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Mathematics and Finance: The Black-Scholes Option Pricing Formula and Beyond

Bernt Øksendal[1,2]

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1 Introduction

There was a time when finance was completely without interest from a mathematical point of view. The mathematical content in finance was – at best – elementary and uninteresting. Today the situation is completely different. All companies which are dealing with finance on a large scale are using advanced mathematical methods. Financial experts are studying mathematics and mathematics researchers are studying finance. Almost every university now has a special program on mathematical finance.

There are several reasons for this new situation. The main reason is the construction and development of *stochastic analysis*: About 60 years ago mathematicians started to combine classical mathematical analysis (integrals, derivatives …) with modern probability theory, developed by Kolmogorov in the 1930's. N. Wiener gave a rigorous construction of Brownian motion (the wiener process) and P. Lévy explored many essential features of this and other stochastic processes. K. Itô constructed the *stochastic integral*, later coined the Itô integral, and started seminal research about the properties of this and related concepts. J. Doob introduced and studied the concept of *martingales*, and together with P.-A. Meyer and others they founded the modern theory of semimartingales. In the first 20 years this research was purely mathematical. Then around 1970 it was disovered by H.P. McKean, P. Samuelsen and others that this new mathematical theory of stochastic analysis could be useful in finance. The final breakthrough came in 1973 when M. Scholes and F. Black published their celebrated *option pricing*

[1]Centre of Mathematics for Applications (CMA), Dept of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N–0316 Oslo, Norway. E-mail: oksendal@math.uio.no

[2]Norwegian School of Economics and Business Administration (NHH), Helleveien 30, N–5045 Bergen, Norway.

formula. This theoretical price formula was based on advanced stochastic analysis, and agreed well with the price that had been established (by trial and failure) through trading on the option market, which had existed for some years already. In 1997 M. Scholes, together with R. Merton who also played an essential role in the option pricing formula and in addition made other fundamental contributions, were awarded the Nobel Prize in Economics for their achievements. (F. Black died in 1995.)

After the Black-Scholes formula was published there has been an enormous research activity within mathematical finance, and it shows no sign of slowing down. We will not attempt to give a comprehensive account of this activity here. But we will try to illustrate the interplay between mathematics and finance by looking at some themes in more detail.

In Section 2 we consider the simplest possible financial market with one risky asset and only two possible scenarios. We show that even in this simple case the option pricing question is nontrivial and requires a subtle equilibrium argument.

In Section 3 we extend the model to the multi-period case.

In Section 4 we explain the more realistic *time-continuous*, Brownian motion based market model setting of the Black-Scholes formula. Even this model is highly stylized compared to real financial markets, but nevertheless it catches some essential aspects of pricing of European options and related issues.

However, as the current financial crisis shows, the established mathematical models, albeit highly advanced, are still inadequate for a satisfactory understanding and handling of real-life financial markets. In particular, it has been pointed out that more emphasis should be put on the possibility of *discontinuities* or *jumps* ("cracks") in the market. There is a tractable mathematical machinery for handling this, namely the stochastic calculus driven by general *Lévy processes*, not just Brownian motion. This leads to models where stock prices may have jumps, which is more realistic than continuous models. On the other hand, such models are mathematically challenging. In Section 5 we discuss this more.

Finally, in Sections 6–8 we present other recent developments which represent research frontiers in mathematical finance today.

2 The Black-Scholes option pricing formula

Consider the following 1-period financial market with two investment possibilities:

(i) We can buy *risk free assets* (e.g. *bonds*) with a fixed interest rate $r \geq 0$. For simplicity we here assume that $r = 0$.

(ii) We can buy *risky assets* (e.g. *stocks*). Let us denote the price of one stock at time t by $S(t)$, where $t = 0$ or $t = T > 0$. Assume that $S(0) = 100$ units, e.g. Danish Crowns (DKK). The price $S(T)$ at the future time T is uncertain at time $t = 0$. We assume that there are only two possible scenarios:

Scenario 1: The price goes *up* to *DKK 115* at time T. We assume that the probability p that this occurs is $\frac{1}{2}$. In other words, P (Scenario 1) $= p = \frac{1}{2}$, where P stands for "probability".

Scenario 2: The price goes *down* to *DKK 95* at time T. The probability $1 - p$ that this occurs is also $\frac{1}{2}$. So we have P (Scenario 2) $= 1 - p = \frac{1}{2}$.

A *European call option* in this market is a contract which gives the buyer of the contract the right – but not the obligation – to buy one stock at the specified future time T and at a specified price K, usually called the *exercise price*. In this example we assume that $K =$ DKK 105. See Figure 1.

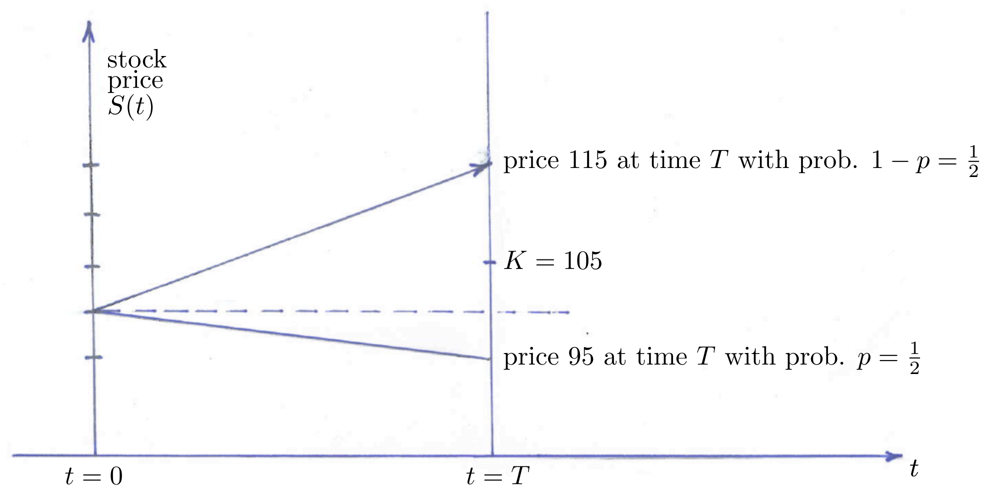

Figure 1

The question is:

What is the "right" price to pay for such a contract/option at time 0?

The answer depends of course on what we mean by "right" price. Some people will say that the right price should be the *expected payoff* at time T. So let us compute this:

Scenario 1: If the price goes up to DKK 115, then the buyer of the option can buy one stock for DKK 105, sell it again for DKK 115 and thus get a payoff of DKK$(115 - 105) =$ DKK 10. This happens with probability $p = \frac{1}{2}$.

Scenario 2: If the price goes down to DKK 95, then the buyer will not exercise the option and the payoff is 0. This also happens with probability $\frac{1}{2}$ $(= 1 - p)$.

We conclude that the *expected payoff* (wth respect to the probability law P) for the buyer is

$$E_P[\text{payoff}] = 10 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{2} = \underline{\underline{5}} \text{ (DKK).} \tag{2.1}$$

(E_P denotes expectation with respect to P). Is this the right price to pay for the option at time 0? Perhaps surprisingly, the answer is *no*, if "right" price is interpreted in an *equilibrium sense*. By this we mean the following:

An *arbitrage* in this market is an investment policy at time 0 which at time T gives a (strictly) positive profit with a (strictly) positive probability and a (strictly negative profit with probability 0. Thus an arbitrage is a kind of "money machine", also called a "free lunch". There is no chance for a loss, and a positive chance for a positive profit. It is a basic equilibrium criterion for a financial market that *arbitrages cannot exist.* If a market had an arbitrage, then everybody would use it and the market would collapse. In view of this, we choose to define the "right" price of an option as the price which does not lead to an arbitrage for buyer or seller.

We claim that *the expected payoff price DKK 5 found earlier gives an arbitrage opportunity to the seller of the option.* Here is how:

If the seller receives DKK 5 at time 0 for the option, she can borrow DKK 95 in the bank and use the total amount, DKK 100, to buy one stock. This stock she keeps till time T and then she sells it. There are now to possibilities:

In Scenario 1 she receives DKK 115 for the stock. With this amount she can pay back the loan to the bank (DKK 95) and she can pay the buyer of the option the promised payoff, DKK 10. This leaves her with a *profit of DKK 10.*

In Scenario 2 she receives DKK 95 for the stock. This is exactly enough to pay back the bank. In this scenario there is nothing to pay to the owner of the option. Thus in this case the *profit (and the loss) is 0.*

We see that with this strategy the seller cannot lose money, and there is a positive probability for a positive profit. Hence paying DKK 5 for the option leads to an *arbitrage for the seller.*

We conclude that, by such an equilibrium requirement, the price DKK 5 is too high.

What, then, is the non-arbitrage price of this option?

A fundamental part of the Black-Scholes option pricing formula states that the non-arbitrage price is given by the expected (and, in general, discounted, but here we have assumed $r = 0$) payoff *with respect to the risk neutral probability measure* Q, not with respect to P. Thus, in our case,

$$\text{price}_{\text{BS}} = E_Q[\text{payoff}] = 10 \cdot q + 0 \cdot (1 - q), \tag{2.2}$$

where $q = Q$ (Scenario 1), i.e. the Q-probability that Scenario 1 occurs.

How do we find this risk neutral probability measure Q?

According to Black-Scholes the measure Q is characterized by the property that the (discounted) stock price is a *martingale* with respect to it. In our setting this simply means that

$$E_Q[S(T)] = S(0), \tag{2.3}$$

where $S(t)$ is the stock price at time $t = 0, T$. This gives the equation

$$115 \cdot q + 95 \cdot (1 - q) = 100,$$

from which we get $q = \frac{1}{4}$. Therefore, according to (2.2) the right price for this option is

$$\text{price}_{\text{BS}} = 10 \cdot \tfrac{1}{4} + 0 \cdot \tfrac{3}{4} = \underline{\underline{2.50}} \text{ (DKK)}. \tag{2.4}$$

More generally, if the interest rate in the bank is $r \geq 0$ and the exercise price at time T is $K > 0$, then the Black-Scholes option pricing formula states that the arbitrage free price for the option is

$$\text{price}_{\text{BS}} = E_Q\big[e^{-rT}(S(T) - K)^+\big], \tag{2.5}$$

where

$$(S(T) - K)^+ = \max\{S(T) - K, 0\}$$

and Q is the *risk neutral* probability measure, characterized by the property that the *discounted* stock price, $e^{-rt}S(t)$, is a *martingale* with respect to Q. In our 1-period market this simply means that

$$E_Q\big[e^{-rT}S(T)\big] = S(0). \tag{2.6}$$

The above example is too simple to be realistic, but nevertheless we have seen that it contains several essential features of real life financial markets. As another illustration of this, let us consider the more general situation where the probability p of Scenario 1 is not $\frac{1}{2}$, but some unknown number between 0 and 1. What can we say about the option price then? Note that *the risk neutral measure* Q *defined by equation (2.6) does not depend on* p. Therefore q is still $\frac{1}{4}$ and formula (2.5) gives *the same price 2.50 DKK. This shows that to decide the option price at* $t = 0$ *it is not necessary to know the probability* p *of Scenario 1.* This result is a useful (and perhaps surprising) consequence of the model. It turns out to remain true in the more elaborate (and realistic) models discussed in the next sections.

3 Multi-period models

A natural first extension of the model in Section 2 is the multi-period model, where trading takes place at specified times t_i, $0 \le i \le N-1$, where

$$0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots < t_N = T.$$

At each trading time t_i the agent has to decide how many stocks, say $\varphi_1(t_i)$, to keep and how many bonds, say $\varphi_0(t_i)$ to keep. However, such a choice cannot be made arbitrarily and freely. It is necessary to put constraints of such a *trading strategy* (or *portfolio*) $\varphi(t) = (\varphi_0(t), \varphi_1(t))$.

(i) First of all, it must be *self-financing*, in the sense that if we decide to, say, buy stocks at time t_i, then we must borrow the corresponding amount in the bank. The precise mathematical way of expressing this is the following: Let

$$V(t) = \varphi_0(t)S_0(t) + \varphi_1(t)S_1(t), \tag{3.1}$$

be the *value* of the portfolio at time t, where $S_0(t)$ and $S(t)$ are the unit prices of the risk free and risky asset, respectively. Then the increase

$$\Delta V(t_i) = V(t_{i+1}) - V(t_i)$$

of the value right after transaction has taken place at time t_i should be coming from the increase of prices only, i.e. we should have

$$\Delta V(t_i) = \varphi_0(t_i)\Delta S_0(t_i) + \varphi_1(t_i)\Delta S_1(t_i) \tag{3.2}$$

where

$$\Delta S_k(t_i) = S_k(t_{i+1}) - S_k(t_i); \qquad k = 0, 1,\ i = 0, \ldots, N-1.$$

Condition (3.2) is called the *self-financing condition.* It is expressing mathematically that no money is coming into the system or going out of the system.

(ii) Second, the portfolio decision $\varphi(t_i)$ at time t_i must be based on the observed prices up to and including that time, and not on any future asset prices. Mathematically this is expressed by requiring the portfolio choice $\varphi(t_i)$ (as a random variable) to be *measurable with respect to the σ-algebra* $\mathcal{F}_{t_i}$ generated by *the previous asset prices* $S_0(s), S_1(s)$; $0 \le s \le t_i$.

If we assume, as in Section 2, that

$$S_0(t) = e^{rt} \qquad (r \ge 0 \text{ constant}), \tag{3.3}$$

then the *martingale condition* corresponding to (2.6) for a *risk neutral measure* Q becomes

$$E_Q\big[e^{-rt_{i+1}}S_1(t_{i+1})|\mathcal{F}_{t_i}\big] = e^{-rt_i}S_1(t_i); \qquad i = 0, 1, \ldots, N-1 \tag{3.4}$$

where $E_Q[\cdot|\mathcal{F}_{t_i}]$ denotes conditional expectation with respect to the σ-algebra $\mathcal{F}_{t_i}$.

An *arbitrage* in this market is a portfolio $\varphi(t)$ satisfying (i) and (ii) and such that the corresponding value process

$$V^\varphi(t) = \varphi(t) \cdot S(t) = \varphi_0(t)S_0(t) + \varphi_1(t)S_1(t)$$

satisfies

$$V^\varphi(0) = 0, \quad V^\varphi(T) \geq 0 \quad \text{a.s.} \quad \text{and} \quad P[V^\varphi(T) > 0] > 0, \tag{3.5}$$

where, as before, P denote s probability and a.s. means "almost surely", i.e. with probability 1. This is in agreement with the arbitrage concept we discussed in Section 2.

One can now prove that such a market is arbitrage free if and only if there exists (at least one) risk neutral measure Q. This result is sometimes called *the first fundamental theorem of asset pricing*. See e.g. [S].

If such a risk neutral measure Q exists, then the price

$$\text{price}_{\text{BS}} := E_Q[e^{-rT}(S_1(T) - K)^+] \tag{3.6}$$

will be an arbitrage free option price of the corresponding European call option.

This multi-period market is called *complete* if for every $\mathcal{F}_T$-measurable random variable F there exists an initial wealth $x \in \mathbb{R}$ and a portfolio $\varphi(t)$ satisfying (i) and (ii) such that

$$F = V_x^\varphi = x + \sum_{i=0}^{N-1} \varphi(t_i) \cdot \Delta S(t_i) \quad \text{a.s.} \tag{3.7}$$

In other words, we should be able to reproduce (*replicate*) any given terminal "payoff" F by choosing the initial wealth x (constant) and the portfolio φ suitably. The *second fundamental theorem of asset pricing* states that *a given arbitrage-free market is complete if and only if there is only one risk neutral measure Q.*

If this is the case there is *only one* arbitrage-free price price_{BS}, namely the one given by (3.6). See e.g. [S].

4 Time-continuous models

The next step in the progression towards more realistic mathematical financial models is to introduce time-continuous markets, where asset prices change all the time (not just at prescribed discrete times t_i) and trading is allowed to take place continuously in $[0,T]$. In this setting the most basic model for the stock price $S(t)$ at time t is the equation

$$\frac{dS_1(t)}{dt} = S_1(t)[\alpha + \sigma \text{ ''noise''}]; \qquad S_1(0) > 0. \tag{4.1}$$

where α and $\sigma \neq 0$ are constants and "noise" represents the uncertainty of the price dynamics. If "noise" is interpreted as "white noise", then in a weak sense we have

$$\text{''noise''} = \frac{\text{''}dB(t)\text{''}}{dt} \tag{4.2}$$

where $B(t)$ is Brownian motion (the Wiener process) at time t. The rigorous interpretation of (4.1) is then that $S_1(t)$ satisfies the stochastic integral equation

$$S_1(t) = S_1(0) + \int_0^t \alpha S_1(s)ds + \int_0^t \sigma S_1(s)dB(s), \tag{4.3}$$

or – in differential form (shorthand notation) –

$$dS_1(t) = \alpha S_1(t)dt + \sigma S_1(t)dB(t); \qquad S_1(0) > 0. \tag{4.4}$$

The last integral on the right hand side of (4.3) is the famous *Itô inegral* mentioned earlier.

Using the *Itô formula*, which is a stochastic chain rule, one can prove that the solution of (4.3) is

$$S_1(t) = S_1(0)\exp((\alpha - \tfrac{1}{2}\sigma^2)t + \sigma B(t)); \qquad t \geq 0. \tag{4.5}$$

(See e.g. [Ø].)

The market $(S_0(t), S_1(t))$ with $S_0(t) = e^{rt}$ and $S_1(t)$ given by (4.5) is called the *Black-Scholes market*, because this was the market in which Black and Scholes proved their option pricing formula [BS]. Basically one can now transform the argument and formulas of the previous sections to this situation and obtain analogous results.

For example, the *value process* $V^\varphi(t)$ corresponding to a portfolio φ is defined by

$$V^\varphi(t) = \varphi(t) \cdot S(t); \qquad t \in [0,T]. \tag{4.6}$$

The portfolio is called *self-financing* if

$$dV^\varphi(t) = \varphi(t) \cdot dS(t). \tag{4.7}$$

A probability measure Q is called *risk neutral* if the discounted price process $e^{-rt}S_1(t)$ is a *Q-martingale*, i.e.

$$E_Q[e^{-rs}S_1(s)|\mathcal{F}_t] = e^{-rt}S_1(t) \quad \text{for all } s \geq t. \tag{4.8}$$

If there exists a risk neutral measure Q, then the market has no arbitrage. (But the converse is not true in this continuous time model. See [DS].)

If there is only one risk neutral measure Q, then the market is *complete*, in the sense that every bounded $\mathcal{F}_T$-measurable random variable F can be replicated, i.e. written as

$$F = x + \int_0^T \varphi(t)dS(t) \tag{4.9}$$

for some $x \in \mathbb{R}$ and some (admissible) portfolio φ. (We are neglecting some technical conditions here.)

One can show that this Black-Scholes market is indeed complete. Thus there is exactly one risk neutral probability measure Q, and the unique non-arbitrage price, $\mathrm{price}_{\mathrm{BS}}(F)$, at $t = 0$ of a contract which pays F at time T is

$$\mathrm{price}_{\mathrm{BS}}(F) = E_Q[e^{-rT}F]. \tag{4.10}$$

5 Models with jumps

Finally we discuss more recent developments, where the possibility of jumps are introduced. A natural – and at the same time mathematical tractable – way of doing this is to add a jump term in the stock price model as follows:

$$dS_1(t) = S_1(t^-)\Big[\alpha dt + \sigma dB(t) + \gamma \int_{\mathbb{R}_0} z\tilde{N}(dt, dz)\Big] \tag{5.1}$$

where α, σ and γ are constants and

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt. \tag{5.2}$$

Here $N([0, t], U)$ is the number of jumps of a given underlying Lévy process $\eta(s)$ at times s up time t with jump size $\Delta\eta(s) := \eta(s) - \eta(s^-) \in U$, U being a Borel set in $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, with closure $\bar{U} \subset \mathbb{R}_0$. And $\nu(U) := E[N([0, 1], U)]$ is the Lévy measure of η. Intuitively, one can regard (5.1) as another interpretation of (4.1), but now with "noise" represented by

$$\text{"noise"} = \text{"}\frac{d\eta(t)}{dt}\text{"}, \tag{5.3}$$

where $\eta(t)$ is the given Lévy process.

There is a corresponding Itô formula for stochastic differential equations of the form (5.1), and using this one can prove that if $\gamma z \geq -1$ for a.a. z with respect to ν, then

$$S_1(t) = S_1(0) \exp\Big(\Big(\alpha - \tfrac{1}{2}\sigma^2 + \int_{\mathbb{R}_0} \{\ln(1 + \gamma z) - \gamma z\}\nu(dz)\Big)t$$

$$+\int_0^t \int_{\mathbb{R}_0} \ln(1+\gamma z)\tilde{N}(ds,dz)\Big); \qquad 0 \le t \le T. \tag{5.4}$$

See e.g. [ØS], Chapter 1.

Thus we see that also in this case $S_1(t)$ behaves like a "distorted" exponential function, but now it might jump (in either direction) at any time t. (The condition $\gamma z \ge -1$ prevents it from jumping to a negative value.)

In contrast to the (continuous) Black-Scholes market in Section 4, the market $(S_0(t), S_1(t))$ with $S_1(t)$ given by (5.4) is typically *incomplete.* This means that there are several (in fact infinitely many) risk neutral measures Q. If we let $\mathcal{M}$ denote the family of all risk neutral measures, then

$$\text{price}_{\text{buyer}} := \inf_{Q\in\mathcal{M}} E_Q[e^{-rT}F] \tag{5.5}$$

and

$$\text{price}_{\text{seller}} := \sup_{Q\in\mathcal{M}} E_Q[e^{-rT}F] \tag{5.6}$$

is called the *buyer's* and the *seller's* price, respectively, at time 0 of a contract which pays the random ($\mathcal{F}_T$-measurable) amount F at time T. Any price in the interval

$$[\text{price}_{\text{buyer}}, \text{price}_{\text{seller}}]$$

will be a non-arbitrage price. Therefore this interval is called the *non-arbitrage interval.* Note that in this situation an arbitrage-free price is no longer unique, and additional coniderations are required to determine the price.

Since we all believe that real markets are incomplete, the jump models appear to be better suited to handle realistic situations. But they are also more complicated mathematically.

6 Market friction

So far we have assumed that all transactions can be carried out immediately, without any costs or delays. In real financial markets this is not the case. Usually there are *transaction costs* of several types involved. For example, one may have costs which are *proportional* to the volume traded. When modeling such situations mathematically one is led to using *singular* stochastic control theory. Another example of a transaction cost type is a *fixed cost* to be paid for any transaction, no matter how big or small. To deal with such situations one would use *impulse control theory.* See [ØS] for more information.

7 Asymmetric information

All the mathematical models we have discussed so far have assumed that all agents involved have access to the same information, namely the information that can be obtained by observing the market prices up to the present moment. This is only an approximation of the real situation. For example, many traders in the financial only know *some* of the previous market values, not all of them. Or they get access to the information with some time delay. In these cases the trader only has *partial information* to her disposal when making the decisions. Another example is when the agent has (legal or illegal) access to information about the *future* value of some financial asset. In this case the agent is called an *insider*.

Dealing with the mathematical modeling of financial markets with partial and/or inside information represents a big mathematical challenge. One has to work with *anticipative stochastic calculus and Malliavin calculus* to deal with such issues. See e.g. [DØP] and the references therein.

8 Risk measures

An axiomatic construction of *risk measures* first appeared about 10 years ago, and it was subsequently extended to what we today call *convex risk measures*. Intuitively, the *risk* $\rho(F)$ of a financial standing F, is the amount we have to add to F to make the standing "acceptable". If we formulate this rigorously, we arrive at a set of axioms that the risk measure ρ should satisfy. In particular, it should be *convex*, i.e.

$$\rho(\lambda F + (1-\lambda)G) \leq \lambda\rho(F) + (1-\lambda)\rho(G)$$

for all financial standings F, G and all numbers $\lambda \in (0,1)$. Intuitively this means that *the risk is reduced by diversification*. Surprisingly, this crucial property does *not* hold for the traditional and most commonly used risk model so far, namely the *value at risk* (VaR). Therefore one should abandon the VaR as a measure of risk and start using convex risk measures instead. When using mathematics to minimize the risk in this setting, one is faced with challenging problems in *stochastic differential game theory* and stochastic control of *forward-backward* stochastic differential equations. See e.g. [MØ], [ØS2], [ØS3].

9 Summary

We have tried to give a glimpse of the short – but highly successful – history of mathematical finance, from the Black-Scholes formula in 1973 to the most recent research developments of today. A striking feature is the fruitful interplay between financial concepts and the corresponding stochastic analysis machinery.

The current financial crises has many reasons. What seems clear in any case, is that there is a need for better understanding of how the financial markets work. To achieve this, it is necessary to continue and enhance the research activity within mathematics and finance and the interplay between the two.

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