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Weyl and Two Kinds of Potential Domains

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Abstract

According to Weyl, "'inexhaustibility' is essential to the infinite". However, he distinguishes two kinds of inexhaustible, or merely potential, domains: those that are "extensionally determinate" and those that are not. This article clarifies Weyl's distinction and explains its enduring logical and philosophical significance. The distinction sheds lights on the contemporary debate about potentialism, which in turn affords a deeper understanding of Weyl.

1 | INTRODUCTION

According to the great mathematician Hermann Weyl, "'inexhaustibility' is essential to the infinite" (Weyl, 1918, p. 23). To recognize the inexhaustible character of the infinite is not only of philosophical interest but is, Weyl writes, essential for placing mathematics on a sound foundation:

> The deepest root of the trouble lies elsewhere: a field of possibilities open into infinity has been mistaken for a closed realm of things existing in themselves. As Brouwer pointed out, this is a fallacy, the Fall and Original Sin of set-theory, even if no paradoxes result from it. (Weyl, 1949, p. 234)

Weyl's insistence on the inexhaustibility of the infinite is an instance of the ancient view, going back to Aristotle, that the only legitimate notion of infinity is that of potential infinity. A good example is Aristotle's view that matter is infinitely divisible. Consider a stick *s*. No matter how small a part of *s* we have produced, it is possible to produce an even smaller part:

(1) Necessarily, for any proper part x of s, possibly x has a proper part.

However, Aristotle denies that s is, or even could be, infinitely divided, that is, that the following situation should obtain:

(2) For any proper part x of s, x has a proper part.

The reason Aristotle adduces is that, if (2) obtained, then matter would absurdly be "divided away into nothing" (*On Generation and Corruption*, 317a3-8, Aristotle, 1941).

Aristotle defends a similar view of the natural numbers. The process of producing numbers—which according to Aristotle is a matter of instantiating the numbers by means of appropriately numerous pluralities of concrete objects—can always be extended:

(3) Necessarily, for any natural number m, possibly there is a successor m'.

In this case too, Aristotle denies that the process can be completed, that is, that we could ever have:

(4) For any natural number m, there is a successor m'.

In short, while there are potentially infinitely many natural numbers, it is, according to Aristotle, incoherent to assume that they form an actual infinity.

This view of infinity remained the dominant view in philosophy and mathematics for a very long time. Thus, as late as 1831, the "prince of mathematics", Gauss wrote:

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics.

The big turning point, only a few decades later, was Cantor, who defended the diametrically opposite view:

> every potential infinite, if it is to be applicable in a rigorous mathematical way, presupposes an actual infinite (Cantor, 1887, pp. 410–411).

This Cantorian orientation is now dominant in mainstream mathematics, with various constructivists as notable exceptions.

While Cantor steadfastly insists that the natural numbers can be completed, at times he ascribes to the domain of all sets a status akin to that of an Aristotelian potential infinity.

> [I]t is necessary ... to distinguish two kinds of multiplicities [...]. For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities* ... If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a *consistent multiplicity* or a 'set'. (Ewald, 1996, pp. 931-932)

In light of this background, it is useful to distinguish two different orientations towards mathematics. According to *actualism*, there is no use for modal notions in mathematics, whether explicit or implicit. *Potentialism*, by contrast, insists that there is a use for modal notions in mathematics. For some mathematical objects are generated successively in such a way that it is impossible to complete the process of generation.[1] Let us call a domain *merely potential* when it is successively generated in a way that cannot be completed.[2] Aristotle took even the domain of natural numbers to be merely potential, resulting in an austere form of potentialism. By contrast, Cantor always allowed the domain of the natural numbers to be completed but at times suggested that the domain of all sets is merely potential. This yields a far more relaxed, set-theoretic form of potentialism.[3]

Let us return to Weyl. By insisting that "inexhaustability is essential to the infinite", Weyl appears to be endorsing a form of potentialism that is closer to Aristotle's than Cantor's. He agrees with Aristotle, as against Cantor, that already the natural numbers are "inexhaustible".[4] And as we will see shortly, Weyl insists that an infinite set be understood, not as the result of infinitely many arbitrary choices of elements to gather together, but as the extension of a well-defined property. In one respect, however, Weyl's potentialism appears less radical than Aristotle's. While Aristotle argues that actual infinities are incoherent, Weyl does no such thing (as far as we know). His misgivings about actual infinities appear to be based on a lack of compelling evidence *for* their existence rather than compelling evidence *against*.[5] In a nutshell, Weyl claims that the "house of analysis is [...] built on sand" (Weyl, 1918, p. 1), not that the house is already in ruins.

Our main topic will be an entirely new idea that Weyl injects into the discussion of potentialism. Not all incompletable or merely potential domains have the same character. We must distinguish between those that are "extensionally determinate"—or, very roughly, properly demarcated—and those that are not (Weyl 1919, 1921).[6] We believe this novel distinction is of great theoretical importance, including for today's debate concerning potentialism. Our overarching aim is therefore to gain a better understanding of the distinction and its significance.

[1] Different views of the modality in question are possible. The most natural option is to take the modality to be some form of metaphysical modality. This is probably true of Aristotle. Weyl's own view of the matter is not entirely clear. Some recent defenses of potentialist approaches invoke a so-called "interpretational modality", where the modal operators shift the interpretation of the language, not the circumstances that this language describes; see Fine (2005), Linnebo (2018b), and Studd (2019).

[2] The word 'merely' is intended to exclude domains that are not only potential but also actual. After all, actuality entails possibility ("*ab esse ad posse*").

[3] See Parsons (1983b), Linnebo (2013), and Studd (2013) for some ways to develop this view.

[4] While this claim is naturally read as an expression of potentialism, there are other passages in Weyl (1918) that might suggest some form of actualism. For example, on p. 87, Weyl describes the natural numbers as a "totality"; see also pp. 8 and 43. We admit that Weyl (1918) is hard to interpret. We therefore place greater emphasis on later works where Weyl's potentialism is more pronounced.

[5] In the early text (Weyl 1910), Weyl writes: "An actual perception of infinite sets – in the sense that their individual elements are simultaneously present as separately observed contents in our consciousness – is unattainable. It does not follow, though, that infinite sets are logically illegitimate. After all, an actual presentation to consciousness of a set with a large number of elements can be unattainable even when the set is finite. So it is true that 'there is no actual infinity' only in the sense that the actual presence to consciousness of infinite manifolds is impossible."

[6] Thus, Weyl's notion of "extensional determinateness" differs from some recent uses of "extensional definiteness" (Linnebo 2013, 2018b; Studd 2019), where a condition or property is said to be extensionally definite just in case its instances are completable (which in turn may be explicated as forming a plurality). By contrast, Linnebo (2018a)'s notion of "extensional definiteness"is very close to Weyl's.

More specifically, we discuss two questions for which Weyl's notion of extensional determinateness is important. First, there is the question of how quantification over a merely potential domain can and should be understood. Since there is no stage at which all members of the domain are available, the ordinary understanding of universal generality in terms of each and every member of the domain being thus-and-so is not an option. As we shall see, Weyl suggests two alternative ways to understand such generality: one associated with Weyl (1918), which justifies quantification, using classical logic, over any extensionally determinate domain, and another inspired by Weyl (1921), which justifies quantification, using intuitionistic logic, over *any* domain, extensionally determinate or not.

Second, we connect Weyl's distinction between two kinds of merely potential domains with a recent distinction between "liberal" and "strict" potentialism, associated with classical and intuitionistic logic, respectively (Linnebo and Shapiro 2019). We argue that the more moderate, liberal form of potentialism is appropriate if and only if the potential domain is extensionally determinate.[7] This connection enables the historical and the contemporary debates to inform each other.

In short, Weyl's work provides a finer classification of domains. In addition to the familiar Aristotelian distinction between completable and incompletable domains, the latter must be subdivided according to whether or not the domain is extensionally determinate. The resulting three-way classification is important, because it corresponds to the three main views in the contemporary debate about potentialism and their logical consequences:

kind of domain	**ext. determinate**	**ext. indeterminate**
completable	actualism	—
incompletable	liberal potentialism (classical logic)	strict potentialism (intuitionistic logic)

2 | WEYL'S NOTION OF EXTENSIONAL DETERMINATENESS

What, then, is Weyl's distinction between merely potential domains that are "extensionally determinate" and those that are not? The intuitive idea of extensional determinateness is introduced as follows. Even though a concept is "clearly and unambiguously defined", Weyl writes, this

> does not imply that this concept is *extensionally determinate*, i.e., that it is meaningful to speak of the *existent* objects falling under it as an ideally closed aggregate which is intrinsically determined and demarcated [an sich bestimmten und begrenzten]. (Weyl, 1919, p. 109)

While hardly precise, the language is certainly suggestive. We have at least some loose and intuitive sense of what it is for some objects to form "an ideally closed aggregate" and for this aggregate to be "intrinsically determined and demarcated". What matters is that the objects in question

[7] This makes it tempting to interpret the almost actualist-sounding passages in Weyl (1918) that we mentioned in fn. 4 as just early and somewhat clumsy expressions of liberal, as opposed to strict, potentialism. In this way, the distinction between two forms of potentialism might help with the interpretation of Weyl (1918).

should be properly circumscribed or "demarcated". The relevant circumscription or demarcation may even involve some idealized form of closure. We shall shortly consider some examples.

First, however, we note that Weyl proceeds to give the mentioned intuitive ideas a more precise logical articulation.

> Suppose P is a property pertinent to the objects falling under a concept C. [...] if the concept C is extensionally determinate, then not only the question "Does a have the property P?" [...] but also the existential question "*Is there* an object falling under C which has the property P?", possesses a sense which is intrinsically clear. (ibid.)

Let us unpack this a bit. Our question is whether some concept C is extensionally determinate. To answer the question, we are asked to assume that a property P is clearly and unambiguously defined on the sort of objects with which C is concerned. This assumption ensures that 'Pa' has an "intrinsically clear" sense, for every object a of the appropriate sort. But the assumption does not, on its own, ensure that quantification restricted to Cs, such as '$(\exists x : C)Px$', preserves this feature of having an intrinsically clear sense. If the Cs have not been properly demarcated, this quantified statement need not have a clear sense. We can now provide a more precise articulation of the claim that the concept C is extensionally determinate:

> **Extensional determinateness (initial analysis)** A concept C is *extensionally determinate* just in case quantification restricted to C preserves the property of having an "intrinsically clear" sense, that is, just in case, for any property P, if Pa has an intrinsically clear sense for each appropriate instance a, so too does '$(\exists x : C)Px$'.

While this characterization of extensional determinateness is more informative, it still relies on the unanalyzed notion of a statement's having an intrinsically clear sense. How should this notion be understood? Weyl writes that a question has an intrinsically clear sense when it "address[es] an existing state of affairs that allows one to answer the question with yes or no" (Weyl, 1921, p. 88).[8] In other words, a statement has an intrinsically clear sense just in case the statement has been assigned a meaning that ensures that it is either true or false, that is, just in case bivalence holds for the statement. This yields a precise logical analysis of the initially rather loose and intuitive notion of extensional determinateness.

There is only one shortcoming. Couched in terms of the semantic notion of bivalence, this analysis is given in a metalanguage, not in the relevant object language. But this shortcoming is easily remedied. A statement's having an "intrinsically clear sense" can be taken to be a matter of the Law of Excluded Middle (LEM) holding for the statement. The associated step from bivalence (in the metalanguage) to LEM (in the object language) is a natural one.[9] Although Weyl does not, as far as we know, explicitly endorse this step, it turns on a distinction that was clarified only later. Let us therefore take the step and plug the resulting analysis of what it is to have an intrinsically clear sense into Weyl's initial analysis of extensional determinateness. This notion thus receives an analysis that is not only logically precise but also expressed in the object language.

[8] Compare (Weyl, 1919, p. 109).

[9] This step presupposes a background of intuitionistic logic, which was not an option Weyl seriously considered until Weyl (1921). Moreover, although natural, the step can be resisted; see (Avron, 2020, note 18, p. 60) and, more generally, Rumfitt (2015).

> **Extensional determinateness (formal analysis)** A concept C is *extensionally determinate* just in case quantification restricted to C preserves the property of LEM holding, that is, just in case, for every property P, if LEM holds for each instance of 'Px', then it holds for '$(\exists x : C)Px$' as well.[10]

This analysis can be expressed more compactly by introducing some natural definitions. First, say that a formula *behaves classically* when LEM holds for the formula. Second, say that a concept C specifies a *domain of classical logic* just in case quantification restricted to C always preserves classical behavior. Our formal analysis now takes the form of (a version of) a slogan proposed by Feferman: "What's extensionally determinate is the domain of classical logic, what's not is that of intuitionistic logic".[11]

We have arrived at a pleasingly sharp and natural articulation of the initially somewhat obscure notion of extensional determinateness. Although certainly sharp, the proposed analysis is also highly abstract. So the analysis leaves considerable room for disagreement about what concepts are in fact extensionally determinate.

3 | WEYL ON WHAT IS AND IS NOT EXTENSIONALLY DETERMINATE

Our next task is to present Weyl's view of what concepts are extensionally determinate. His view is nicely summarized in the following passage:

> The intuition of iteration assures us that the concept "natural number" is extensionally determinate. [...] However, the universal concept "object" is not extensionally determinate—nor is the concept "property," nor even just "property of natural number". (Weyl, 1919, p. 110)

Let us spell out his view in more detail. The simplest example of an extensionally determinate concept is one whose extension is a finite set. For Weyl, a finite set can be described in "*individual* terms", that is, by exhibiting each of its elements (Weyl, 1918, p. 20). Clearly, this suffices to ensure the extensional determinateness of any concept whose extension is a finite set. In fact, suppose that it is determinate whether or not a property P holds of each of the elements of a finite set A, with A extension of some concept C. Then we can run through all the elements of A up to the last one, and in this way determine whether or not there is an element of A with property P. That is, if LEM holds of 'Pa' for each element a of A, then it holds of '$(\exists x : C)Px$' as well.

A more interesting example of an extensionally determinate concept is that of natural number. Since the natural numbers are (potentially) infinite, they cannot be exhaustively listed, unlike the

[10] We note that this preservation property corresponds to a schematic principle known as "bounded omniscience" for C:

$$(\forall x : C)(\psi(x) \vee \neg\psi(x)) \to ((\exists x : C)\psi(x) \vee \neg(\exists x : C)\psi(x))$$

In fact, the above formulation, inspired by Weyl, is intuitionistically equivalent to a formulation we often encounter today, e.g. in Feferman (2010), namely $(\forall x : C)(\psi(x) \vee \neg\psi(x)) \to ((\forall x : C)\psi(x) \vee (\exists x : C)\neg\psi(x))$.

[11] See (Feferman, 2011, p. 23). We have changed Feferman's 'definite' to 'extensionally determinate'. Closely related ideas are found in Dummett as well; see Crosilla (2016); Linnebo (2018a); Crosilla (202x).

members of a finite set. Yet for Weyl, this does not prevent the natural number concept from being extensionally determinate.

> The intuition of iteration assures us that the concept "natural number" is extensionally determinate. (Weyl, 1919, p. 110)

The idea is tolerably clear. The natural numbers are generated from a first element, say 1, by repeated application of the single generating operation of successor. This, however, does not suffice to demarcate the natural number concept. The concept of being a natural number is the concept of being in the *minimal closure* of the successor operation. As usual, it is the axiom schema of mathematical induction that explicates this closure condition and crucially, for Weyl, mathematical induction is justified by a form of intuition.[12]

Next, we ask whether the concept of being a set of natural numbers is extensionally determinate. Suppose we accept the combinatorial conception of set and its idea of running through the natural numbers one by one, making arbitrary choices as to whether or not each number is to be a member of a set. Then an easy affirmative answer would be available. Since the natural numbers have been properly demarcated, so would be the collection of all sets thereof.[13]

Weyl, however, insists that the combinatorial conception of set is permissible only for finite sets.

> Finite sets can be described in two ways: either in *individual* terms, by exhibiting each of their elements, or in *general* terms, on the basis of a rule [gesetzmäßig], i.e., by indicating properties which apply to the elements of the set and to no other objects. In the case of infinite sets, the first way is impossible (and this is the very essence of the infinite). (Weyl, 1918, p. 20)

A few pages later he uses even stronger language.

> The notion of an infinite set as a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and surveyed as a whole by consciousness, is nonsensical: "inexhaustibility" is essential to the infinite. (Weyl, 1918, p. 23)

How, then, can we describe an infinite—and therefore incompletable—set? We must, according to Weyl, rely on a rule that "indicates properties which apply to the elements of the set and to no other objects" (Weyl, 1918, p. 20). This suggests an alternative to the combinatorial conception of set, which is known as *the logical conception*.[14] As we will now see, this alternative allows us to articulate concepts that are extensionally determinate and whose extensions are infinite, incompletable sets. In particular, we can describe extensionally determinate sets of natural

[12] Weyl (1918, p. 48) acknowledges his agreement with Poincaré on this issue. See also Folina (2007) for a discussion of the role of intuition in the early Weyl. Note that Weyl (1918, p. 24) considers an alternative impredicative foundation of the natural number concept as in Dedekind (1888) and claims that it "may indeed contribute to the systematization of mathematics; but it must not be allowed to obscure the fact that our grasp of the basic concepts of set theory depends on a prior intuition of iteration and of the sequence of natural numbers." See also Weyl (1918, p. 48).

[13] Such a view is held by Bernays (1935) and is implicit in the set-theoretic potentialism mentioned in Section 1.

[14] See e.g. Maddy (1983, 1997); Ferreirós (1996).

numbers. The key is to invoke what Weyl calls the "mathematical process" (Weyl, 1918, p. 22), which describes a "process of concept formation" giving rise to extensionally determinate sets.

The mathematical process describes the "production" of sets as extensional counterparts of certain properties. These properties are generated starting from a stock of initial properties and relations that hold of the elements of one or more primitive domains of objects, such as the natural numbers (Weyl, 1918, p. 28). New complex properties and relations are then obtained from these initial ones by repeated application of the ordinary logical operations of negation, conjunction, disjunction and existential quantification, plus an operation of substitution (Weyl, 1918, p. 10).[15] For example, in the important case of mathematical analysis, which is the main focus of Weyl's book, the primitive domain is that of the natural numbers and the initial relation is the successor relation between natural numbers. In this case, new complex relations are obtained by repeated application of the logical operations together with a further fundamental operation, a so-called *principle of iteration*, which accounts for the principle of mathematical induction.

Crucially, in the "process of concept formation" application of the existential quantifier is restricted to the primitive domains, to ensure a bottom-up construction of new sets. In the case of analysis, for example, quantification is restricted to the natural numbers. The reason for this restriction is a fear of vicious circularity, which will be explained shortly.

Once complex properties of elements of the initial domains have been formed, *sets* can be generated as *extensions* of those properties and identified with each other just in case they are coextensional (Weyl, 1918, p. 20). We may call sets that are generated in this way through the mathematical process *predicative sets*. Weyl takes each predicative set to be extensionally determinate, since the initial domains are extensionally determinate, and the properties used to construct the set are obtained by "any number of repetitions and combinations of the given construction principles" (Weyl, 1921, p. 91).

Having established that *each* predicative set of natural numbers, is extensionally determinate, the question arises whether there is an extensionally determinate domain of *all* such sets. Since a set of numbers is specified by a property of numbers, the question is, in effect, whether *extensionally determinate property of the natural numbers* is itself extensionally determinate. The answer, Weyl (1919, pp. 110–13) argues, is negative.[16]

If upheld, this argument has dramatic consequences for mathematical analysis. Here is why. First, a real number is defined as a set of rational numbers (a Dedekind cut). Next, each rational number can be represented by a natural number. Consequently, a real number corresponds to a set—and thus also an extensionally determinate property—of natural numbers. Weyl's argument thus implies that "*the concept 'real number' is not extensionally determinate*" (Weyl, 1919, p. 111; his emphasis).

[15] More precisely, Weyl describes the combination of statements expressing properties and relations by means of the logical operations and substitution. These statements then correspond to complex properties.

[16] His arguments runs as follows. Consider any extensionally determinate ('ED') collection of properties of numbers. This collection must be specified by some ED property, say $\varkappa$, of ED properties of numbers. But given $\varkappa$, we can define an ED property of numbers that "lies outside" of the collection specified by $\varkappa$, e.g. the property defined by $\exists F(\varkappa(F) \wedge Fx)$, where F ranges over ED properties of numbers. This property, Weyl contends, "most certainly differs in sense from every $\varkappa$-property" (Weyl, 1919, p. 110). (He admits, but dismisses as implausible, the possibility of finding an extensionally equivalent $\varkappa$-property.) It follows that $\varkappa$ failed to specify the totality of all ED properties of numbers. Since $\varkappa$ was arbitrary, we conclude that this totality admits of no ED specification.

This, in turn, means that it is illegitimate to define a real number by quantification over all such numbers, as one routinely does in standard analysis.[17] These difficulties are at the heart of Weyl's claim that contemporary analysis is plagued by vicious circles, and that "the house of analysis is [...] built on sand" (Weyl, 1918, p. 1). Weyl's remarkable accomplishment was to prove, in the second chapter of *Das Kontinuum*, that the sets of natural numbers obtained by the mathematical process suffice to develop large parts of 19th century analysis.

4 | HOW TO GENERALIZE OVER A MERELY POTENTIAL DOMAIN

Generalizations over a completable—and thus, according to Weyl, finite—domain are naturally understood in an instance-based manner. For example, a universal generalization $\forall x\, \varphi(x)$ is true because each and every object a in the domain is such that $\varphi(a)$. How, though, should generalizations over an infinite—and therefore merely potential—domain be understood? Here, according to Weyl, the instance-based conception is impermissible.

> But we have to beware of the idea that, when an infinite set is defined, we know not merely the property that is characteristic for its elements, but we also have these elements, so to speak, laid out in front of us, so that, in order to find out whether an object of this or that kind exists in the set, we only need to go through them one by one, like a police officer in his files. For an infinite set this is meaningless. (1921, p. 87)

Suppose Weyl is right. Then we urgently need an alternative conception of generality. We will now look at a series of proposals, ordered by their increasing conceptual distance from the more familiar instance-based conception.

4.1 | Limiting ourselves to extensionally determinate domains

A natural response to the problem is to limit ourselves to generalizations over domains that are extensionally determinate, which are at least properly "demarcated". This is the dominant response in Weyl (1918).[18] As we have seen, Weyl believes that quantification over an extensionally determinate domain behaves classically: provided that bivalence (or LEM) holds for each instance of the formula being generalized, bivalence (or LEM) holds for the resulting generalization as well. As Weyl puts it, this generalization has an "intrinsically clear sense".

In Section 2, we discussed Weyl's articulation of extensional determinateness in precise terms. We will now dig deeper and ask how this articulation might be justified. The obvious question

[17] E.g., as Weyl (1919, pp. 111–12) observes, we will be unable to prove that every non-empty, bounded set of real numbers has a least upper bound, which is a key property of the real numbers.

[18] We should note, though, that Weyl (1918) sometimes quantifies over domains that are not extensionally determinate; for example, his definition of continuity quantifies over the indeterminate domain of all real numbers (p. 81). (See Avron (2020) for a useful discussion of such quantification in Weyl (1918).) But Weyl engages in such quantification only with great care: while collections such as the real numbers are definite in the sense that an object either belongs or does not belong to them, there is no guarantee that bivalence will hold for the quantified statements. This means, in particular, that such quantification cannot be utilized in the mathematical process whereby new entities are defined.

is why, exactly, quantification over an extensionally determinate domain behaves classically. In some retrospective remarks, Weyl (1921) justifies his use in Weyl (1918) of classical logic for such quantification in the following way:

> If I run through the sequence of numbers and terminate if I find a number of property **E**, then this termination will either occur at some point, or it will not; that is, *it is so, or it is not so*, without any wavering and without a third possibility. (Weyl, 1921, p. 97)

As we have seen, Weyl takes all infinite domains, including ones that are extensionally determinate, to be merely potential. Properly analyzed, the statement about running through the sequence of numbers is thus a modal statement about what would happen were we to examine ever larger numbers in search of one with the property **E**. Weyl claims that either we would eventually find such a number (and the search could terminate) or we would not. A unique answer to our modal question is pre-determined.

To obtain a more thorough analysis, it is useful to explicate the claims about potential infinity. As we saw in Section 1, the natural way to do so is modally. But the language of ordinary mathematics is obviously non-modal. To apply the modal analysis, we must therefore connect the non-modal language of ordinary mathematics with the modal language in which potential infinity is explicated. A way to do so has recently been developed.[19] While this account goes far beyond anything we find in writings by Weyl, we believe it provides a useful rational reconstruction of the kind of reasoning that might have motivated Weyl. The central idea is that, when a domain is merely potential, the quantifiers $\forall$ and $\exists$ of the non-modal language of ordinary mathematics correspond to $\Box\forall$ and $\Diamond\exists$, respectively, of the modal language in which the potentiality is explicated. The connectives are translated homophonically. This translation is both intuitive and supported by the discovery that, given certain plausible assumptions, these modal operation-quantifier hybrids behave logically just like classical first-order quantifiers.

To explain this discovery, let us begin with the question of what is the appropriate modal logic. Although the modality we use is primitive or “rock bottom”, it is useful to invoke the heuristic of “possible worlds”. One possible world has access to other possible worlds that contain objects that have been constructed or generated from those in the first world. From the perspective of the earlier world, the “new” objects in the second exist only potentially. For example, the later world might contain more natural numbers than those of the first, say the successor of the largest natural number in the first world.

We also assume that objects are not destroyed in the process of construction or generation. So, to continue the heuristic, it follows from the foregoing that the domains of the possible worlds are non-decreasing along the accessibility relation. So we assume:

$$w_1 \leq w_2 \rightarrow D(w_1) \subseteq D(w_2) \tag{1}$$

where ‘$w_1 \leq w_2$’ says that w_2 is accessible from w_1, and for each world w, $D(w)$ is the domain of w. As is well-known, the conditional (1) entails that the converse Barcan formula is valid. Thus, we adopt:

$$\exists x \Diamond \phi(x) \rightarrow \Diamond \exists x \varphi(x). \tag{CBF}$$

[19] See Linnebo (2010), Linnebo and Shapiro (2019), Studd (2013), and Studd (2019).

For present purposes, we can think of a possible world as determined completely by the mathematical objects it contains. In other words, we assume the converse of (1). We will talk neutrally about the extra mathematical objects existing at a world w_2 but not at an "earlier" world w_1 which accesses w_2, as having been "constructed" or "generated". This motivates the following principle:

> **Partial ordering**: The accessibility relation $\leq$ is a partial order. That is, it is reflexive, transitive, and anti-symmetric.

So the underlying logic is at least S4. So far, then, we have S4 plus (CBF).[20]

We sometimes have a choice of what objects to generate. For many types of construction it makes sense to require that a license to generate certain objects is not revoked at accessible worlds. Numbers and sets provide examples. Since a number is generated from its immediate predecessor by applying the successor operation, the license to generate it cannot be revoked. Likewise, a set is generated from a well-defined membership criterion, which, once available, remains available. In these cases, which are the main focus of Weyl (1918), any two worlds w_1 and w_2 accessible from a common world have a common extension w_3. This is a directedness property known as *convergence* and formalized as follows:

$$\forall w_0 \forall w_1 \forall w_2 (w_0 \leq w_1 \wedge w_0 \leq w_2 \rightarrow \exists w_3 (w_1 \leq w_3 \wedge w_2 \leq w_3))$$

For constructions that have this property, then, we adopt the following principle:

> **Convergence**: The accessibility relation $\leq$ is convergent.

This principle ensures that when we have a choice of what mathematical objects to generate, the order in which we choose to proceed is irrelevant. Whichever object(s) we choose to generate first, the other(s) can always be generated later. It is well known that the convergence of $\leq$ ensures the soundness of the following principle:

$$\Diamond\Box p \rightarrow \Box\Diamond p. \tag{G}$$

The modal propositional logic that results from adding this principle to a complete axiomatization of S4 is known as S4.2.

Finally, we say that a formula φ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\varphi \rightarrow \Box\varphi \qquad\qquad \neg\varphi \rightarrow \Box\neg\varphi$$

Intuitively, a formula is stable just in case it never "changes its mind", in the sense that, if the formula is true (or false) of certain objects at some world, it remains true (or false) of these objects at all "later" worlds as well.

We are now ready to state the key results. Let $\vdash$ be the relation of classical deducibility in a non-modal first-order language $\mathcal{L}$. Let $\mathcal{L}^{\Diamond}$ be the corresponding modal language, and let $\vdash^{\Diamond}$ be deducibility, in this corresponding language, by $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of $\mathcal{L}$.

[20] Recall that S4 and (non-free) first-order logic entails (CBF).

Theorem 1 (**Classical potentialist mirroring**). *For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:*

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

(See Linnebo (2013) for a proof.)

The theorem has a simple moral. Suppose we are interested in logical relations between formulas in the range of the potentialist translation, in a classical (first-order) modal theory that includes S4.2 and the stability axioms. Then we may delete all the modal operators and proceed by the ordinary non-modal logic underlying $\vdash$.[21] In particular, under the stated assumptions, the modalized quantifiers $\Box\forall$ and $\Diamond\exists$ behave logically just as ordinary quantifiers, except that they generalize across all (accessible) possible worlds rather than a single world.

There is an analogous result for intuitionistic logic as well. Let $\vdash_{\text{int}}$ be the relation of intuitionistic deducibility in a first-order language $\mathcal{L}$. Let $\vdash_{\text{int}}^{\Diamond}$ be deducibility in the modal language corresponding to $\mathcal{L}$ by $\vdash_{\text{int}}$, S4.2 and stability axioms for all atomic predications of $\mathcal{L}$.[22] We reformulate the stability axioms as $\Diamond\varphi \to \Box\varphi$, which is classically, but not intuitionistically, equivalent to the earlier formulation.[23]

Theorem 2 (**Intuitionistic potentialist mirroring**). *For any formulas $\varphi_1, \ldots, \varphi_n$, and ψ of $\mathcal{L}$, we have:*

$$\varphi_1, \ldots, \varphi_n \vdash_{int} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash_{int}^{\Diamond} \psi^{\Diamond}.$$

Let us return to Weyl and his claim that quantification restricted to an extensionally determinate domain has the property of preserving classical behavior. We contend that this claim is supported by the classical mirroring theorem. As mentioned, this contention is not intended as an exegesis, only as a rational reconstruction of the reasoning that might have motivated Weyl. We do, however, claim that Weyl's extensionally determinate domains satisfy all of the assumptions of the theorem. First, when a merely potential domain is extensionally determinate, the relevant space of possibilities is properly "demarcated". This ensures that the modal operators are well-defined. Second, as seen above, Weyl (1918) takes a broadly realist view of this space of possibilities. This entitles him to use a *classical* modal logic. (Without this realist assumption, Weyl might still be able to invoke the intuitionistic version of the theorem.) Third, since an extensionally determinate domain is "an ideally closed aggregate which is intrinsically determined and demarcated" (Weyl, 1919, p. 109), it is reasonable to take such a domain to admit of convergent generation. Without convergence, we would have failed to demarcate a unique domain. Certainly, the kinds of generation that are the focus of Weyl (1918), namely, that of numbers and sets, are convergent. This justifies the axiom (G) and thus also the classical modal logic S4.2.Finally, for these kinds of construction, the stability axioms are justified as well. For example, when a number is (or is not) an element of some set, it will remain so (or remain not so) as the construction unfolds.

[21] There are interesting issues concerning comprehension axioms in higher-order frameworks. See Linnebo and Shapiro (2019), §7.

[22] The intuitionistic modal predicate system must be formulated with some care, since the two modal operators are not inter-definable. See Simpson (1994) for the details.

[23] This reformulation enables a slight improvement of the analogous theorem proved in Linnebo and Shapiro (2019).

The time has come to assess the approach of limiting ourselves to quantification over extensionally determinate domains. The approach has one major advantage, namely that it licences classical first-order logic for some very important domains, such as that of natural numbers or of predicative sets. We wish to highlight two limitations as well, which flow from the non-trivial assumptions of the theorem.

First, there are cases where the "space of possibilities" fails to be properly "demarcated". As we have seen, the domain of predicative sets of numbers provides an example. Second, the possibilities in question can fail to be convergent. A stark example arises in connection with Brouwer's "free choice sequences", which figure in Weyl (1921)'s preferred approach to the continuum. A free choice sequence is a potentially infinite sequence of natural numbers whose entries are chosen freely, one after the other, but where a choice can never be undone. Consider a choice sequence α whose first entry has not yet been chosen. That entry might be 0 or it might be 1. Once a choice is made, though, this will preclude the alternative choice. This yields a case of divergent possibilities, where a constructional possibility that once existed is removed by some future construction.[24] Thus, the restriction to extensionally determinate domains, which we assume to admit of convergent generation, constitutes a genuine limitation.

How serious are these limitations? The answer will obviously depend on which domains one accepts as extensionally determinate. While Weyl was very strict here, others have been more liberal.[25] Even if we follow Weyl, though, the disadvantage is shown to be less serious by the observation, due to Feferman and others, that substantial portions of modern analysis can be developed on the basis of very weak systems, in fact, systems no stronger than Peano arithmetic. Weyl's *Das Kontinuum* is a fundamental text in this respect, which has opened up new paths of research in mathematical logic.[26] As Feferman (1993) aptly puts it, "a little bit goes a long way".

4.2 | Schematic generality

Next we wish to consider a conception of generality that has its roots in the work of Bertrand Russell. Although he did not conceive of his domains as merely potential, Russell faced a challenge analogous to the one articulated at the beginning of Section 4. The problem is that in the ramified theory of types, every quantifier has an order index and ranges only over entities of that order. This severely limits the generality that can be achieved by a single quantifier. Quantification over monadic properties of objects provides an illustration. Although we can quantify over all such properties of any fixed order, there is, on Russell's theory, no such thing as quantification over absolutely all such properties, irrespective of order.

To soften the blow of this expressive limitation, Russell made an intriguing distinction between 'all' and 'any':

[24] Other examples arise in connection with a model-theoretic version of potentialism developed in Hamkins (2018).

[25] While Weyl contemplated the possibility of iterating the mathematical process "arbitrarily often" (Weyl, 1918, p. 29), he preferred not to do so for mathematical reasons. Subsequently, Kreisel, Feferman and Schütte proposed formal systems that may also be thought to encode a related notion of extensionally determinate domain (Kreisel 1958; Feferman 1964; Schütte 1965b, 1965a). The resulting progression of systems of ramified second order arithmetic indexed by ordinals extends well beyond the system hinted at by Weyl (1918). In particular, Feferman and Schütte (independently) proved that the proof-theoretic strength of this progression of systems is bounded by the well-known proof-theoretic ordinal Γ_0. See Feferman (2005); Crosilla (2017) for discussion. Far more liberal yet is the view associated with Bernays and set-theoretic potentialism, which takes even the power-set operation to preserve extensional definiteness (cf. fn. 13).

[26] See, for example, the reverse mathematics program (Simpson, 1999).

> We can speak of any property of x, but not of all properties, because new properties would be thereby generated. (Russell, 1908, p. 230)

The idea is that 'any' expresses the kind of generality achieved by a free variable, which can take any value whatsoever, irrespective of order, whereas 'all' (or, for that matter, 'every') expresses the generality achieved by a universal quantifier, which is necessarily restricted to a single order. Broadly similar ideas are defended by Weyl's *Doktorvater*, David Hilbert.[27] A quantifier requires a domain, which in any interesting case would have to be infinite. But infinite domains are unacceptable in finitary mathematics, which is epistemologically privileged. A free variable, by contrast, does not require any domain. Hilbert writes of the free-variable expression of the law of commutativity, "$a + b = b + a$", that it

> is in no wise an immediate communication of something signified but is rather a certain formal structure whose relation to the old finitary statements
>
> $$2 + 3 = 3 + 2$$
>
> $$5 + 7 = 7 + 5$$
>
> consists in the fact that, when a and b are replaced in the formula by the numerical symbols 2, 3, 5, 7, the individual finitary statements are thereby obtained, i.e., by a proof procedure, albeit a very simple one. (Hilbert, 1926, p. 196)

Thus, a formula with free variables does not express a statement but can nevertheless be endorsed (to speak deliberately loosely) when we have a "proof procedure" which, when applied to any instance of the schematic generalization, yields a proof of that instance.[28] What are the pros and cons of this schematic conception of generality? One major advantage is that it is available even where the domain fails to be extensionally determinate. All that is required is that the individual instances of the schematic generalization be well-defined, or, as Weyl might have put it, have an "intrinsically clear" sense. Another advantage is that the conception poses no threat to the use of classical logic. Provided that classical logic is licensed in the underlying language in which the instances are expressed, the introduction of this new expressive device yields no new reason to depart from classical logic.

The conception is afflicted with one serious disadvantage, however. It provides only a very limited ability to generalize. All we get is the effect of Π_1-generalizations. Thus, as Hilbert puts it, there are statements that "from our finitary perspective [are] *incapable of negation*" (1926, p. 194). Since $\varphi(x)$ has the effect of $\forall x \varphi(x)$, there is no good negation; for $\neg\varphi(x)$ has the effect of $\forall x\, \neg\varphi(x)$, not the desired $\neg\forall x \varphi(x)$. In sum, while schematic generality is a widely available form of generalization, including in contexts of classical logic, its utility is severely compromised by the expressive limitations, which prevents schematic generality from working as desired in logically complex formulas.

[27] Similar ideas can also be found in the context of a form of set-theoretic potentialism; see Parsons (1977) (who invokes Russell on p. 524), Glanzberg (2004) (who draws on Russell on pp. 559–61), and Parsons (2006) (who draws on Russell in Sect. 8.5 and mentions Hilbert in fn. 34).

[28] As we will see shortly, the idea that free-variable formulas do not express judgments is found already in Weyl (1921), which probably influenced Hilbert on this issue.

4.3 | An alternative inspired by Weyl (1921)

Weyl (1921) suggests a way forward. The central idea is nicely illustrated in a remarkable passage where Weyl discusses whether there is a natural number that has some decidable property *P*. He writes:

> Only the finding *that has actually occurred* of a determinate number with the property *P* can give a justification for the answer "Yes," and—since I cannot run a test through all numbers—only the insight, that it lies in the *essence* of number to have the property not-*P*, can give a justification for the answer "No"; even for God no other ground for decision is available. *But these two possibilities do not stand to one another as assertion and negation.*[29]

We wish to make three observations about this passage.[30] First, as Weyl notes in the final sentence, on his interpretation, the quantifiers are not dual to one another, unlike in classical logic. '∃' expresses the existence of a witness to a generalization. The dual notion would be the absence of counterexamples to a universal generalization. But on the interpretation Weyl proposes, '∀' expresses something far stronger than the absence of counterexamples, namely, that the generalization (and thus also the absence of counterexamples) is underwritten by relevant essences. There is a good reason for this change. In a potentialist setting, the absence of counterexamples is ephemeral. Even if there are no counterexamples at one stage, this might change as the generative process unfolds. By contrast, the conception of universal generality to which Weyl appeals is not instance-based. The truth of a generalization over, say, the natural numbers is underwritten by the essence of number, with no need for a recourse to any individual number. Since the essence of number is available at the very beginning of the relevant generative process, this truth-ground is straightforwardly available to potentialists.

Suppose we grant Weyl this interpretation of '∀'. Might we not interpret '∃' as the dual notion? On this interpretation, '∃' would state that the existence of a witness to the generalization is not precluded by any essences. But this too is an ephemeral matter. Even if the existence of a witness isn't precluded at one stage, it might be precluded later. For the generation of more objects, and of more facts about these, might give rise to new essences, which narrow down the space of possibilities that are left open.[31]

Second, while Hilbert offers assertibility conditions for his schematic generalizations, namely that we have a "proof procedure", Weyl is naturally read as proposing truth-conditions. The only justification one can give for a negative existential generalization over the natural numbers—or, for that matter, for a universal generalization—must appeal to the essence of number. This can hardly be understood as just an assertibility condition that entitles us to endorse a schematic generalization when we are in a sufficiently fortuitous epistemic position. After all, "even for God no other ground for decision is available". Rather, Weyl's point appears to be that the only

[29] (Weyl, 1921, p. 97; emphasis in original). We follow a slightly adapted translation from Parsons (2015).

[30] Weyl's appeal to essences here may well be inspired by Husserl. For the role of essences in Husserl's philosophy of mathematics see, e.g., (Hartimo, 2021, Ch. 4).

[31] Free choice sequences provide an example. The possibility of a sequence taking a certain value on a certain argument might be open at one stage of the construction but shut down at a later stage where an alternative value has been assigned. Once a value is assigned, the process becomes constrained to respect this assignment. The assignment thus becomes part of "the essence" of the sequence.

facts that might render a generalization over a merely potential domain true are ones about relevant essences.

This leads to our third observation. On the interpretation of the quantifiers suggested by Weyl, both types of generalization can be accounted for at a stage of a generative process solely on the basis of material that is available at that stage, or, as one might also put it, are made true by material available at the stage. This also ensures that both forms of generalization are stable as the generative process unfolds, not just ephemeral. An existential generalization is accounted for, or made true, by the availability of a witness at the relevant stage. And this witness will not go away as the generative process unfolds. A universal generalization is accounted for, or made true, by being underwritten by essences of entities available at the relevant stage. Just as in the case of the witnesses, these essences will not go away as the generative process unfolds. Both types of generalization, then, admit of local truthmakers, available at a single stage of the generative process, and therefore preserved as the process unfolds.

Attentive readers will have noticed the cautious wording of our three observations. We do not unreservedly ascribe all these ideas to Weyl. His (1921) reads more as an enthusiastic—and, we think, inspiring—progress report than a carefully worked-out account. As the scholarly literature has revealed, Weyl's discussion is problematic in several respects. For one thing, Weyl denies that quantified formulas express proper judgments, much as we saw in the case of Hilbert. Rather, such formulas are "judgment abstracts" (97). This calls into question the possibility of nested quantification, which is obviously required for mathematical analysis. For another, Weyl insists that "[t]he expression 'there is' commits us to Being and law, while 'every' releases us into Becoming and freedom" (96). This suggests an additional—and problematic[32]—asymmetry between the quantifiers, which surfaces in his treatment of choice sequences. While the witness to an existential generalization over such sequences must be a lawlike sequence, universal generalizations are concerned with free choice sequences. We choose to set aside such complications, because our present aim is not exegetical.

Of course, the task remains of developing the exciting material found in Weyl's article into a worked-out theory. One attempt to do so can be found in (Linnebo 2022), which develops a truthmaker semantics inspired by Weyl (1921). On this semantics, an existential generalization is made true by a witness, and a universal generalization can be made true by states recording appropriate essences, often working in concert with states with instance-based information. In light of the non-duality of the quantifiers, it should come as no surprise that this semantics validates intuitionistic logic rather than classical. As is well known, many constructive approaches use a verificationist semantics, which assimilates truth to proof. What is striking about the Weyl-inspired semantics, by contrast, is that there is no commitment to verificationism. The truthmaking need not be a matter of proof but can be based on an entirely realist conception of essences.[33]

Let us take stock of what has been achieved by the conception of generality inspired by Weyl (1921). Some people will no doubt regard it as a disadvantage that the conception validates intuitionistic logic, not classical. On the flip side, though, we find two major and indisputable advantages. First, since Weyl's interpretation of the quantifiers allows them to have local truthmakers, the "space of possibilities" need not be properly demarcated. Weyl's conception is therefore available also for domains that fail to be extensionally determinate, such as the domain of all generated sets of natural numbers. Second, the possibilities in question need not be convergent, as was required

[32] See van Dalen (1995); van Atten et al. (2002).

[33] See Fine (1994) for an influential such conception.

in Weyl (1918)'s strategy of limiting ourselves to generalizations over extensionally determinate domains. Since both types of generalization are made true by material available at the relevant stage, it is unproblematic if the stages diverge. This means that Weyl's 1921 account, unlike his 1918 account, can handle free choice sequences, in which he had become interested.

4.4 | Summary and assessment

A more comprehensive summary of our main claims throughout Section 4 is provided by the following table, which assesses the three conceptions of generality that we have discussed with respect to three different success criteria, namely, whether the conception (i) is available for any type of domain; (ii) permits generalizations of arbitrary logical complexity; and (iii) validates classical or intuitionistic logic.

conception of generality	**available for any type of domain?**	**complexity of generalizations**	**logic validated**
restricting to ED	no	full	classical
schematic	yes	Π_1 only	classical
Weyl (1921)-inspired	yes	full	intuitionistic

As can be seen, each conception requires some sacrifice. If we restrict ourselves to quantification over extensionally determinate domains, we make the major sacrifice of forsaking the possibility of quantifying over any type of domain. Next, on the schematic conception, we make a huge sacrifice of expressive power by limiting ourselves to Π_1-generalizations. On the conception inspired by Weyl (1921), by contrast, we make the comparatively minor sacrifice of limiting classical logic to extensionally determinate domains, while otherwise reasoning intuitionistically; but in return we secure quantification over any type of domain and complete expressive power. Small wonder, then, that Weyl (1921) was so excited about his new conception, writing, for instance, that he had finally found "the magic word" (p. 97).

5 | WEYL AND LIBERAL VS. STRICT POTENTIALISM

Our final aim is to provide some examples of how historical studies and the contemporary debate can inform one another. The illumination can proceed in either direction. We first show how Weyl's writings shed light on an important open question in the contemporary discussion of potentialism. Then we show how a distinction found in the contemporary debate—between "liberal" and "strict" potentialism—sheds light on the major shift in Weyl's view from 1918 to 1921.

Liberal potentialism is the view that mathematical objects are successively generated in an incompletable process of generation (Linnebo and Shapiro 2019). Each object is generated at some stage or other. But there is no stage at which every object has been generated. Does something analogous hold for truths? This would amount to truths too being successively "'made true". Liberal potentialists refrain from any such requirement on truths. A modal truth may well be true only in virtue of the entire space of possibilities without being "made true" at any single stage. *Strict potentialism*, on the other hand, states not only that mathematical objects are successively generated, but also that every truth is "made true" at some stage of the incompletable process of generation.

Of course, the loose talk about statements being "made true" at a stage needs to be made precise. In constructive mathematics, a statement is sometimes regarded as made true by producing a proof of it. This allows a universal generalization to be made true even though not all of its instances have been generated. But the associated verificationism is widely seen as problematic. Thankfully, it is possible to avoid verificationism while also being more precise. One option is to use realizability semantics, in the sense of Kleene (1945), where truths successively obtain "realizers" that encode a computational verification of the truth in question. Another option is to use the mentioned truthmaker semantics of Linnebo (2022), where each truth is verified by a state that is available at some stage or other.

With the distinction between liberal and strict potentialism in place, we turn to an important question that arises. What does it take for liberal potentialism to be acceptable, without destabilizing into either actualism or strict potentialism? The answer, or at least the beginning of an answer, falls out of our discussion of Weyl. *Liberal potentialism is permissible just in case the potential domain is extensionally determinate*. We defend this contention as follows. First, suppose a potential domain is extensionally determinate. Then the possibilities in question have been sharply demarcated. This ensures that the modal operators have been properly defined and makes it possible for a modal statement to be true solely in virtue of the entire space of possibilities. This is a powerful conclusion. If we take a realist attitude towards the possibilities in question, the modal logic can be classical. If some further assumptions hold as well (namely, convergence and stability of atomic predications), then the classical mirroring theorem applies, which justifies classical quantification over the domain.

Conversely, suppose that a potential domain isn't extensionally determinate. Then we have not fully demarcated the possibilities in question. So it does not make sense for a statement to be true solely in virtue of the entire space of possibilities. If we want to quantify over the potential domain, we must be strict potentialists and require that each true generalization be made true by material available at some stage or other. Provided that this locally available material suffices to make the generalization true, it does not matter whether the entire space of possibilities has been properly demarcated. A locally available truthmaker ensures robustness in the face of global indeterminacy.

It may be objected that our answer is uninformative because there remain disputes about which domains are extensionally determinate. The existence of such disputes has been noted.[34] We nonetheless believe it constitutes progress to connect the acceptability of liberal potentialism with Weyl's notion of extensional determinateness. This connection provides an abstract criterion for liberalism to be acceptable. Such a criterion is not rendered worthless just because there can be controversy, in particular cases, about whether or not the criterion is satisfied. In particular, the criterion serves as a salutary reminder to take seriously the non-trivial presuppositions of liberal potentialism, which the strict variety avoids.[35]

Having shown how our historical study can shed light on the contemporary debate, we turn to an example of the reverse direction of illumination. Weyl's thinking clearly undergoes a major shift between 1918 and 1921. How should this shift be understood? A key part of the answer, we claim, is that Weyl moved from liberal to strict potentialism. Weyl (1918)'s primary approach

[34] See footnotes 13 and 25 and and the text to which these are attached. Other thinkers who accept a lot of extensional determinateness are set-theoretic potentialists who presuppose a sufficiently determinate metaphysical or logical modality, e.g. Putnam (1967) and Hellman (1989).

[35] Arguably, Linnebo (2018b) fails to take these presuppositions sufficiently seriously; see Studd (2019, § 7.5).

to quantification over merely potential domains was to insist that the domain be extensionally determinate. (Later predicativists have followed him in this.) This allows him to be a liberal potentialist about these domains and to quantify over them using classical logic.[36] Weyl (1921) discovers another option, based on his new conception of the quantifiers and the accompanying strict potentialism.

While Weyl (1921) still deems his 1918 option acceptable, he unreservedly favors the new one:

> both attempts at providing a foundation for analysis portrayed here are equally possible, even though the one of Brouwer may, from the outset, have the advantage that the formation of concepts is not tied down and does more justice to the intuitive nature of the continuum. (Weyl, 1921, p. 98)

Let us connect Weyl's assessment in this passage with our own analysis. As we have seen, there is a trade-off between the 1918 and the 1921 options. Limiting ourselves to extensionally determinate domains is superior with respect to the strength of one's logic, which on this approach can be classical. However, the new 1921 conception of generality is superior in two other respects, namely its availability for domains that fail to be extensionally determinate (thus allowing "the formation of concepts [not to be] tied down") and its ability to handle the divergent possibilities associated with free choice sequences (thus doing "more justice to the intuitive nature of the continuum"). Inspired by Brouwer[37], Weyl therefore enthusiastically embraces the new option of using a non-instance-based conception of generality and intuitionistic logic:

> So I now abandon my own attempt and join Brouwer. I tried to find solid ground in the impending dissolution of the State of analysis [...] without forsaking the order upon which it is founded, by carrying out its fundamental principle purely and honestly. I believe I was successful–as far as this is possible. For *this order is in itself untenable*, as I have now convinced myself, and Brouwer—that is the revolution! (Weyl, 1921, pp. 88-89)

Weyl's revolution, then, is at heart a turn from liberal to strict potentialism. The global truthmaking available when limiting ourselves to quantification over extensionally determinate domains is replaced by a local form of truthmaking that is robust enough to handle divergent possibilities and global indeterminacy, but that requires intuitionistic logic.

[36] Some authors (Feferman, 2005, pp. 598 and 621) appear to conflate this liberal potentialism (say, concerning the domain of natural numbers) with actualism. While we admit that liberal potentialism is closer to actualism than its strict analogue, we insist that there remains an important difference between liberal potentialism and actualism. Compare Hamkins (2018, §7).

[37] Dirk van Dalen (1995, p. 147) argues that Brouwer's influence on Weyl was prompted by a number of conversations between the two mathematicians during the summer of 1919, on the occasion of Brouwer's vacation in Switzerland. These conversations are reported in a letter from Brouwer to Fraenkel and, according to van Dalen, explain the puzzling fact that Weyl was already familiar with some of the most recent aspects of Brouwer's mathematics, which had been carried out in isolation during the world war.

6 | CONCLUDING SUMMARY

Weyl introduced an unprecedented notion into the discussion of potentialism, namely a distinction between merely potential domains that are extensionally determinate and those that are not. We have argued that this distinction has great and enduring value. For one thing, Weyl's novel distinction marks a watershed concerning our understanding of quantification over a merely potential domain. There is a powerful argument for the availability of quantification using classical logic that goes through when the domain is extensionally determinate but that fails otherwise. Even in the latter case, though, it can be shown that quantification using intuitionistic logic is permissible.

For another, the notion of extensional determinateness holds the key to the important question of when liberal potentialism is acceptable, without destabilizing into either actualism or strict potentialism. The answer, we have argued, is that liberal potentialism is permissible just in case the potential domain is extensionally determinate; otherwise, we must be strict potentialists. As advertised, our analysis is summarized by the following diagram:

kind of domain	**ext. determinate**	**ext. indeterminate**
completable	actualism	—
incompletable	liberal potentialism (classical logic)	strict potentialism (intuitionistic logic)

The question of the scope of liberal potentialism is mathematically important. In the case of arithmetic, this corresponds to the fundamental question of what kind of view might justify a classical theory of arithmetic, such as first-order Peano Arithmetic or a weak second-order arithmetic such as ACA_0, but not much more, such as full impredicative second-order Peano Arithmetic. A strict potentialist about arithmetic would be entitled only to Heyting Arithmetic, while an actualist would be entitled to full second-order Peano arithmetic or beyond. Our answer is that liberal potentialism concerning the natural numbers justifies first-order, but not full second-order, arithmetic. And this form of potentialism, in turn, is acceptable just in case the domain of natural numbers is merely potential but extensionally determinate. In the case of set theory, the analogous question is what kind of view might justify a classical set theory such as first-order Zermelo-Fraenkel, but not much more, such as the impredicative Morse-Kelley theory of sets and classes. Again, our answer is that liberal potentialism concerning sets justifies first-order Zermelo-Fraenkel, and that this form of potentialism, in turn, is acceptable just in case the domain of sets is merely potential but extensionally determinate.

This analysis obviously leaves us with some hard questions about which domains are extensionally determinate, and in particular, whether the powerset operation should be seen as preserving extensional determinateness, which Weyl vehemently denied.[38] Although these questions obviously cannot be answered here, we believe the link we have forged between liberal potentialism and Weyl's notion of extensional determinateness constitutes progress.[39]

[38] One of the authors (LC) is inclined to agree with Weyl on this question, while the other (ØL) is inclined to side with Bernays (1935) as against Weyl.

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