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A general maximum principle for anticipative stochastic control and applications to insider trading

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Abstract

In this paper we suggest a general stochastic maximum principle for optimal control of anticipating stochastic differential equations driven by a Lévy type of noise. We use techniques of Malliavin calculus and forward integration. We apply our results to study a general optimal portfolio problem of an insider. In particular, we find conditions on the insider information filtration which are sufficient to give the insider an infinite wealth. We also apply the results to find the optimal consumption rate for an insider.

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1 Introduction

In the classical Black-Scholes model, and in most problems of stochastic analysis applied to finance, one of the fundamental hypotheses is the homogeneity of information that market participants have. This homogeneity does not reflect reality. In fact, there exist many types of agents in the market who have different levels of information. In this paper, we are focusing on agents who have additional information (insiders), and show that, it is important to understand how an optimal control is affected by particular pieces of such information.

In the following, let $\{B(t)\}_{0\leq t\leq T}$ be a Brownian motion and $\widetilde{N}(dz,ds) = N(dz,ds) - ds\nu(dz)$ be a compensated Poisson random measure associated with a Lévy process with Lévy measure

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ν on the (complete) filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$ with $T > 0$ fixed time horizon. In the sequel, we assume that the Lévy measure ν fulfills

$$\int_{\mathbb{R}_0} z^2\, \nu(dz) < \infty,$$

where $\mathbb{R}_0 := \mathbb{R}\backslash \{0\}$.

Here we suppose that we are given a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$, with

$$\mathcal{F}_t \subseteq \mathcal{G}_t, \;\; t \in [0, T], \tag{1.1}$$

representing the information available to the agent at time t. This information is used at decision making level yielding a $\mathbb{G}$-predictable strategy or control.

Suppose that the state process $X(t) = X^{(u)}(t, \omega);\ 0 \leq t \leq T, \omega \in \Omega$, characterizing the agent's wealth, is a controlled jump diffusion in $\mathbb{R}$ of the form:

$$\left\{\begin{array}{lcl} d^-X(t) & = & b(t, X(t), u(t))\, dt \,+\, \sigma(t, X(t), u(t))\, d^-B(t) \\ & & +\int_{\mathbb{R}_0} \theta(t, X(t), u(t), z)\, \widetilde{N}(dz, d^-t); \\ X(0) & = & x \in \mathbb{R}. \end{array}\right. \tag{1.2}$$

Since $B(\cdot)$ and $\widetilde{N}(A, \cdot),\ A \subseteq \mathbb{R}_0$ Borel, need not be semimartingales with respect to $\{\mathcal{G}_t\}_{0\leq t\leq T}$, the two last integrals in (1.2) are *anticipating* stochastic integral that we interpret as *forward* integrals. The choice of forward integration, as an anticipative extension of the Itô integration, is motivated by the possible applications to optimal portfolio problems for insiders as in Section 6 see for e.g., [3, 7, 6]. However, the applications are not restricted to this area and include all situations of optimization problems in anticipating environments (see e.g., [15, 20]).

The control process

$$u : [0, T] \times \Omega \longrightarrow U,$$

is called an *admissible control* if (1.2) has a unique (strong) solution $X = X^{(u)}$ such that $u(\cdot)$ is predictable with respect to the filtration $\{\mathcal{G}_t\}_{0\leq t\leq T}$. We let $\mathcal{A}_{\mathbb{G}}$ denote a given family of admissible controls assumed to be $\mathbb{G}$-predictable and such that (1.2) has a strong solution.

More specifically, the problem we are dealing with is the following. Suppose that we are given a performance functional of the form

$$J(u) := E\left[\int_0^T f(t, X(t), u(t))\, dt \,+\, g(X(T))\right],\ u \in \mathcal{A}_{\mathbb{G}}, \tag{1.3}$$

with

$$\begin{array}{lcl} f & : & [0, T] \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}\,, \\ g & : & \mathbb{R} \times \Omega \longrightarrow \mathbb{R}, \end{array}$$

where f is an $\mathbb{F}$-adapted process for each $x \in \mathbb{R}$, $u \in U$ and g is an $\mathcal{F}_T$-measurable random variable for each $x \in \mathbb{R}$ satisfying

$$E\left[\int_0^T |f(t, X(t), u(t))| \, dt \, + \, |g(X(T))|\right] \, < \, \infty, \text{ for all } u \in \mathcal{A}_{\mathbb{G}}.$$

The goal is to find the optimal control $u^* \in \mathcal{A}_{\mathbb{G}}$ such that

$$\Phi_{\mathbb{G}} := \sup_{u \in \mathcal{A}_{\mathbb{G}}} J(u) = J(u^*) \,. \tag{1.4}$$

Special cases of this problem have been studied by many authors. See e.g. [1, 3, 4, 7, 11, 12, 14, 15] and the references therein.

The purpose of this paper is two-fold.

First, we want to establish a general maximum principle for the optimal anticipative control problem (1.2)-(1.4), without any a priori semimartingale assumptions for the inside information filtration $\{\mathcal{G}_t\}_{0 \le t \le T}$ (see Theorem 3.1 and Theorem 4.1).

Second, we want to use these general results to investigate the following problem in insider trading: How much information does an insider need in order to generate an infinite value of $\Phi_{\mathbb{G}}$?

The following example by Pikovski and Karatzas in [14] illustrates the situation. Suppose the financial market has two investments opportunities:

1. a risk free asset with unit price
$$S_0(t) = 1; \;\; t \in [0, T] \,,$$
2. a risky asset with unit price
$$dS_1(t) = S_1(t) \left[\mu dt + \sigma dB(t)\right]; \;\; S(0) > 0; \;\; t \in [0, T]$$

($\mu, \sigma > 0$ constants). If the trader chooses a portfolio $\pi(t)$ representing the fraction of wealth to be invested in the risky asset at time t, the corresponding wealth process $X(t)$, $t \in [0, T]$, will have the dynamics

$$d^- X_\pi(t) = X_\pi(t)\pi(t) \left[\mu dt + \sigma d^- B(t)\right]; \;\; X_\pi(0) > 0.$$

If the information flow accessible to the insider trader is given by a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0 \le t \le T}$ such that $\mathcal{G}_t \supseteq \mathcal{F}_t$, this means that π is required to be $\mathbb{G}$-adapted (thus the Itô integration cannot be applied and the forward integration is chosen to be used instead). Suppose the insider wants to maximize the expected logarithmic utility of the terminal wealth, i.e. to find $\Phi_{\mathbb{G}}$ and π^* (if it exists) such that

$$\Phi_{\mathbb{G}} := \max_{\pi \in \mathcal{A}_{\mathbb{G}}} E\left[\ln\left(X_\pi(T)\right)\right] = E\left[\ln\left(X_{\pi^*}(T)\right)\right]$$

In [14] it is proved that if

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(B(T)\right), \; t \in [0, T] \,,$$

then $\Phi_{\mathbb{G}} = \infty$ and π^* does not exist.

In this paper we generalize this situation in several directions:

a) We include jumps in the risky asset model

b) We study more general utility functions

c) We study more general insider filtrations.

These points were already partially discussed in [7] from the point of view of the existence of an optimal portfolio for a given insider. The present paper, we repeat, focuses on the study of conditions on the amount of information $\mathbb{G} = \{\mathcal{G}_t\}_{0 \le t \le T}$ needed to obtain $\Phi_{\mathbb{G}} = \infty$ and the non-existence of an optimal insider portfolio. For example, in a context as in the case of [14], we can see that if

$$\mathcal{F}_t \vee \sigma \left(B(t + \delta_n(t)); \;\; n = 1, 2, \ldots \right)$$

where

$$\delta_n(t) = (\frac{1}{n})^p \text{ for some } p \in (0, 1)$$

then π^* does not exist (see Corollary 6.6).

The main result, which represents a stochastic maximum principle, is presented in full generality (see Theorem 3.1). However it is difficult to apply because of the appearance of some terms, which all depend on the control. We then consider the special case (see Theorem 4.1) when the coefficients of the controlled process X do not depend on X; we call such processes *controlled Itô-Lévy processes*. In this case, we give a condition for the existence of an optimal control. More specific results are obtained in the cases when the insider filtration is either

i) a *D-commutable filtration* (Subsection 5.1 and Theorem 5.2) or

ii) a *smoothly anticipative filtration* (Subsection 5.2.)

Besides the application of these results to optimal portfolio problems, we also consider applications to optimal insider consumption. In this case we show that there exists an optimal insider consumption, and in some special cases the optimal consumption can be expressed explicitly.

The paper is structured as follows: In Section 2, we briefly recall some basic concepts of Malliavin calculus and its connection to the theory of forward integration. In Section 3, we use Malliavin calculus to obtain a maximum principle for this general non-Markovian insider information stochastic control problem. Section 4 considers the special case of Itô-Lévy processes. In Section 5 some specific classes of insider information are considered. Finally, in Section 6 and 7, we apply the results from the previous sections to study optimal insider portfolio and optimal insider consumption problems respectively.

2 Framework

In this Section we briefly recall some basic concepts of Malliavin calculus and its connection to the theory of forward integration. We refer to [17] or [8] for more information about Malliavin calculus. As for the theory of forward integration the reader may consult [18, 24, 25] and [6].

2.1 Malliavin Calculus for Lévy Processes

In the sequel consider a Brownian motion $\{B(t)\}_{0\leq t\leq T}$ on the filtered probability space

$$(\Omega^{(B)}, \mathcal{F}^{(B)}, \{\mathcal{F}_t^{(B)}\}_{0\leq t\leq T}, P^{(B)}),$$

where $\{\mathcal{F}_t^{(B)}\}_{0\leq t\leq T}$ is the $P^{(B)}$-augmented filtration generated by $\{B(t)\}_{0\leq t\leq T}$ with $\mathcal{F}^{(B)} = \mathcal{F}_T^{(B)}$.
Further we assume that a Poisson random measure $N(dt, dz)$ associated with a Lévy process is defined on the stochastic basis

$$(\Omega^{(\widetilde{N})}, \mathcal{F}^{(\widetilde{N})}, \{\mathcal{F}_t^{(\widetilde{N})}\}_{0\leq t\leq T}, P^{(\widetilde{N})}).$$

We denote by $\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ the compensated Poisson random measure, where ν is the Lévy measure of the Lévy process. See [2, 26] for more information about Lévy processes.

The starting point of Malliavin calculus is the following observation which goes back to K. Itô [13]: Square integrable functionals of $B(t)$ and $\widetilde{N}(dt, dz)$ enjoy the chaos representation property, that is

(i) If $F \in L^2(\mathcal{F}^{(B)}, P^{(B)})$ then

$$F = \sum_{n\geq 0} I_n^{(B)}(f_n) \tag{2.1}$$

for a unique sequence of symmetric $f_n \in L^2(\lambda^n)$, where λ is the Lebesgue measure and

$$I_n^{(B)}(f_n) := n! \int_0^T (\int_0^{t_n} \dots (\int_0^{t_2} f_n(t_1, \cdots, t_n) dB(t_1)) dB(t_2) \dots dB(t_n), \quad n \in \mathbb{N}$$

the n-fold iterated stochastic integral with respect to $B(t)$. Here $I_n^{(B)}(f_0) := f_0$ for constants f_0.

(ii) Similarly, if $G \in L^2(\mathcal{F}^{(\widetilde{N})}, P^{(\widetilde{N})})$, then

$$G = \sum_{n\geq 0} I_n^{(\widetilde{N})}(g_n), \tag{2.2}$$

for a unique sequence of kernels g_n in $L^2((\lambda \times \nu)^n)$, which are symmetric with respect to $(t_1, z_1), \dots, (t_n, z_n)$. Here $I_n^{(\widetilde{N})}(g_n)$ is defined as

$$I_n^{(\widetilde{N})}(g_n) := n! \int_0^T \int_{\mathbb{R}_0} \int_0^{t_n} \int_{\mathbb{R}_0} \dots (\int_0^{t_2} \int_{\mathbb{R}_0} g_n(t_1, z_1, \cdots, t_n, z_n)) \widetilde{N}(dt_1, dz_1) \dots \widetilde{N}(dt_n, dz_n),$$

$n \in \mathbb{N}$.

If $F \in L^2(\mathcal{F}^{(B)}, P^{(B)})$ has chaos expansion (2.1) the *Malliavin derivative* D_t of F in the direction of the Brownian motion is defined as

$$D_t F = \sum_{n\geq 1} n I_{n-1}^{(B)}(\widetilde{f}_{n-1}), \tag{2.3}$$

where $\widetilde{f}_{n-1}(t_1, \cdots, t_{n-1}) := f_n(t_1, \cdots, t_{n-1}, t)$, provided that

$$\sum_{n\geq 0} n n! \, \|f_n\|^2_{L^2(\lambda^n)} < \infty. \tag{2.4}$$

Similarly, for all $G \in L^2(\mathcal{F}^{(\widetilde{N})}, P^{(\widetilde{N})})$ with chaos representation (2.2) such that

$$\sum_{n\geq 0} n n! \, \|g_n\|^2_{L^2((\lambda\times\nu)^n)} < \infty, \tag{2.5}$$

the Malliavin derivative $D_{t,z}$ of G the direction of $\widetilde{N}(dt, dz)$ is introduced as

$$D_{t,z} G := \sum_{n\geq 1} n I_{n-1}^{(\widetilde{N})}(\widetilde{g}_{n-1}), \tag{2.6}$$

where $\widetilde{g}_{n-1}(t_1, z_1, \cdots, t_{n-1}, z_{n-1}) := g_n(t_1, z_1, \cdots, t_{n-1}, z_{n-1}, t, z)$.

In the following we denote by $\mathbb{D}^B_{1,2}$ the stochastic Sobolev space of square integrable Brownian functionals such that (2.4) is fulfilled. The symbol $\mathbb{D}^{\widetilde{N}}_{1,2}$ stands for the corresponding space with respect to $\widetilde{N}(dt, dz)$.

We recall that the *Skorohod integral* with respect to B respectively $\widetilde{N}(\delta t, dz)$ is defined as the adjoint operator of $D_\cdot : \mathbb{D}^B_{1,2} \longrightarrow L^2(\lambda\times P^{(B)})$ resp. $D_{\cdot,\cdot} : \mathbb{D}^{\widetilde{N}}_{1,2} \longrightarrow L^2(\lambda \times \nu\times P^{(\widetilde{N})})$. Thus if we denote by

$$\int_0^T (\cdot)\, \delta B_t \text{ and } \int_0^T \int_{\mathbb{R}_0} (\cdot)\, \widetilde{N}(\delta t, dz)$$

the corresponding adjoint operators the following duality relations are satisfied:

(i)

$$E_{P^{(B)}}\left[F \int_0^T \varphi(t) \delta B_t\right] = E_{P^{(B)}}\left[\int_0^T \varphi(t) D_t F \, dt\right] \tag{2.7}$$

for all $F \in \mathbb{D}^B_{1,2}$ and all Skorohod integrable $\varphi \in L^2(\lambda\times P^{(B)})$ (i.e. φ in the domain of the adjoint operator).

(ii)

$$E_{P^{(\widetilde{N})}}\left[G \int_0^T \int_{\mathbb{R}_0} \psi(t,z) \widetilde{N}(\delta t, dz)\right] = E_{P^{(\widetilde{N})}}\left[\int_0^T \int_{\mathbb{R}_0} \psi(t,z) D_{t,z} G \nu(dz) dt\right] \tag{2.8}$$

for all $G \in \mathbb{D}^{\widetilde{N}}_{1,2}$ and all Skorohod integrable $\psi \in L^2(\lambda \times \nu\times P^{(\widetilde{N})})$.

In what follows our reference stochastic basis will be

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P),$$

where $\Omega = \Omega^{(B)}\times\, \Omega^{(\widetilde{N})}$, $\mathcal{F} = \mathcal{F}^{(B)} \times \mathcal{F}^{(\widetilde{N})}$, $\mathcal{F}_t = \mathcal{F}_t^{(B)} \times \mathcal{F}_t^{(\widetilde{N})}$, $P = P^{(B)} \times P^{(\widetilde{N})}$.

Later on in the paper we will employ the duality relations (2.7) and (2.8) in connection with P. We will need the following result from [9].

Theorem 2.1 ***[Decomposition uniqueness for Skorohod-semimartingales]*** *Let $\{X(t)\}_{0\le t\le T}$ be a Skorohod-semimartingale of the form*

$$X_t = \zeta + \int_0^t \alpha(s)ds + \int_0^t \beta(s)\delta B_s + \int_0^t \int_{\mathbb{R}_0} \gamma(s,z)\widetilde{N}(dz,\delta s),$$

where $\alpha(t) \in L^2(P)$ for all t. Then if

$$X_t = 0 \text{ for all } \; 0 \le t \le T.$$

we have

$$\zeta = 0, \alpha = 0, \; \beta = 0, \; \gamma = 0 \; a.e.$$

2.2 Malliavin calculus and forward integral

In this Section we briefly recall some basic concepts of Malliavin calculus and forward integrations related to this paper. We refer to [18, 24, 25] and [6] for more information about these integrals.

2.2.1 Forward integral and Malliavin calculus for $B(\cdot)$

This Section constitutes a brief review of the forward integral with respect to the Brownian motion. Let $\{B(t)\}_{0\le t\le T}$ be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$, and $T > 0$ a fixed horizon.

Definition 2.2 *Let $\phi : [0,T] \times \Omega \to \mathbb{R}$ be a measurable process. The forward integral of ϕ with respect to $\{B(t)\}_{0\le t\le T}$ is defined by*

$$\int_0^T \phi(t,\omega)\, d^- B(t) = \lim_{\epsilon\to 0} \int_0^T \phi(t,\omega) \frac{B(t+\epsilon) - B(t)}{\epsilon}\, dt, \tag{2.9}$$

if the limit exist in probability, in which case ϕ is called forward integrable.

Note that if ϕ is càdlàg and forward integrable, then

$$\int_0^T \phi(t,\omega)\, d^- B(t) = \lim_{\Delta t\to 0} \sum_j \phi(t_j)\Delta B(t_j). \tag{2.10}$$

where $\Delta B(t_j) = B(t_{j+1}) - B(t_j)$ and the sum is taken over the points of a finite partition of $[0,T]$.

Definition 2.3 *Let $\mathcal{M}^B$ denote the set of stochastic processes $\phi : [0,T] \times \Omega \to \mathbb{R}$ such that:*

1. *$\phi \in L^2(\lambda \times P)$, $\phi(t) \in \mathbb{D}^B_{1,2}$ for almost all t and satisfies*

$$\mathbb{E}\left(\int_0^T |\phi(t)|^2\, dt + \int_0^T \int_0^T |D_u\phi(t)|^2\, dudt\right) < \infty\,.$$

   We will denoted by $\mathbb{L}^{1,2}[0,T]$ the class of such processes.

2. *$D_{t+}\phi(t) := \lim_{s\to t+} D_s\phi(t)$ exists in $L^1(\lambda \times P)$ uniformly in $t \in [0,T]$.*

We let $\mathbb{M}^B_{1,2}$ be the closure of the linear span of $\mathcal{M}^B$ with respect to the norm given by

$$\|\phi\|_{\mathbb{M}^B_{1,2}} := \|\phi\|_{\mathbb{L}^{1,2}[0,T]} + \|D_{t+}\phi(t)\|_{L^1(\lambda\times P)}$$

Then we have the relation between the forward integral and the Skorohod integral (see [15, 8]):

Lemma 2.4 *If $\phi \in \mathbb{M}^B_{1,2}$ then it is forward integrable and*

$$\int_0^T \phi(t)d^-B(t) = \int_0^T \phi(t)\delta B(t) + \int_0^T D_{t+}\phi(t)dt\,. \tag{2.11}$$

Moreover

$$\mathbb{E}\left[\int_0^T \phi(t)d^-B(t)\right] = \mathbb{E}\left[\int_0^T D_{t+}\phi(t)dt\right]\,. \tag{2.12}$$

Using (2.11) and the duality formula for the Skorohod integral see e.g. [8], we deduce the following result.

Corollary 2.5 *Suppose $\phi \in \mathbb{M}^B_{1,2}$ and $F \in \mathbb{D}^{(B)}_{1,2}$ then*

$$\begin{aligned}\mathbb{E}\left[F\int_0^T \phi(t)d^-B(t)\right] &= \mathbb{E}\left[F\int_0^T \phi(t)\delta B(t) + F\int_0^T D_{t+}\phi(t)dt\right]\\ &= \mathbb{E}\left[\int_0^T \phi(t)D_tF\,dt + \int_0^T F\,D_{t+}\phi(t)dt\right]\end{aligned} \tag{2.13}$$

Proposition 2.6 *Let $\mathcal{H}$ be a given fixed σ-algebra and $\varphi : [0,T]\times\Omega \to \mathbb{R}$ be a $\mathcal{H}$-measurable process. Set $X(t) = E[B(t)|\,\mathcal{H}]$. Then*

$$E\left[\int_0^T \varphi(t)d^-B(t)\middle|\,\mathcal{H}\right] = E\left[\int_0^T \varphi(t)d^-X(t)\right] \tag{2.14}$$

Proof. Using uniform convergence on compacts in $L^1(P)$ and the definition of forward integration in the sense of Russo-Vallois (see [24]) we observe that

$$\begin{aligned} E\,[\int_0^T \varphi(t) d^- B(t)\,|\mathcal{H}] =& E\,[\lim_{\epsilon\to 0^+}\int_0^T \varphi(t)\frac{B(t+\epsilon)-B(t)}{\epsilon}dt\,|\mathcal{H}] \\ =& L^1(P) - \lim_{\epsilon\to 0^+} E\,[\int_0^T \varphi(t)\frac{B(t+\epsilon)-B(t)}{\epsilon}dt\,|\mathcal{H}] \\ =& \lim_{\epsilon\to 0^+}\int_0^T \varphi(t) E\,[\frac{B(t+\epsilon)-B(t)}{\epsilon}\,|\mathcal{H}]\,dt \\ =& \lim_{\epsilon\to 0^+}\int_0^T \varphi(t)\frac{X(t+\epsilon)-X(t)}{\epsilon}dt \\ =& \int_0^T \varphi(t) d^- X(t), \quad \text{in the ucp sense} \end{aligned}$$

and the result follows. ∎

Definition 2.7 *Let $\mathbb{H} = \{\mathcal{H}_t\}_{0\le t\le T}$ be a given filtration and $\varphi : [0,T]\times\Omega \to \mathbb{R}$ be a $\mathbb{H}$-adapted process. The conditional forward integral of φ with respect to $B(\cdot)$ is defined by*

$$\int_0^T \varphi(t)\,E\,[\,d^- B(t)\,|\mathcal{H}_{t^-}] = \lim_{\epsilon\to 0}\int_0^T \varphi(t)\frac{E\,[\,B(t+\epsilon)-B(t)\,|\mathcal{H}_{t^-}]}{\epsilon}\,dt, \tag{2.15}$$

if the convergence holds uniformly on compacts in probability (i.e. ucp sense), where $\mathcal{H}_{t^-} = \bigvee_{s<t}\mathcal{H}_s$.

Remark 2.8 *Note that Definition 2.7 is different from Proposition 2.6 except if $\mathcal{H}_t = \mathcal{H}$ for all t*

2.2.2 Forward integral and Malliavin calculus for $\widetilde{N}(\cdot,\cdot)$

In this Section, we review the forward integral with respect to the Poisson random measure $\widetilde{N}$.

Definition 2.9 *The forward integral*

$$J(\phi) := \int_0^T \int_{\mathbb{R}_0} \phi(t,z)\widetilde{N}(dz, d^- t)\,,$$

with respect to the Poisson random measure $\widetilde{N}$, of a càdlàg stochastic function $\phi(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}$, with $\phi(t,z) = \phi(\omega,t,z)$, $\omega\in\Omega$, is defined as

$$J(\phi) = \lim_{m\to\infty}\int_0^T \int_{\mathbb{R}_0} \phi(t,z)1_{U_m}(z)\widetilde{N}(dz,dt)\,,$$

if the limit exist in $L^2(P)$. Here $U_m, m = 1,2,\ldots,$ is an increasing sequence of compact sets $U_m \subseteq \mathbb{R}\backslash\{0\}$ with $\nu(U_m) < \infty$ such that $\lim_{m\to\infty} U_m = \mathbb{R}\backslash\{0\}$. The integral on the right is for each m defined ω-wise in the usual way, as limits of integrals of simple integrands.

Definition 2.10 *Let $\mathcal{M}^{\widetilde{N}}$ denote the set of stochastic functions $\phi : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that:*

1. $\phi(t,z,\omega) = \phi_1(t,\omega)\phi_2(t,z,\omega)$ *where $\phi_1(\omega,t) \in \mathbb{D}_{1,2}^{\widetilde{N}}$ is càdlàg and $\phi_2(\omega,t,z)$ is adapted such that*
$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \phi_2(t,z)\nu(dz)dt\right] < \infty\,,$$
2. $D_{t+,z}\phi := \lim_{s\to t+} D_{s,z}\phi$ *exists in $L^2(\lambda \times \nu \times P)$,*
3. $\phi(t,z) + D_{t+,z}\phi(t,z)$ *is Skorohod integrable.*

We let $\mathbb{M}_{1,2}^{\widetilde{N}}$ be the closure of the linear span of $\mathcal{M}^{\widetilde{N}}$ with respect to the norm given by

$$\|\phi\|_{\mathbb{M}_{1,2}^{\widetilde{N}}} := \|\phi\|_{L^2(\lambda\times\nu\times P)} + \|D_{t+,z}\phi(t,z)\|_{L^2(\lambda\times\nu\times P)}$$

Then we have the relation between the forward integral and the Skorohod integral (see [6, 8]):

Lemma 2.11 *If $\phi \in \mathbb{M}_{1,2}^{\widetilde{N}}$ then it is forward integrable and*

$$\int_0^T \int_{\mathbb{R}_0} \phi(t,z)\widetilde{N}(dz,d^-t) = \int_0^T \int_{\mathbb{R}_0} D_{t+,z}\phi(t,z)\nu(dz)dt + \int_0^T \int_{\mathbb{R}_0} (\phi(t,z)+D_{t+,z}\phi(t,z))\widetilde{N}(dz,\delta t)\,. \tag{2.16}$$

Moreover,

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \phi(t,z)\widetilde{N}(dz,d^-t)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} D_{t+,z}\phi(t,z)\nu(dz)dt\right]\,. \tag{2.17}$$

Then by (2.16) and duality formula for Skorohod integral for Poisson process see [8], we have

Corollary 2.12 *Suppose $\phi \in \mathbb{M}_{1,2}^{\widetilde{N}}$ and $F \in \mathbb{D}_{1,2}^{\widetilde{N}}$, then*

$$\begin{aligned}
&\mathbb{E}\left[F\int_0^T \int_{\mathbb{R}_0} \phi(t,z)\widetilde{N}(dz,d^-t)\right]\\
&= \mathbb{E}\left[F\int_0^T \int_{\mathbb{R}_0} D_{t+,z}\phi(t,z)\nu(dz)dt\right] + \mathbb{E}\left[F\int_0^T \int_{\mathbb{R}_0} (\phi(t,z)\,+\,D_{t+,z}\phi(t,z))\widetilde{N}(dz,\delta t)\right]\\
&= \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \phi(t,z)D_{t,z}F\nu(dz)dt\right] + \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} (F\,+\,D_{t,z}F)D_{t+,z}\phi(t,z)\nu(dz)dt\right].
\end{aligned} \tag{2.18}$$

3 A Stochastic Maximum Principle for insider

In view of the optimization problem (1.4) we require the following conditions 1–5 on the coefficients and on the family of admissible controls $\mathcal{A}_{\mathbb{G}}$:

1. The functions $b : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\theta : [0,T] \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega \to \mathbb{R}$, $f : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \Omega \to \mathbb{R}$ are contained in C^1 with respect to the arguments $x \in \mathbb{R}$ and $u \in U$ for each $t \in \mathbb{R}$ and a.a $\omega \in \Omega$.

2. For all $r, t \in (0,T)$, $t \leq r$ and all bounded $\mathcal{G}_t-$measurable random variables $\alpha = \alpha(\omega)$, $\omega \in \Omega$, the control

$$\beta_\alpha(s) := \alpha(\omega)\chi_{[t,r]}(s), \; 0 \leq s \leq T\,, \tag{3.1}$$

is an admissible control i.e., belongs to $\mathcal{A}_{\mathbb{G}}$ (here $\chi_{[t,r]}$ denotes the indicator function on $[t,r]$).

3. For all $u, \beta \in \mathcal{A}_{\mathbb{G}}$ with β bounded, there exists a $\delta > 0$ such that

$$u + y\beta \in \mathcal{A}_{\mathbb{G}}\,, \; \text{for all } y \in (-\delta, \delta) \tag{3.2}$$

and such that the family

$$\left\{ \frac{\partial}{\partial x} f(t, X^{u+y\beta}(t), u(t) + y\beta(t)) \frac{d}{dy} X^{u+y\beta}(t) \right. \\ \left. + \frac{\partial}{\partial u} f(t, X^{u+y\beta}(t), u(t) + y\beta(t))\beta(t) \right\}_{y \in (-\delta,\delta)}$$

is $\lambda \times P$-uniformly integrable and

$$\left\{ g'(X^{u+y\beta}(T)) \frac{d}{dy} X^{u+y\beta}(T) \right\}_{y \in (-\delta,\delta)}$$

is P-uniformly integrable.

4. For all $u, \beta \in \mathcal{A}_{\mathbb{G}}$ with β bounded the process

$$Y(t) = Y_\beta(t) = Y_\beta^u(t) = \left. \frac{d}{dy} X^{(u+y\beta)}(t) \right|_{y=0}$$

exists and follows the stochastic differential equation

$$\begin{aligned} dY_\beta^u(t) = & Y_\beta(t^-) \left[\frac{\partial}{\partial x} b(t, X^u(t), u(t))\, dt + \frac{\partial}{\partial x} \sigma(t, X^u(t), u(t))\, d^- B(t) \right. \\ & \left. + \int_{\mathbb{R}_0} \frac{\partial}{\partial x} \theta(t, X^u(t), u(t), z)\, \widetilde{N}(dz, d^- t) \right] \\ & + \beta(t) \left[\frac{\partial}{\partial u} b(t, X^u(t), u(t))\, dt + \frac{\partial}{\partial u} \sigma(t, X^u(t), u(t))\, d^- B(t) \right. \\ & \left. + \int_{\mathbb{R}_0} \frac{\partial}{\partial u} \theta(t, X^u(t), u(t), z)\, \widetilde{N}(dz, d^- t) \right] \\ Y(0) = & 0 \end{aligned} \tag{3.3}$$

5. Suppose that for all $u \in \mathcal{A}_{\mathbb{G}}$ the processes

$$\begin{aligned}
K(t) &:= g'(X(T)) + \int_t^T \frac{\partial}{\partial x} f(s, X(s), u(s))\, ds &(3.4)\\
D_t K(t) &:= D_t g'(X(T)) + \int_t^T D_t \frac{\partial}{\partial x} f(s, X(s), u(s))\, ds\\
D_{t,z} K(t) &:= D_{t,z} g'(X(T)) + \int_t^T D_{t,z} \frac{\partial}{\partial x} f(s, X(s), u(s))\, ds\\
H_0(s,x,u) &:= K(s) \Big(b(s,x,u) + D_{s+}\sigma(s,x,u) + \int_{\mathbb{R}_0} D_{s+,z}\theta(s,x,u,z)\,\nu(dz) \Big)\\
&\quad + D_s K(s) \sigma(s,x,u) &(3.5)\\
&\quad + \int_{\mathbb{R}_0} D_{s,z} K(s) \Big\{ \theta(s,x,u,z) + D_{s+,z}\theta(s,x,u,z) \Big\} \nu(dz)\\
G(t,s) &:= \exp\Bigg(\int_t^s \Big\{ \frac{\partial b}{\partial x}(r, X(r), u(r)) - \frac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2 (r, X(r), u(r)) \Big\}\, dr\\
&\quad + \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), u(r))\, dB^-(r)\\
&\quad + \int_t^s \int_{\mathbb{R}_0} \Big\{ \ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r), u(r), z)\Big) - \frac{\partial \theta}{\partial x}(r, X(r), u(r), z) \Big\} \nu(dz)\, dr\\
&\quad + \int_t^s \int_{\mathbb{R}_0} \Big\{ \ln\Big(1 + \frac{\partial \theta}{\partial x}\big(r, X(r^-), u(r^-), z\big)\Big) \Big\} \widetilde{N}(dz, d^- r) \Bigg) &(3.6)
\end{aligned}$$

$$\begin{aligned}
p(t) &:= K(t) + \int_t^T \frac{\partial}{\partial x} H_0(s, X(s), u(s)) G(t,s)\, ds &(3.7)\\
q(t) &:= D_t p(t) &(3.8)\\
r(t,z) &:= D_{t,z} p(t); \quad t \in [0,T],\ z \in \mathbb{R}_0\,. &(3.9)
\end{aligned}$$

are well-defined.

Now let us introduce the *general Hamiltonian of an insider*

$$H : [0,T] \times \mathbb{R} \times U \times \Omega \longrightarrow \mathbb{R}$$

by

$$\begin{aligned}
H(t,x,u,\omega) := p(t) &\Big(b(t,x,u,\omega) + D_{t+}\sigma(t,x,u,\omega) + \int_{\mathbb{R}_0} D_{t+,z}\theta(t,x,u,\omega)\,\nu(dz) \Big)\\
&+ f(t,x,u,\omega) + q(t)\sigma(t,x,u,\omega)\\
&+ \int_{\mathbb{R}_0} r(t,z) \Big\{ \theta(t,x,u,z,\omega) + D_{t+,z}\theta(t,x,u,z,\omega) \Big\} \nu(dz) \qquad (3.10)
\end{aligned}$$

We can now state a general stochastic maximum principle for our control problem (1.4):

Theorem 3.1 *Retain the conditions 1–5. Assume that* $\widehat{u} \in \mathcal{A}_{\mathbb{G}}$ *is a* critical point *of the performance functional* $J(u)$ *in (1.4), that is*

$$\frac{d}{dy} J(\widehat{u} + y\beta)\Big|_{y=0} = 0 \tag{3.11}$$

for all bounded $\beta \in \mathcal{A}_{\mathbb{G}}$. *Then*

$$E\left[\frac{\partial}{\partial u}\widehat{H}(t, \widehat{X}(t), \widehat{u}(t))\Big| \mathcal{G}_t\right] + E[A] = 0 \quad a.e. \ in \quad (t, \omega), \tag{3.12}$$

where A *is given by Equation (7.21)*

$$\begin{aligned} \widehat{X}(t) =& X^{(\widehat{u})}(t), \\ \widehat{H}(t, \widehat{X}(t), u) =& p(t)\left(b(t, \widehat{X}, u) + D_{t+}\sigma(t, \widehat{X}, u) + \int_{\mathbb{R}_0} D_{t+,z}\theta(t, \widehat{X}, u)\,\nu(dz)\right) \\ &+ f(t, \widehat{X}, u) + q(t)\sigma(t, \widehat{X}, u) \\ &+ \int_{\mathbb{R}_0} r(t, z)\Big\{\theta(t, \widehat{X}, u, z) + D_{t+,z}\theta(t, \widehat{X}, u, z)\Big\}\nu(dz) \end{aligned} \tag{3.13}$$

with

$$\begin{aligned} \widehat{p}(t) =& \widehat{K}(t) + \int_t^T \frac{\partial}{\partial x}\widehat{H}_0(s, \widehat{X}(s), \widehat{u}(s))\widehat{G}(t, s)\,ds\,, \\ \widehat{K}(t) :=& g'(\widehat{X}(T)) + \int_t^T \frac{\partial}{\partial x} f(s, \widehat{X}(s), \widehat{u}(s))ds, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \widehat{G}(t, s) :=& \exp\Bigg(\int_t^s \left\{\frac{\partial b}{\partial x}\Big(r, \widehat{X}(r), u(r)\Big) - \frac{1}{2}\left(\frac{\partial \sigma}{\partial x}\right)^2\Big(r, \widehat{X}(r), u(r)\Big)\right\} dr \\ &+ \int_t^s \frac{\partial \sigma}{\partial x}\Big(r, \widehat{X}(r), u(r)\Big)\,dB^-(r) \\ &+ \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial \theta}{\partial x}\Big(r, \widehat{X}(r), u(r), z\Big)\right) - \frac{\partial \theta}{\partial x}\Big(r, \widehat{X}(r), u(r), z\Big)\right\}\nu(dz)dt \\ &+ \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial \theta}{\partial x}\Big(r, \widehat{X}(r^-), u(r^-), z\Big)\right)\right\}\widetilde{N}(dz, d^-r)\Bigg) \\ \widehat{H}(t, x, u) =& \widehat{K}(t)\left(b(t, x, u) + D_{t+}\sigma(t, x, u) + \int_{\mathbb{R}_0} D_{t+,z}\theta(t, x, u)\,\nu(dz)\right) \\ &+ D_t\widehat{K}(t)\sigma(t, x, u) + f(t, x, u) \\ &+ \int_{\mathbb{R}_0} D_{t,z}\widehat{K}(t)\Big\{\theta(t, x, u, z) + D_{t+,z}\theta(t, x, u, z)\Big\}\nu(dz) \end{aligned}$$

Conversely, suppose there exists $\widehat{u} \in \mathcal{A}_{\mathbb{G}}$ *such that (3.12) holds. Then* $\widehat{u}$ *satisfies (3.11).*

Proof. See Appendix. ■

4 Controlled Itô-Lévy processes

The main result of the previous Section (Theorem 3.1) is difficult to apply because of the appearance of the terms $Y(t)$, $D_{t+}Y(t)$ and $D_{t+,z}Y(t)$, which all depend on the control u. However, consider the special case when the coefficients do not depend on X, i.e., when

$$\begin{aligned} b(t,x,u,\omega) = b(t,u,\omega), \quad \sigma(t,x,u,\omega) = \sigma(t,u,\omega) \\ \text{and } \theta(t,x,u,z,\omega) = \theta(t,u,z,\omega). \end{aligned} \tag{4.1}$$

Then the equation (1.2) gets the form

$$\begin{cases} d^{-}(X)(t) &= b(t,u(t),\omega)dt + \sigma(t,u(t),\omega)d^{-}B_t \\ & \quad + \int_{\mathbb{R}_0} \theta(t,u(t),z,\omega)\widetilde{N}(dz,d^{-}t); \\ X(0) &= x \in \mathbb{R} \end{cases} \tag{4.2}$$

We call such processes *controlled Itô-Lévy processes.*

In this case, Theorem 3.1 simplifies to the following

Theorem 4.1 *Let $X(t)$ be a controlled Itô-Lévy process as given in Equation (4.2). Retain the conditions 1–5 as in Theorem 3.1.*
Then the following are equivalent:

1. *$\widehat{u} \in \mathcal{A}_{\mathbb{G}}$ is a critical point of $J(u)$,*
2. $$E\left[L(t)\alpha + M(t)D_{t+}\alpha + \int_{\mathbb{R}_0} R(t,z)D_{t+,z}\alpha\,\nu(dz)\right] = 0$$

   for all $\mathcal{G}_t$-measurable $\alpha \in \mathbb{D}_{1,2}$ and all $t \in [0,T]$, where

$$\begin{aligned} L(t) =& K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + \frac{\partial f(t)}{\partial u} \\ &+ \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\Big)\nu(dz) + D_tK(t)\frac{\partial \sigma(t)}{\partial u}, \end{aligned} \tag{4.3}$$

$$M(t) = K(t)\frac{\partial \sigma(t)}{\partial u} \tag{4.4}$$

and

$$R(t,z) = \{K(t) + D_{t,z}K(t)\}\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\Big). \tag{4.5}$$

Proof.

1. It is easy to see that in this case, $p(t) = K(t)$, $q(t) = D_tK(t)$, $r(t,z) = D_{t,z}K(t)$ and the general Hamiltonian H given by (3.10) is reduced to H_1 given as follows

$$\begin{aligned} H_1(s,x,u,\omega) :=& K(s)\left(b(s,u,\omega) + D_{s+}\sigma(s,u,\omega) + \int_{\mathbb{R}_0} D_{s+,z}\theta(s,u,\omega)\nu(dz)\right) \\ &+ D_sK(s)\sigma(s,u,\omega) + f(s,x,u,\omega) \\ &+ \int_{\mathbb{R}_0} D_{s,z}K(s)\Big\{\theta(s,u,z,\omega) + D_{s+,z}\theta(s,u,z,\omega)\Big\}\nu(dz). \end{aligned}$$

Then, performing the same calculus lead to

$$
\begin{aligned}
A_1 =& A_3 = A_5 = 0, \\
A_2 =& E\left[\int_t^{t+h} \left\{K(t)\left(\frac{\partial b(s)}{\partial u} + D_{s+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \gamma(s)}{\partial u}\nu(dz)\right) + \frac{\partial f(s)}{\partial u}\right.\right. \\
&\left.\left. + \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \gamma(s)}{\partial u}\Big)\nu(dz) + D_sK(s)\frac{\partial \sigma(s)}{\partial u}\right\}\alpha ds\right], \\
A_4 =& E\left[\int_t^{t+h} K(s)\frac{\partial \sigma(s)}{\partial u}D_{s+}\alpha ds\right], \\
A_6 =& E\left[\int_t^{t+h}\int_{\mathbb{R}_0}\{K(s) + D_{s,z}K(s)\}\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \gamma(s)}{\partial u}\Big)\nu(dz)D_{s+,z}\alpha ds\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{d}{dh}A_2\bigg|_{h=0} =& E\left[\left\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + \frac{\partial f(t)}{\partial u}\right.\right. \\
&\left.\left. + \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz) + D_tK(t)\frac{\partial \sigma(t)}{\partial u}\right\}\alpha\right], \\
\frac{d}{dh}A_4\bigg|_{h=0} =& E\left[K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\alpha\right], \\
\frac{d}{dh}A_6\bigg|_{h=0} =& E\left[\int_{\mathbb{R}_0}\{K(t) + D_{t,z}K(t)\}\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz)D_{t+,z}\alpha\right].
\end{aligned}
$$

This means that

$$
\begin{aligned}
0 =& E\left[\left\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + \frac{\partial f(t)}{\partial u}\right.\right. \\
&\left. \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz) + D_tK(t)\frac{\partial \sigma(t)}{\partial u}\right\}\alpha \\
&\left. + K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\alpha + \left\{\int_{\mathbb{R}_0}\{K(t) + D_{t,z}K(t)\}\Big(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \gamma(t)}{\partial u}\Big)\nu(dz)\right\}D_{t+,z}\alpha\right],
\end{aligned}
$$

and the first part of the result follows.

2. The converse part follows from the arguments used in the proof of Theorem 3.1.

By this the proof is complete. ■

5 Applications to some special cases of filtrations

We consider the case of an insider who has an additional information compared to the standard normally informed investor.

- It can be the case of an insider who always has advanced information compared to the honest trader. This means that if $\mathbb{G} = \{\mathcal{G}_t\}_{0\le t\le T}$ and $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ represent respectively the flows of informations of the insider and the honest investor then we can write that $\mathcal{G}_t \supset \mathcal{F}_{t+\delta(t)}$ where $\delta(t) > 0$;

- It can also be the case of a trader who has at the initial date particular information about the future (initial enlargement of filtration). This means that if $\mathbb{G} = \{\mathcal{G}_t\}_{0\le t\le T}$ and $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ represent respectively the flows of informations of the insider and the honest investor then we can write that $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ where L is a random variable.

5.1 D-commutable filtrations

In the following we need the notion of *D-commutativity* of a σ-algebra.

Definition 5.1 *A σ-algebra $\mathcal{A} \subseteq \mathcal{F}$ is called D-commutable if for all $F \in \mathbb{D}_{1,2} = \mathbb{D}^B_{1,2} \cap \mathbb{D}^{\widetilde{N}}_{1,2}$ the conditional expectation $E[F\,|\mathcal{A}]$ belongs to $\mathbb{D}_{1,2}$ and*

$$D_t E[F\,|\mathcal{A}] = E[D_t F\,|\mathcal{A}], \tag{5.1}$$

$$D_{t,z} E[F\,|\mathcal{A}] = E[D_{t,z} F\,|\mathcal{A}] \tag{5.2}$$

Theorem 5.2 *Suppose that $\widehat{u} \in \mathcal{A}_{\mathbb{G}}$ is a critical point for $J(u)$. Assume that $\mathcal{G}_t$ is D-commutable for all t. Further require that for all t the set of smooth $\mathcal{G}_t$-measurable random variables is dense in $L^2(\mathcal{G}_t)$ and that $E[M(t)\,|\mathcal{G}_t]$ and $E[R(t,z)\,|\mathcal{G}_t]$ are Skorohod integrable. Then for any $t_0 \in [0,T)$*

$$\begin{aligned} 0 = &\int_0^T E[L(t)\,|\mathcal{G}_{t_0}]\, h(t)\, dt + \int_0^T E[M(t)\,|\mathcal{G}_{t_0}]\, h(t)\, \delta B_t \\ &+ \int_0^T \int_{\mathbb{R}_0} E[R(t,z)|\, \mathcal{G}_{t_0}]\, h(t)\, \widetilde{N}(\delta t, dz). \end{aligned} \tag{5.3}$$

for all $h \in L^2([0,T])$ with $\operatorname{supp} h \subseteq [t_0, T]$.

Proof. Without loss of generality, we give the proof for the Brownian motion case only. The pure jump case and mixed case follow similarly. Define $\langle X, Y\rangle = E[XY]$.
Let fix a $t_0 \in [0,T)$. Then, by assumption, it follows that for all $\mathcal{G}_{t_0}$-measurable smooth α and $h \in L^2([0,T])$ with

$$\operatorname{supp} h \subseteq [t_0, T], \quad t_0 \le t \le T,$$

$$0 = \left\langle \int_0^T E[L(t)\,|\mathcal{G}_{t_0}]\, h(t) dt, \alpha \right\rangle + \left\langle E[\int_0^T M(t) h(t) \delta B_t\,|\mathcal{G}_{t_0}], \alpha \right\rangle.$$

On the other hand the duality relation (2.7) implies

$$\begin{aligned}\left\langle E\left[\int_0^T M(t)h(t)\delta B_t \middle| \mathcal{G}_{t_0}\right], \alpha\right\rangle &= E\left[\int_0^T M(t)h(t)\delta B_t\, E\left[\alpha | \mathcal{G}_{t_0}\right]\right] \\ &= E\left[\int_0^T M(t)h(t)\left(D_t E\left[\alpha | \mathcal{G}_{t_0}\right]\right) dt\right] \\ &= E\left[\int_0^T M(t)h(t)E\left[D_t\alpha | \mathcal{G}_{t_0}\right] dt\right] \\ &= E\left[\int_0^T E\left[M(t)h(t) | \mathcal{G}_{t_0}\right] D_t\alpha\, dt\right] \\ &= \left\langle \int_0^T E\left[M(t) \mid \mathcal{G}_{t_0}\right] h(t)\delta B_t, \alpha\right\rangle\end{aligned}$$

for all $\mathcal{G}_{t_0}$-measurable smooth α. So

$$E\left[\int_0^T M(t)h(t)\delta B_t \middle| \mathcal{G}_{t_0}\right] = \int_0^T E\left[M(t) \mid \mathcal{G}_{t_0}\right] h(t)\delta B_t.$$

Hence, by our density assumption, we obtain that

$$0 = \int_0^T E\left[L(t) \,|\mathcal{G}_{t_0}\right] h(t)dt + \int_0^T E\left[M(t) \,|\mathcal{G}_{t_0}\right] h(t)\delta B_t.$$

By this the proof is complete. ■

To provide some concrete examples let us confine ourselves to the following type of filtrations $\{\mathcal{G}_t\}_{0\le t\le T}$. Given an increasing family of $\{G_t\}_{t\in[0,T]}$ of Borel sets $G_t \supset [0,t]$. Define

$$\{\mathcal{G}_t\}_{0\le t\le T} \text{ where } \mathcal{G}_t = \mathcal{F}_{G_t} = \sigma\left\{\int_0^T \chi_U(s)dB(s);\, U\subset G_t,\ U \text{ Borel}\right\} \vee \mathcal{N} \tag{5.4}$$

where $\mathcal{N}$ is the collection of P-null sets. Then Conditions (5.1) and (5.2) hold (see Proposition 3.12 in [8]). Examples of filtrations of type (5.4) are

$$\begin{aligned}\mathcal{G}_t^1 &= \mathcal{F}_{t+\delta(t)}, \\ \mathcal{G}_t^2 &= \mathcal{F}_{[0,t]\cup O},\end{aligned}$$

where O is an open set contained in $[0,T]$.
It is easily seen that filtrations of type (5.4) satisfy conditions of Theorem 5.2 as well. Hence, we have

Theorem 5.3 *Suppose that $\{\mathcal{G}_t\}_{0\le t\le T}$ is given by (5.4). Then $u=\widehat{u}$ is a critical point for $J(u)$ if and only if Equation (5.3) holds.*

From this, we get

Theorem 5.4 *Suppose that* $\{\mathcal{G}_t\}_{0\leq t\leq T}$ *is of type (5.4). Then there exists a critical point* $u=\widehat{u}$ *for the performance functional* $J(u)$ *in (1.3) if and only if the following three conditions hold:*

$$
\begin{aligned}
&(i)\ E\left[L(t)|\,\mathcal{G}_t\right]=0,\\
&(ii)\ E\left[M(t)|\,\mathcal{G}_t\right]=0,\\
&(iii)\ E\left[R(t,z)|\,\mathcal{G}_t\right]=0.
\end{aligned}
$$

where L, M *and* R *are given by (4.3), (4.4) and (4.5).*

Proof. This follows from the uniqueness of decomposition of Skorohod-semimartingale processes of type (5.3) (See Theorem 3.3 in [9].) ∎

Remark 5.5 *Not all filtrations satisfy conditions (5.1) and (5.2). An important example is the following: Choose the* σ*-field* $\mathcal{H}$ *to be* $\sigma(B(T))$*, where* $\{B(t)\}_{0\leq t\leq T}$*, is the Wiener process (Brownian motion) starting at* 0 *and* $T>0$ *is fixed. Then,* $\mathcal{H}$ *is not* D*-commutable. In fact, let* $F=B(t_0)$ *for some* $t_0<T$ *and choose* s *such that* $t_0<s<T$*. Then*

$$D_sE\left[B(t_0)|\mathcal{H}\right]=D_s\left(\frac{t_0}{T}B(T)\right)=\frac{t_0}{T},$$

while

$$E\left[D_sB(t_0)|\mathcal{H}\right]=E\left[0|\mathcal{H}\right]=0.$$

A similar argument works to prove that (5.1) and (5.2) are not satisfied for $\mathcal{G}_t=\mathcal{F}_t\vee\sigma(B_T)$ either. It follows that the technique used in the preceding Section cannot be applied to the σ-algebras of the type $\mathcal{F}_t\vee\sigma(B_T)$, and hence we need a different approach to discuss such cases.

5.2 Smoothly anticipative filtrations

In this Section, we consider σ-algebras which do not necessarily satisfy conditions (5.1) and (5.2). The starting point is again statement 2 of Theorem 4.1.

Definition 5.6 *We say that the filtration* $\{\mathcal{G}_t\}_{0\leq t\leq T}$ *is smoothly anticipative if for all* $t_0\in[0,T]$ *there exists a set* $\mathcal{A}=\mathcal{A}_{t_0}\subseteq\mathbb{D}_{1,2}\cap L^2(\mathcal{G}_{t_0})$ *and a measurable set* $\mathcal{M}\subset[t_0,T]$ *such that* $E\left[L(t)|\,\mathcal{G}_{t_0}\right]\cdot\chi_{[0,T]\cap\mathcal{M}}$, $E\left[M(t)|\,\mathcal{G}_{t_0}\right]\cdot\chi_{[0,T]\cap\mathcal{M}}$ *and* $E\left[R(t,z)|\,\mathcal{G}_{t_0}\right]\cdot\chi_{[0,T]\cap\mathcal{M}}$, $t\in[0,T],\ z\in\mathbb{R}_0$, *are Skorohod integrable and*

(i) $D_t\alpha$ *and* $D_{t,z}\alpha$ *are* $\mathcal{G}_{t_0}$*-measurable, for all* $\alpha\in\mathcal{A},\ t\in\mathcal{M}$.

(ii) $D_{t+}\alpha=D_t\alpha$ *and* $D_{t+,z}\alpha=D_{t,z}\alpha$ *for all* $\alpha\in\mathcal{A}$ *and a.a.* $t,z,\ t\in\mathcal{M}$.

(iii) $Span\mathcal{A}$ *is dense in* $L^2(\mathcal{G}_{t_0})$.

Theorem 5.7 *Suppose* $\{\mathcal{G}_t\}_{0\leq t\leq T}$ *is smoothly anticipative. Suppose* $\widehat{u}\in\mathcal{A}_{\mathbb{G}}$ *is a critical point of* $J(u)$*. Then for all* $h(t)=\chi_{[t_0,s)}(t)\chi_{\mathcal{M}}(t),\ t\in[0,T]$ *(and some* $s\in[0,T]$*)*

$$
\begin{aligned}
0=E\Bigg[&\int_0^T E\left[L(t)\ |\mathcal{G}_{t_0}\right]h(t)dt+\int_0^T E\left[M(t)\ |\mathcal{G}_{t_0}\right]h(t)\delta B_t\\
&+\int_0^T\int_{\mathbb{R}_0}E\left[R(t,z)|\,\mathcal{G}_{t_0}\right]h(t)\,\widetilde{N}(\delta t,dz)\Bigg|\,\mathcal{G}_{t_0}\Bigg].
\end{aligned}
\tag{5.5}
$$

Proof. By Theorem 4.1 we know that, for every t

$$E\left[L(t)\alpha + M(t)D_{t+}\alpha + \int_{\mathbb{R}_0} R(t,z)D_{t+,z}\alpha\,\nu(dz)\right] = 0$$

Let $\alpha = E\left[F|\,\mathcal{G}_{t_0}\right]$ for all $F \in \mathcal{A}$. Further, choose $h \in L^2([0,T])$ with $h(t) = \chi_{[t_0,s)}(t)\chi_{\mathcal{M}}(t)$. By assumption, we see that

$$\begin{aligned}
0 = &\left\langle \int_0^T E\left[L(t)\,|\mathcal{G}_{t_0}\right]h(t)dt, \alpha\right\rangle + \left\langle E\left[\int_0^T M(t)h(t)\delta B_t \middle|\, \mathcal{G}_{t_0}\right], \alpha\right\rangle \\
&+ \left\langle E\left[\int_0^T \int_{\mathbb{R}_0} R(t,z)h(t)\,\widetilde{N}(\delta t, dz)\middle|\, \mathcal{G}_{t_0}\right], \alpha\right\rangle.
\end{aligned}$$

On the other hand, the duality relation (2.7) and (ii) imply that

$$\begin{aligned}
\left\langle E\left[\int_0^T M(t)h(t)\delta B_t \middle|\, \mathcal{G}_{t_0}\right], \alpha\right\rangle &= E\left[\int_0^T M(t)h(t)\delta B_t\, E\left[F|\,\mathcal{G}_{t_0}\right]\right] \\
&= E\left[\int_0^T M(t)h(t)\left(D_t E\left[F|\,\mathcal{G}_{t_0}\right]\right)dt\right] \\
&= E\left[\int_0^T E\left[M(t)|\,\mathcal{G}_{t_0}\right]h(t)\left(D_t E\left[F|\,\mathcal{G}_{t_0}\right]\right)dt\right] \\
&= E\left[\int_0^T E\left[M(t)\,|\,\mathcal{G}_{t_0}\right]h(t)\delta B_t\, E\left[F|\,\mathcal{G}_{t_0}\right]\right] \\
&= \left\langle \int_0^T E\left[M(t)\,|\,\mathcal{G}_{t_0}\right]h(t)\delta B_t, \alpha\right\rangle.
\end{aligned}$$

In the same way, we show that

$$\left\langle E\left[\int_0^T \int_{\mathbb{R}_0} R(t,z)h(t)\,\widetilde{N}(\delta t, dz)\middle|\, \mathcal{G}_{t_0}\right], \alpha\right\rangle = \left\langle \int_0^T \int_{\mathbb{R}_0} E\left[R(t,z)\,|\,\mathcal{G}_{t_0}\right]h(t)\,\widetilde{N}(\delta t, dz), \alpha\right\rangle.$$

Then it follows from (iv) that

$$\begin{aligned}
0 = E\Bigg[&\int_0^T E\left[L(t)\,|\mathcal{G}_{t_0}\right]h(t)dt + \int_0^T E\left[M(t)\,|\mathcal{G}_{t_0}\right]h(t)\delta B_t \\
&+ \int_0^T \int_{\mathbb{R}_0} E\left[R(t,z)|\,\mathcal{G}_{t_0}\right]h(t)\,\widetilde{N}(\delta t, dz)\Bigg|\, \mathcal{G}_{t_0}\Bigg].
\end{aligned}$$

for all $h \in L^2([0,T])$ with $\operatorname{supp} h \subseteq (t_0, T]$. ∎

Theorem 5.8 *[Brownian motion case] Assume that the conditions in Theorem 5.7 are in force and $\theta = 0$. In addition, we require that $E\left[M(t)|\,\mathcal{G}_{t-}\right] \in \mathbb{M}_{1,2}^B$ and is forward integrable with respect to $E\left[\,d^-B(t)\,|\mathcal{G}_{t-}\right]$. Then*

$$\begin{aligned}
0 = &\int_0^T E\left[L(t)\,|\mathcal{G}_{t-}\right]h_0(t)dt + \int_0^T E\left[M(t)\,|\mathcal{G}_{t-}\right]h_0(t)E\left[\,d^-B\,|\mathcal{G}_{t-}\right] \\
&- \int_0^T D_{t+}E\left[M(t)|\,\mathcal{G}_{t-}\right]h_0(t)dt
\end{aligned} \tag{5.6}$$

for all bounded deterministic functions $h_0(t),\ t \in [0,T]$.

Proof. We apply the preceding result to $h(t) = h_0(t)\chi_{[t_i,t_{i+1}]}(t)$, where $0 = t_0 < t_1 < \ldots < t_i < t_{i+1} = T$ is a partition of $[0, T]$. From Equation (5.5), we have

$$0 = \int_{t_i}^{t_{i+1}} E\left[L(t) \;|\mathcal{G}_{t_i}\right] h(t)dt + E\left[\int_{t_i}^{t_{i+1}} E\left[M(t) \;|\mathcal{G}_{t_i}\right] h(t)\delta B_t \bigg| \mathcal{G}_{t_i}\right]$$
$$+ E\left[\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}_0} E\left[R(t,z)|\, \mathcal{G}_{t_i}\right] h(t)\, \widetilde{N}(\delta t, dz) \bigg| \mathcal{G}_{t_i}\right]. \tag{5.7}$$

By Lemma 2.4 and by assumption, we know that

$$\int_{t_i}^{t_{i+1}} E\left[M(t) \;|\mathcal{G}_{t_i}\right] h_0(t)\delta B_t = \int_{t_i}^{t_{i+1}} E\left[M(t) \;|\mathcal{G}_{t_i}\right] h_0(t)d^- B(t)$$
$$- \int_{t_i}^{t_{i+1}} D_{t^+} E\left[M(t)|\, \mathcal{G}_{t_i}\right] h_0(t)dt. \tag{5.8}$$

Substituting (5.8) into (5.7) and summing over all i and taking the limit as $\Delta t_i \to 0$, we get

$$0 = \lim_{\substack{\Delta t_i \to 0 \\ n\to\infty}} \left\{ \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} E\left[L(t) \;|\mathcal{G}_{t_i}\right] h_0(t)dt \right.$$
$$+ \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} E\left[M(t) \;|\mathcal{G}_{t_i}\right] h_0(t) \frac{E\left[B(t_{i+1}) - B(t_i)|\, \mathcal{G}_{t_i}\right]}{\Delta t_i} \Delta t_i$$
$$\left. - \sum_{i=1}^{n} \int_{t_i}^{t_{i+1}} D_{t^+} E\left[M(t) \;|\mathcal{G}_{t_i}\right] h_0(t)dt \right\},$$

in the topology of uniform convergence in probability. Hence, by Definition 2.7, we get the result. ■

Important examples of filtrations satisfying the conditions of Theorem 5.7 are based on σ-algebras that are first chaos generated (see [19]). Namely, we consider σ-algebras of the form

$$\sigma(I_1(h_i), i \in \mathbb{N}, h_i \in L^2([0,T])) \vee \mathcal{N}, \tag{5.9}$$

where $\mathcal{N}$ is the collection of $P-$null sets. Concrete examples of these σ-algebras are

$$\mathcal{G}_t^3 = \mathcal{F}_t \vee \sigma(B(T)) \tag{5.10}$$

or (see (5.15) below)

$$\mathcal{G}_t^4 = \mathcal{F}_t \vee \sigma\left(B(t + \delta_n(t))\right); \;\; n = 1, 2, \ldots \tag{5.11}$$

We first study the case (5.10).

Lemma 5.9 *Suppose that $\mathcal{G}_t = \mathcal{G}_t^3 = \mathcal{F}_t \vee \sigma(B(T))$. Then*

$$E\left[B(t) \mid \mathcal{G}_{t_0}\right] = \frac{T-t}{T-t_0} B(t_0) + \frac{t-t_0}{T-t_0} B(T) \text{ for all } t > t_0.$$

In particular

$$E\left[B(t+\varepsilon) \mid \mathcal{G}_t\right] = B(t) + \frac{\varepsilon}{T-t}\left(B(T) - B(t)\right)$$

Proof. We have that

$$E\left[B(t)\mid\mathcal{G}_{t_0}\right]=\int_0^{t_0}\varphi(t,s)dB(s)+C(t)B(T).$$

On one hand, we have

$$\begin{aligned} t=E\left[E\left[B(t)\mid\mathcal{G}_{t_0}\right]B(T)\right]=&E\left[\left(\int_0^{t_0}\varphi(t,s)dB(s)\right)B(T)\right]+C(t)T\\ =&\int_0^{t_0}\varphi(t,s)ds+C(t)T. \end{aligned}\tag{5.12}$$

On the other hand

$$\begin{aligned} u=E\left[E\left[B(t)\mid\mathcal{G}_{t_0}\right]B(u)\right]=&E\left[\left(\int_0^{t_0}\varphi(t,s)dB(s)\right)B(u)\right]+C(t)u\\ =&\int_0^{u}\varphi(t,s)ds+C(t)u,\ \text{for all } u<t. \end{aligned}\tag{5.13}$$

Differentiating Equation (5.13) with respect to u, it follows that

$$\varphi(t,u)+C(t)=1.$$

Substituting φ by its value in Equation(5.12), we obtain $C(t)=\frac{t-t_0}{T-t_0}$ and then $\varphi(t,s)=\frac{T-t_0}{T-t_0}$. Therefore, the result follows. ■

Corollary 5.10 *Suppose that $\mathcal{G}_t=\mathcal{G}_t^3=\mathcal{F}_t\vee\sigma(B(T))$. Then*

$$E\left[\, d^-B\,|\mathcal{G}_{t^-}\right]=\frac{B(T)-B(t)}{T-t}dt.$$

Combining this with Theorem 5.8 we get

Theorem 5.11 *Suppose $\mathcal{G}_t=\mathcal{G}_t^3=\mathcal{F}_t\vee\sigma(B(T))$ and $\theta=0$. Suppose the conditions of Theorem 5.8 hold. Then $u=\widehat{u}$ is a critical point for $J(u)$ in (1.3) if and only if*

$$E\left[\, L(t)\,|\mathcal{G}_{t^-}\right]+E\left[\, M(t)\,|\mathcal{G}_{t^-}\right]\frac{B(T)-B(t)}{T-t}=0 \text{ for a.a. } t\in[0,T]\,. \tag{5.14}$$

Next we study the case (5.11). For each $t\in[0,T)$, let $\{\delta_n\}_{n=0}^{\infty}=\{\delta_n(t)\}_{n=0}^{\infty}$ be a given decreasing sequence of numbers $\delta_n(t)\geq 0$ such that

$$t+\delta_n(t)\in[t,T] \text{ for all } n.$$

Define

$$\mathcal{G}_t=\mathcal{G}_t^4=\mathcal{F}_t\vee\sigma\left(B(t+\delta_n(t));\ \ n=1,2,\ldots\right)\tag{5.15}$$

Then, at each time t, the σ-algebra $\mathcal{G}_t^4$ contains full information about the values of the Brownian motion at the future times $t+\delta_n(t)$; $n=1,2,\ldots$ The amount of information that this represents, depends on the density of the sequence $\delta_n(t)$ near 0. Define

$$\rho_k(t)=\frac{1}{\delta_{k+1}^2}\left(\delta_k-\delta_{k+1}\right)\ln\left(\ln\left(\frac{1}{\delta_k-\delta_{k+1}}\right)\right);\ \ k=1,2,\ldots\tag{5.16}$$

We may regard $\rho_k(t)$ as a measure of how small $\delta_k - \delta_{k+1}$ is compared to δ_{k+1}. If $\rho_k(t) \to 0$, then $\delta_k \to 0$ slowly, which means that the controller has at time t many immediate future values of $B(t + \delta_k(t))$; $k = 1, 2, \cdots$, at her disposal when making her control value decision. For example, if

$$\delta_k(t) = \left(\frac{1}{k}\right)^p \text{ for some } p > 0,$$

then we see that

$$\lim_{k\to\infty} \rho_k(t) = \begin{cases} 0 & \text{if} \quad p < 1 \\ 1 & \text{if} \quad p = 1 \\ \infty & \text{if} \quad p > 1 \end{cases} \tag{5.17}$$

Lemma 5.12 *Suppose that $\mathcal{G}_t = \mathcal{G}_t^4$ as in (5.15) and that*

$$\lim_{k\to\infty} \rho_k(t) = 0 \quad \textit{a.s., uniformly in } t \in [0, T). \tag{5.18}$$

Then

$$E\left[d^- B(t) \,|\, \mathcal{G}_{t-}\right] = d^- B(t);\ t \in [0, T)$$

Proof. For each $\varepsilon > 0$, choose $\delta_k = \delta_k^{(\varepsilon)}$ such that

$$\delta_{k+1} < \varepsilon \le \delta_k.$$

Then

$$\begin{aligned}
&\frac{1}{\varepsilon} E\left[B(t+\varepsilon) - B(t) \,|\, \mathcal{G}_{t-}\right] \\
=&\frac{1}{\varepsilon} E\left[B(t+\varepsilon) - B(t) \,|\, \mathcal{F}_{t+\delta_{k+1}(t)} \vee \sigma\left(B(t + \delta_k(t))\right)\right] \\
=&\frac{1}{\varepsilon}\left[\frac{\delta_k - \varepsilon}{\delta_k - \delta_{k+1}} B(t + \delta_{k+1}) + \frac{\varepsilon - \delta_{k+1}}{\delta_k - \delta_{k+1}} B(t + \delta_k) - B(t)\right] \\
=&\frac{1}{\varepsilon}\left[B(t + \delta_{k+1}) - B(t) + \frac{\varepsilon - \delta_{k+1}}{\delta_k - \delta_{k+1}} \left\{B(t + \delta_k) - B(t + \delta_{k+1})\right\}\right] \\
=&\frac{\delta_{k+1}}{\varepsilon} \cdot \frac{1}{\delta_{k+1}} \left[B(t + \delta_{k+1}) - B(t)\right] + \frac{\varepsilon - \delta_{k+1}}{\varepsilon(\delta_k - \delta_{k+1})} \left[B(t + \delta_k) - B(t + \delta_{k+1})\right]
\end{aligned}$$

Note that

$$\frac{\varepsilon - \delta_{k+1}}{\varepsilon(\delta_k - \delta_{k+1})} \le \frac{1}{\delta_{k+1}}$$

and, by the law of iterated logarithm for Brownian motion (See e.g [23], p. 56),

$$\begin{aligned}
&\overline{\lim_{k\to\infty}} \frac{1}{\delta_{k+1}} \left|B(t + \delta_k) - B(t + \delta_{k+1})\right| \\
&= \overline{\lim_{k\to\infty}} \frac{1}{\delta_{k+1}} \left[(\delta_k - \delta_{k+1}) \ln\left(\ln\left(\frac{1}{\delta_k - \delta_{k+1}}\right)\right)\right]^{\frac{1}{2}} = 0 \text{ a.s.,}
\end{aligned}$$

uniformly in t, by assumption (5.18).
Therefore, since

$$\frac{\delta_{k+1}}{\delta_k} \leq \frac{\delta_{k+1}}{\varepsilon} \leq 1, \text{ for all } k$$

and

$$\frac{\delta_{k+1}}{\delta_k} \to 1 \text{ a.s.}, \; k \to \infty, \text{ again by (5.18)},$$

we conclude that, using Definition 2.7,

$$\begin{aligned}\int_0^T \varphi(t)E\left[d^-B(t) \mid \mathcal{G}_{t^-}\right] &= \lim_{\varepsilon\to 0}\int_0^T \varphi(t)\frac{E\left[B(t+\varepsilon)-B(t) \mid \mathcal{G}_{t^-}\right]}{\varepsilon}\,dt \\ &= \lim_{k\to\infty}\int_0^T \varphi(t)\frac{B(t+\delta_{k+1})-B(t)}{\delta_{k+1}}\,dt = \int_0^T \varphi(t)d^-B(t)\end{aligned}$$

in probability, for all bounded forward-integrable $\mathbb{G}$-adapted process φ. This proves the lemma. ∎

Combining this with Theorem 5.8 we get

Theorem 5.13 *Suppose $\mathcal{G} = \mathcal{G}_t^4$ as in (5.15) and $\theta = 0$. Suppose that (5.18) and the conditions of Theorem 5.8 hold. Then $u = \widehat{u}$ is a critical point for $J(u)$ in (1.3) if and only if*

$$E\left[\, L(t)\,|\mathcal{G}_{t^-}\right]dt + E\left[\, M(t)\,|\mathcal{G}_{t^-}\right]d^-B(t) - D_{t^+}E\left[\, M(t)\,|\mathcal{G}_{t^-}\right]dt = 0\;;\;\; t\in[0,T]\,. \tag{5.19}$$

6 Application to optimal insider portfolio

Consider a financial market with two investments possibilities:

1. A risk free asset, where the unit price $S_0(t)$ at time t is given by

$$dS_0(t) = r(t)S_0(t)\,dt, \;\; S_0(0) = 1. \tag{6.1}$$

2. A risky asset, where the unit price $S_1(t)$ at time t is given by the stochastic differential equation

$$dS_1(t) = S_1(t^-)\left[\mu(t)dt + \sigma_0(t)dB^-(t) + \int_{\mathbb{R}_0}\gamma(t,z)\widetilde{N}(d^-t,dz)\right], \;\; S_1(0) > 0. \tag{6.2}$$

Here $r(t) \geq 0$, $\mu(t), \sigma_0(t)$, and $\gamma(t,z) \geq -1+\epsilon$ (for some constant $\epsilon > 0$) are given $\mathbb{G}$-predictable, forward integrable processes, where $\mathbb{G} = \{\mathcal{G}_t\}_{0\leq t\leq T}$ is a given filtration such that

$$\mathcal{F}_t \subset \mathcal{G}_t \text{ for all } t\in[0,T] \tag{6.3}$$

Suppose a trader in this market is an insider, in the sense that she has access to the information represented by $\mathcal{G}_t$ at time t. This means that if she chooses a portfolio $u(t)$, representing the *amount* she invests in the risky asset at time t, then this portfolio is a $\mathbb{G}$-predictable stochastic process.
The corresponding wealth process $X(t) = X^{(u)}(t)$ will then satisfies the (forward) SDE

$$\begin{aligned} d^{-}X(t) =& \frac{X(t)-u(t)}{S_0(t)}dS_0(t) + \frac{u(t)}{S_1(t)}d^{-}S_1(t) \\ =& X(t)r(t)dt + u(t)\Big[\left(\mu(t)-r(t)\right)dt + \sigma_0(t)dB^{-}(t) \\ & + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^{-}t,dz)\Big], \quad t \in [0,T], \qquad (6.4) \\ X(0) =& x > 0. \qquad (6.5) \end{aligned}$$

By choosing $S_0(\cdot)$ as a numeraire, we can, without loss of generality, assume that

$$r(t) = 0 \qquad (6.6)$$

from now on. Then Equations (6.4) and (6.5) simplify to

$$\begin{cases} d^{-}X(t) &= u(t)\left[\mu(t)dt + \sigma_0(t)dB^{-}(t) + \int_{\mathbb{R}_0} \gamma(t,z)\widetilde{N}(d^{-}t,dz)\right], \\ X(0) &= x > 0. \end{cases} \qquad (6.7)$$

This is a controlled Itô-Lévy process of the type discussed in Section 4 and we can apply the results of that Section to the problem of the insider to maximize the expected utility of the terminal wealth, i.e., to find $\Phi_{\mathbb{G}}(x)$ and $u^* \in \mathcal{A}_{\mathbb{G}}$ such that

$$\Phi_{\mathbb{G}}(x) = \sup_{u \in \mathcal{A}_{\mathbb{G}}} E\left[U\left(X^{(u)}(T)\right)\right] = E\left[U\left(X^{(u^*)}(T)\right)\right], \qquad (6.8)$$

where $U : \mathbb{R}_+ \to \mathbb{R}$ is a given utility function, assumed to be concave, strictly increasing and C^1. In this case the processes $K(t),\ L(t),\ M(t)$ and $R(t,z)$, given respectively by Equations (3.4), (4.3), (4.4) and (4.5), take the form

$$\begin{aligned} K(t) =& U'\left(X(T)\right), \qquad (6.9) \\ L(t) =& U'\left(X(T)\right)\left[\mu(t) + D_{t+}\sigma_0(t) + \int_{\mathbb{R}_0} D_{t+,z}\gamma(t,z)\,\nu(dz)\right] \qquad (6.10) \\ & + \int_{\mathbb{R}_0} D_{t,z}U'\left(X(T)\right)\left[\gamma(t,z) + D_{t+,z}\gamma(t,z)\right]\,\nu(dz) + D_tU'\left(X(T)\right)\sigma_0(t), \\ M(t) =& U'\left(X(T)\right)\sigma_0(t), \qquad (6.11) \\ R(t,z) =& \left\{U'\left(X(T)\right) + D_{t,z}U'\left(X(T)\right)\right\}\left\{\gamma(t,z) + D_{t+,z}\gamma(t,z)\right\}. \qquad (6.12) \end{aligned}$$

6.1 Case $\mathcal{G}_t = \mathcal{F}_{G_t},\ G_t \supset [0,t]$. See (5.4).

In this case, $\mathcal{G}_t$ satisfies conditions (5.1) and (5.2). Therefore, Theorem 5.4 of Section 4 gives the following:

Theorem 6.1 *Suppose that $P\{\lambda\{t\in[0,T];\ \sigma_0(t)\neq 0\}>0\}>0$ where λ denotes the Lebesgue measure on $\mathbb{R}$ and that $\mathcal{G}_t$ is given by (5.4). Then, there does not exist an optimal portfolio $u^*\in\mathcal{A}_{\mathbb{G}}$ for the insider's portfolio problem (6.8).*

Proof. Suppose an optimal portfolio exists. Then we have seen that in either case, the conclusion is that

$$E\left[L(t)|\,\mathcal{G}_t\right]=E\left[M(t)|\,\mathcal{G}_t\right]=E\left[R(t,z)|\,\mathcal{G}_t\right]=0$$

for a.a. $t\in[0,T]$, $z\in\mathbb{R}_0$. In particular,

$$E\left[M(t)|\,\mathcal{G}_t\right]=E\left[U'\left(X(T)\right)\big|\,\mathcal{G}_t\right]\sigma_0(t)=0,\ \text{for a.a } t\in[0,T].$$

Since $U'>0$, this contradicts our assumption about U. Hence an optimal portfolio cannot exist. ■

Remark 6.2 *In the case that $\mathcal{G}_t=\mathcal{G}_t^i$, $i=1$ or $i=3$ it is known that $B(\cdot)$ is not a semimartingale with respect to $\mathbb{G}=\{\mathcal{G}_t\}_{0\leq t\leq T}$ and hence an optimal portfolio cannot exist, by Theorem 3.8 in [3] and Theorem 15 in [7]. It follows that $S_1(\cdot)$ is not a $\mathbb{G}$-semimartingale either and hence we can even deduce that the market has an arbitrage for the insider in this case, by Theorem 7.2 in [5]*

6.2 Case $\mathcal{G}_t=\mathcal{F}_t\vee\sigma(B(T))$. See (5.10).

In this case, $\mathcal{G}_t$ is not D-commutable (see Remark 5.5). Therefore we apply results from Section 5.2. We have seen that

$$E\left[\,d^-B\,|\mathcal{G}_{t-}\right]=\frac{B(T)-B(t)}{T-t}dt$$

(Corollary 5.10). It follows that

Theorem 6.3 *Assume that $\mu(t)=\mu_0$, $\sigma_0(t)=\sigma_0$ and $\gamma(t,z)=0$ and conditions in Theorem 5.7 hold. In addition, require that*

1. $E\left[M(t)|\,\mathcal{G}_{t-}\right]\in\mathbb{M}_{1,2}^B$
2. $\underline{\lim}_{t\uparrow T}E\left[\left|D_{t^+}E\left[M(t)|\,\mathcal{G}_{t-}\right]\right|\right]<\infty.$
3. $\underline{\lim}_{t\uparrow T}E\left[|L(t)|\right]<\infty.$

Then, there does not exist a critical point of the performance functional $J(u)$ in (1.3).

Proof. Assume that there is a critical point of the performance functional $J(u)$ in (1.3). It follows from Theorems 4.1, 5.7 and 5.8 that Equation 5.6 holds. Replacing $K(t)$, $L(t)$, and $M(t)$ by their given expressions in Equations (6.9), (6.10) and (6.11), Equation (5.6) becomes

$$\begin{aligned}0=&\,E\left[\mu_0U'(X(T))+\sigma_0D_tU'(X(T))\,|\mathcal{G}_{t-}\right]+\,E\left[U'(X(T))\sigma_0\,|\mathcal{G}_{t-}\right]\frac{B(T)-B(t)}{T-t}\\&-\,D_{t^+}E\left[\sigma_0U'(X(T))\big|\,\mathcal{G}_{t-}\right],\ \text{a.e } t\end{aligned}\tag{6.13}$$

Taking the limit as $t\uparrow T$, the second term in Equation (6.13) goes to ∞. Therefore, there is no critical point for the performance functional $J(u)$ in (1.3). ■

Remark 6.4 *This result is a generalization of a result in [14], where the same conclusion was obtained in the special case when*

$$U(x) = \ln(x)$$

6.3 Case $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(B(t+\delta_n(t));\ \ n = 1, 2, \ldots\right)$. See (5.11).

In this case, we have seen that if (5.18) holds then

$$E\left[d^- B(t) \mid \mathcal{G}_{t^-}\right] = d^- B(t)$$

(see Lemma 5.12). Therefore, we get

Theorem 6.5 *Suppose that, with $\mathcal{G}_t$ as above, (5.18) and the conditions of Theorem 5.8 are satisfied. Then u is a critical point for $J(u) = E\left[U(X^u(T))\right]$ if and only if*

$$E\left[L(t)|\,\mathcal{G}_{t^-}\right] - D_{t^+}E\left[M(t)|\,\mathcal{G}_{t^-}\right] = 0, \tag{6.14}$$

and

$$E\left[M(t)|\,\mathcal{G}_{t^-}\right] = 0, \ \textit{for a.a } t \in [0, T]. \tag{6.15}$$

Proof. This follows from Theorem 5.13 and the uniqueness of the decomposition of forward processes. ■

Corollary 6.6 *Suppose $\mathcal{G}_t$ is as in Theorem 6.5 and that $P\left(\lambda\left\{t \in [0, T];\ \sigma_0(t) \neq 0\right\} > 0\right) > 0$ where λ denotes the Lebesgue measure on $\mathbb{R}$. Then, there does not exist an optimal portfolio $u^* \in \mathcal{A}_{\mathbb{G}}$ for the performance $J(u) = E\left[U(X^u(T))\right]$.*

Proof. This follows from Equation (6.15) and the properties of the utility function U. ■

7 Application to optimal insider consumption

Suppose we have a cash flow $X(t) = X^{(u)}(t)$ given by

$$\begin{cases} dX(t) &= (\mu(t) - u(t))\,dt + \sigma(t)dB(t) + \displaystyle\int_{\mathbb{R}_0} \theta(t,z)\widetilde{N}(dt,dz), \\ X(0) &= x \in \mathbb{R}. \end{cases} \tag{7.1}$$

Here $\mu(t)$, $\sigma(t)$ and $\theta(t,z)$ are given $\mathbb{G}$-predictable processes and $u(t) \geq 0$ is our *consumption* rate, assumed to be adapted to a given insider filtration $\mathbb{G} = \{\mathcal{G}_t\}_{0 \leq t \leq T}$ where $\mathcal{F}_t \subset \mathcal{G}_t$ for all t. Let $f(t,u,\omega);\ t \in [0,T],\ u \in \mathbb{R},\ \omega \in \Omega$ be a given $\mathcal{F}_T$-measurable utility process. Assume that $u \to f(t,u,\omega)$ is strictly increasing, concave and C^1 for a.a (t,ω).
Let $g(x,\omega);\ x \in \mathbb{R},\ \omega \in \Omega$ be a given $\mathcal{F}_T$-measurable random variable for each x. Assume that $x \to g(x,\omega)$ is concave for a.a ω. Define the performance functional J by

$$J(u) = E\left[\int_0^T f(t,u(t),\omega)\,dt + g\left(X^{(u)}(T),\omega\right)\right];\ u \in \mathcal{A}_{\mathbb{G}},\ u \geq 0. \tag{7.2}$$

Note that $u \to J(u)$ is concave, so $u = \widehat{u}$ maximizes $J(u)$ if and only if $\widehat{u}$ is a critical point of $J(u)$.

Theorem 7.1 *(Optimal insider consumption I).*
$\widehat{u}$ is an optimal insider consumption rate for the performance functional J in Equation (7.2) if and only if

$$E\left[\frac{\partial}{\partial u}f(t,\widehat{u}(t),\omega)\middle|\,\mathcal{G}_t\right] = E\left[g'\left(X^{(\widehat{u})}(T),\omega\right)\middle|\,\mathcal{G}_t\right]. \tag{7.3}$$

Proof. In this case we have

$$\begin{aligned}K(t) &= g'\left(X^{(u)}(T)\right)\\ L(t) &= -g'\left(X^{(u)}(T)\right) + \frac{\partial}{\partial u}f(t,\widehat{u}(t))\\ M(t) &= R(t,z) = 0\end{aligned}$$

Therefore Theorem 4.1 gives $\widehat{u}$ is a critical point for $J(u)$ if and only if
$0 = E\left[L(t)|\,\mathcal{G}_t\right] = E\left[\frac{\partial}{\partial u}f(t,\widehat{u}(t))\middle|\,\mathcal{G}_t\right] + E\left[-g'\left(X^{(\widehat{u})}(T)\right)\middle|\,\mathcal{G}_t\right].$ ■

Since $X^{(\widehat{u})}(T)$ depends on $\widehat{u}$, Equation (7.3) does not give the value of $\widehat{u}(t)$ directly. However, in some special cases $\widehat{u}$ can be found explicitly:

Corollary 7.2 *(Optimal insider consumption II).*
Assume that

$$g(x,\omega) = \lambda(\omega)x \tag{7.4}$$

for some $\mathcal{G}_T$-measurable random variable $\lambda > 0$.
Then the optimal consumption rate $\widehat{u}(t)$ is given by

$$E\left[\frac{\partial}{\partial u}f(t,u,\omega)\middle|\,\mathcal{G}_t\right]_{u=\widehat{u}(t)} = E\left[\lambda|\,\mathcal{G}_t\right]. \tag{7.5}$$

Thus we see that an optimal consumption rate exists, for *any* given insider information filtration $\{\mathcal{G}_t\}_{0\le t\le T}$. It is not necessary to be in a semimartingale setting.

Another example in the same direction is the following.

Theorem 7.3 *(Complete future information).*
Suppose we have complete future information, i.e.,

$$\mathcal{G}_t = \mathcal{F}_T \text{ for all } t\in[0,T]. \tag{7.6}$$

Suppose we have the exponential utilities, i.e.

$$f(t,u,\omega) = -K_1(t,\omega)e^{-\alpha u},\;\; g(x,\omega) = -K_2(\omega)e^{-\alpha x} \tag{7.7}$$

for some measurable process $K_1(t,\omega) > 0$ and some $\mathcal{F}_T$-measurable random variable $K_2(\omega) > 0$ and some constant $\alpha > 0$.

Then the optimal consumptions rate $\widehat{u}(t)$, if it exists, satisfies the equation

$$\widehat{u}(t) = \frac{1}{\alpha} \ln \left(\frac{K_1(t)}{K_2} \right) + X^{(0)}(T) - \int_0^T \widehat{u}(s) ds \tag{7.8}$$

where

$$X^{(0)}(T) = x + \int_0^T \mu(s) ds + \int_0^T \sigma(s) dB(s) + \int_0^T \int_{\mathbb{R}_0} \theta(s,z) \widetilde{N}(ds,dz)$$

is the terminal wealth when there is no consumption.

In particular, if $K_1(t) = K_1$ does not depend on t, then $\widehat{u}(t) = \widehat{u}$ does not depend on t and we get

$$\widehat{u}(t) = \widehat{u} = \frac{1}{1+T} \left(\frac{1}{\alpha} \ln \left(\frac{K_1}{K_2} \right) + X^{(0)}(T) \right); \quad t \in [0,T]\,. \tag{7.9}$$

Proof. By (7.3) we get

$$-\alpha K_1(t) e^{-\alpha \widehat{u}(t)} = -\alpha K_2 e^{-\alpha X(T)}$$

or

$$\widehat{u}(t) = \frac{1}{\alpha} \ln \left(\frac{K_1(t)}{K_2} \right) + X(T) = \widehat{u}(t) = \frac{1}{\alpha} \ln \left(\frac{K_1(t)}{K_2} \right) + X^{(0)}(T) - \int_0^T \widehat{u}(s) ds,$$

which proves (7.8.) If $K_1(t) = K_1$ does not depend on t, then by (7.8) $\widehat{u}(t) = u(t)$ does not depend on t either and (7.9) follows. ∎

For related results (based on a different method) on optimal insider consumption see [22].

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Appendix: Proof of Theorem 3.1

Proof.

1. Since $\widehat{u} \in \mathcal{A}_{\mathbb{G}}$ is a critical point for $J(u)$, there exists a $\delta > 0$ as in Equation (3.2) for all bounded $\beta \in \mathcal{A}_{\mathbb{G}}$. Thus

$$
\begin{aligned}
0 &= \frac{d}{dy} J(\widehat{u} + y\beta)\Big|_{y=0} \qquad (7.10)\\
&= E\left[\int_0^T \left\{ \frac{\partial}{\partial x} f(t, X(t), u(t))\widehat{Y}(t) + \frac{\partial}{\partial u} f(t, X(t), u(t))\beta(t) \right\} dt + g'(X(T))\widehat{Y}(T)\right],
\end{aligned}
$$

where $\widehat{Y} = Y_\beta^{\widehat{u}}$ is as defined in Equation (3.3).

We study the two summands separately. By Corollary 2.5 and 2.12 and the product rule, we get

$$
\begin{aligned}
& E\left[g'(X(T))Y(T)\right] \\
=& E\Bigg[g'(X(T))\Bigg(\int_0^T \left\{ \frac{\partial b(t)}{\partial x} Y(t) + \frac{\partial b(t)}{\partial u}\beta(t)\right\} dt \\
&+ \int_0^T \left\{ \frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right\} d^- B(t) \\
&+ \int_0^T \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right\} \widetilde{N}(dz, d^- t)\Bigg)\Bigg] \\
=& E\left[\int_0^T g'(X(T)) \left\{ \frac{\partial b(t)}{\partial x} Y(t) + \frac{\partial b(t)}{\partial u}\beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T D_t g'(X(T)) \left\{ \frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T g'(X(T)) D_{t+} \left(\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right) dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} D_{t,z} g'(X(T)) \left\{ \frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right\} \nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{ g'(X(T)) + D_{t,z} g'(X(T))\right\} D_{t+,z} \left(\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right) \nu(dz)dt\right]
\end{aligned}
$$

$$
\begin{aligned}
=& E\left[\int_0^T \left\{ g'(X(T))\frac{\partial b(t)}{\partial x} + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\frac{\partial \theta(t)}{\partial x}\nu(dz)\right\} Y(t)dt\right] \\
&+ E\left[\int_0^T \left\{ g'(X(T))\frac{\partial b(t)}{\partial u} + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\frac{\partial \theta(t)}{\partial u}\nu(dz)\right\} \beta(t)dt\right] \\
&+ E\left[\int_0^T g'(X(T)) D_{t+}\frac{\partial \sigma(t)}{\partial x} Y(t)dt\right] + E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial x} D_{t+}Y(t)dt\right] \\
&+ E\left[\int_0^T g'(X(T)) D_{t+}\frac{\partial \sigma(t)}{\partial u} \beta(t)dt\right] + E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial u} D_{t+}\beta(t)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} D_{t+,z}\frac{\partial \theta(t)}{\partial x} Y(t)\nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} D_{t+,z}\frac{\partial \theta(t)}{\partial u} \beta(t)\nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right] \\
=& E\left[\int_0^T \left\{ g'(X(T))\left(\frac{\partial b(t)}{\partial x} + D_{t+}\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial x}\nu(dz)\right) + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial x}\right.\right. \\
&+ \left.\left. \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\left(\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right)\nu(dz)\right\} Y(t)dt\right] \\
&+ E\left[\int_0^T \left\{ g'(X(T))\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + D_t g'(X(T))\frac{\partial \sigma(t)}{\partial u}\right.\right. \\
&+ \left.\left. \int_{\mathbb{R}_0} D_{t,z} g'(X(T))\left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right)\nu(dz)\right\} \beta(t)dt\right] \\
&+ E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial x} D_{t+}Y(t)dt\right] \\
&+ E\left[\int_0^T g'(X(T))\frac{\partial \sigma(t)}{\partial u} D_{t+}\beta(t)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left\{g'(X(T)) + D_{t,z}g'(X(T))\right\} \left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right]
\end{aligned}
$$

Similarly, we have using both Fubini and duality theorems,

$$
\begin{aligned}
& E\left[\int_0^T \frac{\partial}{\partial x} f(t) Y(t) dt\right] \\
= & E\left[\int_0^T \frac{\partial}{\partial x} f(t)\left(\int_0^t\left\{\frac{\partial b(s)}{\partial x} Y(s)+\frac{\partial b(s)}{\partial u} \beta(s)\right\} ds\right.\right. \\
& +\int_0^t\left\{\frac{\partial \sigma(s)}{\partial x} Y(s)+\frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} d^{-} B(s) \\
& \left.\left.+\int_0^t \int_{\mathbb{R}_0}\left\{\frac{\partial \theta(s)}{\partial x} Y(s)+\frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \widetilde{N}\left(dz, d^{-} s\right)\right) dt\right] \\
= & E\left[\int_0^T\left(\int_0^t \frac{\partial f(t)}{\partial x}\left\{\frac{\partial b(s)}{\partial x} Y(s)+\frac{\partial b(s)}{\partial u} \beta(s)\right\} ds\right) dt\right] \\
+ & E\left[\int_0^T\left(\int_0^t D_s \frac{\partial f(t)}{\partial x}\left\{\frac{\partial \sigma(s)}{\partial x} Y(s)+\frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} ds\right) dt\right] \\
+ & E\left[\int_0^T\left(\int_0^t \frac{\partial f(t)}{\partial x} D_{s+}\left\{\frac{\partial \sigma(s)}{\partial x} Y(s)+\frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} ds\right) dt\right] \\
+ & E\left[\int_0^T\left(\int_0^t \int_{\mathbb{R}_0} D_{s,z} \frac{\partial f(t)}{\partial x}\left\{\frac{\partial \theta(s)}{\partial x} Y(s)+\frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \nu(dz) ds\right) dt\right] \\
+ & E\left[\int_0^T\left(\int_0^t \int_{\mathbb{R}_0}\left\{\frac{\partial f(t)}{\partial x}+D_{s,z} \frac{\partial f(t)}{\partial x}\right\} \times\right.\right. \\
& \left.\left. D_{s+,z}\left(\frac{\partial \theta(s)}{\partial x} Y(s)+\frac{\partial \theta(s)}{\partial u} \beta(s)\right) \nu(dz) ds\right) dt\right] \\
= & E\left[\int_0^T\left(\int_s^T \frac{\partial f(t)}{\partial x} dt\right)\left\{\frac{\partial b(s)}{\partial x} Y(s)+\frac{\partial b(s)}{\partial u} \beta(s)\right\} ds\right] \\
& + E\left[\int_0^T\left(\int_s^T D_s \frac{\partial f(t)}{\partial x} dt\right)\left\{\frac{\partial \sigma(s)}{\partial x} Y(s)+\frac{\partial \sigma(s)}{\partial u} \beta(s)\right\}\right] \\
& + E\left[\int_0^T\left(\int_s^T \frac{\partial f(t)}{\partial x} dt\right) D_{s+}\left\{\frac{\partial \sigma(s)}{\partial x} Y(s)+\frac{\partial \sigma(s)}{\partial u} \beta(s)\right\} ds\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0}\left(\int_s^T D_{s,z} \frac{\partial f(t)}{\partial x} dt\right)\left\{\frac{\partial \theta(s)}{\partial x} Y(s)+\frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \nu(dz) ds\right] \\
& + E\left[\int_0^T \int_{\mathbb{R}_0}\left(\int_s^T\left\{\frac{\partial f(t)}{\partial x}+D_{s,z} \frac{\partial f(t)}{\partial x}\right\} dt\right) \times\right. \\
& \left. D_{s+,z}\left\{\frac{\partial \theta(s)}{\partial x} Y(s)+\frac{\partial \theta(s)}{\partial u} \beta(s)\right\} \nu(dz) ds\right]
\end{aligned}
$$

Changing the notation $s \to t$, this becomes

$$
\begin{aligned}
&= E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \left\{\frac{\partial b(t)}{\partial x} Y(t) + \frac{\partial b(t)}{\partial u}\beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T \left(\int_t^T D_t \frac{\partial f(s)}{\partial x} ds\right) \left\{\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right\}\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z} \frac{\partial f(s)}{\partial x} ds\right) \left\{\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right\} \nu(dz)dt\right] \\
&+ E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) D_{t+} \left\{\frac{\partial \sigma(t)}{\partial x} Y(t) + \frac{\partial \sigma(t)}{\partial u}\beta(t)\right\} dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \left(\int_t^T \left\{\frac{\partial f(s)}{\partial x} + D_{t,z}\frac{\partial f(s)}{\partial x}\right\} ds\right)\right. \\
&\quad \left.\left(D_{t+,z} \left\{\frac{\partial \theta(t)}{\partial x} Y(t) + \frac{\partial \theta(t)}{\partial u}\beta(t)\right\}\right) \nu(dz)dt\right] \qquad (7.11) \\
&= E\left[\int_0^T \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial b(t)}{\partial x} + D_{t+}\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial x}\nu(dz)\right)\right.\right. \\
&\quad + \left(\int_t^T D_t \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial x} \\
&\quad \left.\left. + \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z} \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right) \nu(dz)\right\} Y(t)dt\right] \\
&\quad + E\left[\int_0^T \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right)\right.\right. \\
&\quad + \left(\int_t^T D_t \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial u} \\
&\quad \left.\left. + \int_{\mathbb{R}_0} \left(\int_t^T D_{t,z} \frac{\partial f(s)}{\partial x} ds\right) \left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right) \nu(dz)\right\} \beta(t)dt\right] \\
&\quad + E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial x} D_{t+}Y(t)dt\right] \\
&\quad + E\left[\int_0^T \left(\int_t^T \frac{\partial f(s)}{\partial x} ds\right) \frac{\partial \sigma(t)}{\partial u} D_{t+}\beta(t)dt\right] \\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} + D_{t,z}\frac{\partial f(s)}{\partial x} ds\right)\right\} \left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
&\quad + E\left[\int_0^T \int_{\mathbb{R}_0} \left\{\left(\int_t^T \frac{\partial f(s)}{\partial x} + D_{t,z}\frac{\partial f(s)}{\partial x} ds\right)\right\} \left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right]
\end{aligned}
$$

Recall that

$$K(t) \;:=\; g'(X(T)) + \int_t^T \frac{\partial}{\partial x} f(s, X(s), u(s))ds$$

and combining (3.11)-(7.11), it follows that

$$
\begin{aligned}
0 = E\Bigg[\int_0^T \Bigg\{ & K(t)\left(\frac{\partial b(t)}{\partial x} + D_{t+}\frac{\partial \sigma(t)}{\partial x} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial x}\nu(dz)\right) + D_t K(t)\frac{\partial \sigma(t)}{\partial x} \\
&+ \int_{\mathbb{R}_0} D_{t,z}K(t)\left(\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right)\nu(dz)\Bigg\} Y(t)dt\Bigg] \\
&+ E\Bigg[\int_0^T \Bigg\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t)}{\partial u}\nu(dz)\right) + D_t K(t)\frac{\partial \sigma(t)}{\partial u} \\
&+ \int_{\mathbb{R}_0} D_{t,z}K(t)\left(\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right)\nu(dz) + \frac{\partial f(t)}{\partial u}\Bigg\}\beta(t)dt\Bigg] \\
&+ E\left[\int_0^T K(t)\frac{\partial \sigma(t)}{\partial x}D_{t+}Y(t)dt\right] \\
&+ E\left[\int_0^T K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\beta(t)dt\right] \qquad (7.12) \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \{K(t) + D_{t,z}K(t)\}\left\{\frac{\partial \theta(t)}{\partial x} + D_{t+,z}\frac{\partial \theta(t)}{\partial x}\right\} D_{t+,z}Y(t)\nu(dz)dt\right] \\
&+ E\left[\int_0^T \int_{\mathbb{R}_0} \{K(t) + D_{t,z}K(t)\}\left\{\frac{\partial \theta(t)}{\partial u} + D_{t+,z}\frac{\partial \theta(t)}{\partial u}\right\} D_{t+,z}\beta(t)\nu(dz)dt\right]
\end{aligned}
$$

We observe that for all $\beta_\alpha \in \mathcal{A}_{\mathbb{G}}$ given as $\beta_\alpha(s) := \alpha\chi_{[t,t+h]}(s)$, for some $t, h \in (0,T)$, $t+h \le T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_t$-measurable. Then $Y^{(\beta_\alpha)}(s) = 0$ for $0 \le s \le t$ and hence (7.12) becomes

$$
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 0 \qquad (7.13)
$$

Where

$$
\begin{aligned}
A_1 = E\Bigg[\int_t^T \Bigg\{ & K(t)\left(\frac{\partial b(s)}{\partial x} + D_{s+}\frac{\partial \sigma(s)}{\partial x} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial x}\nu(dz)\right) \\
&+ \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz) + D_s K(s)\frac{\partial \sigma(s)}{\partial x}\Bigg\} Y^{(\beta_\alpha)}(s)ds\Bigg] \\
A_2 = E\Bigg[\int_t^{t+h} \Bigg\{ & K(t)\left(\frac{\partial b(s)}{\partial u} + D_{s+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial u}\nu(dz)\right) + \frac{\partial f(s)}{\partial u} \\
&+ \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz) + D_s K(s)\frac{\partial \sigma(s)}{\partial u}\Bigg\}\alpha ds\Bigg] \\
A_3 = E\Bigg[\int_t^T & K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y^{(\beta_\alpha)}(s)ds\Bigg] \\
A_4 = E\Bigg[\int_t^{t+h} & K(s)\frac{\partial \sigma(s)}{\partial u}D_{s+}\alpha ds\Bigg]
\end{aligned}
$$

$$A_5 = E\left[\int_t^T \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\} \Big(\frac{\partial\theta(s)}{\partial x} + D_{s+,z}\frac{\partial\theta(s)}{\partial x}\Big)\nu(dz) D_{s+,z}Y^{(\beta_\alpha)}(s)ds\right]$$

$$A_6 = E\left[\int_t^{t+h} \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\} \Big(\frac{\partial\theta(s)}{\partial u} + D_{s+,z}\frac{\partial\theta(s)}{\partial u}\Big)\nu(dz) D_{s+,z}\alpha ds\right]$$

Note by the definition of Y, with $Y(s) = Y^{(\beta_\alpha)}(s)$ and $s \geq t+h$, the process $Y(s)$ follows the dynamics

$$dY(s) = Y(s^-)\left[\frac{\partial b}{\partial x}(s)ds + \frac{\partial\sigma}{\partial x}(s)d^-B(s) + \int_{\mathbb{R}_0} \frac{\partial\theta}{\partial x}(s,z)\widetilde{N}(dz, d^-s)\right], \tag{7.14}$$

for $s \geq t+h$ with initial condition $Y(t+h)$ in time $t+h$. By Itô's formula for forward integral, this equation can be solved explicitly and we get

$$Y(s) = Y(t+h)G(t+h,s), \;\; s \geq t+h \tag{7.15}$$

where, in general, for $s \geq t$,

$$\begin{aligned} G(t,s) \;\; := \;\; \exp\Bigg(& \int_t^s \left\{\frac{\partial b}{\partial x}(r, X(r), u(r), \omega) - \frac{1}{2}\left(\frac{\partial\sigma}{\partial x}\right)^2 (r, X(r), u(r), \omega)\right\} dr \\ & + \int_t^s \frac{\partial\sigma}{\partial x}(r, X(r), u(r), \omega)\, dB^-(r) \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial\theta}{\partial x}(r, X(r), u(r), \omega)\right) - \frac{\partial\theta}{\partial x}(r, X(r), u(r), \omega)\right\}\nu(dz)dt \\ & + \int_t^s \int_{\mathbb{R}_0} \left\{\ln\left(1 + \frac{\partial\theta}{\partial x}\left(r, X(r^-), u(r^-), \omega\right)\right)\right\}\widetilde{N}(dz, d^-r)\Bigg). \end{aligned}$$

Note that $G(t,s)$ does not depend on h, but $Y(s)$ does. Defining H_0 as in (3.5), it follows that

$$A_1 = E\left[\int_t^T \frac{\partial H_0}{\partial x}(s)Y(s)ds\right].$$

Differentiating with respect to h at $h=0$, we get

$$\frac{d}{dh}A_1\bigg|_{h=0} = \frac{d}{dh}E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)Y(s)ds\right]_{h=0} + \frac{d}{dh}E\left[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)Y(s)ds\right]_{h=0}.$$

Since $Y(t)=0$, we see that

$$\frac{d}{dh}E\left[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)Y(s)ds\right]_{h=0} = 0$$

Therefore, by (7.15),

$$\begin{aligned} \frac{d}{dh}A_1\bigg|_{h=0} &= \frac{d}{dh}E\left[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)Y(t+h)G(t+h,s)ds\right]_{h=0} \\ &= \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)Y(t+h)G(t+h,s)\right]_{h=0} ds \\ &= \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)G(t,s)Y(t+h)\right]_{h=0} ds, \end{aligned}$$

where, $Y(t+h)$ is given by

$$Y(t+h) = \int_t^{t+h} Y(r^-)\left[\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)d^-B(r) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(r,z)\widetilde{N}(dz,d^-r)\right]$$
$$+ \alpha \int_t^{t+h} \left[\frac{\partial b}{\partial u}(r)dr + \frac{\partial \sigma}{\partial u}(r)d^-B(r) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(r,z)\widetilde{N}(dz,d^-r)\right].$$

Therefore, by the two preceding equalities,

$$\frac{d}{dh}A_1\Big|_{h=0} = A_{1,1} + A_{1,2},$$

where

$$A_{1,1} = \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)G(t,s)\alpha \int_t^{t+h}\left\{\frac{\partial b}{\partial u}(r)dr + \frac{\partial \sigma}{\partial u}(r)d^-B(r)\right.\right.$$
$$\left.\left. + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(r,z)\widetilde{N}(dz,d^-r)\right\}\right]_{h=0} ds,$$

and

$$A_{1,2} = \int_t^T \frac{d}{dh}E\left[\frac{\partial H_0}{\partial x}(s)G(t,s)\int_t^{t+h} Y(r^-)\left\{\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)d^-B(r)\right.\right.$$
$$\left.\left. + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(r,z)\widetilde{N}(dz,d^-r)\right\}\right]_{h=0} ds.$$

Applying again the duality formula, we have

$$\begin{aligned}
A_{1,1} = &\int_t^T \frac{d}{dh}E\left[\alpha \int_t^{t+h}\left\{\frac{\partial b}{\partial u}(r)F(t,s) + \frac{\partial \sigma}{\partial u}(r)D_rF(t,s) + F(t,s)D_{r^+}\frac{\partial \sigma}{\partial u}(r)\right.\right.\\
&+ \int_{\mathbb{R}_0}\left\{\left(\frac{\partial \theta}{\partial u}(r,z) + D_{r^+,z}\frac{\partial \theta}{\partial u}(r,z)\right)D_{r,z}F(t,s)\right.\\
&\left.\left.\left. + D_{r^+,z}\frac{\partial \theta}{\partial u}(r,z)F(t,s)\right\}\nu(dz)\right\}dr\right]_{h=0} ds\\
= &\int_t^T E\left[\alpha\left\{\left(\frac{\partial b}{\partial u}(t) + D_{t^+}\frac{\partial \sigma}{\partial u}(t) + \int_{\mathbb{R}_0} D_{t^+,z}\frac{\partial \theta}{\partial u}(t,z)\nu(dz)\right)F(t,s)\right.\right.\\
&\left.\left. + \frac{\partial \sigma}{\partial u}(t)D_tF(t,s) + \int_{\mathbb{R}_0}\left(\frac{\partial \theta}{\partial u}(t,z) + D_{t^+,z}\frac{\partial \theta}{\partial u}(t,z)\right)D_{t,z}F(t,s)\nu(dz)\right\}\right] ds,
\end{aligned}$$

where we have put

$$F(t,s) = \frac{\partial H_0}{\partial x}(s)G(t,s)$$

Since $Y(t) = 0$ we see that

$$A_{1,2} = 0.$$

We conclude that

$$\frac{d}{dh}A_1\Big|_{h=0} = A_{1,1} \tag{7.16}$$

Moreover, we see that

$$
\begin{aligned}
\frac{d}{dh}A_2\Big|_{h=0} =& E\Bigg[\Bigg\{K(t)\left(\frac{\partial b(t)}{\partial u} + D_{t+}\frac{\partial \sigma(t)}{\partial u} + \int_{\mathbb{R}_0} D_{t+,z}\frac{\partial \theta(t,z)}{\partial u}\nu(dz)\right) \\
&+ \frac{\partial f(t)}{\partial u} + D_t K(t)\frac{\partial \sigma(t,z)}{\partial u} \\
&+ \int_{\mathbb{R}_0} D_{t,z}K(t)\Big(\frac{\partial \theta(t,z)}{\partial u} + D_{t+,z}\frac{\partial \theta(t,z)}{\partial u}\Big)\nu(dz)\Bigg\}\alpha\Bigg], && (7.17) \\
\frac{d}{dh}A_4\Big|_{h=0} =& E\left[K(t)\frac{\partial \sigma(t)}{\partial u}D_{t+}\alpha\right], && (7.18) \\
\frac{d}{dh}A_6\Big|_{h=0} =& E\left[\int_{\mathbb{R}_0}\{K(t)+D_{t,z}K(t)\}\Big(\frac{\partial \theta(t,z)}{\partial u} + D_{t+,z}\frac{\partial \theta(t,z)}{\partial u}\Big)\nu(dz)D_{t+,z}\alpha\right]. && (7.19)
\end{aligned}
$$

On the other hand, by differentiating A_3 with respect to h at $h=0$, we get

$$
\begin{aligned}
\frac{d}{dh}A_3\Big|_{h=0} =& \frac{d}{dh}E\left[\int_t^{t+h} K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y(s)ds\right]_{h=0} \\
&+ \frac{d}{dh}E\left[\int_{t+h}^{T} K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y(s)ds\right]_{h=0}.
\end{aligned}
$$

Since $Y(t)=0$, we see that

$$
\begin{aligned}
\frac{d}{dh}A_3\Big|_{h=0} &= \frac{d}{dh}E\left[\int_{t+h}^{T} K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}\Big(Y(t+h)G(t+h,s)\Big)ds\right]_{h=0} \\
&= \int_t^T \frac{d}{dh}E\left[K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}\Big(Y(t+h)G(t+h,s)\Big)\right]_{h=0} ds \\
&= \int_t^T \frac{d}{dh}E\Bigg[K(s)\frac{\partial \sigma(s)}{\partial x}\Big(D_{s+}G(t+h,s)\cdot Y(t+h) \\
&\quad + D_{s+}Y(t+h)\cdot G(t+h,s)\Big)\Bigg]_{h=0} ds \\
&= \int_t^T \frac{d}{dh}E\left[K(s)\frac{\partial \sigma(s)}{\partial x}\Big(Y(t+h)D_{s+}G(t,s) + D_{s+}Y(t+h)G(t,s)\Big)\right]_{h=0} ds.
\end{aligned}
$$

Using the definition of $\widehat{p}$ and $\widehat{H}$ given respectively by (3.14) and (3.13) in the theorem, it follows by (7.13) that

$$
E\left[\frac{\partial}{\partial u}\widehat{H}(t,\widehat{X}(t),\widehat{u}(t))\Big|\mathcal{G}_t\right] + E[A] = 0 \quad \text{a.e. in} \quad (t,\omega), \tag{7.20}
$$

where

$$
A = \frac{d}{dh}A_3\Big|_{h=0} + \frac{d}{dh}A_4\Big|_{h=0} + \frac{d}{dh}A_5\Big|_{h=0} + \frac{d}{dh}A_6\Big|_{h=0} \tag{7.21}
$$

2. Conversely, suppose there exists $\widehat{u} \in \mathcal{A}_{\mathbb{G}}$ such that (3.12) holds. Then by reversing the previous arguments, we obtain that (7.13) holds for all $\beta_\alpha(s) := \alpha\chi_{[t,t+h]}(s) \in \mathcal{A}_{\mathbb{G}}$, where

$$\begin{aligned}
A_1 =& E\Big[\int_t^T \Big\{K(t)\Big(\frac{\partial b(s)}{\partial x} + D_{s+}\frac{\partial \sigma(s)}{\partial x} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial x}\nu(dz)\Big) \\
&+ \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz) + D_sK(s)\frac{\partial \sigma(s)}{\partial x}\Big\} Y^{(\beta_\alpha)}(s)ds\Big] \\
A_2 =& E\Big[\int_t^{t+h} \Big\{K(t)\Big(\frac{\partial b(s)}{\partial u} + D_{s+}\frac{\partial \sigma(s)}{\partial u} + \int_{\mathbb{R}_0} D_{s+,z}\frac{\partial \theta(s)}{\partial u}\nu(dz)\Big) + \frac{\partial f(s)}{\partial u} \\
&+ \int_{\mathbb{R}_0} D_{s,z}K(s)\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz) + D_sK(s)\frac{\partial \sigma(s)}{\partial u}\Big\} \alpha ds\Big] \\
A_3 =& E\Big[\int_t^T K(s)\frac{\partial \sigma(s)}{\partial x}D_{s+}Y^{(\beta_\alpha)}(s)ds\Big] \\
A_4 =& E\Big[\int_t^{t+h} K(s)\frac{\partial \sigma(s)}{\partial u}D_{s+}\alpha ds\Big]
\end{aligned}$$

$$\begin{aligned}
A_5 =& E\Big[\int_t^T \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\}\Big(\frac{\partial \theta(s)}{\partial x} + D_{s+,z}\frac{\partial \theta(s)}{\partial x}\Big)\nu(dz)D_{s+,z}Y^{(\beta_\alpha)}(s)ds\Big] \\
A_6 =& E\Big[\int_t^{t+h} \int_{\mathbb{R}_0} \{K(s) + D_{s,z}K(s)\}\Big(\frac{\partial \theta(s)}{\partial u} + D_{s+,z}\frac{\partial \theta(s)}{\partial u}\Big)\nu(dz)D_{s+,z}\alpha ds\Big]
\end{aligned}$$

for some $t, h \in (0,T)$, $t+h \le T$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_t-$measurable. Hence, these equalities hold for all linear combinations of β_α. Since all bounded $\beta \in \mathcal{A}_{\mathbb{G}}$ can be approximated pointwise boundedly in (t,ω) by such linear combinations, it follows that (7.13) holds for all bounded $\beta \in \mathcal{A}_{\mathbb{G}}$. Hence, by reversing the remaining part of the previous proof, we conclude that

$$\frac{d}{dy}J_1(\widehat{u} + y\beta)\Big|_{y=0} = 0, \text{ for all } \beta,$$

and then $\widehat{u}$ satisfies (3.11).

■