On Corner Velocity Interpolation and Mixed Finite Elements

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Abstract

This paper presents the corner velocity interpolation as bilinear and trilinear mixed finite elements on quadrilaterals and hexahedra. The corner velocity interpolations are bilinear or trilinear vertex based barycentric coordinate interpolations on quadrilaterals and hexahedra. The benefit of this velocity interpolation is that it ensure reproduction of uniform flow. We provide edge based basis functions ensuring the same interpolation, and show how this basis functions would perform as separate velocity elements. The resulting $H(\text{div})$ finite elements contains the constants.

1 Introduction

Mixed finite elements are classes of finite element methods solving for two primary unknowns simultaneously. The methods have proven to yield accurate solutions to systems of equations arising within various fields of engineering. Our application is flow in heterogeneous porous media, which can be modeled by a second-order elliptic equation. The mixed formulation then solves for the pressure and some related velocity field. Contrary to standard finite elements, mixed methods preserve mass on a local scale. Also, the mixed formulation provides continuity of normal fluxes, and the method comes with a rich theory for use in error analysis.

A conforming finite element method seeks an approximate solution in a finite dimensional subspace of the true solution space. Here, for the vector variable in the mixed formulation, we will seek a discrete solution in a subspace of the space $H(\text{div}, \Omega)$. Raviart and Thomas [24, 28] proposed the first family of $H(\text{div}, \Omega)$-conforming elements for rectangular grids in two space dimensions, namely the $RT_r$ elements, with $r$ indicating the polynomial degree of functions in the approximation space. The lowest-order $RT$ elements are among the simplest possible finite element spaces. However, they are seen to exhibit optimal-order convergence in $H(\text{div}, \Omega)$ for 2D rectangular
meshes and 3D cubic meshes. The extension of the RT\(_r\) elements to 3D was proposed by Nédélec [18], hence, the elements are commonly denoted RT\(_N\)\(_r\) elements. Among other \(H(\text{div}, \Omega)\)-conforming families of mixed elements, we have the spaces \(BDM_r, r \geq 1\), and \(BDFM_{r+1}, r \geq 0\). The former can be viewed as a natural extension of the RT\(_{r-1}\) elements. See, e.g., [4] and references therein for a definition of these spaces.

Due to arbitrary complex reservoir geology, reliable flow simulations will in general require more flexibility than parallelogram shaped meshes. For deformed meshes, like quadrilaterals or hexahedra, the construction of a robust \(H(\text{div}, \Omega)\)-conforming mixed method is more challenging. This issue has been addressed by many authors over the last years, among many others see for instance [2, 3, 17, 26, 29].

For irregular meshes, the construction of a finite element subspace of \(H(\text{div}, \Omega)\) relies on a multi-linear map to a reference space \(\mathcal{R}\). The space of vector shape functions are constructed with the aid of a unit element in \(\mathcal{R}\). Each function is associated to one face of the cell, and they provide pointwise continuity of normal components across inter-element faces, respectively. Hence, the shape functions form a basis for the approximation of \(H(\text{div}, \Omega)\). The \textit{Piola transformation} relates the space of reference shape functions to the approximation space on the arbitrary irregular cell in physical space so that fluxes are preserved. For each cell, the shape functions interpolate the face fluxes to obtain a velocity field on the cell interior.

For general quadrilaterals, an analysis of vector fields defined via the Piola mapping by Arnold et al. [2], demonstrates a degradation of \(H(\text{div}, \Omega)\) convergence compared to rectangular meshes. The RT\(_0\) elements on shape-regular quadrilaterals, do in fact not converge in \(H(\text{div}, \Omega)\). For a vector field \(v\), the \(L^2(\Omega)\) estimate is of optimal-order \(h\), with some additional regularity requirements, though. In the \(L^2(\Omega)\) estimate of the divergence of \(v\), on the other hand, accuracy of the interpolation is lost, and hence convergence is lost. This result was actually not fully resolved before revealed by the analysis of [2].

However, in \(L^2(\Omega)\), the RT\(_0\) elements retain optimal-order convergence for both the scalar and the vector fields [26]. Also, what usually is established, is convergence in \(H(\text{div}, \Omega)\) for smooth grids, or for a sequence of \(h^2\)-uniform grids, i.e.: grids asymptotically reaching parallelogram meshes. The discrete fluxes from the RT element can immediately also be post-processed with an alternative cell interpolation, related to the ABF elements proposed in [2], to retain full \(H(\text{div})\) convergence on general rough meshes, c.f., [12]. Moreover, in [2], they show that for full \(H(\text{div})\) convergence of finite elements built on the Piola mapping, the lowest-order discrete space on \(\mathcal{R}\) has to contain the constants. By exact representation of uniform flow, we ensure reproduction of constants on the physical space instead. This implies that for a constant
velocity field, the numerical scheme returns the exact solution. Henceforth, we will refer to a constant velocity field as uniform flow.

Similar results was obtained for the $RT_N_0$ space on hexahedral meshes in [3]. They prove convergence in $H(\text{div}, \Omega)$ for shape-regular asymptotically parallelepipeds meshes. Numerical experiments support this result by indicating no $H(\text{div}, \Omega)$ convergence for meshes of trapezoidal shape. These meshes are conceptually similar to the trapezoidal meshes used in [2], which yield the same numerical results for 2D $RT_0$ elements. The lack of convergence for the $RT_N_0$ space on general hexahedral meshes is also demonstrated in, e.g., [25]. This deficiency of the $RT_N_0$ velocity space, can be associated to the lack of reproduction of uniform flow. In [17], it is shown that, on a general hexahedron, a constant flow field does not imply linear face fluxes. Hence, the $RT_N_0$ velocity space obtained via the Piola mapping, which implicitly yields a linear flux reconstruction, does not contain the constant functions. The findings in [20] generalize this observation. There, they prove that for a general hexahedron with bilinear faces, both a local reconstruction of velocity based on the six face fluxes and exact representation of uniform flow (constant velocity field), can not be satisfied in $H(\text{div}, \Omega)$.

Here we propose multi-linear mixed finite elements on quadrilaterals and hexahedra based on the corner velocity interpolation (CVI). The CVI method was originally proposed to improve the accuracy within streamline simulations on irregular grids [7]. Simple streamline calculations were introduced in the groundwater literature by Pollock [21]; based on flux interpolation in $RT_0$, streamlines are traced element-by-element using simple analytical formulas. An extension of this method to irregular grids and finite element methods in 2D was proposed in [5]. Their method was based on a post-processing step, with mass conservative fluxes recovered on a dual control-volume grid. A further extension of this method to finite volume methods and 3D was given in [22, 23]. An extension of the streamline simulations to corner-point grids was proposed in [10]. A major obstacle in streamline formulations on irregular grids, is the fact that constant vectors (uniform flow) are not preserved in the $RT_0$-extensions. To circumvent this problem, Hægland et al. devised the CVI method [7]. Alternatively, higher-order interpolation schemes have been investigated, e.g., based on the $BDM_1$ or higher-order $BDM$ spaces [9, 13, 14, 15, 16]. A drawback of many of these extensions of streamline tracing to irregular grids, e.g., the CVI method, is that simple analytic formulas for the streamline tracing are no longer possible. The use of numerical integration to trace streamlines has been investigated by, e.g., [8]. See also [6] for an overview of these issues.

The CVI method is based on the usual bilinear or trilinear barycentric vertex based interpolation. The velocity field is interpolated on each cell from reconstructed corner velocities. Since barycentric interpolation retains con-
stant fields, the CVI elements also preserve uniform flow on general meshes both in 2D and 3D. Here, we provide edge based shape functions for the CVI interpolation. We employ these functions as the velocity space in a mixed formulation. In section 3.1, we find that the CVI space can be regarded as a perturbation of the $RT_0$ elements. Numerical results for general quadrilateral meshes are given in section 4. Section 3.2 provides the CVI shape functions for hexahedral elements.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, be a bounded domain, with polygonal boundary $\partial \Omega$. We limit the discussion to the steady-state incompressible single phase flow problem

$$
- \text{div}(K(x) \text{grad } p) = g, \quad \text{for } x \in \Omega,
$$
$$
p(x) = p_0, \quad \text{for } x \in \partial \Omega.
$$

For applications in reservoir simulation, Equation (1) is to be viewed as a model equation for the pressure, and it is based on an underlying principle of conservation of mass. We denote by $p$ the pressure, and by $K$ the symmetric positive definite diffusion tensor. To account for a general reservoir geology, we allow for the diffusion coefficients to be discontinuous. The geology of this kind of problems also require the use of rough grids, with general hexahedral cells. Finally, any sources or sinks present are represented by $g \in L^2(\Omega)$.

A mixed finite element formulation of (1) is generally based on a variational principle utilizing the space of $L^2(\Omega)$ vector functions. We define the space as follows.

$$
H(\text{div}, \Omega) = \{ v \in (L^2(\Omega))^d \mid \text{div } v \in L^2(\Omega) \},
$$

where $L^2(\Omega)$ is the set of square Lebesgue integrable functions on $\Omega$, with norm defined via the inner product, $\| \cdot \|^2_{L^2(\Omega)} = (\cdot, \cdot)_{L^2(\Omega)}$. For brevity, we denote the $L^2(\Omega)$ inner product by $(\cdot, \cdot)$ henceforth. The norm in $H(\text{div}, \Omega)$ is provided by

$$
\| v \|^2_{H(\text{div}, \Omega)} = \| v \|^2_{L^2(\Omega)} + \| \text{div } v \|^2_{L^2(\Omega)}.
$$

Finally, let $< \cdot, \cdot >$ be the $L^2(\partial \Omega)$ inner product.

Let $q = -K \text{grad } p$ denote the unknown fluid velocity. A variational formulation of the problem (1) then reads: Find a pair $(q, p) \in H(\text{div}, \Omega) \times L^2(\Omega)$ such that

$$
(K^{-1} q, v) - (p, \text{div } v) = -< v \cdot n, p_0 >, \quad \forall v \in H(\text{div}, \Omega),
$$
$$
(div q, u) = (g, u), \quad \forall u \in L^2(\Omega).
$$
2.1 Quadrilateral Meshes

For the sake of simplicity we assume $\Omega \subset \mathbb{R}^2$, until further. Let $\{T_h\}$ denote a family of partitions of $\Omega$ into regular cells, i.e.; all cells are convex, the angles are uniformly bounded away from zero and $\pi$, and the ratio between the length of the smallest edge and the diameter of the cell is uniformly bounded from below. Then an element $E \in T_h$ is of general quadrilateral shape, with $h$ being the maximum element edge. Finally, denote by $E_h$ the set of all element edges in $T_h$.

In a mixed setting, the finite element space $V_h \subset H(\text{div}, \Omega)$ is generally defined in terms of shape functions on an element $\hat{E}$ in a reference space $\mathcal{R}$. An arbitrary quadrilateral cell $E \in T_h$ is thus the image of the reference element $\hat{E}$ via the bilinear map $F = F_E : \hat{E} \to E$, which is smooth and invertible, see Figure 1. Here, $\hat{E} = (0, 1) \times (0, 1)$ is the unit square.

We denote by $x_{ij}, i, j = 0, 1$, the vertices of element $E$ as shown in Figure 1. Let $(\hat{x}, \hat{y}) \in \hat{E}$. The bilinear map reads

$$F_E(\hat{x}, \hat{y}) = x_{00}(1 - \hat{x})(1 - \hat{y}) + x_{10}\hat{x}(1 - \hat{y}) + x_{11}\hat{x}\hat{y} + x_{01}(1 - \hat{x})\hat{y}. \quad (3)$$

Any point $\mathbf{x} \in E$ can thus be expressed as

$$\mathbf{x}(\hat{x}, \hat{y}) = \sum_{ij=1}^{2} x_{ij} \phi_{ij}(\hat{x}, \hat{y}), \quad (4)$$

where the $\phi_{ij}$’s are the nodal functions of the bilinear mapping, or bilinear barycentric functions. Given the nodes $x_{kl}$, we have $\phi_{ij}(x_{kl}) = \delta_{(ij)(kl)}$, with $i, j, k, l = 1, 2$, and further

$$\sum_{ij=1}^{2} \phi_{ij}(\hat{x}, \hat{y}) = 1. \quad (5)$$

The latter is needed to exactly represent uniform flow.
The Jacobian matrix of $F_E$ is denoted by $D = DF_E(\hat{x})$ with a strictly positive Jacobian $J = \det D > 0$, $\forall \hat{x} \in \hat{E}$.

Also, we will henceforth assume the partitions $\{T_h\}$ to be $h^2$-uniform or smooth. That is, we assume there exists a constant $\sigma$ independent of $h$, such that

$$|F_E(\hat{x}, \hat{y})| = |x_{00} + x_{11} - (x_{01} + x_{10})| \leq \sigma h^2. \quad (6)$$

This condition is met if the grid refinement asymptotically leads to parallelogram cells. In particular, this is the case if an initial mesh is refined $j$ times by dividing the sides of the previous mesh at the face midpoints. To see this, consider the quadrilateral cell in Figure 1. The cell is divided into four subelements by dividing each edge into two equal halves. Let all cells of $T_h$ be refined in a similar manner. Thus, since the map (3) is linear along edges, this yields a uniform refinement. As $h$ decreases, any element $E \in T_h$ approaches a parallelogram. Let $x_{ij}, i, j = 1, 2$, specify a given cell, and let a subcell be defined by $x'_{ij}, i = 1, 2$. Then for the subsequent refinement level

$$|F_E(\hat{x}, \hat{y})| = |x'_{00} + x'_{11} - (x'_{01} + x'_{10})| = \frac{1}{4} |x_{00} + x_{11} - (x_{01} + x_{10})|. \quad (7)$$

Hence, when the element size has been halved $j$ times, $|F_{\hat{x}\hat{y}}|$ is reduced by a factor of $(1/2)^2j$, and the mesh size $h = H^{2j}$, where $H$ is the mesh size of the original mesh. A further discussion of $h^2$-uniform grids can be found in [11].

## 3 CVI Finite Element Space

Let $V_h \times Q_h \subset H(\text{div}, \Omega) \times L^2(\Omega)$. The following problem is then a discrete formulation of (2): Find a pair $(q_h, p_h) \in V_h \times Q_h$ such that

$$(K^{-1}q_h, v) - (p_h, \text{div} v) = -<v \cdot n, p_0>, \quad \forall v \in V_h,$$

$$(\text{div} q_h, u) = (g, u), \quad \forall u \in Q_h. \quad (8)$$

Let $\hat{v}$ be a vector field in $H(\text{div}, \hat{E})$. Then, the Piola transformation $P = P_E$ yields the natural way to represent a vector field $v = P_E \hat{v} \in H(\text{div}, E)$ by

$$v(x) = P(\hat{x})\hat{v}(\hat{x}) = \frac{1}{J} D\hat{v} \circ F^{-1}(x). \quad (9)$$

The Piola transformation relates the two vector fields so that the normal fluxes are preserved. If $p = \hat{p} \circ F^{-1}$ for some $\hat{p} : \hat{E} \to \mathbb{R}$, we have

$$\int_E v \cdot n p ds = \int_{\hat{E}} \hat{v} \cdot \hat{n} \hat{p} d\hat{s}, \quad (10)$$

where $n$ and $\hat{n}$ are the outward unit normal vectors on $\partial E$ and $\partial \hat{E}$, respectively. The construction of finite element subspaces of $H(\text{div}, \Omega)$ necessitates
Figure 2: The corners and faces of a quadrilateral cell expressed in terms of the reference space coordinates $0 \leq \hat{x}, \hat{y} \leq 1$.

continuity of normal fluxes. Let $V_h \subset H(\text{div}, \hat{E})$ denote the space of shape functions on a reference element $\hat{E}$. Thus, by applying (9) to some function $\hat{\psi} \in V_h$, we obtain a shape function $\psi \in V_h$ on $E$.

Henceforth, we let $P_{i,j}$ denote a piecewise polynomial of degree at most $(i, j)$ in $(\hat{x}, \hat{y})$, respectively, associated with $T_h$. The classical family of finite elements for $H(\text{div})$ approximation is the Raviart-Thomas elements $\mathcal{RT}_r := P_{r+1}(\hat{E}) \times P_{r,r+1}(\hat{E}), r \geq 0$.

Here we consider the lowest-order Raviart-Thomas space $\mathcal{RT}_0$. Let $P_{0,0}$ denote piecewise constants on $(0,1) \times (0,1)$. On the reference square $\hat{E}$ the $\mathcal{RT}_0$ velocity space is defined as the four-dimensional space given as all vector fields of the following form.

$$\mathcal{RT}_0(\hat{E}) = (P_{0,0} + \hat{x} P_{0,0}) \times (P_{0,0} + \hat{y} P_{0,0}).$$

Element wise, this gives rise to the following shape functions as functions of the reference space coordinates.

$$\psi_i(\hat{x}) = \mathcal{P}(\hat{x}, \hat{y}) \hat{\psi}_i(\hat{x}),$$

for $i = 1, \ldots, 4$. Here $\hat{\psi}_1 = [0,1 - \hat{y}]^T$, $\hat{\psi}_2 = [\hat{x},0]^T$, $\hat{\psi}_3 = [0,\hat{y}]^T$ and $\hat{\psi}_4 = [1 - \hat{x},0]^T$.

Finally, the space of scalar variables $Q_h$ is simply given as

$$Q_h := \{ u \in L_2 : u|_E \in P_0(E), \forall E \in T_h \}. \quad (12)$$

3.1 CVI Shape Functions

We will now define the CVI elements. Note, that for simplicity of exposition, we let $\mathcal{P}(\hat{x}, \hat{y}) = \mathcal{P} \hat{x}, \hat{y}$, with $\hat{x}, \hat{y} = 0$ or 1, from this point on, while $\phi_{ij} = \phi_{ij}(\hat{x}, \hat{y})$, $i, j = 0, 1$, is defined in Equation (4). Further, the numbering of the corners and faces are shown in Figure 2.
Definition 1.

\[ \psi_{cvi}^1 = \mathcal{P}_{00} \begin{pmatrix} 0 \\ \phi_{00} \end{pmatrix} + \mathcal{P}_{10} \begin{pmatrix} 0 \\ \phi_{10} \end{pmatrix}, \]  
\[ \psi_{cvi}^2 = \mathcal{P}_{10} \begin{pmatrix} \phi_{10} \\ 0 \end{pmatrix} + \mathcal{P}_{11} \begin{pmatrix} \phi_{11} \\ 0 \end{pmatrix}, \]  
\[ \psi_{cvi}^3 = \mathcal{P}_{11} \begin{pmatrix} 0 \\ \phi_{11} \end{pmatrix} + \mathcal{P}_{01} \begin{pmatrix} 0 \\ \phi_{01} \end{pmatrix}, \]  
\[ \psi_{cvi}^4 = \mathcal{P}_{00} \begin{pmatrix} \phi_{00} \\ 0 \end{pmatrix} + \mathcal{P}_{01} \begin{pmatrix} \phi_{01} \\ 0 \end{pmatrix}. \]

The shape functions \( \psi_{cvi}^i \) are constructed to fulfill the next lemma.

**Lemma 1.** Let \( \mathbf{n}_j, j = 1, \ldots, 4, \) be the (positive) normal vector to edge \( e_j \) of an element, with length equal to the length of \( e_j. \) Then

\[ \psi_{cvi}^i(e_j) \cdot \mathbf{n}_j = \delta_{ij}. \]

**Proof.** We prove Lemma 1 for \( \psi_{cvi}^4. \) The other \( \psi_{cvi}^i, \) for \( i = 1, \ldots, 3, \) follow in a similar manner. From (9), we have \( \mathcal{P}_{00} = \frac{1}{J_{00}} \mathbf{D}_{00}, \) with \( \mathbf{D}_{00}(\phi_{00}, 0)^T = (x_{10} - x_{00}) \phi_{00}. \) Now, \( (x_{10} - x_{00}) \cdot \mathbf{n}_1 = 0, \) and \( (x_{10} - x_{00}) \cdot \mathbf{n}_4 = (x_{10} - x_{00}) \times (x_{01} - x_{00}) = J_{00}, \) while \( \phi_{00} \) is zero along \( e_2 \) and \( e_3. \) Similarly, for \( \mathcal{P}_{01} \) we have, \( (x_{11} - x_{01}) \cdot \mathbf{n}_3 = 0, \) and \( (x_{11} - x_{01}) \cdot \mathbf{n}_4 = J_{01}, \) while \( \phi_{01} \) is zero along \( e_1 \) and \( e_2. \) Hence, \( \psi_{cvi}^4 \cdot \mathbf{n}_4 = (\phi_{00} + \phi_{01}), \) which equals 1 along \( e_4, \) while \( \psi_{cvi}^i(e_j) \cdot \mathbf{n}_j = 0 \) for \( j = 2, 3, 4. \)

To more clearly see the relation to the \( \mathcal{RT}_0 \) elements, we provide the following lemma.

**Lemma 2.** Let \( \hat{\psi}_i, i = 1, \ldots, 4, \) be the \( \mathcal{RT}_0 \) shape functions on the element \( \hat{E}, \) see Equation (11). Then, we have

\[ \psi_{cvi}^1(\hat{x}, \hat{y}) = \mathcal{P}_{00} \hat{\psi}_1 + (\mathcal{P}_{10} - \mathcal{P}_{00})[0, \hat{x}(1 - \hat{y})]^T, \]
\[ \psi_{cvi}^2(\hat{x}, \hat{y}) = \mathcal{P}_{10} \hat{\psi}_2 + (\mathcal{P}_{11} - \mathcal{P}_{10})[\hat{x}\hat{y}, 0]^T, \]
\[ \psi_{cvi}^3(\hat{x}, \hat{y}) = \mathcal{P}_{01} \hat{\psi}_3 + (\mathcal{P}_{11} - \mathcal{P}_{01})[0, \hat{x}\hat{y}]^T, \]
\[ \psi_{cvi}^4(\hat{x}, \hat{y}) = \mathcal{P}_{00} \hat{\psi}_4 + (\mathcal{P}_{01} - \mathcal{P}_{00})[(1 - \hat{x})\hat{y}, 0]^T, \]

and for parallelogram cells, the CVI element degenerates to the \( \mathcal{RT}_0 \)-element.

**Proof.** The proof of Lemma 2 follows immediately from Definition 1, and the fact that on parallelogram cells, the Piola mapping is constant.

Finally, the CVI basis elements also fulfill exact representation of uniform flow, or equivalently, they preserve constant vector fields.
Lemma 3. Let \( e_k, k = 1, 2 \), be the unit vector, and \( n_i \) the (positive) edge normal with length equal to the edge length. Then the CVI basis functions for \( i = 1, \ldots, 4 \), fulfil

\[
e_k = \sum_{i=1}^{4} f_{ki} \psi_{i}^{cvi},
\]

with flux \( f_{ki} = (e_k \cdot n_i) \).

Proof. For simplicity of notation, let \( \xi_i \) be the edge vector along edge \( e_i \), such that \( \xi_1 = (x_{10} - x_{00}) \), and let the vertices be indexed from 1 to 4 in counter-clockwise direction, so that \( i = 1 \) corresponds to vertex \( x_{00} \). Finally, let \( i = 0 \) or 5 correspond to \( i = 4 \) or 1. With this notation,

\[
f_{ki} \psi_{1}^{cvi} = (e_k \cdot n_1) P_1 \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} + (e_k \cdot n_1) P_2 \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix},
\]

where \( P_1 = \frac{1}{J(x_1)} [\xi_1, \xi_4] \), and \( P_2 = \frac{1}{J(x_2)} [\xi_1, \xi_2] \). Now,

\[
\sum_{i=1}^{4} f_{ki} \psi_{i}^{cvi} = \sum_{i=1}^{4} \frac{(e_k \cdot n_i) \xi_{i-1}}{J(x_i)} \phi_i + \sum_{i=1}^{4} \frac{(e_k \cdot n_i) \xi_{i+1}}{J(x_{i+1})} \phi_{i+1}
\]

\[
= \sum_{i=1}^{4} \frac{(e_k \cdot n_{i+1}) \xi_i}{J(x_{i+1})} \phi_{i+1} + \sum_{i=1}^{4} \frac{(e_k \cdot n_i) \xi_{i+1}}{J(x_{i+1})} \phi_{i+1}
\]

\[
= \sum_{i=1}^{4} \left[ (e_k \cdot n_{i+1}) \xi_i + (e_k \cdot n_i) \xi_{i+1} \right] \frac{\phi_{i+1}}{J(x_{i+1})}.
\]

Since \( e_k \) is the unit vector,

\[
(e_k \cdot n_{i+1}) \xi_i + (e_k \cdot n_i) \xi_{i+1} = \frac{1}{2} (\xi_i n_{i+1}^T + \xi_{i+1} n_i^T) e_k.
\]

Note that all vectors are taken in positive direction with regards to the reference space. Thus, with \( \xi_i = [\xi_i^1, \xi_i^2]^T = \pm [n_i^2, -n_i^1]^T \), we have

\[
(\xi_i n_{i+1}^T + \xi_{i+1} n_i^T) / 2 = J(x_{i+1}) I,
\]

with \( I \) the identity matrix. Summing up and using eq. (5),

\[
\sum_{i=1}^{4} f_{ki} \psi_{i}^{cvi} = \sum_{i=1}^{4} J(x_{i+1}) I e_k \frac{\phi_{i+1}}{J(x_{i+1})} = e_k \sum_{i=1}^{4} \phi_i = e_k.
\]

For each cell, the CVI basis functions can also be viewed as barycentric edge based \( H(\text{div}) \) coordinates. In this sense, Equation (17) is analogous to Equation (5).
3.2 Hexahedral 3D Mesh

We limit the 3D discussion to shape-regular hexahedral cells, with six faces and eight vertices. Extension of face based elements from 2D to 3D is unfortunately a bit more complicated from quadrilaterals to hexahedra, than from triangulations to tetrahedra. In 2D, both quadrilaterals and triangulations have two vertices per edge, while in 3D, a hexahedral face has four vertices compared to three on a tetrahedral face. This implies that the adjacent edge vectors to a vertex do not provide us enough information for a definition of the face for a hexahedral element. As we will show below, we need to provide additional information.

Due to the result of [20], which proves that for a general hexahedron with bilinear faces, both a local reconstruction of velocity based on the six faces and reproduction of uniform flow cannot be satisfied in $H(\mathrm{div}, \Omega)$, the CVI shape functions are limited to grids with planar faces henceforth. We note that even when this is the case, the $RT_0$ elements are generally not able to preserve uniform flow. An example is a grid of truncated pyramids [17].

Let $\hat{x} = (\hat{x}, \hat{y}, \hat{z}) \in \mathbb{R}^3$ denote a point in the reference space. A hexahedron in physical space is represented as the image of a reference element $\hat{E}$ under the trilinear map $H_E : \hat{E} \rightarrow E$. Here, $\hat{E} = \{\hat{x} | 0 \leq \hat{x}, \hat{y}, \hat{z} \leq 1\}$ is the unit cube. Let $x_{ijk}, i, j, k = 0, 1$, be the vertices of the physical hexahedron $E$, where the index refers to the corresponding $(\hat{x}, \hat{y}, \hat{z})$ value. Then, under the map $H_E$, any point $x \in E$ is given as

$$x(\hat{x}, \hat{y}, \hat{z}) = \sum_{ijk=1}^{2} x_{ijk} \phi_{ijk}(\hat{x}, \hat{y}, \hat{z}), \quad (18)$$

where $\phi_{ijk}(\hat{x}, \hat{y}, \hat{z})$ is the trilinear node functions on $\hat{E}$, or barycentric basis functions. That is; $\phi_{ijk}(x_{lmn}) = \delta_{(ijk)(lmn)}$ and $\sum \phi_{ijk} = 1$ on $\hat{E}$.

We number the columns of the Jacobian matrix of $H_E$ from 1 to 3, i.e. $D = [D^1, D^2, D^3]$, and we evaluate the Jacobian matrix at the eight vertices.
$x_{ijk}, i, j, k = 0, 1$. Further, we denote the faces by the four vertex indices defining the face, e.g., the face with the four vertices $x_{0jk}, j, k = 0, 1$, will be indexed by $0jk$. We denote the face area $A_{ijk}$ and the unit normal $\bar{n}_{ijk}$ in the same manner. Note that the unit normals are always orientated in the positive reference space direction.

For a definition of face elements in accordance with an analogue of Lemma 1, note that the Jacobian determinant of the trilinear mapping $H E$ evaluated at a vertex $x_{ijk}$, does not provide us with enough information. Therefore, we define the diagonal matrix

$$
J_{ijk} = \text{diag}(D^1_{ijk} \cdot \bar{n}_{ijk} A_{ijk}, D^2_{ijk} \cdot \bar{n}_{ijk} A_{ijk}, D^3_{ijk} \cdot \bar{n}_{ijk} A_{ijk}),
$$

for $i, j, k = 0, 1$. Here, hats indicate indices pointing to faces. Similarly, we define

$$
\tilde{P}_{ijk} = D_{ijk} J_{ijk}^{-1}.
$$

Due to the limitation to planar faces, observe that $v \cdot \bar{n}_{ijk} A_{ijk}$ on a face $e_{ijk}$ corresponds to the face flux. Now, on the six faces of a hexahedron, the CVT shape functions can be defined by

**Definition 2.**

$$
\psi_{\alpha jk}^{cvi} = \sum_{jk=0}^{1} \tilde{P}_{\alpha jk} \begin{pmatrix} \phi_{\alpha jk} \\ 0 \\ 0 \end{pmatrix}, \quad \text{for face } e_{\alpha jk}, \alpha = 0, 1,
$$

$$
\psi_{iak}^{cvi} = \sum_{ik=0}^{1} \tilde{P}_{iak} \begin{pmatrix} 0 \\ \phi_{iak} \\ 0 \end{pmatrix}, \quad \text{for face } e_{iak}, \alpha = 0, 1,
$$

$$
\psi_{ija}^{cvi} = \sum_{ij=0}^{1} \tilde{P}_{ija} \begin{pmatrix} 0 \\ 0 \\ \phi_{ija} \end{pmatrix}, \quad \text{for face } e_{ija}, \alpha = 0, 1.
$$

With Definition 2, we can immediately provide a lemma analogous to Lemma 1.

**Lemma 4.** Let the six faces of a hexahedron be denoted by $e_{lmn}$, with $l, m, n = 0, 1$. Then

$$
\psi_{ijk}^{cvi}(e_{lmn}) \cdot n_{lmn} A_{lmn} = \delta_{(ijk)(lmn)}.
$$

**Proof.** It holds to prove Lemma 4 for one face. For $\psi_{0jk}^{cvi}$, we start with the vertex at $x_{000}$, where $D_{000}(\phi_{000}, 0, 0)^T = (x_{100} - x_{000}) \phi_{000}$. Further, from the Definition (19) $(x_{100} - x_{000}) \cdot n_{0jk} A_{0jk} = (J_{000})_{1,1}$, and $(x_{100} - x_{000}) \cdot n_{00k} =
\[(x_{100} - x_{000}) \cdot n_{ij0} = 0, \text{ while } \phi_{000} \text{ is zero on face } e_{1jk}, e_{i1k} \text{ and } e_{ij1}. \] A similar derivation applies to the other three vertices of \(e_{0jk}\). It follows that \(\psi_{0jk}^\text{cv} \cdot n_{0jk} A_{0jk} = \sum_{j=0}^{1} \phi_{0jk}, \) which is 1 on face \(e_{0jk}\), while \(\psi_{0jk}^\text{cv} \cdot n_{lmn} = 0\) for the other five faces.

In [19], a first version of one of the CVI shape functions for hexahedral cells was defined. There, a counterexample on coercivity of \((K^{-1} \mathbf{v}, \mathbf{v})\) in the full \(H(\text{div})\) norm was presented.

A lemma analogous to Lemma 3 can be stated also for the CVI space in 3D. Due to the properties of the barycentric coordinates, we have that the CVI elements preserve uniform flow. The details are omitted here.

### 3.3 Deficiencies

The CVI finite elements do represent uniform flow exactly, but the price to pay is that the divergence of the CVI elements is not contained in the discrete pressure space. This means that

\[\text{div}(\mathbf{v}^\text{cv}) \not\subset Q_h.\]

Define the interpolation, \(P_h : L^2 \to Q_h\), and as \(L^2\)-interpolation, \(\Pi^\text{cv} : H(\text{div}) \to \text{CVI}\), by \((\text{div}(\Pi^\text{cv} \mathbf{v} - \mathbf{v}), u)\) for all \(u \in Q_h\). Then, we do not meet the commutative hypothesis, i.e.; \(P_h \text{div} \mathbf{v} \neq \text{div} \Pi^\text{cv} \mathbf{v}\) for \(\mathbf{v} \in H(\text{div})\). For classical mixed finite elements, like the Raviart-Thomas elements or the Brezzi-Douglas-Marini elements, this is usually fulfilled. These properties play an essential part in the analysis of the classical mixed methods, however, this hypothesis may be too strong a requirement.

### 4 Numerical Examples

In this section, we illustrate the convergence of the CVI mixed finite element space by simulations on smooth grids. A definition of smooth meshes was given in section 2. We employ a sequence of the mesh shown in Figure 4.

The errors are measured in discrete \(L^2\) norms for both variables. Let \(A_E\) be the area of a grid cell \(E \in T_h\). Then, for the pressure, we define

\[\|p - p_h\|_{(L^2,h)} = \left( \frac{\sum_{E \in T_h} A_E (p_E - p_h,E)^2}{\sum_{E \in T_h} A_E} \right)^{1/2}.\]

Let \(W_e\) be the area associated to an edge \(e \in E_h\). Here, \(W_e\) is set to be half the area sum of the two cells adjacent to edge \(e\). Then, the discrete \(L^2\) norm for the normal velocities are given by

\[\|q - q_h\|_{(L^2,h)} = \left( \frac{\sum_{e \in E_h} W_e ((q_e - q_h,e)/|e_h|)^2}{2 \sum_{e \in E_h} W_e} \right)^{1/2}.\]
where \( q_e \) is the normal velocity across the edge \( e \). We also provide the results measured in discrete maximum norm.

We note that since, by construction, the CVI method preserves a constant flow field, it is expected to return the exact solution for a linear pressure field. Moreover, for parallelogram grids, the CVI scheme degenerates exactly to the \( RT_0 \) interpolation. Thus, numerical results should be identical for the two discretizations for such grids. This is observed in the simulations.

First, we apply the smooth solution
\[
\begin{align*}
  u(x, y) &= \cosh(\pi x) \cos(\pi y), \\
  \end{align*}
\]
(23)
where we assume the permeability to be the identity matrix. The discrete \( L^2 \) convergence is seen in Figure 5(a). We observe that both the pressure and the normal velocity converge as \( h^2 \) in discrete \( L^2 \) norm. The results in maximum norm is seen in Figure 5(b).

![Figure 5](image)

(a) Results in discrete \( L^2 \) norm.
(b) Results in discrete \( L^\infty \) norm.

**Figure 5:** Numerical convergence of solution (23). \( N \) is the number of elements in each direction.

The next cases apply to a more realistic simulation where we let the permeability vary throughout the domain. Denote the four regions in our
We may impose different conductivities in the four regions, which again may render a singularity at the corner where they meet.

\[ u(r, \theta) = r^{x} \left\{ \begin{array}{ll} \cos \alpha (\theta - \pi/3) & \text{for } \theta \in [0, 2\pi/3], \\ d \cos \alpha (4\pi/3 - \theta) & \text{for } \theta \in [2\pi/3, 2\pi], \end{array} \right. \] (24)

where \( \alpha = (3/\pi) \arctan \sqrt{1 + 2/\kappa} \) and \( d = \cos(\alpha \pi/3)/\cos(2\alpha \pi/3) \). Denote by \( \kappa = k_1/k_2 \) the conductivity ratio. Here, \( k_1 \) is the permeability for region 1 and \( k_2 \) for the rest of the medium. We have \( \kappa \geq 0 \), which yields \( \alpha \in [0.75, 1.5] \). This parameter describes the regularity of the solution, that is; we find that the solution (24) belongs to the space \( H^{1+\alpha-\epsilon}_{\pi} \) for an arbitrary \( \epsilon > 0 \). Here, the subscript \( \pi \) means that we operate in interpolated Hilbert spaces [27].

Let \( \kappa = 10^{-3} \) and \( 10^{2} \), and thus we have \( \alpha \approx 1.4787 \) and \( \alpha \approx 0.7547 \), respectively. The convergence behavior in discrete \( L^2 \) norm and discrete maximum norm can be found in Figures 7(a) and 7(b), respectively.

Further, we let the conductivities in regions 1 and 3 equal, and similarly, regions 2 and 4 have the same conductivities. We still assume an isotropic permeability. This accommodates a solution where \( \alpha \in [0, 1.5] \), and moreover it satisfies \( u(r, \theta) = -u(r, \theta - \pi) \). The solution reads

\[ u(r, \theta) = r^{y} \left\{ \begin{array}{ll} \cos \alpha (\theta - \pi/3) & \text{for } \theta \in [0, 2\pi/3], \\ d \sin \alpha (5\pi/6 - \theta) & \text{for } \theta \in [2\pi/3, \pi], \end{array} \right. \] (25)

and here \( \alpha = (6/\pi) \arctan (1/\sqrt{1+2\kappa}) \) and \( d = \cos(\alpha \pi/3)/\sin(\alpha \pi/6) \). We choose conductivity ratios of 3, 10 and 100 to illustrate the convergence behavior for various regularities of the solution (25). This is seen in Figures 8 and 9 for estimates in discrete \( L^2 \) and \( L^{\infty} \) norm, respectively.
Figure 7: Numerical convergence for solution (24) with $\kappa = 10^{-3}$ and $10^2$, respectively. $N$ is the number of grid cells.

Figure 8: Numerical $L^2$ convergence for solution (25) with $\kappa = 3, 10$ and $10^2$, respectively. $N$ is the number of grid cells.

Figure 9: Numerical $L^\infty$ convergence for solution (25) with $\kappa = 3, 10$ and $10^2$, respectively. $N$ is the number of grid cells.
We can summarize the observed convergence rates for solution (23), (24) and (25), the latter two for various values of the parameter \( \alpha \), as follows. Measured in discrete \( L^2 \) norm we find, for the potential, the relation
\[
\| p - p_h \|_{(L^2,h)} \sim h^{\min\{2,2\alpha}\}}. \tag{26}
\]
For the normal velocity we have
\[
\| q - q_h \|_{(L^2,h)} \sim h^2, \quad \text{for } p \in H^2, \tag{27}
\]
\[
\| q - q_h \|_{(L^2,h)} \sim h^{\alpha}, \quad \text{else.}
\]
This is in consistency with the finite element theory [27]. Similar results for some control-volume multipoint flux approximation methods can be found in [1].

For geophysical applications, rough grids honoring the geology are often required for accurate simulations. Thus, it is appropriate to apply such meshes to the CVI method presented here. By a rough grid, or \( h \)-perturbed grid, we mean a sequence of grids not approaching parallelograms as the grid is refined, see Figure 10. At every refinement level, each element is randomly distorted by \( \mathcal{O}(h) \), with \( h \) being the maximum element edge. The CVI method is seen not to yield a convergent approximation to any of the solutions for \( h \)-perturbed meshes, whereas the \( \mathcal{RT}_0 \) elements converges in discrete \( L^2 \) norm also for the rough meshes. Note that, according to analysis, if measured in full \( H(\text{div},\Omega) \) norm, the \( \mathcal{RT}_0 \) velocity space will not convergence for rough meshes either.

Finally, we remark that the implementation of the CVI velocity space is much more comprehensive than the \( \mathcal{RT}_0 \) elements.

5 Discussion

In this work, we have presented an analysis of the CVI elements as velocity elements in a mixed finite element setting. The CVI shape functions are
derived both in 2D and 3D. Numerical convergence behavior is provided for the CVI space on smooth quadrilateral grids.

In 2D, we have seen that the CVI elements fulfill Lemma 1 - 3. That is; the CVI velocity space lies in $H(\text{div})$, and is closely related to the $\mathcal{RT}_0$ space. In fact, for parallelogram shaped meshes, the CVI space degenerates to the $\mathcal{RT}_0$ space. Finally, by Lemma 3, the method preserves uniform flow. For the CVI functions in 3D, we can show that the same property applies.

However, since the divergence of the CVI velocity space is not contained in the space of constant pressures, it does not meet the commutative qualities, which is an essential part in classical analysis of the $\mathcal{RT}_0$ mixed method. Thus, compared to the $\mathcal{RT}_0$ method, we would expect some drawbacks of the proposed CVI method. This is also observed in the numerical simulations, where the CVI elements exhibit convergence in discrete $L^2$ norm for $h^2$-uniform grids, contrary to $h$-perturbed grids, where the elements do not converge.

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References


