HOMOGENEITY OF THE PURE STATE SPACE OF THE
CUNTZ ALGEBRA

OLA BRATTELI AND AKITAKA KISHIMOTO

Abstract. If \( \omega_1, \omega_2 \) are two pure gauge-invariant states of the Cuntz algebra \( \mathcal{O}_d \), we show that there is an automorphism \( \alpha \) of \( \mathcal{O}_d \) such that \( \omega_1 = \omega_2 \circ \alpha \). If \( \omega \) is a general pure state on \( \mathcal{O}_d \) and \( \varphi_0 \) is a given Cuntz state, we show that there exists an endomorphism \( \alpha \) of \( \mathcal{O}_d \) such that \( \varphi_0 = \omega \circ \alpha \).

1. Introduction

Let \( \mathfrak{A} \) be a simple separable C*-algebra, and let \( \pi_1, \pi_2 \) be representations of \( \mathfrak{A} \) on Hilbert spaces \( H_1, H_2 \). The representations \( \pi_1, \pi_2 \) are said to be algebraically equivalent if \( \pi_1(\mathfrak{A})'' \) and \( \pi_2(\mathfrak{A})'' \) are isomorphic von Neumann algebras. If there is an automorphism \( \alpha \) of \( \mathfrak{A} \) such that \( \pi_1 \) and \( \pi_2 \circ \alpha \) are quasi-equivalent, then \( \pi_1, \pi_2 \) are clearly algebraically equivalent. Powers proved in [Pow67] that if \( \mathfrak{A} \) is a UHF algebra the converse is true. His method extends readily to the case that \( \mathfrak{A} \) is an AF-algebra, [Bra72]. See also section 12.3 in [KR86]. In the special case that \( \pi_1 \) (and therefore \( \pi_2 \)) is irreducible, Kadison’s transitivity theorem therefore implies that if \( \mathfrak{A} \) is a UHF algebra the converse is true. His method extends readily to the case that \( \mathfrak{A} \) is an AF-algebra, [Bra72]. As a beginning of a possible resolution of the question for purely infinite algebras, we here prove the statements in the abstract. Recall from [Cun77] that the Cuntz algebra \( \mathcal{O}_d \) is the C*-algebra generated by \( d \) operators \( s_1, \ldots, s_d \) satisfying

\[
\begin{align*}
    s_j^* s_i &= \delta_{ij} I \\
    \sum_{i=1}^{d} s_i s_i^* &= I
\end{align*}
\]

There is an action \( \gamma \) of the group \( U(d) \) of unitary \( d \times d \) matrices on \( \mathcal{O}_d \) given by

\[
\gamma_g(s_i) = \sum_{j=1}^{d} g_{ji}s_j
\]

for \( g = [g_{ij}]_{i,j=1}^{d} \) in \( U(d) \). In particular the gauge action \( \tau = \gamma|_T \) is defined by

\[
\tau_z(s_i) = zs_i, \quad z \in T \subset \mathbb{C}.
\]

If \( \text{UHF}_d \) is the fixed point subalgebra under the gauge action, then \( \text{UHF}_d \) is the closure of the linear span of all Wick ordered polynomials of the form

\[
\prod_{i=1}^{k} s_{j_i} s_{j_i}^* 
\]
UHF\(d\) is isomorphic to the UHF algebra of Glimm type \(d^\infty\):

\[
\text{UHF}_d \cong M_{d^{\infty}} = \bigotimes_1^\infty M_d
\]

in such a way that the isomorphism carries the Wick ordered polynomial above into the matrix element

\[
e^{(1)}_{i_1j_1} \otimes e^{(2)}_{i_2j_2} \otimes \cdots \otimes e^{(k)}_{i_kj_k} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots.
\]

The gauge action \(\tau\) is in fact characterized by the fact that its fixed point algebra is isomorphic to \(\text{UHF}_d\), i.e. if \(\alpha\) is another faithful action of \(T\) on \(O_d\) such that the fixed point algebra \(O^\alpha_d\) is isomorphic to \(\text{UHF}_d\), then either \(z \mapsto \alpha_z\) or \(z \mapsto \alpha_z^{-1}\) is conjugate to \(\tau\). This follows from [BK99, Corollary 4.1]. (Since \(\text{UHF}_d\) is simple and \(\alpha\) is faithful, the crossed product \(O_d\times\alpha T\) is stably isomorphic to \(\text{UHF}_d\), [KT78], and in particular it is simple. Since \(O^\alpha_d \cong P_\alpha(0)(O_d\times\alpha T)P_\alpha(0)\), \(\left[P_\alpha(0)\right]\) is just \([1]_1\) when \(K_0(O_d\times\alpha T)\) is identified with \(K_0(O^\alpha_d)\). By the Pimsner-Voiculescu exact sequence it follows that \(\hat{\alpha}_*\) on \(K_0(O_d\times\alpha T) = \mathbb{Z}^{[1]}\) is multiplication by \(d\) or \(1/d\).) Because of this, our main result Theorem 5 can be given the following more universal form:

**Corollary 1.** Let \(\varphi_1\) and \(\varphi_2\) be pure states on \(O_d\), and assume that there exist actions \(\alpha_i\) of \(T\) on \(O_d\) such that \(O^{\alpha_i}_d \cong \text{UHF}_d\) and \(\varphi_i \circ \alpha_i = \varphi_i\) for \(i = 1, 2\). Then there exists an automorphism \(\beta\) of \(O_d\) such that

\[
\varphi_1 = \varphi_2 \circ \beta.
\]

The question whether any pure state on \(O_d\) is invariant under a gauge action like this is left open.

The restriction of \(\gamma_g\) to \(\text{UHF}_d\) is carried into the action

\[
\text{Ad}(g) \otimes \text{Ad}(g) \otimes \cdots
\]

on \(\bigotimes_1^\infty M_d\). We define the canonical endomorphism \(\lambda\) on \(\text{UHF}_d\) (or on \(O_d\)) by

\[
\lambda(x) = \sum_{j=1}^d s_j x s_j^*.
\]

and the isomorphism carries \(\lambda\) over into the one-sided shift

\[
x_1 \otimes x_2 \otimes x_3 \otimes \cdots \rightarrow \mathbb{1} \otimes x_1 \otimes x_2 \otimes \cdots
\]

on \(\bigotimes_1^\infty M_d\).

If \(\eta_1, \ldots, \eta_d\) are complex scalars with \(\sum_{j=1}^d |\eta_j|^2 = 1\), we can define a state on \(O_d\) by

\[
\varphi_\eta(s_{i_1} \cdots s_{i_k} s_{i_k}^* \cdots s_{i_1}^*) = \eta_{i_1} \cdots \eta_{i_k} \overline{\eta_{i_k}} \cdots \overline{\eta_{i_1}}\]

[Cun77], [Eva80], [BIP96], [BJ97], [BJKW].

This state is pure, and non-gauge invariant, and the \(U(d)\) action is transitive on these states, which are called Cuntz states. The restriction of \(\varphi_\eta\) to \(\text{UHF}_d\) identifies with the pure product state given by infinitely many copies of the vector state defined by the vector \((\eta_1, \ldots, \eta_d)\) on \(M_d\).
In this paper we will also consider the one-one correspondence between the set $U(O_d)$ of unitaries in $O_d$ and the set $\text{End}(O_d)$ of unital endomorphisms of $O_d$. If $u \in U(O_d)$ then $\alpha_u(s_i) = us_i$ defines an endomorphism, and if $\alpha \in \text{End}(O_d)$ the corresponding unitary is $u = \sum_{i=1}^{d} \alpha(s_i)s_i^*$. It has been proved by Rørdam that

$$U_a = \{u \in U(O_d) | \alpha_u \text{ is an inner automorphism}\}$$

is a dense subset of $U(O_d)$, [Ror93]. We give a shorter proof of this, and also show that

$$U_a = \{u \in U(O_d) | \alpha_u \text{ is an automorphism}\}$$

is a dense $G_\delta$ subset of $U(O_d)$ such that the complement $U(O_d) \setminus U_a$ is also dense.

By using the above correspondence between $U(O_d)$ and $\text{End}(O_d)$, it follows (see the proof of Proposition 8) that if $\omega$ is a pure state and $\varphi_0$ a Cuntz state there exists an endomorphism $\alpha$ of $O_d$ such that $\varphi_0 = \omega \circ \alpha$. Although the automorphism group is dense in $\text{End}(O_d)$ (in the topology of pointwise convergence), the question whether $\alpha$ can be chosen to be an automorphism is left open (in this approach).

2. Transitivity of the automorphism group on the pure gauge-invariant states

In this section we prove the first main result mentioned in the abstract.

Let $UHF_d$ be the UHF algebra of type $d^\infty$ and let $(A_n)$ be an increasing sequence of $\mathbb{C}^*$-subalgebras of $UHF_d$ such that $UHF_d = \bigcup_n A_n$ and $A_n \cong M_{d^n}$. We first use Power’s transitivity on $UHF_d$ to find an approximate factorization for any pure state on $UHF_d$.

**Lemma 2.** Let $\varphi$ be a pure state of $UHF_d$ and $\varepsilon > 0$. Then there exists a pure state $\varphi'$ of $UHF_d$, an increasing sequence $\{B_n\}$ of finite type I subfactors of $UHF_d$, and an increasing subsequence $\{k_n\}$ in $\mathbb{N}$ such that $\varphi'|B_n$ is a pure state of $B_n$ and $A_{k_n} \subset B_n \subset A_{k_{n+1}}$ for every $n$, and

$$\|\varphi - \varphi'\| < \varepsilon .$$

**Proof.** Since the automorphism group $\text{Aut}(UHF_d)$ of $UHF_d$ acts transitively on the set of pure states of $UHF_d$, [Pow67], there exists an increasing sequence $\{D_n\}$ of finite type I subfactors of $UHF_d$ such that $D_n \cong M_{d^n}$ and $\varphi|D_n$ is pure for every $n$. Then we can find sequences $\{u_n\}$ and $\{v_n\}$ of unitaries in $UHF_d$ and increasing sequences $\{k_n\}$ and $\{\ell_n\}$ in $\mathbb{N}$ such that

$$A_{k_n} \subset \text{Ad}(v_1 u_1)(D_{\ell_1}) \subset A_{k_2} \subset \text{Ad}(v_2 u_2 v_1 u_1)(D_{\ell_2}) \subset A_{k_3} \subset \cdots$$

$$u_n \in UHF_d \cap \text{Ad}(v_{n-1} u_{n-1} \ldots v_1 u_1)(D_{\ell_{n-1}})^\prime$$

$$v_n \in UHF_d \cap A_{k_n}$$

$$\|u_n - 1\| < \varepsilon / 2^{n+2} \quad \|v_n - 1\| < \varepsilon / 2^{n+2}$$

where $D_0 = C1$. (Let $k_1 = 1$. Then we choose $u_1$ and $\ell_1$ such that $A_{k_1} \subset \text{Ad} u_1(D_{\ell_1})$ and $\|u_1 - 1\| < \varepsilon / 8$. Further we choose $k_2$ and $v_1$ such that $v_1 \in UHF_d \cap A_{k_1}, \|v_1 - 1\| < \varepsilon / 8$, and $\text{Ad}(v_1 u_1)(D_{\ell_1}) \subset A_{k_2}$. We just repeat this process.) Then the limit $w = \lim v_n u_n \ldots v_1 u_1$ exists and is a unitary such that $\|w - 1\| < \varepsilon / 2$ and

$$A_{k_1} \subset \text{Ad} w(D_{\ell_1}) \subset A_{k_2} \subset \text{Ad} w(D_{\ell_2}) \subset \cdots$$
Let \( \varphi' = \varphi \circ \text{Ad} w^* \). Then \( \varphi' \) is a pure state with \( \| \varphi - \varphi' \| < \varepsilon \) and \( \varphi' \mid \text{Ad} w(D_{\ell_n}) \) is a pure state for every \( n \). Put \( B_n = \text{Ad} w(D_{\ell_n}) \).

We next show that for any pair of pure states \( \varphi_1, \varphi_2 \) on \( \text{UHF}_d \), there is a tensor product decomposition of \( \text{UHF}_d \) such that \( \varphi_1, \varphi_2 \) have approximate factorizations with respect to certain sub-decompositions (necessarily different for \( \varphi_1 \) and \( \varphi_2 \)):

**Lemma 3.** Let \( \varphi_1 \) and \( \varphi_2 \) be pure states of \( \text{UHF}_d \) and let \( \varepsilon > 0 \). Then there exist pure states \( \varphi'_1, \varphi'_2, \) and \( \psi \) of \( \text{UHF}_d \), an increasing sequence \( \{k_n\} \) in \( \mathbb{N} \) and an increasing sequence \( \{B_n\} \) of finite type I subfactors of \( A \) such that

\[
\| \varphi_i - \varphi'_i \| < \varepsilon, \\
\varphi'_1|B_{2n+1} \text{ is pure} \\
\varphi'_2|B_{2n} \text{ is pure} \\
\psi|B_{6k-1} \cap B'_6k-3 = \varphi'_1|B_{6k-1} \cap B'_6k-3 \\
\psi|B_{6k+2} \cap B'_6k = \varphi'_2|B_{6k+2} \cap B'_6k \\
\psi|B_{6k} \cap B'_6k-1 \text{ is pure,} \\
\psi|B_{6k-3} \cap B'_6k-4 \text{ is pure,} \\
k_{n+1} - k_n \to \infty \\
A_{k_1} \subset B_1 \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset \cdots
\]

**Proof.** It follows from the previous lemma that there exist pure states \( \varphi'_i \), increasing sequences \( \{B_m^i\} \) of finite type I subfactors of \( A \), and an increasing sequence \( \{k_n\} \) in \( \mathbb{N} \) such that

\[
\| \varphi_i - \varphi'_i \| < \varepsilon, \\
\varphi_i|B_{m^i} \text{ is pure for } i = 1, 2, \\
A_{k_i} \subset B_{m^i} \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset \cdots
\]

By passing to subsequences of \( \{k_n\} \) and \( \{B_m^i\} \) and setting \( B_n = B_{m^1} \) if \( n \) is odd and \( B_n = B_{m^2} \) if \( n \) is even, we may assume that

\[
\varphi'_1|B_{2n+1} \text{ is pure} \\
\varphi'_2|B_{2n} \text{ is pure} \\
k_{n+1} - k_n \to \infty \\
A_{k_1} \subset B_1 \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset \cdots
\]

Then \( \varphi'_i \) has a tensor product decomposition into pure states on the matrix subalgebras \( B_{2n+1} \cap B_{2n+1}^i \), and \( \varphi'_2 \) likewise on the subalgebras \( B_{2n} \cap B_{2n}^2 \). Thus we can define a pure state \( \psi \) by requiring that it decomposes under the tensor product decomposition

\[
\cdots \otimes (B_{6k-4} \cap B'_{6k-6}) \otimes (B_{6k-3} \cap B'_{6k-4}) \otimes (B_{6k-1} \cap B'_{6k-3}) \\
\otimes (B_{6k} \cap B'_{6k-1}) \otimes (B_{6k+2} \cap B'_{6k}) \otimes \cdots
\]
into states given by:

\[ \psi|B_{6k-1} \cap B_{6k-3}' = \varphi'_1|B_{6k-1} \cap B_{6k-3}' , \]
\[ \psi|B_{6k+2} \cap B_{6k} = \varphi'_2|B_{6k+2} \cap B_{6k}' , \]
\[ \psi|B_{6k} \cap B_{6k-1}' \text{ is an arbitrary pure state,} \]
\[ \psi|B_{6k-3} \cap B_{6k-4}' \text{ is an arbitrary pure state.} \]

\[ \square \]

Recall that \( \tau \) is the gauge action of \( T \) on \( \mathcal{O}_d \), i.e.,

\[ \tau_z(s_i) = z s_i , \quad z \in T . \]

Let \( \varepsilon \) be the conditional expectation of \( \mathcal{O}_d \) onto \( \text{UHF}_d \) defined by

\[ \varepsilon(x) = \int_T \tau_z(x) \frac{|dz|}{2\pi} , \quad x \in \mathcal{O}_d . \]

Note that if \( \varphi \) is a gauge-invariant state of \( \mathcal{O}_d \), then

\[ \varphi = \varphi|_{\text{UHF}_d} \circ \varepsilon . \]

Recall that \( \lambda \) is canonical endomorphism of \( \mathcal{O}_d \): \( \lambda(x) = \sum_{i=1}^d s_i x s_i^* , \quad x \in \mathcal{O}_d \), and that the restriction of \( \lambda \) to \( \text{UHF}_d \) is the one-sided shift \( \sigma \).

**Lemma 4.** If \( \varphi \) is a gauge-invariant state on \( \mathcal{O}_d \) then the following conditions are equivalent:

(i) \( \varphi \) is pure

(ii) \( \varphi|_{\text{UHF}_d} \) is pure and \( \varphi|_{\text{UHF}_d} \circ \sigma^n \) is disjoint from \( \varphi \) for \( n = 1, 2, \ldots \)

**Proof.** (i)⇒(ii). Since \( \varphi \) is pure, and gauge-invariant, it follows that \( \varphi|_{\text{UHF}_d} \) is pure. Let \( p \) be the support projection of \( \varphi \) in \( \mathcal{O}_d^* \). Since \( p \) is minimal, and \( \varphi \) is gauge-invariant, it follows that for any \( a \in \text{UHF}_d \) and any multi-index \( I = (i_1, i_2, \ldots, i_n) \) with \( |I| = n \geq 1 \),

\[ ps_I p = \varphi(a s_I) p = 0 , \]

where \( s_I = s_{i_1} s_{i_2} \ldots s_{i_n} \). Thus we obtain that

\[ p(\text{UHF}_d) \lambda^n(p) = 0 , \]

which implies that \( \varphi|_{\text{UHF}_d} \circ \sigma^n \) is disjoint from \( \varphi \).

(ii)⇒(i). Let \( p \) be the support projection of \( \varphi|_{\text{UHF}_d} \) in \( \text{UHF}_d^* \subset \mathcal{O}_d^* \). It suffices to show that for any multi-indices \( I, J \)

\[ ps_I s_J^* p \in \mathbb{C} p \]

since the linear span of \( s_I s_J^* \) is dense in \( \mathcal{O}_d \). If \( |I| \neq |J| \), we have that \( ps_I s_J^* p = 0 \) by using the fact that \( \varphi|_{\text{UHF}_d} \circ \sigma^n \) is disjoint from \( \varphi \) for \( n = |I| - |J| \). If \( |I| = |J| \), we have that \( ps_I s_J^* p = \varphi(s_I s_J^*) p \) since \( \varphi|_{\text{UHF}_d} \) is pure. \[ \square \]

**Lemma 5.** Let \( \varphi_1 \) and \( \varphi_2 \) be gauge-invariant pure states of \( \mathcal{O}_d \) such that all \( \varphi_i|_{\text{UHF}_d} \circ \sigma^n , \quad i = 1, 2, \quad n = 0, 1, 2, \ldots \) are mutually disjoint. Then there exists an automorphism \( \alpha \) of \( \mathcal{O}_d \) such that \( \alpha \circ \tau_z = \tau_z \circ \alpha , \quad z \in T \) and \( \varphi_1 = \varphi_2 \circ \alpha . \)
Proof. By Lemma 4, \( \psi_1 = \varphi_1|_{\text{UHF}_d} \) and \( \psi_2 = \varphi_2|_{\text{UHF}_d} \) are pure states on \( \text{UHF}_d \).

Applying Lemma 3 on \( \psi_1, \psi_2 \) in lieu of \( \varphi_1, \varphi_2 \), with \( \varepsilon = 1 \), we obtain pure states \( \psi_1', \psi_2' \) and \( \psi \) of \( \text{UHF}_d \) with the properties given there. Since \( \psi_1 \) is equivalent to \( \psi_1' = \psi_1' \circ \varepsilon \) is a pure state of \( \mathcal{O}_d \) by Lemma 4 and this state is equivalent to \( \varphi_1 = \psi_1 \circ \varepsilon \). By Kadison’s transitivity theorem we have a unitary \( u \in \text{UHF}_d \) such that \( \psi_1' = \psi_1 \circ \text{Ad} \ u \); it follows that \( \varphi_1 = \psi_1 \circ \text{Ad} \ u \).

It is not automatic that \( \psi \) satisfies the condition that all \( \psi \circ \sigma^n, \ n = 0, 1, 2, \ldots \) are mutually disjoint and are disjoint from \( \psi_1' \circ \sigma^n \). But using the freedom in constructing \( \psi_1, \psi_2 \) of \( \text{UHF}_d \), we can certainly impose this condition.

Thus we obtain three pure states \( \psi_1', \psi_2', \psi \) of \( \text{UHF}_d \) such that \( \psi \circ \sigma^n, \psi_1' \circ \sigma^n \) are mutually disjoint and \( \psi_1' \) and \( \psi \) are spotwise asymptotically equal as specified in Lemma 3. It now suffices to prove the lemma for the pairs \( (\psi' \circ \varepsilon, \psi \circ \varepsilon) \) and \( (\psi_2' \circ \varepsilon, \psi \circ \varepsilon) \). Thus replacing \( \varphi_1, \varphi_2 \) by one of these pairs, we may assume the lemma satisfy the additional condition that there exists an increasing sequence \( \{k_n\} \) in \( \mathbb{N} \) and an increasing sequence \( \{B_{k_n}\} \) of finite type I subfactors of \( \text{UHF}_d \) such that

\[
\begin{align*}
A_{k_1} &\subset B_1 \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset B_3 \subset \\
\varphi_1|_{B_{3n+1}} &\text{ is pure} , \\
\varphi_1|_{B_{3n+3} \cap B_{3n+1}'} &\text{ is pure} \\
k_{3n+3} - k_{3n+2} &\to \infty .
\end{align*}
\]

We shall construct a sequence \( \{v_n\} \) of unitaries in \( \text{UHF}_d \) such that \( \alpha = \lim_{n \to \infty} \text{Ad}(v_nv_{n-1} \ldots v_1) \) defines an automorphism of \( \mathcal{O}_d \) with \( \varphi_1 = \varphi_2 \circ \alpha \). To ensure the existence of the limit we choose the unitaries such that they mutually commute and \( \sum ||\lambda(v_n) - v_n|| < \infty \). Since \( \alpha \) commutes with the gauge action \( \tau \), this will complete the proof.

We fix a large \( N \in \mathbb{N} \). We choose \( n_1 \) so large that the support projections \( e_1^{(1)} = \text{supp}(\varphi_1|_{B_{3n_1+1}}) \) are almost orthogonal and \( k_{3n_1+3} - k_{3n_1+2} > 2^{2(N+1)} \). Let \( w_1 \) be a partial isometry in \( B_{3n_1+1} \) with \( w_1^*w_1 = e_1^{(1)} \), \( w_1w_1^* = e_2^{(1)} \). By the polar decomposition of the approximate unitary

\[
w_1 + (1 - e_2^{(1)})w_1^*(1 - e_1^{(1)}) + (1 - e_1^{(1)})(1 - e_2^{(1)}) ,
\]

we obtain a unitary \( v_1 \in B_{3n_1+1} \) such that

\[
v_1e_1^{(1)} = v_1w_1^{(1)} = e_2^{(1)}w_1 = e_2^{(1)}v_1 \in B_{3n_1+1}
\]

and \( v_1(1 - e_2^{(1)})(1 - e_1^{(1)}) \approx (1 - e_2^{(1)})(1 - e_1^{(1)}) \).

We next choose \( n_2 > n_1 \) so large that

\[
\sigma^n \circ \text{supp}(\varphi_1|_{B_{3n_2+1} \cap B_{3n_1+3}'}) , \quad i = 1, 2, \quad n = -2^{N-1} , -2^{-N+1} + 1 , \ldots , 0 , \ldots , 2^{N+1}
\]

are almost orthogonal and \( k_{3n_2+2} - k_{3n_1+1} > 2^{2(N+2)} \). (Though \( \sigma \) is an endomorphism, \( \sigma^n \) on \( B_{3n_2+1} \cap B_{3n_1+3}' \) is well defined for \( n = 1, 2, \ldots , k_{3n_2+2} \).) Let \( w_2 \) be a partial isometry in \( B_{3n_2+1} \cap B_{3n_1+3}' \) such that

\[
w_2^*w_2 = e_1^{(2)} = \text{supp}(\varphi_1|_{B_{3n_2+1} \cap B_{3n_1+3}'})
\]

and

\[
w_2w_2^* = e_2^{(2)} = \text{supp}(\varphi_2|_{B_{3n_2+1} \cap B_{3n_1+3}'}) ,
\]

respectively.
and let $\zeta$ be a partial isometry in $A_{k_{3n+2}+1} \cap A'_{k_{3n+3}}$ such that $\zeta^* \zeta = e_1^{(2)}$ and $\zeta \zeta^* = \sigma(e_1^{(2)})$.

Assume for the moment that $\sigma^\ell(e_1^{(2)}), i = 1, 2; \ell = -2^{N+1}, -2^{N+1}+1, \ldots, 2^{N+1}$ are all orthogonal and set

$$e_{ij} = \begin{cases} 
\sigma^{i-1}(\zeta) \sigma^{j-2}(\zeta) \ldots \sigma^j(\zeta) & i > j \\
\sigma^i(e_1^{(2)}) & i = j \\
\sigma^i(\zeta^*) \sigma^{i+1}(\zeta^*) \ldots \sigma^{j-1}(\zeta^*) & i < j 
\end{cases}$$

for $i, j = -2^{N+1}, \ldots, 2^{N+1}$. Then $(e_{ij})$ is a family of matrix units such that $\sigma(e_{ij}) = e_{i+1,j+1}$ when $|i|, |i+1|, |j|, |j+1| \leq 2^{N+1}$. Let

$$E = e_1^{(2)} + \sum_{\ell=1}^{2^{N+1}-1} (1 - e_1^{(2)}) \left( \frac{2^{N+1} - \ell}{2^{N+1}} e_{\ell,\ell} + \frac{\ell}{2^{N+1}} e_{-2^{N+1},\ell-2^{N+1}} + \frac{1}{2^{N+1}} \sqrt{(2^{N+1} - \ell)^2 (e_{-2^{N+1},\ell} + e_{-2^{N+1},\ell})} \right) (1 - e_1^{(2)})$$

as in [Kis95]. Then $E$ is a projection in $D_2 = A_{k_{3n+2}+2^{N+1}} \cap A'_{k_{3n+3}-2^{N+1}}$ and satisfies

$$\|\sigma(E) - E\| \sim \frac{1}{2^{N+1}}.$$

Let $w = w_2 + (1 - e_2^{(2)}) \left( \sum_{\ell=1}^{2^{N+1}} \left( \sigma^\ell(w_2) + \sigma^{-\ell}(w_2) \right) \right) (1 - e_1^{(2)})$ and

$$v = wE + (1 - F)w^*(1 - E) + (1 - F)(1 - E)$$

where $F = wEw^*$.

By the orthogonality assumption on $\sigma^\ell(e_1^{(2)})$, $v$ is a unitary in $D_2$ and satisfies

$$\|\sigma(v) - v\| \approx \|\sigma(E) - E\|,$$

$$v_1^{(2)} = w_2 e_1^{(2)} = e_2^{(2)} w_2 = e_2^{(2)} v .$$

Note also that $v$ commutes with $v_1$ and $e_1^{(2)}$.

Now, the projections $\sigma^\ell(e_1^{(2)}), i = 1, 2, \ell = -2^{N+1}, \ldots, 2^{N+1}$ are not actually orthogonal but choosing $v_2$ so large that they are very close to being orthogonal, we may obtain a unitary $v_2$ in $D_2$ by polar decomposition of $v$ such that $v_2$ satisfies the same conditions as above, i.e.,

$$v_2 = v_2 e_1^{(2)} = e_2^{(2)} v_2 = e_2^{(2)} v_2 \in B_{3n+2+1} \cap B'_{3n+3},$$

$$\|\lambda(v_2) - v_2\| \sim 2^{-\frac{N+1}{2}}$$

and $v_2 \in D_2$.

Since

$$\text{supp}(\varphi_1|_{B_{3n+2}+1}) = \text{supp}(\varphi_1|_{B_{3n+3}}) \cap \text{supp}(\varphi_1|_{B_{3n+3}+1}) \supp(\varphi_1|_{B_{3n+3}+1})$$

$$= e_1^{(1)} \rho e_1^{(2)}$$
with $p = \text{supp}(\varphi_1|_{B_{n_1+3} \cap B_{3n_k+1}^c}) = \text{supp}(\varphi_2|_{B_{n_1+3} \cap B_{3n_k+1}^c})$, and since the operators $v_1e_1^{(1)} = e_2^{(1)}v_1p$, and $v_2e_1^{(2)} = e_2^{(2)}v_2p$ commute, we obtain that

$$v_1 v_2 \cdot \text{supp}(\varphi_1|_{B_{3n_2+1}}) = v_1 v_2 e_1^{(1)} e_2^{(2)}.$$ 

Here we have also used the fact that $v_1$ commutes with $e_2^{(2)}$. We repeat this procedure. Thus we obtain an increasing sequence $\{u_k\}$ in $\mathbf{N}$ and a sequence $\{v_k\}$ of mutually commuting unitaries such that

$$\|\lambda(v_k) - v_k\| \sim 2^{-\frac{k}{n+1}},$$

$v_k e_1^{(k)} = e_2^{(k)}v_k \in B_{3n_1+1} \cap B_{3n_k+1}^c$ where

$$e_1^{(k)} = \text{supp}(\varphi_1|_{B_{3n_k+1} \cap B_{3n_k+1}^c}).$$

and such that $\text{Ad}(v_1 \ldots v_l)$ maps $\text{supp}(\varphi_1|_{B_{3n_k+1}})$ into $\text{supp}(\varphi_2|_{B_{3n_k+1}})$. Then the limit $\alpha = \lim k \text{Ad}(v_k \ldots v_l)$ defines the desired automorphism.

**Theorem 6.** Let $\varphi_1$ and $\varphi_2$ be gauge-invariant pure states of $\mathcal{O}_d$. Then there exists an automorphism $\alpha$ of $\mathcal{O}_d$ such that $\varphi_1 = \varphi_2 \circ \alpha$.

**Proof.** If $\varphi_1$ is disjoint from $\varphi_2$, then it follows that $(\varphi_i|_{UHF_d}) \circ \sigma^n = \varphi_i \circ \lambda^n|_{UHF_d}$, $i = 1, 2$, $n = 0, 1, 2, \ldots$ are mutually disjoint (by Lemma 4); thus the assertion follows from Lemma 5. If $\varphi_1$ is equivalent to $\varphi_2$, there is a unitary $u \in \mathcal{O}_d$ such that $\varphi_1 = \varphi_2 \text{Ad} u$ (by Kadison’s transitivity).

3. Pure states mapped into Cuntz states by endomorphisms

There is a one-to-one correspondence between the set $\mathcal{U}(\mathcal{O}_d)$ of unitaries of $\mathcal{O}_d$ and the set $\text{End}(\mathcal{O}_d)$ of unital endomorphisms of $\mathcal{O}_d$; if $u \in \mathcal{U}(\mathcal{O}_d)$, the endomorphism $\alpha_u$ is defined by $\alpha_u(s_i) = us_i$ and if $\alpha \in \text{End}(\mathcal{O}_d)$, $\alpha$ corresponds to the unitary $u$ defined by $u = \sum_{i=1}^d \alpha(s_i) s_i^*$. Define

$$\mathcal{U}_u = \{u \in \mathcal{U}(\mathcal{O}_d)| \alpha_u \text{ is an inner automorphism}\}$$

$$\mathcal{U}_a = \{u \in \mathcal{U}(\mathcal{O}_d)| \alpha_u \text{ is an automorphism}\}$$

$$\mathcal{U}_a = \mathcal{U}(\mathcal{O}_d) \setminus \mathcal{U}_u.$$ 

**Proposition 7.** Let $\mathcal{U}_i, \mathcal{U}_a, \mathcal{U}_a$ be as above.

(i) $\mathcal{U}_u$ is a dense subset of $\mathcal{U}(\mathcal{O}_d)$.

(ii) $\mathcal{U}_a$ is a dense $G_\delta$ subset of $\mathcal{U}(\mathcal{O}_d)$.

(iii) $\mathcal{U}_a$ is a dense $F_\sigma$ subset of $\mathcal{U}(\mathcal{O}_d)$.

**Proof.** M. Rørdam proved (i) in [Ror93] and the other statements are more or less known.

We shall give a proof of (i). We again denote by $\lambda$ the canonical endomorphism of $\mathcal{O}_d : \lambda(x) = \sum_{i=1}^d s_i x s_i^*$, $x \in \mathcal{O}_d$. Since the unitary corresponding to $\text{Ad} u$ is $u \lambda(u^*)$,
it suffices to show that $v\lambda(u^*)$, $v \in \mathcal{U}(\mathcal{O}_d)$, is dense in $\mathcal{U}(\mathcal{O}_d)$. If UHF$_d$ denotes the C*-subalgebra generated by $s_{i_1}s_{i_2}\ldots s_{i_n}s^*_{j_1}\ldots s^*_{j_l}$, then we mentioned in the introduction that UHF$_d$ is isomorphic to the UHF algebra $\bigotimes_{n} M_d$ and $\lambda|\text{UHF}_d$ corresponds to the one-sided shift on $\bigotimes_{n} M_d$. Thus $\lambda|\text{UHF}_d$ satisfies the Rohlin property, [BKRS93], [Kis95]. In particular for any $n$ and $\varepsilon > 0$ there is an orthogonal family $e_0, e_1, \ldots, e_{n-1}$ of projections in UHF$_d$ such that

$$\sum_{i=0}^{d^n-1} e_i = 1$$

$$\|\lambda(e_i) - e_{i+1}\| < \varepsilon$$

with $e_0 = e_0$. The similar properties hold for $\text{Ad} u \circ \lambda$, i.e., if UHF$_d^n$ denotes the C*-subalgebra generated by $u s_{i_1} s_{i_2} \ldots u s_{i_n} s^*_{j_1} \ldots s^*_{j_l}$, then $\text{Ad} u \circ \lambda|\text{UHF}_d^n$ corresponds to the one-sided shift on $\bigotimes_{n} M_d$. Hence for any $n$ and $\varepsilon > 0$ there is an orthogonal family $f_0, f_1, \ldots, f_{d^n-1}$ of projections in UHF$_d^n$ such that

$$\sum_{i=0}^{d^n-1} f_i = 1$$

$$\|\text{Ad} u \circ \lambda(f_i) - f_{i+1}\| < \varepsilon$$

with $f_0 = f_0$. Suppose we have chosen such projections $e_i, f_i$ for the same $n$.

Since $K_0(\mathcal{O}_d) = \mathbb{Z}/(d-1)\mathbb{Z}$, we have that $[e_0] = 1 = [f_0]$ in $K_0(\mathcal{O}_d)$ and so obtain a partial isometry $w \in \mathcal{O}_d$ such that $w^* w = e_0$, $ww^* = f_0$. We find unitaries $v_1, v_2 \in \mathcal{O}_d$ such that $\text{Ad} v_1 \lambda(e_i) = e_{i+1}$, $\text{Ad} v_2 \text{Ad} u \lambda(f_i) = f_{i+1}$, and $\|v_1 - 1\| \approx 0$, $\|v_2 - 1\| \approx 0$ (depending on $\varepsilon$). Let

$$z = w^* (L_{v_2 u} R_{v_1} \lambda)^{d^n}(w)$$

where $R_{v_1}$ is the right multiplication by $v_1^*$ and $L_{v_2 u}$ is the left multiplication by $v_2 u$. Since $(L_{v_2 u} R_{v_1} \lambda)^i(w)$ is a partial isometry with initial projection $e_i$ and final projection $f_i$, $z$ is a unitary in $e_0 \mathcal{O}_d e_0$. Since $K_1(\mathcal{O}_d) = 0$ and $\mathcal{O}_d$ has real rank zero, we find a sequence $z_0, z_1, \ldots, z_{d^n-1}$ of unitaries in $e_0 \mathcal{O}_d e_0$ such that $z_0 = z$, $z_{d^n-1} = 1$.

Define a unitary $v$ by

$$v = \sum_{i=0}^{d^n-1} (L_{v_2 u} R_{v_1} \lambda)^i(w z_i)$$

Then since

$$v - (L_{u v_2 u} R_{v_1} \lambda)(v) = \sum_{i=1}^{d^n-1} (L_{v_2 u} R_{v_1} \lambda)^i(w z_i - w z_{i-1}) + wz_0 - (L_{v_2 u} R_{v_1} \lambda)^{d^n}(w)$$

it follows that

$$\|v - L_{v_2 u} R_{v_1} \lambda(v)\| < 4/d^n$$

or

$$\|v - u \lambda(v)\| < 4/d^n$$

This completes the proof of (i).
Since \( U_a \supset U_i \), \( U_a \) is dense. That \( U_a \) is a \( G_\delta \) set follows from
\[
U_a = \bigcap_n \bigcup_j \left\{ u \in U(O_d); \|a_u(x_i) - x_j\| < \frac{1}{n} \right\}
\]
where \( \{x_i\} \) is a dense sequence in \( O_d \).

If \( U_a \) contains a non-empty open set, then it follows that \( U_a = U(O_d) \) or \( U_a = \emptyset \). Because for any unitaries \( u, w \) of \( O_d \) we find a unitary \( v \) such that \( w\lambda(v) \approx vu \).

(Apply the previous argument for the endomorphism \( Ad u \circ \lambda \) instead of \( \lambda \) and the unitary \( wu^* \).) Since \( U_a \lambda(v^*) = U_d \) for any unitary \( v \in O_d \), the above fact implies that \( U_a \) contains an arbitrary unitary. But we know that \( U_a \neq \emptyset \). For example if \( u = \sum s_is_j^*s_j^* \), then \( \alpha_u = \lambda \) and \( \lambda(O_d)' \approx M_d \). Thus we obtain that \( U_a \) is dense.

For a unit vector \( \xi \in C^d \) we have defined the Cuntz state \( f_\xi \) of \( O_d \) by
\[
f_\xi(s_{i_1}\cdots s_{i_m}s_{j_1}^*\cdots s_{j_1}^*) = \xi_{i_1}\cdots \xi_{i_m}\overline{\xi_{j_1}}\cdots \overline{\xi_{j_1}}
\]
It follows that \( f_\xi \) is a unique pure state of \( O_d \) satisfying
\[
f_\xi \left( \sum_{i=1}^d \xi_is_i \right) = 1.
\]

Let \( F \) be the linear span of \( s_is_j^* \), \( i, j = 1, \ldots, d \). Then \( F \) is isomorphic to \( M_d \) and each unitary \( u \) in \( F \) defines an automorphism \( \alpha_u \) of \( O_d \). This group of automorphisms acts transitively on the compact set of Cuntz states.

We denote by \( f_0 \) the Cuntz state \( f_\xi \) with \( \xi = (1, 0, \ldots, 0) \).

**Proposition 8.** If \( \varphi \) is a pure state of \( O_d \), there is a unital endomorphism \( \alpha \) of \( O_d \) such that \( \varphi \circ \alpha = f_0 \), where \( f_0 \) is the Cuntz state defined above. Furthermore \( \alpha \) may be chosen so that \( \pi_\varphi \circ \alpha(O_d)' \) contains the one-dimensional projection onto \( CS_\varphi \).

**Proof.** It suffices to show that if \( \varphi \) is a pure state there is a unitary \( u \in O_d \) such that
\[
\varphi(us_1) = 1.
\]

Since \( O_d \) has real rank zero, there is a decreasing sequence \( (e_n) \) of projections in \( O_d \) such that \( \varphi \) is the unique state satisfying \( \varphi(e_n) = 1 \) for \( n = 1, 2, \ldots \), i.e., \( (e_n) \) converges to the support projection of \( \varphi \) in \( O_d^\tau \). We may further assume that \( [e_n] = 0 \) in \( K_0(O_d) \).

Pick up a projection \( e = e_n \) such that \( \varphi(e) = 1 \) and \( e < 1 \). Then \( es_1^* \) is a partial isometry with initial projection \( s_1es_1^* \) and final projection \( e \). Let \( w \) be a partial isometry such that \( w^*w = 1 - s_1es_1^* \) and \( ww^* = 1 - e \). Then \( u = es_1^* + w \) is a unitary in \( O_d \) such that
\[
us_1e = (es_1^* + w)s_1e = e.
\]
Thus we have that \( \varphi(us_1) = 1 \).

To prove the last statement we shall modify \( u \) so that \( \varphi \) is the unique state satisfying
\[
\varphi(us_1) = 1.
\]

We have chosen \( e = e_n \). We let
\[
h = \sum_{k=1}^\infty 2^{-k}e_{n+k}.
\]
Then $h$ is self-adjoint with $0 \leq h \leq 1$ and $\varphi$ is the only state satisfying $\varphi(h) = 1$.

Let $u_1 = e^{2\pi i h}u$.

Then $u_1s_1e = e^{2\pi i h}e$ and the assertion follows. 

References


Mathematics Institute, University of Oslo, PB 1053 Blindern, N-0316 Oslo, Norway

Department of Mathematics, Hokkaido University, Sapporo, 060 Japan