Non-slender RC Columns Limits for Braced, End-loaded Members

Jostein Hellesland
Professor, Mechanics Division, Department of Mathematics
University of Oslo, P. Box 1054-Blindern, N-0316 Oslo, Norway

ABSTRACT
Second order effects of axial forces on displacements in compression members are often negligible and may justify design to be based on forces obtained by conventional first order theory. Extensive nonlinear analysis results are presented to document slenderness limits below which this is so for braced, end moment loaded compression members with various end restraints. Major factors that affect lower slenderness limit predictions of such members are investigated and a new lower slenderness limit is proposed and verified against the nonlinear results. Slenderness is defined in terms of a so-called normalized slenderness that, in addition to the geometrical slenderness, is a function of axial force and reinforcement. The limit itself is a function of the first order end moment ratio. The formulation is rational and the limits are generally found to be more reliable than existing lower limit formulations.

KEYWORDS
Columns (supports); Compression members; Non-slender members; Reinforced concrete; Lower slenderness limits; Second order effects; Slenderness effects; Nonlinear analysis; Braced.
INTRODUCTION

Second order (secondary) effects of axial forces on displacements in slender compression members (columns, struts, etc.) are small and can be neglected in a great many structures. For this reason, and to simplify analysis work, most design codes and standards for reinforced concrete structures give lower slenderness limits for compression members, and allow them to be designed for forces obtained by conventional first order theory when these limits are not exceeded.

The use of effective lengths, to account for different boundary conditions, have been standard in such limits for a very long time. For the 1971 revision of the ACI 318 code, a limit for end-loaded braced members was proposed (MacGregor, Breen and Pfrang 1970) that appear for the first time also to be given as a function of the first order moment gradient along the member. This allowed for a significantly more rational assessment of such members in all states of single and double curvature than before. The proposed limit found wide acceptance and was adopted in many codes and standards internationally, in the exact same form, or in very similar forms. Several efforts have since been made to develop more refined limits that also included the effect of important parameters such as axial load and axial load as well as reinforcement. Several of these have been adopted in national standards. Despite the inclusion of additional parameters, these limits give rather very different results for some combinations of influencing parameters.

An overview of a number of these limits is given in Hellesland (200A). The paper also discusses criteria and other aspects that affect lower slenderness limit predictions in general, presents a so-called “normalized” slenderness measure, and deals otherwise in some detail with compression members that are either 1) unbraced against sidesway or 2) braced against sidesway and subjected to transverse loading between ends.

The present paper is limited to compression members that are braced against sidesway and subjected to end moments only. No transverse loading between ends is considered. The main objective of the paper is to present nonlinear analysis results documenting slenderness values corresponding to given criteria, investigate factors that affect lower slenderness limit predictions, and to present reliable lower slenderness limit formulations for such members. Factors considered include
- criterion for lower slenderness limit;
- first order moment gradient (through the end moment ratio);
- end restraints and their reflection through effective lengths;
- axial force and reinforcement;
- sustained loading.

The study is limited to members with symmetrical reinforcement and uniform cross section, reinforcement and axial load along the member axis. Existing nonlinear analysis data, available in the literature, are employed to document sustained load effects.
RESEARCH SIGNIFICANCE

Codified lower slenderness limits for compression members are very important in that they are accepted as documentation of when second order effects may be ignored. Existing slenderness limits in various design standards and codes vary considerably. Results of elastic and extensive nonlinear analyses are presented that allow assessment of slenderness limits and provide a basis for development of more rational limits. A limit is proposed that is extensively verified and that may be used with increased confidence at low and high axial loads and for all states of single and double curvature bending and, it is believed, that may allow member slenderness effects to be ignored in many more cases than allowed by current limits.

PROBLEM DEFINITION

Fig. 1(a) illustrates typical displacement shapes and first order moment distributions (full lines) for braced, end-loaded members with rotationally restrained ends. Case b, with a controlled (imposed) relative end displacement, behaves similar to case a. Also shown are total moment distributions (dashed lines) that include second order (slenderness) effects sufficient to produce maximum total moments, \( M_{t \text{ max}} \), that exceed the numerically largest first order end moments, \( M_2 \). Unlike in unrestrained (hinged) members, where end moments stay constant (e.g., \( M_{t2} = M_2 \)) with increasing axial load, total end moments may change significantly in restrained members, thereby inflicting corresponding changes in moments of the restraining elements (beams, etc.). Such consequences, which is important enough, is not considered in the development of existing slenderness limits previously reviewed (Hellesland 200A), or in those of the present paper. Rather, it is the development of maximum moment between member ends that is of concern. If this moment exceeds the larger first order end moment, the member's load capacity will become less than it would have in the absence of slenderness effects. The normal intention of lower slenderness limits, and of those considered in this paper, is to ensure that slenderness effects do not reduce the member's load carrying capacity by more than an acceptable amount.

ELASTIC MOMENT GRADIENT EFFECTS

The development of a maximum moment between ends will be delayed by a moment gradient. Elastic theory is used to clarify how this increase is affected by end restraints and at which end the larger first order end moment is applied. The elastic moment magnifier, \( \delta_M \),

\[
\delta_M = \frac{M_{t \text{ max}}}{M_2}
\]

defined by the ratio between the maximum total moment, \( M_{t \text{ max}} \), and the nu-
merically largest first order end moment, $M_2$, gives a measure of the second order effects. For an elastic, initially straight member with constant bending stiffness $EI$ along the length, small deformations (rotations) and negligible shear deformations, slenderness values at a moment magnification of 5% ($\delta_M = 1.05$) are shown in Fig. 2. Elastic results for three cases are shown. They are defined by the inserts labeled $a$ (hinged at both ends), and $b$ and $c$ (hinged at one end and restrained by a very stiff beam at the other). The abscissa is the first order end moment ratio defined by

$$\mu_o = \frac{M_1}{M_2}$$

(2)

between the smaller ($M_1$) and the larger ($M_2$) first order end moment. The ratio is defined as positive for members bent in single curvature by these moments, and negative for members bent in double curvature.

Note that the figure ordinate includes the effective length factors $k$. Use of these, $k=1.0, 0.7$ and 0.7 for case $a$, $b$ and $c$, respectively, are seen to reflect different end restraints well for nearly uniform bending, but not as well for more non-uniform bending. Curves $b$ and $c$ are for identical members and coincide at $\mu_o = 1$ and -1. In case $c$, the larger end moment is applied (by imposed rotation) at the very stiff restraint. Due to the moment relief at this end (moment transfer to the restraint due to second order effects), this case will be less unfavorably affected by slenderness effects than case $b$, in which the larger moment acts at the hinge, where no moment relief takes place. Curve $b$ can for all practical purposes be considered a reasonable lower bound on results for any end restraint combination.

Some members may have loads close to the critical load (at the ordinate value of 1.0 in the figure), and consequently have considerable second order effects, before the maximum total moment exceeds the larger first order end moment. The extremes of such response, represented by the peaks of curve $a$ and $c$, are associated with rather sudden changes in the members’ displacement shapes (characterized by “unwrapping” or “unwinding” into the respective buckling modes). This calls for caution when selecting approximations at $\mu_o$-values for which such response may take place.

The rather significant difference between curves $b$ and $c$ is due to the strong difference between the rotational end restraints at the two ends. Infinite and zero end restraint stiffnesses represent a theoretical case that in practice is not easily attainable. For more practical differences, the curves will be closer together. Curve $a$, for hinged ends, is reasonably representative also for cases with other, but reasonably similar end restraints.

The straight line approximation labeled $d$ in Fig. 2, located between curve $a$ and curve $b$, should be a reasonably representative approximate lower bound for practical cases with realistic end restraints. The approximation labeled $e$ may be too conservative for practical cases. (The base value (at $\mu_o=1$) of the approximation is about that obtained (0.218) by assuming a sine curve for the deflected
shape). These approximations will be considered further relative to nonlinear results.

**CRITERION**

A general lower slenderness limit is here defined as that at which second order (slenderness) effects do not reduce a member’s load carrying capacity by more than 5% below the cross-sectional capacity (“non-slender member strength”). Non-negligible detrimental effects of sustained (long term) loading must, if relevant, be included in capacity reduction assessments.

The wording ‘capacity reduction’ is ambiguous in that it may mean reduction in moment capacity, axial load capacity or both. For members in which the maximum total moment result at the same section as the maximum first order moment, different interpretations may give significantly different results (Hellesland 200A). Similar results for braced, end-loaded members with moment gradients, such as considered in this paper, show that the difference for different interpretations become significantly reduced with increasing first order moment gradient. This can be seen in Fig. 3, that shows approximate results (trends) for the three cases defined by

a) 5% reduction in moment capacity for a constant axial load;

b) 5% reduction in axial load capacity for an applied constant axial load eccentricity (implies also a 5% reduction in moment capacity);

c) 5% reduction in axial load capacity for an applied constant moment.

The load applications that correspond to these “sub-criteria” are defined by the correspondingly labeled inserts (a,b and c) in the figure. The results were obtained using the approximate ACI moment magnifier approach in conjunction with a $P - M$ diagram for a typical medium reinforced, rectangular cross section. For computational details, see Hellesland (200A).

The lower terminations of the case c curves correspond to an applied moment ($M_2$) approximately equal to the pure moment capacity. As defined, case c is not relevant for greater applied moments. With increasing moment gradient, towards and into double curvature states, the curves for case b and c approach that based on criterion a. This points to the use of criterion a as a single, conservative criterion. However, creep effects due to sustained loading are not considered explicitly above and need further consideration. Possible effects of creep are strongest in highly compressed members with smaller load eccentricities. For such members, for which the axial load capacity may be the most relevant strength parameter, criteria such as b or c may be considered the most appropriate. The applicability of the various criteria is discussed in more detail in Hellesland (200A).

In the present nonlinear analyses, criterion a will be used in combination with short term material properties. Effects of creep on lower limits will be checked in a separate section.
NORMALIZED SLENDERNESS

A slenderness parameter, “normalized” with respect to the effect of axial force and reinforcement, is defined by

$$\lambda_n = \lambda \sqrt{\frac{\nu}{1 + k_t \omega_t}}$$

in which

$$\nu = \frac{P_u}{\phi f_{ce}^c A_g} ; \quad \omega_t = \frac{f_y A_{st}}{f_{ce}^c A_g} ; \quad \lambda = \frac{kL}{r_g}$$

is the nominal, factored design axial load level, the total mechanical reinforcement ratio and the geometrical slenderness, respectively. Further, \( k_t \) is a reinforcement contribution factor that for a given section may be approximated by

$$k_t = \frac{4.3}{1000 \varepsilon_y} \left( \frac{r_s}{r_g} \right)^2$$

Unlike \( k_t \), the product \( k_t \omega_t \),

$$k_t \omega_t = \frac{4.3E_s}{1000 f_{ce}^c} \frac{A_{st}}{A_g} \left( \frac{r_s}{r_g} \right)^2$$

is not dependent on steel grade. Above, \( f_y \) and \( \varepsilon_y \) are the steel yield strength and strain, \( f_{ce}^c \) is the nominal structural concrete compressive strength (often denoted \( f_c^c \) in the literature), \( r_g \) and \( r_s \) are the radii of gyration of the gross section \( (A_g) \) and the total reinforcement \( (A_{st}) \), respectively, both about the centroidal axis. Values of \( r_g \), \( r_s \) and \( r_s/r_g \) for typical cross-sections and reinforcement arrangements are given in Fig. 4. In practice, the last case is considered rather unlikely.

The basis for \( \lambda_n \), denoted normalized slenderness to distinguish it from other slenderness measures, as well as some simplifications of Eq. 5 that involve some conservativeness, are discussed in more detail in Hellesland (200A).

NONLINEAR ANALYSES

Method

Slenderness values at specified moment capacity reductions were computed for initially straight, reinforced concrete compression members subjected to constant axial loads and moments applied at member ends. The members are either a) statically determinate with rotationally unrestrained (hinged) ends, or, b) statically indeterminate with one end unrestrained and the other rotationally restrained by an extremely stiff beam. The larger first order end moment is applied at the unrestrained end in order to obtain approximate lower bound results (comparable to the elastic curve \( b \) in Fig. 2).
A computer program, based on the finite difference approach, was tailor made for the problem and included both material and geometric nonlinear effects (Aas-rum 1992). An overview of the major steps in the iterative analysis is given in the Appendix. A member may become unstable either due to primary material failure (exhaustion of the cross-section capacity) or primary instability failure prior to material failure. The basis for the analysis was nominal moment–curvature relationships for given sections, reinforcement and nominal axial loads \( P_n = P_n / \phi \). The same strength reduction factor \( \phi \) was assumed at all sections. This is a common, but conservative approach.

The standard assumptions of plane sections remaining plane, full bond and zero concrete tensile strength were adopted. The common parabola-rectangle stress-strain diagram was chosen for concrete in compression and a standard elasto-plastic diagram for reinforcing steel in tension and compression. The peak concrete compression stress in the concrete diagram is the previously defined \( f'_{cm} \). Non-mechanical concrete strains (creep, shrinkage) were not included. Consequently, results obtained are so-called “short term” results.

Parameters

Both 5 and 10% capacity reductions were considered. Corresponding slendernesses were obtained for a wide range of parameters. Unless otherwise mentioned, the results presented are for members with rectangular cross-section and reinforcement in two opposite layers perpendicular to the plane of bending or equivalent (e.g., corner reinforcement). This section is labeled RC. Further, \( h' / h = 0.8 \), \( \varepsilon_y = 0.0025 \), concrete strain at peak stress \( \varepsilon_{co} = 0.002 \), ultimate concrete strain \( \varepsilon_{cu} = 0.0035 \) and \( k_t \) as defined by Eq. 5. For this “standard” case, \( k_t = 3.3 \). In \( \lambda_n \) presentations, the elastic effective length factors of 1.0 (case a) and 0.7 (case b) are used for the two kinds of member restraints considered.

Results are included for both a small amount, \( \omega_t = 0.2 \), and a large amount, \( \omega_t = 1.0 \), of reinforcement. These mechanical reinforcement ratios covers a wide range from approximately minimum reinforcement (1-1.5%) to an, in practice, upper limit (6-8%).

Moment gradient, axial force and reinforcement

The three major factors affecting the slenderness limits are moment gradient, axial load and reinforcement. The relative effect of moment gradients in reinforced concrete members, Fig. 5, are stronger than that found in the elastic analyses (from Fig. 2, curve a). This is an expected result. The difference can be quite substantial. It is greatest for low axial load and low reinforcement ratios, i.e. for cases with moment–curvature relationships that deviate most from the linear elastic one.

The effect of axial load and reinforcement on geometrical slenderness (\( \lambda = kL / r_g \)) is seen more clearly in Fig. 6(a). The reinforcement is the least important
parameter of these factors, but as seen, still quite substantial. The ACI axial load limit of 80% of the “squash load” for tied members is shown by the vertical lines in the figure.

**Normalized slenderness versus axial load**

A normalized slenderness representation of Fig. 6(a) is shown in the top portion of Fig. 6(b). In Fig. 6(b), bottom, similar results are shown for a uniform moment case ($\mu_o=1.0$). The rather significant increase (lift) in results caused by a small increase in moment gradient (from $\mu_o=1.0$ to 0.8) is worth noting. The uniform moment distribution represents a rather academic case. A more “practical uniformity” is probably better represented by a ratio of about $\mu_o=0.8$.

The chosen $k_t$ is a compromise value for load levels above and below the balanced load, and are seen to accomplish bringing curves for different reinforcements amounts reasonably close together. The relative difference between results at axial load levels below and above the balanced load (at about $\nu=0.4$), will decrease with increasing moment gradient due to the relative higher rate of increase versus moment gradient for low axial load levels than for high (cf Fig. 6(b)). Tension stiffening, if it had been included, and a modification of the normalized slenderness, to be discussed later, will help “linearize results” (towards horizontal lines). For further discussion of the $k_t$-definition (Eq. 5), including its capability of reflecting the influence of various section details (such as $h'/h$, reinforcement arrangement, $\varepsilon_y$, concrete quality and section shapes), see Hellesland (200A).

**Normalized slenderness versus moment gradient**

Normalized slenderness results versus the first order end moment ratio ($\mu_o$) are shown in Figs. 7 and 8. In the hinged member case (a), end moments are given directly by the applied moments. In the rather extreme, statically indeterminate case (b), the maximum moment is applied at the hinged end and a rotation is imposed at the end with the very stiff restraint to give a desired moment at that end. In this case, the smaller moment $M_i$ and the ratio will be dependent on the nonlinear material properties. It is in terms of such ratios that results are plotted. This is emphasized, since in design, moment ratios are computed using elastic stiffness assumptions. For instance, whereas half of a moment applied at the hinged end would be distributed to a clamped end in the elastic case, more than half would be so in the nonlinear case (due to the higher stiffness in the column portion with the smaller moments). Therefore, in terms of an elastic end moment ratio, the correct slenderness in this example would be obtained by entering the figures with a smaller (more negative) $\mu_o$ than the elastic $\mu_o=0.5$. A reading at the elastic $\mu_o=0.5$ will in other words be conservative. Although this complicates any direct use of the presented results for case b, they still serve as a good indication of a lower bound on results.

Results for circular sections and rectangular sections with distributed rein-
forcement ($RD$) are very similar to those above. The results for the $RD$ sections in Fig. 8 are for high axial load levels only. It should be noted that this figure does not include hinged member results. They will to the same extent seen before, be located above those shown.

The effect of different ultimate concrete strains are shown in Fig. 7 (upper, right). Use of the lower $\varepsilon_{cu}=0.003$ will allow a somewhat greater slenderness limit than the higher $\varepsilon_{cu}=0.0035$. However, the difference is at most about 5% at low reinforcement levels and less at higher levels.

For uniform bending ($\mu_0=1.0$), slenderness values at 10% capacity reduction is roughly about 40% greater than those at 5% capacity reduction. This difference decreases with increasing moment gradient (decreasing $\mu_o$).

**APPROXIMATE LIMITS. COMPARISONS**

An approximate limit giving predictions below or about equal to the nonlinear results for case $b$ (“clamped-hinged” case), is given by $\lambda_n = 22 - 12 \mu_0$. Completely “clamped” and “hinged” restraints are almost impossible to obtain in practice, and is therefore as mentioned before somewhat academic. A limit providing an approximate lower bound in cases with more practical end restraints (a very small restraint stiffness instead of the the theoretical hinge at one end and a very small restraint flexibility instead of the theoretical full fixity at the other), is that given by

$$\lambda_n = 24 - 14\mu_0 \tag{7}$$

and shown in Fig. 6(b) and Figs. 7 - 8. It corresponds to curve $d$ in the elastic case (Fig. 2). Eq. 7 may seem to be somewhat unconservative relative to the 5% limits for the combination of nearly uniform moment distributions and low axial load levels. However, already at the only slightly non-uniform distribution at $\mu_0 = 0.8$, this generally no longer so.

ACI 318-02 permits slenderness effects to be ignored in “compression members in non-sway frames” when the slenderness is less than

$$\lambda = 34 - 12 \mu_0 \leq 40 \tag{8}$$

The upper restriction (40) was first included in ACI 318-95. At that time, also the previous classification “compression members braced against sidesway” was replaced by the one above. In the present paper, the term “braced (against sidesway)” is used. The ACI limit is included in Fig. 6(a) and Figs. 7 - 8. Generally it is seen to be very conservative at lower axial load levels. At high levels, it may become unconservative, as seen for the heavily reinforced case in Fig. 8. If instead the member had been lightly reinforced, the degree of unconservativeness would become considerably greater. More so for single than for double curvature cases because of a very modest moment gradient effect in the ACI limit.
NS 3473 (NSF 1989) has adopted a limit for braced members without transverse loads between ends that has strong similarities with the one in Eq. 7. It is defined by

$$\lambda_N = 18 - 8 \mu_o$$

where $\lambda_N$ is denoted “load dependent slenderness” in the code. A premise for the use of this limit is that all sections along the member is designed for the larger first order moment ($M_2$) and that the end moment ratio is taken equal to 1.0 if $M_2$ is less than a specified minimum value.

This $\lambda_N$ differs from the $\lambda_n$ (Eq. 3) only by differences in safety philosophy (partial material factors rather than strength reduction factor) and by the chosen $k_t$-values. For the $RC$ and $RD$-sections considered here, NS 3473 implies $k_t=4$ and $8/3 (=2.67)$, respectively. Irrespective of concrete cover and steel quality. In comparison, $k_t=3.3$ (RC) and 2.2 (RD) are used in the presentations of this paper. For the same $\lambda$, $\nu$ and $\omega_t$ as in $\lambda_n$, $\lambda_N$ will provide predictions given by straight lines between about $\lambda_n=10.4$ ($\mu_o = 1$) and 27 ($\mu_o = -1$) for $\omega_t = 0.2$ and between about $\lambda_n=10.8$ and 28 for $\omega_t = 1.0$. These predictions are not included in the figures (in order not to overcrowd). The moment gradient effect in NS 3473 is chosen considerably more modest than in Eq. 7. This so for two reasons. First, documentation was not as extensive at the time (1987) the Norwegian limit was derived (Hellesland 1990). Thus some prudence was called for. The derivation was based on a combination of elastic moment gradient studies of hinged and restrained members, and nonlinear analyses of members with assumed displacement shape. Second, the gradient effect was chosen conservatively in order to allow indirectly for some unintentional eccentricities (due to imperfections etc.).

Both the ACI and the NS limits are related to 5% capacity reduction. The ACI limit was derived from the moment magnifier expression “assuming that 5% increase in moments due to slenderness is acceptable” (ACI 318-02 Commentary). For a given axial load, this implies a moment capacity reduction of about 5% (below the cross-sectional moment capacity). The latter was the criterion used (Hellesland 1990) when deriving the NS 3473 limit. In comparison, other codes allow greater reductions. For instance, Eurocode 2 (CEN 1991) relates its limit to a 10% “moment increase beyond first order moments” and the CEB-FIP Model Code 1990 (CEB 1993) to a reduction in bearing capacity of not more than 10%.

**SUSTAINED LOADING**

Creep at a given sustained load level has normally the most pronounced effects on compression members with high axial forces and small relative load eccentricities ($e_2/h$). Some results for such members are briefly reviewed below and compared to approximate predictions.

In an early study (Manuel 1966; Manuel and MacGregor 1967), that is relevant relative to criterion $c$ (“load capacity reduction for constant moment” in Fig. 3),
framed columns were analyzed. They were either symmetrically restrained by beams at both ends and bent in single, uniform curvature ($\mu_o = 1$), or restrained by a beam at one end and clamped at the other end and bent in double curvature (elastic $\mu_o = -0.5$). The full beam (“moment”) loading was applied first and then sustained throughout the load history. Axial loads, applied at column ends to represent stories above, were incremented to specified loads ($P_s$) that were sustained for 25 years and subsequently incremented quickly to column failure. The nominal eccentricities were at or below $e_2/h=0.1$. Nonlinear creep was included, with a high creep factor of about $\phi_{25_{yr}}=3.5$ - 3.8 at $\sigma_c/f'_c=0.4$ - 0.5 (estimated from total stress-strain curves in the paper). Given results in the study were reexamined in terms of nominal sectional load strengths (with half and full gross section stiffness, respectively, for beams and columns).

For columns in uniform first order curvature, there was a significant reduction (about 40%) in slenderness limits caused by sustained loading at about $P_s/P_n=0.6$. Even so, it was found for this sustained load level that an axial load capacity of $0.95P_n$ could be developed for $\lambda_n = 23 \, (L/h = 12, k = 0.87, \lambda = 36, \omega_1=0.294, k_1=4.94 \, (\text{Eq. 5}))$. In comparison, the approximate prediction of $\lambda_n = 10$ by Eq. 7 is very conservative. Compared to $\lambda = 36$ above, the ACI limit of $\lambda = 22$ is also quite conservative.

For columns in first order double curvature, it was found that a load capacity of $0.95P_n$ could be developed for about $\lambda_n=34$ to 30 ($L/h= 24$ to 20, $k=0.64, \lambda=53$ to 44) for sustained load levels of $P_s/P_n=0.4$ to 0.6, respectively. This can be seen in Fig. 9, which is produced here based on data deduced from Manuel (1966). For the double curvature case with the elastic $\mu_o = -0.5$, the approximate prediction by Eq. 7 of $\lambda_n = 31$ is close to the numerical result for $P_s/P_n=0.55$. This is considered acceptable. The comparable ACI limit of $\lambda=40$ , to be compared with $\lambda=53$ to 44, is more conservative.

Considering the very high creep factor and the unfavorable beam loading, one might have expected smaller numerical slenderness values than those reported above. That this is not so is due to a favorable moment transfer that took place from column ends to the end restraints (beams, support). In isolation, this tends to strengthen a column. For a lower, more practical creep factor, the numerical results would increase and the approximation become more conservative.

Other relevant results, for both unrestrained and restrained members, have been reported by Mari and Hellesland (2002). They were obtained by numerical nonlinear analysis (FEM) that considered both material and geometric nonlinearities, including cracking and also tension stiffening. The columns were first subjected to a period (50 years) of sustained loading at a specified fraction ($\psi$) of the axial load and moment capacities. Subsequently they were loaded to failure in a short time load application conforming to one of the 3 load applications defined in conjunction with the sub-criteria ($a, b, c$) discussed previously.

Selected 5% capacity reductions results for $\psi=0.6$, a creep factor of $\phi_{0.0_{yr}}=2$,
small and very small nominal eccentricities (mostly $e_z/h=0.1$) and low and high reinforcement ratios are compared in Fig. 10 to the approximate limit given by Eq. 7. For unrestrained columns ($k = 1$), the approximate limit is generally conservative at all considered end moment ratios ($\mu_o = 1.0, 0, -0.9$). Thus, there is room in the approximate limit for additional creep effects (greater creep factor) than used in the study. For columns hinged at one end and clamped at the other (elastic $k = 0.7$, $\mu_o = -0.5$), the approximate limit is just acceptable for the creep factor used in the study. With added creep, it becomes more unconservative. It should be noted, however, that the “hinged-clamped” restraint condition is unfavorable, as discussed previously, and also that, in accordance with normal practice, no concrete strength increase due to aging was included (in either study above). This is conservative assumption. With aging effects included, added creep effects can be allowed.

In a proposed revision of Eurocode 2 (CEN 2002), creep is included through an explicit parameter (in the slenderness limit) that has the same relative effect at both high and low load levels. This is a drawback. The review above of sustained load effects at small eccentricities, documents reasonably well that the approximate Eq. 7 allows for practical creep effects at sustained loads as high as 50-60% of nominal sectional capacities. This is would seem to be quite acceptable for most cases in regular structures.

PROPOSAL

Introductory comments

It has been found that Eq. 7 reflects important parameters well. It is given independent of creep, but allows for normal creep in both unrestrained and restrained members.

The strength reduction factor $\phi$ (Eq. 4) is in ACI 318-02 load (or strain) dependent, varying between 0.9 for lower axial loads (tension controlled sections) to 0.65 for loads at or above the balanced load (compression controlled sections. For spiral reinforced members, 0.65 is to be replaced by 0.7). It will be proposed to replace $\phi$ by a constant (“stiffness reduction”) factor $\phi_k$ with a value between 0.9 and 0.65. This yields a modified normalized slenderness that may be defined by

$$\lambda_{n_0} = \frac{kL}{r_g} \sqrt{\frac{P_u/f'_{cm} A_g}{\phi_k(1 + k_i \omega_i)}}$$

(10)

The subscript ‘n’ has been altered to ‘no’ for the sake of distinction and ease of discussion. This modification has its parallel in the ACI 318-02 moment multiplier expression, where $\phi_k=0.75$ is adopted. In addition to simplify the slenderness definition, it also has the desired effect of making results such as those in Fig. 6(b), fall on more horizontal lines if plotted in terms of $\lambda_{n_0}$; at lower axial load levels (at which $\phi_k$ is less than $\phi$) results will be lifted, whereas at higher ax-
ial loads (at which $\phi_k$ is greater than $\phi$) they will be somewhat lowered. This lowering is expected to be more than compensated for by the conservativeness of using the same strength reduction factor at all sections in the numerical analyses. Additional details and discussion on this are given in Hellesland (200A).

ACI 318-02, NS 3473 and other codes reviewed elsewhere (Hellesland 200A), allow the end moment ratio ($\mu_0 = M_1/M_2$) to be calculated with moments obtained from a conventional first order analysis based on recommended stiffness assumptions and on the intended geometry, i.e., without unintentional eccentricities (to reflect imperfections, uncertainties, etc.) included. Is this acceptable also in conjunction with Eq. 7, which reflects a considerable stronger moment gradient effect than NS 3473, and even more so than ACI 318-02? Possible uncertainties in moment calculations may be considered adequately covered by reduction factors and generally conservative design assumptions. Or is additional caution called for considering that slenderness predictions by Eq. 7 may increase by a factor of about 3.8 when the end moment ratio changes from 1 to -1? These questions are not addressed further here. However, possible adaptations to allow for additional uncertainties, if that should be considered necessary, is discussed below.

For computation of $\lambda_{no}$ ($\lambda_n$) at a preliminary design stage, conservative assumptions, such as minimum reinforcement and axial load due to an unfavorable load case, maybe introduced. If necessary, checking with more refined values can be performed at a more advanced stage in the design process.

**Modified slenderness proposal**

A normalized slenderness is defined by $\lambda_{no}$, Eq. 10. In lieu of more accurate values, it is considered acceptable to take $f_{cn}'=0.8 f'_c$ for all concrete strengths, and $\phi_v=0.75$. The relative “reinforcement contribution factor” $k_i$ can be taken according to Eq. 5, or according to conservative simplifications (Hellesland 200A).

**Lower limit proposal. Alternative 1**

For compression members braced against sidesway, and without transverse loading between ends, it is permitted to ignore slenderness effects if the slenderness is less than

$$\lambda_{no} = 24 - 14 \mu_0$$

(11)

provided that all sections along the member and supports are designed for the larger first order end moment. The moment ratio $\mu_0$ is between the smaller ($M_1$) and the larger ($M_2$) first order design end moments as obtained from a conventional first order analysis. The ratio is to be taken positive for members bent in single curvature by these moments, and negative for members bent in double curvature.

An added restraint might be that the end moment ratio is to be taken equal to unity when the maximum first order moment is less than some specified lower value. This is similar to one option given in the ACI 318-02 for the computation
of the moment gradient factor $C_m$. It has the disadvantage that it introduces a discontinuity, and a strong one in columns with high moment gradients, as $M_2$ "passes" the lower value, and it does not cover uncertainties at eccentricities in excess of the minimum one.

**Lower limit proposal. Alternative 2**

Several other approaches may be suitable in allowing for additional uncertainties, if that should be considered necessary. One is to replace $\mu_0$ in Eq. 11 by an “imperfection (or uncertainty) adjusted” $\mu_1$ that allows explicitly for uncertainties of various kinds. This $\mu_1$ can be defined by adding an “unintentional” uniform moment distribution ($P_u \Delta e$) to the nominal distribution. Thus,

$$\mu_1 = \frac{M_1 + P_u \Delta e}{M_2 + P_u \Delta e} = \frac{\epsilon_1 + \Delta \epsilon}{\epsilon_2 + \Delta \epsilon}$$  \hspace{1cm} (12)

where the larger first order end moment $M_2$ (eccentricity $\epsilon_2$) is always taken positive. The smaller moment $M_1$ ($\epsilon_1$) is taken positive when the member is bent into single curvature by the first order moments and negative when bent the into double curvature.

This approach will not alter the inclination of the line of axial thrust and would seem like a rational approach. It is illustrated in Fig. 11 for different $\epsilon_2$-values when $\Delta \epsilon = 0.05h$.

This $\mu_1$ - definition is also a suitable instrument in allowing, if desired, for additional creep effects in members with moment gradients, beyond those already allowed for. The effect on the slenderness limit of a chosen "creep addition" to $\Delta \epsilon$ is like that of creep. It is strongest at smaller eccentricities, and tapers off with increasing eccentricity. Also, the effect decreases with increasing uniformity of the moment distribution (when creep effects can be significant and still not be of concern relative to criterion $b$ or $c$).

**Lower limit proposal. Alternative 3**

Another approach is to reflect uncertainties implicitly through adoption of a conservative moment gradient relationship. It may be derived based on Eq. 11 with $\mu_0$ replaced by $\mu_1$. For the case with $\Delta \epsilon = 0.05h$ illustrated in Fig. 11, a limit given by

$$\lambda_{n0} = 20 - 10 \mu_0 \hspace{1cm} (\lambda_{n0} = 22 - 12 \mu_0)$$  \hspace{1cm} (13)

might be sufficiently conservative. The limit in parenthesis can be obtained in a similar fashion if the smaller unintentional eccentricity of $\Delta \epsilon = 0.02h$ is adopted. The smallest $\epsilon_2$ in the comparison in the figure is $0.1h$. This value corresponds approximately to the upper axial load limit ($0.8P_o$ for tied members), and represents the smallest moment capacity a member will be provided with. Comparisons for smaller $\epsilon_2$ - values have less practical interest.

An advantage of this approach is that it is somewhat simpler than the one above. It reflects uncertainties well at small eccentricities, at which they have
the strongest effects. A disadvantage is that it will have the same conservative impact also at larger eccentricities (or axial loads).

**SUMMARY AND CONCLUSIONS**

Factors affecting lower slenderness limits have been discussed and results presented of extensive nonlinear analyses of braced compression members without transverse loads between ends (i.e., end-loaded members).

The current ACI limit is found to be very conservative for members with low axial load levels. On the other hand, it may become very unconservative for lightly reinforced members with high axial loads.

A new slenderness limit is proposed in terms of a rational slenderness measure, labeled normalized slenderness, that includes effects of axial force and reinforcement. The limit is found to allow for normal creep effects and is considered to be applicable for all concrete qualities, and will allow member slenderness effects to be ignored in a great many more cases, it is believed, than allowed by the current ACI limit. It is felt that reduced design efforts due to this, but most of all the added reliability of the formulation, more than compensates for the added complexity, which still is rather reasonable. The inclusion of the additional parameters also has the advantage of adding focus on and awareness of parameters that is important in design of compression members.

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**APPENDIX. NOTATION**

$A_g, A_{st}$ = area of gross section and of total reinforcing steel

$I_g, I_s$ = second moment of area about centroidal axis of gross section and of total reinforcing steel

$L$ = length of compression member

$M_n$ = nominal moment capacity (of section) at given $P_n$

$M_1, M_2$ = smaller and larger factored first order (design) end moment

$P_n, P_u$ = nominal axial load capacity and ultimate factored (design) axial load

$f_c, f_{cn} =$ cylinder and nominal structural compressive strength of concrete

$f_y$ = yield strength of reinforcing steel

$h, h'$ = section depth and distance between reinforcement in opposite faces

$k$ = effective length factor of compression member

$r$ = radius of gyration of cross section, normally taken as $r_g$

$r_g = (I_g/A_g)^{1/2}$ = radius of gyration of gross concrete section

$r_s = (I_s/A_{st})^{1/2}$ = radius of gyration of total reinforcing steel

$\delta_M$ = exact elastic moment magnifier

$\varepsilon_y = f_y/E_s =$ yield strain of reinforcing steel

$\phi, \phi_k =$ strength and stiffness reduction factors

$\lambda = kL/r_g =$ geometric slenderness

$\lambda_n, \lambda_{no}$ = normalized slendernesses

$\mu_o = M_1/M_2$ first order end moment ratio

$\omega_t =$ total mechanical reinforcement ratio
APPENDIX. NUMERICAL ANALYSIS

The differential equation, \( v'' = -\kappa \), where \( v'' \) is the second derivative of the displacement, \( \kappa = 1/R \) the curvature and \( R \) the radius of curvature, was discretized using the central difference expression

\[
(v_{i+1} - 2v_i + v_{i-1})/(\Delta x)^2 = -\kappa_i \quad i = 1, 2, \ldots, m - 1
\]

for the member divided into \( m-1 \) elements of equal length \( (\Delta x) \). Section \( i = 0 \) is at end 1 and \( i = m \) at end 2. Subject to the forced boundary conditions \( v_0 = v_m = 0 \), Eq. 14 results in a stiffness relationship from which the displacements at all sections can be computed iteratively. It becomes

\[
\begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& \ddots & \ddots & \ddots \\
& -1 & 2 & -1 \\
& -1 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_{m-2} \\
v_{m-1}
\end{bmatrix}
= 
\begin{bmatrix}
(\Delta x)^2 \cdot \kappa_1 \\
(\Delta x)^2 \cdot \kappa_2 \\
\vdots \\
(\Delta x)^2 \cdot \kappa_{m-2} \\
(\Delta x)^2 \cdot \kappa_{m-1}
\end{bmatrix}
\]

A. Determine slenderness limits

This description gives the main steps of only the two cases defined by i) both ends being unrestrained (hinged) and ii) one end unrestrained and the other (end 2) restrained by a very stiff restraint. For details see Aasrum (1992) or Hellesland (2002). The program is very fast, and a fine subdivision (normally \( \Delta x = h/30 \)) and narrow tolerances are adopted. Steps:

1. Choose cross–section and reinforcement (symmetrical).

2. Choose axial load \( (P_n = P_u / \phi) \) and compute the moment–curvature relationship \( (M_{nx} - \kappa) \) of the section. The maximum moment value, i.e., the cross-sectional moment capacity, is denoted \( M_n \) (= max \( M_{nx} \)). It is obtained at predefined ultimate strains (at \( \varepsilon_{cu} \), or at \( \varepsilon_{su} = 0.010 \) if this gives lower moment resistance than \( \varepsilon_{cu} \)).

3. Set the larger first order end moment \( M_2 \) (at end 2) equal to a specified fraction (0.95 or 0.90) of \( M_n \). Since the end is unrestrained (hinged), \( M_{i2} = M_2 \).

4. At end 1 with the smaller (in absolute value) first order end moment, choose a value for the total end moment \( M_{i1} \), and compute (by iteration as described below in routine B) the largest member length, \( L_{max} \), for which it is possible to establish equilibrium. Continue the procedure with this length and the corresponding displacements \( v_i \).

If the member is unrestrained at both ends, set \( M_1 = M_{i1} \) and go to Step 7. Otherwise, go to next step.

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5. Set the first order rotation at end 1 equal to the total rotation, \( \theta_1 = \theta_{\odot} = (2v_1 + (\Delta x)^2 \kappa_0)/2\Delta x \), as the rotation at the very stiff restraint is insignificantly affected by second order effects.

6. Compute the first order end moment \( M_1 \) that corresponds to this \( \theta_1 \) (by iteration as described below in routine C).

7. Compute the first order end moment ratio \( \mu_0 = M_1/M_2 \) and slenderness parameters \( \lambda = kL/r \) and \( \lambda_n \) using elastic effective length factors.

8. Repeat from step 1, 2, 3 or 4 (as desired).

**B. Maximum \( L \) (at instability), iteration scheme**

1. Assume \( L \), compute total moments at all discrete sections from \( M_{t,i} = M_{t,i} + (M_{t,2} - M_{t,1})i\Delta x/L + P\nu_i \) (set \( \nu_i = 0 \) in the first iteration) and determine corresponding curvature values from the \( M - \kappa \) relationship in step A2.

2. Solve the equilibrium equations, Eq. 15, for the displacements \( \nu_i \), and repeat steps 1 to 2 until there is no significant change in the \( \nu_i \)’s from two consecutive iterations (adopted tolerance 0.01%).

3. Repeat from Step 1 with new \( L \) until the largest length \( L_{\text{max}} \) at which equilibrium can be established is obtained (adopted tolerance 0.1\%). This length corresponds to stability failure (at or prior to material failure).

**C. End moment \( M_1 \), iteration scheme**

1. Assume \( M_1 \), compute first order moments at all discrete sections from \( M_i = M_i + (M_2 - M_1)i\Delta x/L \) and determine corresponding curvatures from the applicable \( M - \kappa \) relationship (step A2).

2. Solve the equilibrium equations, Eq. 15, for first order values of the displacements \( \nu_i \), and compute the end rotation \( \theta_1 = (2v_1 + (\Delta x)^2 \kappa_0)/2\Delta x \).

3. Repeat from Step 1 with a new \( M_1 \) until the difference between the calculated \( \theta_1 \) in the step above and \( \theta_1 \) in step A5 is negligible (adopted tolerance \( \Delta \theta_1 = 0.00001 \)).
Figure 1: Typical first order and total moment distributions for rotationally restrained, braced columns in single and double curvature

Figure 2: Slenderness limits vs. moment gradient – Exact elastic results with $\delta_M=1.05$ for different end restraints, and approximate curves (ordinate values at $\mu_o=1.0$: curve $a=0.197$, curve $b$ and $c=0.185$, approximate curves $d=0.218$)
**Figure 3:** Approximate slenderness limits at 5% capacity reduction according to various criteria (based on the moment multiplier with \( C_m = 0.6 + 0.4\mu_o \) and a \( P - M \) diagram for a medium reinforced member \((\omega_t = 0.6, h'/h = 0.8, \varepsilon_y = 0.0024)\))

**Figure 4:** Typical radii of gyration of gross cross-sections and of symmetrical reinforcing bar arrangements
**Figure 5:** Slenderness limits vs. moment gradient – Effect of different axial load levels and reinforcement ratios (at 5% capacity reduction) and comparison with elastic results (at $\delta_M=1.05$).

**Figure 6:** a) Geometric slenderness and b) Normalized slenderness vs. axial load level for unrestrained members in single curvature.
Figure 7: Normalized slenderness vs. moment gradient: At left, low load level ($\nu=0.2$), and at right, intermediate load level ($\nu=0.6$). At top, light reinforcement ($\omega_l=0.2$), and at bottom, heavy reinforcement ($\omega_l=1.0$). Rectangular section with corner reinforcement ($RC$, $k_i=3.3$).
Figure 8: Normalized slenderness vs. moment gradient for strongly reinforced member with high $\nu=1.2$ and very high $\nu=1.5$ load level. Rectangular section with distributed reinforcement ($RD, k_f = 2.2$)

Figure 9: Effect of sustained load on slenderness limits for 5 and 10% reduction in axial load capacity (deduced from results given in Manuel (1966)
Figure 10: Comparison of approximate and accurate slenderness limits at 5% capacity reduction for members subjected to high sustained loading (60% of the nominal capacity) prior to short term loading to failure in accordance with sub-criterion a, b or c (from results given in Mari and Hellesland (2002))

Figure 11: Comparison of alternative slenderness limits