \in [0,T]$, P–a.s.

The linearity of b with respect to V and c implies that $\partial_h V(t)$ and $\Delta V(t) = V(t, c(t)) - V(t, c^*(t))$ satisfy the same FBSDE and with initial condition $\partial_h V(0) = \Delta V(0) = 0$, therefore $\partial_h V(t) = \Delta V(t)$, $\forall t \in [0,T]$, P–a.s.

□

We can then derive the following sufficient conditions for optimality.

Theorem 3.4. *Suppose that the conditions of Theorem 3.1 are satisfied. Let (V^*, c^*) be an admissible strategy. Let (X^*, Y^*, K^*) be the stochastic differential utility of the investor. If* (3.9) *and* (3.10) *are satisfied then (V^*, c^*) is optimal.*

Proof. Let (V, c) be an arbitrary admissible control, with corresponding (X, Y, K). Put

$$\Delta L(0, H, c) = L(0, H, c) - L(0, H^*, c^*).$$

From the Proposition 3.3, we get:

$$\Delta X(t) \le \partial_h X^*(t), \text{ and } \Delta V(t) = \partial_h V(t), \tag{3.21}$$

P–a.s for all $0 \le t \le T$, with $V(T) = H$, $V^*(T) = H^*$. So we have

$$\Delta L(0, H, c) \le \partial_h X(0) - \lambda \mathbb{E}\Big[R(T) \partial_h V^{(\pi,c)}(T) - \int_0^T R(s) \Delta c(s) ds\Big] \tag{3.22}$$

Equation (3.8), (3.9) and (3.10) imply that the right hand side of (3.22) is zero at the optimal point (V^*, c^*) satisfying (3.9) and (3.10). Thus

$$\Delta L(0, H, c) \le 0 \tag{3.23}$$

i.e.,

$$L(0, H, c) \le L(0, H^*, c^*), \ \forall (H, c) \in \mathcal{I}^{Q^*}. \tag{3.24}$$

□

In order to guaranty the unity of the solution of the problem, we assume that.

Assumption A8. There exists $\alpha_0 > 0$ such that

$$\begin{aligned} &\bigtriangledown_c f(t,c,x,y,k) \le C|C|^{-q},\ \forall C \ge \alpha_0,\\ &\bigtriangledown_c b(t,c,v,\pi) \le C_1,\ C_1 > 0,\\ &\bigtriangledown_c h(v) \le C|v|^{-q},\ \forall v \ge \alpha_0, \end{aligned}$$

where $1-p \in]0,1[$.

The following theorem is proved in El Karoui et al. [6] for the continuous case, and the proof in discontinuous case follows in the same way.

Theorem 3.5. *Assume that the Assumptions A1–A8 are satisfied. There exists a unique* $(H^*, c^*) \in \mathcal{I}^Q$ *that attains the maximum of the Problem B.*

4. W–divergences, W–projections and solution of Problem B

In this section, we briefly introduce the notion of W–divergence distances and recall some important results about W–projections. For further information and proofs, the reader is directed to Liese and Vajda [17] and the references therein.

The novelty is that our primary function can consider as a vector valued function. The assumptions on f, U and b guarantee the concavity of the value function and this allows us to applied the results of convex analysis for vector valued functions (see Luenberger [18]). We will also give the solution to the Problem B in term of W–divergence and using the results obtained in Section 3. In fact, we shall show the optimization of the Problem B is reduced to the optimization problem (3.1) where Q^* is W–projection of P on $\mathcal{P}_e$.

Definition 4.1. i) Let $W : (0,\infty) \to \mathbb{R}$ be a convex function. Then the W–divergence between Q and P (or the W–divergence of Q w.r.t. P) is defined as

$$W(Q|P) := \begin{cases} \mathbb{E}_P[W(\frac{dQ}{dP})], & \text{if } Q \ll P,\\ \infty, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $\lim_{x\to 0} W(x) = W(0)$.

ii) A measure $Q^* \in \mathcal{P}$ is called the W–projection of P on $\mathcal{P}$ if

$$W(Q^*|P) = \inf_{Q\in\mathcal{P}} W(Q|P) = W(\mathcal{P}\|P). \tag{4.2}$$

We assume that W is a continuous, strictly convex and differential function. The following result concerning existence and uniqueness of W–projection was proven by Liese [16].

Theorem 4.2. *Assume that* $\mathcal{P}$ *is closed with respect to the variational distance and* $\lim_{x\to\infty} \frac{W(x)}{x} = \infty$. *Then there exists at least one W–projection of P to $\mathcal{P}$.*

If in additional, W is strictly convex and $\inf_{Q\in\mathcal{P}} W(Q|P) < \infty$. *Then there is exactly one W–projection of P on $\mathcal{P}$.*

Remark. Let $W'(0) = -\infty$. If there exists a measure $Q \in \mathcal{P}$ such that $Q \sim P$ and $W(Q|P) < \infty$. Then the W–projection Q^* of P is equivalent to P.

For $Q \in \mathcal{P}_e$ and $v > \bar{v}$, define

$$\begin{aligned}
\tilde{W}_Q(v) = \sup_{(H,c)} \Big\{ \mathbb{E}\Big[\int_0^T f(t, c(t), X(t), Y(t), K(t, \cdot))dt + U(H) \Big], R(T)H \in L^1(Q), \\
\int_0^T R(t)c(t)dt \in L^1(Q), \mathbb{E}_Q[R(T)H + \int_0^T R(s)c(s)ds] \le v \Big\} \\
= \sup_{(H,c)} \Big\{ \mathbb{E}[W_1(H,c)], R(T)H \in L^1(Q), \int_0^T R(t)c(t)dt \in L^1(Q), \\
\mathbb{E}_Q[W_2(H,c)] \le v, \mathbb{E}[W_1(H,c)^-] < \infty \Big\}, \qquad (4.3)
\end{aligned}$$

where

$$\begin{aligned}
W_1(H,c) &= \int_0^T f(t, c(t), X(t), Y(t), K(t, \cdot))dt + U(H), \\
W_2(H,c) &= R(T)H + \int_0^T R(s)c(s)ds.
\end{aligned}$$

From the assumptions on f, U and b, we know that W_1 is a utility function. Let now introduce the convex dual functional W^* define on $\mathbb{R}_+$ by

$$W\Big(\lambda \frac{dQ}{dP}\Big) = W_Q(\lambda) = \sup_{(H,c)} \Big\{ W_1(H,c) - \lambda \frac{dQ}{dP} W_2(H,c) \Big\} \qquad (4.4)$$

We can now give a representation of $\tilde{W}_Q(v)$.

Theorem 4.3. *Let $Q \in \mathcal{P}_e$ and $\mathbb{E}_Q[H_\lambda R_T + \int_0^T R(s)c_\lambda(s)ds] < \infty$, for $\forall \lambda > 0$. Then*

(i) $\tilde{W}_Q(v) = \inf_{\lambda > 0}\{\mathbb{E}[W_Q(\lambda)] + \lambda v\}$.

(ii) *There exists a unique solution, denoted by $\lambda_Q(v) \in (0, \infty)$, of the equation $\mathbb{E}_Q[W_2(H_\lambda^*, c_\lambda^*)] = v$. Furthermore, we also have $\tilde{W}_Q(v) = \mathbb{E}[W_1(H_\lambda^*, c_\lambda^*)]$, where $(H_\lambda^*, c_\lambda^*)$ is optimal claim under pricing measure Q.*

Proof. (i) Let $R(T)H$ and $\int_0^T R(s)c(s)ds \in L^1(Q)$ with $\mathbb{E}_Q[W_2(H,c)] < v$ and $\mathbb{E}[W_1(H,c)] < \infty$. For $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}[W_1(H,c)] &\le \mathbb{E}[W_1(H,c) + \lambda(v - \mathbb{E}_Q[W_2(H,c)]) \\
&\le \mathbb{E}[\sup_{(H,c)} (W_1(H,c) - \lambda \frac{dQ}{dP} W_2(H,c))] + \lambda v \\
&= \mathbb{E}[W_1(H^0, c^0) - \lambda \frac{dQ}{dP} W_2(H^0, c^0)] + \lambda v \qquad (4.5) \\
&= \mathbb{E}[W_Q(\lambda)] + \lambda v, \qquad (4.6)
\end{aligned}$$

It follow from Theorem 3.1, Theorem 3.5 and Corollary 3.2 that the inequality holds as equality if and only if

$$H = H^*_\lambda = V^*(T) = (\bigtriangledown_v U)^{-1}\Big(\lambda \frac{\Gamma^*(T)}{G(T)}\Big); \tag{4.7}$$

$$c = c^*_\lambda = (\bigtriangledown_v f)^{-1}\Big(t, \lambda \frac{\Gamma^*(T)}{G(T)}, X(t), Y(t), K(t,z)\Big). \tag{4.8}$$

In fact, for $Q^* \in \mathcal{P}_e$, since $(H^*_\lambda, c^*_\lambda)$ is maximal value of the problem

$$\sup_{(H,c)} \mathbb{E}[W_1(H,c) - \lambda \frac{dQ^*}{dP} W_2(H,c)],$$

we have

$$\begin{aligned}
&\mathbb{E}\Big[W_1(H^*_\lambda, c^*_\lambda) - \lambda \frac{dQ^*}{dP} W_2(H^*_\lambda, c^*_\lambda)\Big] \\
&= \sup_{(H,c)} \mathbb{E}\Big[W_1(H,c) - \lambda \frac{dQ^*}{dP} W_2(H,c)\Big] \\
&\geq \mathbb{E}\Big[W_1(H,c) - \lambda \frac{dQ^*}{dP} W_2(H,c)\Big], \ \forall (H,c) \in \mathcal{I}^Q,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\mathbb{E}\Big[W_1(H^*_\lambda, c^*_\lambda) - \lambda \frac{dQ^*}{dP} W_2(H^*_\lambda, c^*_\lambda)\Big] \\
&\geq \mathbb{E}\Big[\sup_{(H,c)} \{W_1(H,c) - \lambda \frac{dQ^*}{dP} W_2(H,c)\}\Big] \\
&= \mathbb{E}[W_{Q^*}(\lambda)],
\end{aligned}$$

i.e.,

$$\mathbb{E}\Big[W_1(H^*_\lambda, c^*_\lambda) - \lambda \frac{dQ^*}{dP} W_2(H^*_\lambda, c^*_\lambda)\Big] = \mathbb{E}[W_{Q^*}(\lambda)].$$

(ii) The first statement of (ii) follows from [6, Proposition 5.4]. The equality is obtained using the fact that $W_Q(H^*_{\lambda_Q}, c^*_{\lambda_Q}) = v$.

To check that $\mathbb{E}[W_1(H^*_{\lambda_Q}, c^*_{\lambda_Q})]^- < \infty$, it suffices to see that, from

$$W_1(H,c) - \lambda \frac{dQ}{dP} W_2(H,c) \leq W_1(H^*_{\lambda_Q}, c^*_{\lambda_Q}) - \lambda \frac{dQ}{dP} W_2(H^*_{\lambda_Q}, c^*_{\lambda_Q}),$$

we have

$$\mathbb{E}\Big[W_1(H^*_{\lambda_Q}, c^*_{\lambda_Q}) - \lambda \frac{dQ}{dP} W_2(H^*_{\lambda_Q}, c^*_{\lambda_Q})\Big]^- < \infty,$$

and the rest follows as in [9, Lemma 4.1]. □

Remark. (i) $(H^*_{\lambda_Q}, c^*_{\lambda_Q})$ can be interpreted as optimal claim which is fanciable under pricing measure Q.

(ii) If for $Q \in \mathcal{P}_e$, there exists $\lambda > 0$, with $\mathbb{E}[W_Q(\lambda)] < \infty$, then $\tilde{W}_Q(v) < \infty$ for all $v > \bar{v}$. Moreover, if for $Q \in \mathcal{P}_e$ with $\tilde{W}_Q(v) < \infty$, the assumptions of Theorem 4.3 are fulfilled and $\mathbb{E}[W_Q(\lambda)] < \infty$.

Definition 4.4. A measure $Q^* = Q(v) \in \mathcal{P}_e$ is called minimax measure for v and $\mathcal{P}_e$ if it minimizes $Q \mapsto \tilde{W}_Q(v)$ over all $Q \in \mathcal{P}_e$, i.e.,

$$\tilde{W}(v) = \tilde{W}_{Q^*}(v) = \inf_{Q \in \mathcal{P}_e} \tilde{W}_Q(v).$$

Denote by $\partial \tilde{W}(v)$ the sub–differential of the function $\tilde{W}$ at v . Let $W^*(v) = W_{\lambda_0}(v)$, then the corresponding W–divergence is $W_\lambda(\cdot|\cdot)$. The following result, similar to [9, Proposition 4.3], gives a dual characterization of our problem.

Before we state on of the main theorem of this section we need the following assumption.

Assumption A8 There exists $v > \bar{v}$ with $\tilde{W}(v) < \infty$,

$$\mathbb{E}_Q[W_2(H^*_\lambda, c^*_\lambda)] < \infty, \ \forall \lambda > 0, \forall Q \in \mathcal{P}_e,$$

where H^*_λ and c^*_λ are given by (3.11) and (3.12) (with $Q^* = Q$).

Theorem 4.5. *Let $v > \bar{v}$, $\lambda_0(v) \in \partial \tilde{W}(v)$ and $\lambda_0(v) > 0$. Then*

(i) $\tilde{W}_{Q^*}(x) = \inf_{Q \in \mathcal{P}_e} \tilde{W}_Q(v) = \tilde{W}(\mathcal{P}_e \| P) = W_{\lambda_0}(Q^*|P) + \lambda_0(v).v$

(ii) *If $Q^* \in \mathcal{P}_e$ is an W_{λ_0}–projection of P on $\mathcal{P}$, then Q^* is a minimax measure and $\lambda_0(v) = \lambda_{Q^*}(v)$.*

(iii) *If $Q^* \in \mathcal{P}_e$ is a minimax measure, then Q^* is an W_{λ_0}–projection of P on $\mathcal{P}$, $\lambda_{Q^*}(v) \in \partial \tilde{W}(v)$ and we have*

$$\tilde{W}_{Q^*}(v) = \inf_{Q \in \mathcal{P}_e} \tilde{W}_Q(v) = \sup_{(H,c)} \{\mathbb{E}[W_1(H,c)], \sup_{Q \in \mathcal{P}_e(v)} \mathbb{E}_Q[W_2(H,c)] \le v\} \tag{4.9}$$

where $\mathcal{P}_e(v) = \{Q \in \mathcal{P}_e : \tilde{W}_Q(v) < \infty\}$

Proof. (i) We have from the preceding theorem that

$$\begin{aligned}\tilde{W}_{Q^*}(v) &= \inf_{Q \in \mathcal{P}} \inf_{\lambda > 0} \Big\{\mathbb{E}\Big[W\Big(\lambda \frac{dQ}{dP}\Big)\Big] + \lambda v\Big\} \\ &= \inf_{\lambda > 0} \{W_\lambda(Q^*|P) + \lambda v\}\end{aligned}$$

Define $\phi : (0, \infty) \to \mathbb{R} \cup \{\infty\}$ by

$$\phi(\lambda) := W_\lambda(Q|P).$$

We want to show that ϕ is a convex function. To this end, let $\epsilon > 0$ and Q_1, $Q_2 \in \mathcal{P}$, such that

$$\begin{aligned}\phi(\lambda_1) + \epsilon &\ge \mathbb{E}\Big[W\Big(\lambda_1 \frac{dQ_1}{dP}\Big)\Big], \\ \phi(\lambda_2) + \epsilon &\ge \mathbb{E}\Big[W\Big(\lambda_2 \frac{dQ_2}{dP}\Big)\Big].\end{aligned}$$

Hence,

$$\begin{aligned}\alpha\phi(\lambda_1) + (1-\alpha)\phi(\lambda_2) + \epsilon &\geq \alpha\mathbb{E}\Big[W\Big(\lambda_1\frac{dQ_1}{dP}\Big)\Big] + (1-\alpha)\mathbb{E}\Big[W\Big(\lambda_2\frac{dQ_2}{dP}\Big)\Big] \\ &\geq \mathbb{E}\Big[W\Big(\alpha\lambda_1\frac{dQ_1}{dP} + (1-\alpha)\lambda_2\frac{dQ_2}{dP}\Big)\Big] \\ &\geq \inf_{Q\in\mathcal{P}_e}\Big\{W\Big(\{\alpha\lambda_1 + (1-\alpha)\lambda_2\}\frac{dQ}{dP}\Big)\Big\} \\ &= \phi(\alpha\lambda_1 + (1-\alpha)\lambda_2),\end{aligned}$$

for $\alpha \in (0,1)$. In fact for the third inequality, it suffices to see that

$$\frac{\alpha\lambda_1}{\alpha\lambda_1 + (1-\alpha)\lambda_2}Q_1 + \frac{(1-\alpha)\lambda_2}{\alpha\lambda_1 + (1-\alpha)\lambda_2}Q_2 = \tilde{Q} \in \mathcal{P}_e.$$

The first statement of the theorem will then follow from some results due to Rockafellar [22].

By [22, Theorem 23.5], $\inf_{\lambda>0}\{\phi(\lambda) + \lambda v\}$ attains its infimum at $\lambda = \lambda_0(v)$ if and only if $-v \in \partial\phi(\lambda_0(v))$. Using [22, Theorem 7.4 and Corollary 23.5.1], this is equivalent to $\lambda_0(v) \in \partial\tilde{W}(v)$.

(ii) This follows from Theorem 4.3.

(iii) The first statement follows from Theorem 4.3. Assume that $\mathbb{E}_Q[W_2(H_{\lambda_Q}, c_{\lambda_Q})] < \infty$, $\forall\lambda > 0$, $\forall Q \in \mathcal{P}$, holds with $(H^*_\lambda, c^*_\lambda)$ given by (4.7) and (4.8), then the same reasoning as in Göll and Rüschendorf [9] leads to

$$\{Q \in \mathcal{P} : \tilde{W}_Q(v) < \infty\} = \{Q \in \mathcal{P} : W_{\lambda_Q(v)}(Q|P) < \infty\},$$

and the equation follows from Theorem 4.3 and Proposition below. □

Proposition 4.6. *Let $Q^* \in \mathcal{P}_e$ satisfy $W(Q^*|P) < \infty$. Then Q^* is the W–projection of P on Q if and only if*

$$\int W'\Big(\frac{dQ^*}{dP}\Big)(dQ^* - dQ) \leq 0, \tag{4.10}$$

for all $Q \in \mathcal{P}$ with $W(Q|P) < \infty$.

Corollary 4.7. *Assume that the hypotheses of Theorem 4.5 are in force. Assume that $\tilde{W}$ is differential in v. Then Q^* is a minimax measure if and only if Q^* is the W_{λ_0}–projection, where $\lambda_0 = \bigtriangledown\tilde{W}(v)$.*

Proposition 4.8. *Assume that $\bar{v} = 0$ and W_1 is bounded from above. Then $\tilde{W}$ is differentiable in every $v > 0$.*

Proof. The proof follows using the same argument as in Göll and Rüschendorf [9], for the sake of completeness, we give the details. We will show that the function $G(\lambda) = W_{\lambda_0}(Q^*|P) = W_{\lambda_0}(Q\|P)$ is strictly convex and the result will follow by applying [22, Theorem 26.3].

For any $\lambda > 0$, let $(Q^n)_{n\geq 0} \in \mathcal{P}_e$ be a such that $W_\lambda(Q^n\|P)$ converges a.s to the infimum of the value $W_\lambda(Q\|P)$ over $Q \in \mathcal{P}_e$. (The existence of such sequence follows from the convexity of the $\mathcal{P}_e$ and [15, Lemma 3.3]).

Since the set $\mathcal{K}_{\mathcal{P}_e}$ are weakly compact, the sequence $\left(\frac{dQ_n}{dP}\right)$ has a cluster point $\left(\frac{d\bar{Q}}{dP}\right) \in \mathcal{K}_{\mathcal{P}_e}$.

The sequence $W_\lambda(Q_n|P)^-$ is uniformly integrable. In fact, this follows from properties of the "convex conjugate", the de la Vallée–Poussin theorem and [15, Proposition 3.1]. We then have

$$\lim_{n\to\infty} W_\lambda(Q_n|P)^- = \lim W_\lambda(\bar{Q}|P)^-.$$

Since W_1 is bounded from above, it follows that W_λ is bounded from above. It then follows from the dominated convergence theorem that

$$\lim_{n\to\infty} W_\lambda(Q_n|P) = W_\lambda(\bar{Q}|P),$$

and hence

$$\inf_{Q\in\mathcal{K}_{\mathcal{P}_e}} W_\lambda(Q|P) = W_\lambda(\bar{Q}|P).$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}_+$, $\gamma \in (0,1)$. There are $\bar{Q}_1, \bar{Q}_2 \in \mathcal{P}_e$ with

$$G(\lambda_1) = W_\lambda(\bar{Q}_1|Q) = \mathbb{E}\Big[W\Big(\lambda_1\frac{d\bar{Q}_1}{dP}\Big)\Big], \text{ and } G(\lambda_2) = \mathbb{E}\Big[W\Big(\lambda_2\frac{d\bar{Q}_2}{dP}\Big)\Big].$$

Therefore,

$$\begin{aligned}\gamma G(\lambda_1) + (1-\gamma)G(\lambda_2) &= \gamma_1\mathbb{E}\Big[W\Big(\lambda_1\frac{d\bar{Q}_1}{dP}\Big)\Big] + (1-\gamma_2)\mathbb{E}\Big[W\Big(\lambda_2\frac{d\bar{Q}_2}{dP}\Big)\Big]\\ &> \mathbb{E}\Big[W\Big(\gamma\lambda_1\frac{d\bar{Q}_1}{dP} + (1-\gamma)\lambda_2\frac{d\bar{Q}_2}{dP}\Big)\Big]\\ &\geq \inf_{Q\in\mathcal{P}_e}\mathbb{E}\Big[W((\gamma\lambda_1 + (1-\gamma)\lambda_2)\frac{d\bar{Q}}{dP})\Big]\\ &= G(\gamma\lambda_1 + (1-\gamma)\lambda_2).\end{aligned}$$

The strict inequality follows from the fact that W is strictly convex and the inequality follows since the set $\mathcal{P}_e$ is convex. Therefore G is strictly convex and the proof is complete. □

As in [9, Proposition 4.7], it is possible to give a way to determine the Lagrange multiplier $\lambda_0 \in \partial\tilde{W}(v)$ and hence the W–divergence distance related to the minimax measure.

Define

$$\begin{aligned}\mathbb{G} = \{\varphi\cdot S(T) :&\varphi^i = H^i\cdot\mathbf{I}_{[s_i,s_{i+1}]}, H^i \text{ bounded }, \mathcal{F}_{s_i}\text{–measurable}\}\\ &\cup\{\mathbf{I}_B : P(B) = 0\}.\end{aligned}$$

Theorem 4.9. *Let $Q^* \in \mathcal{P}_e$, $\lambda > 0$, with $W_\lambda(Q^*|P) < \infty$, such that for a S–integrable process φ,*

$$W_2(H^*_\lambda, c^*_\lambda) = v + \int_0^T \varphi(t)dS(t) \qquad P\text{–a.s}, \tag{4.11}$$

and

$$W'_{Q^*}(\lambda) = W'(\lambda\frac{dQ^*}{dP}) = c + \int_0^T \varphi(t)dS(t) \tag{4.12}$$

P–a.s, for $\int_0^T \varphi(t)dS(t) \in \mathbb{G}$. Then Q^* is the minimax measure for v and $\lambda \in \partial U(v)$.

Proof. Since $\mathbb{E}_{Q^*}[W_2(H^*_\lambda, c^*_\lambda)] = v$, it follows from Theorem 4.3 that $\lambda = \lambda_{Q^*}(v)$. The condition (4.12) guaranties that Q^* is W_λ–projection of P on $\mathcal{P}_e$, (see [9, Proposition 3.3]). Hence by Proposition 4.6 we have that for all measure $Q \in \mathcal{P}_e$ such that $W_\lambda(Q|P) < \infty$, one gets $\mathbb{E}_Q[W_2(H^*_\lambda, c^*_\lambda)] \leq v$ and one can conclude that $\tilde{W}_{Q^*}(v) = \mathbb{E}[W_1(H^*_\lambda, c^*_\lambda)] \leq \tilde{W}_Q(v)$.

From Assumption A8 and (ii) of Remark after Theorem 4.3, one has,

$$\{Q \in \mathcal{P}_e; W_\lambda(Q|P) < \infty\} = \{Q \in \mathcal{P}_e; \tilde{W}_Q(v) < \infty\},$$

and whence Q^* is a minimax measure for v and $\mathcal{P}_e$. It follows from Theorem 4.5 that $\lambda = \lambda_{Q^*}(v) \in \partial U(v)$. □

Finally, we derive a Theorem characterizing the optimal solution of the Problem B in incomplete market.

Theorem 4.10. *Assume that the Assumptions A1–A8 are satisfy and $\bar{v} > -\infty$. Moreover, suppose that the W_λ–projection Q^* of P on $\mathcal{P}_e$ exists. Then*

(i) *The utility optimization Problem B has the solution $(H^*_\lambda, c^*_\lambda)$ given by*

$$H^*_\lambda = (\bigtriangledown_v U)^{-1}\Big(\lambda\frac{dQ^*}{dP}\frac{1}{G(T)}\Big) \tag{4.13}$$

and

$$c^*_\lambda = (\bigtriangledown_v f)^{-1}\Big(t, \lambda\frac{dQ^*}{dP}\frac{1}{G(T)}, X(t), Y(t), K(t,z)\Big) \tag{4.14}$$

This solution is unique P–a.s.

(ii) *The maximal value of the utility is given by*

$$X^*(H^*_\lambda, c^*_\lambda) = W_{\lambda_0}(Q^*|P) + \lambda_0 \cdot v.$$

(iii) *The contingent claim H^*_λ and the consumption c^*_λ given by (4.13) and (4.14) have the following properties*

$$H^*_\lambda + \int_0^T c^*_\lambda(t)dt \in L^1(Q) \text{ for all } Q \in \mathcal{P}_e$$

$$U(H^*_\lambda) + \int_0^T f(t, c(t), X(t), Y(t), K(t,\cdot))dt \in L^1(Q)$$

and

$$\mathbb{E}_{Q^*}\Big[H^*_\lambda + \int_0^T c^*_\lambda(t)dt\Big] = \max_{Q\in\mathcal{P}_e} \mathbb{E}_Q\Big[H_\lambda + \int_0^T c_\lambda(t)dt\Big]$$

Proof. For $H \geq \bar{v}$ and $c = (c(t))_{t\geq 0}$ satisfying the constraint of the Problem B, we have from Theorem 4.3 that for Q^* and $\lambda > 0$

$$\begin{aligned}\mathbb{E}[W_1(H,c)] &\leq \mathbb{E}[W_1(H,c)] - \lambda(v - \mathbb{E}_{Q^*}[W_2(H,c)]) \\ &\leq \mathbb{E}[\sup_{(H,c)}\{W_1(H,c) - \lambda\frac{dQ^*}{dP}W_2(H,c)\}] + \lambda v \\ &= \mathbb{E}[W_{Q^*}(\lambda)] + \lambda v\end{aligned}$$

i.e.,

$$\mathbb{E}[W_1(H,c)] \leq \inf_{\lambda>0}(\mathbb{E}[W_{Q^*}(\lambda)] + \lambda v)$$

Noting that the function $\lambda \mapsto \mathbb{E}[W_{Q^*}(\lambda)] + \lambda v$ attains its minimum at $\lambda = \lambda_{Q^*} = \lambda_{Q^*}(v)$, we have from Theorem 3.1, Theorem 3.4 and Theorem 4.3 that $\mathbb{E}[W_2(H^*_{\lambda_{Q^*}}, c^*_{\lambda_{Q^*}})] = v$ and this gives

$$\begin{aligned}\mathbb{E}[W_1(H,c)] &\leq \mathbb{E}[W_{Q^*}(\lambda_{Q^*}(v)) + \lambda_{Q^*}(v)\cdot v] \\ &= \mathbb{E}[W_1(H^*_{\lambda_{Q^*}}, c^*_{\lambda_{Q^*}})].\end{aligned}$$

The last equation shows that $\mathbb{E}_{Q^*}[W_2(H^*_{\lambda_{Q^*}}, c^*_{\lambda_{Q^*}})] = \sup_{Q\in\mathcal{P}_e}\mathbb{E}[W_2(H_\lambda, c_\lambda)]$. This concludes the proof that $(H^*_{\lambda_{Q^*}}, c^*_{\lambda_{Q^*}})$ is optimal and $X^*(H^*_\lambda, c^*_\lambda) = W_{\lambda_0}(Q^*|P) + \lambda_{Q^*}\cdot v$.

Now we proof the uniqueness of the solution. Assume that $\tilde{H} > v$ and $\tilde{c} > 0$ solves the Problem B. Then $\mathbb{E}_{Q^*}[W_2(\tilde{H},\tilde{c})] \leq v$ and hence $\mathbb{E}[W_1(\tilde{H},\tilde{c})] \geq \mathbb{E}[W_1(H^*_\lambda, c^*_\lambda)]$. This inequality holds strictly unless $\tilde{H} = H^*_\lambda$ and $\tilde{c} = c^*_\lambda$. This follows from the fact that $(H^*_\lambda, c^*_\lambda)$ maximizes $\mathbb{E}[W_1(H,c)]$ under constraint $\mathbb{E}_{Q^*}[W_2(H,c) \leq v$ and from the uniqueness Theorem 3.5. But a strict inequality is also a contradiction to the fact that $(H^*_\lambda, c^*_\lambda)$ is optimizer and thus $H^*_\lambda = \tilde{H}$ and $c^*_\lambda = \tilde{c}$.

The part (iii) of the theorem follows from Proposition 4.6. In fact, let $(H^0_{\lambda_{Q^*}}, c^0_{\lambda_{Q^*}}) \in \mathcal{I}^Q$ such that

$$\begin{aligned}W_1(H^0_{\lambda_{Q^*}}, c^0_{\lambda_{Q^*}}) - \lambda\frac{dQ^*}{dP}W_2(H^0_{\lambda_{Q^*}}, c^0_{\lambda_{Q^*}}) &= \sup_{(H,c)}\{W_1(H,c) - \lambda\frac{dQ^*}{dP}W_2(H,c)\} \\ &= W_{Q^*}(\lambda).\end{aligned}$$

Then

$$W'_{Q^*}(\lambda) = W'(\lambda\frac{dQ^*}{dP}) = -W_2(H^0_{\lambda_{Q^*}}, c^0_{\lambda_{Q^*}}).$$

Applying Proposition 4.6, we get the result by the uniqueness of Theorem 3.5 and Theorem 4.3(i). □

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