Semi-analytical buckling strength analysis of plates with arbitrary stiffener orientations

by

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Abstract

Buckling of plates with arbitrarily oriented, snipped stiffeners are studied. The main objective is to present and validate an approximate, semi-analytical computational model for such plates subjected to in-plane loading. Estimation of the buckling strength is made using the von Mises’ yield criterion for the membrane stress as the strength limit. The formulations derived are implemented in a Fortran computer code, and numerical results are obtained for a variety of plate and stiffener geometries. The model may handle complex plate geometries, by using inclined stiffeners to enclose irregular plate shapes. The method allows for a very efficient analysis. Relatively high numerical accuracy is achieved with low computational efforts. The results are, in most cases, found to be conservative compared to fully nonlinear finite element analysis results.

Key words: Stiffened plates; Arbitrary stiffener orientations; Buckling strength; Elastic buckling load; Semi-analytical method; Rayleigh-Ritz method

1 Introduction

Stiffened plates are fundamental building blocks in many structures, for example in ships, steel bridges, aircrafts and offshore installations. Due to the great amount of individual stiffened plates one may have in some structures, a computationally efficient analysis tool is advantageous in a design situation.

Explicit design formulas [1, 2, 3] have traditionally been used as a standard in design codes and regulations to provide strength estimates of stiffened plates. Such formulas are relatively simple to use, but their applicability is normally limited to plates with regular stiffener orientations. When non-regular stiffener arrangements are required, standard design formulas are not applicable and other methods must be applied. The finite element method could be used, but at present, nonlinear
finite element analyses are mostly restricted to research. Assessing the collapse strength using such analyses is still time consuming and impractical for most design purposes.

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As an alternative to finite element analyses and explicit design formulas, computationally efficient semi-analytical methods are becoming more common. This has, for instance, been the case at the ship classification and engineering services company Det Norske Veritas (DNV), that for a long time has developed rules and guidelines for the ship and offshore industry. DNV, in cooperation with the University of Oslo (UiO), has developed a basis for more advanced nonlinear semi-analytical analysis models (Steen [4, 5, 6]) and studied a number of related plate buckling problems. Further work has been carried out jointly by DNV and the Norwegian University of Science and Technology (NTNU). Some of the most relevant work from that effort is published by Byklum et al. [7, 8, 9, 10]. That work, and the work described here, have recently culminated in a tailor made, computerised software code (PULS), described in more detail by Steen et al. [10, 11]. This computer code, for assessing buckling and ultimate strength limits of flat stiffened plates, has been accepted and implemented into rules and guidelines within DNV. Furthermore, it has been accepted by the International Association of Classification Societies (IACS) as part of the new tanker rules (Joint Tanker Project – JTP) to be issued in 2006.

The amount of published literature on semi-analytical methods for analysis of stiffened plates is growing. For example, Paik et al. [12, 13] and Hughes et al. [14] have presented methods for capacity predictions of plates with an inelastic material. Semi-analytical methods have also been used for linear elastic buckling analysis, for instance by Hughes et al. [15] that developed a method for analysing the torsional buckling behaviour of flanged stiffeners, and by Saadatpour et al. [16] and Bradford et al. [17] that considered skew plates.

All the semi-analytical methods mentioned above, are restricted to plates with regular stiffeners parallel to the boundaries or to unstiffened plates with rectangular shape or arbitrary quadrilateral shape. In the present work, the main objective has been to develop a computationally efficient semi-analytical model for buckling strength analysis of stiffened plates with arbitrarily oriented, sniped stiffeners. Analyses by the present model can be performed for plates with simply supported, clamped and partially clamped boundary conditions, or combinations of these. The model may also handle interior supports, along lines with arbitrary orientations and lengths. By using inclined stiffeners or strong translational springs to enclose triangular, trapezoidal and other plate shapes, the present model may handle complex plate geometries. Failure modes associated with the stiffener are not considered.

2 Plate and computational model

2.1 Plate definition and boundary conditions

Typical cases of stiffened plates are the plate enclosed by the strong longitudinal and transverse girders in Fig. 1a, and the web of the girder in Fig. 1b. Girder stiffeners, that may also be oriented horizontally, will typically be snipped at their ends, and will not, unlike continuous stiffeners such as in
Figure 1. Examples of stiffened plates: (a) Stiffened plate enclosed by longitudinal and transverse girders, and (b) girder stiffened by sniped stiffeners.

Figure 2. (a) Stiffened plate subjected to in-plane shear stress and in-plane, linear varying compression or tension stress, and (b) cross-section of an eccentric stiffener.

Fig. 1a, be subjected to external axial loading (in the stiffener direction). Sniped stiffeners may also be used in conjunction with cases where a rather non-regular stiffener arrangement is required, such as for instance in the stern and in the bow of a ship hull.

In order to model such cases, the plate defined in Fig. 2 is considered. It may be subjected to in-plane shear stress and linear varying in-plane compression or tension stress. It may have none, one or more stiffeners with sniped ends, and the stiffener orientations may be arbitrary. The stiffeners may have different cross-section profiles, and may be eccentric, as in Fig. 2b, or symmetric about the middle plane of the plate.

The stiffeners are modelled as simple beams. Possible sideways displacements and the torsional rigidity of the stiffeners are not accounted for. Further, the influence of axial stiffness of stiffeners on the internal membrane stress distribution in the plating is neglected. The assumption that the stiffeners can be modelled as beams is clearly a simplification, which is discussed in more detail later.

A usual assumption, which is also adopted in the present paper, is that the plate edges are forced to remain straight due to the neighbouring plates. Further, the plate is supported in the out-of-plane direction at all the outer boundaries. A boundary or a part of a boundary may be simply supported, clamped or something in between, as in Fig. 3. Rotational and transversal restraints can be added along specified lines in the interior of the plate.
2.2 Computational model – major steps

The computations are carried out in two major steps. In the first step, labelled the elastic buckling stress limit (ESL) state, the elastic buckling load (first eigenvalue) and corresponding buckling mode of the stiffened plate are calculated. In the second step, labelled the buckling strength limit (BSL) state, a buckling strength assessment is made, accounting for a specified out-of-plane imperfection and, in approximate manner, for yielding in the plate.

The main aspects of the model are described below. Some basic formulations are also included for the sake of completeness, and for the convenience of the reader. A more detailed presentation is given in Brubak [18].

3 Material law and kinematic relationships

The usual plane stress assumption for thin isotropic plates is adopted. The well known Hooke's law for this case is defined by

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu \epsilon_y)$$  
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu \epsilon_x)$$  
$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy}$$

where $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ are the in-plane stresses, and $\epsilon_x$, $\epsilon_y$ and $\gamma_{xy}$ the in-plane strains, defined positive in tension, and the material coefficients $E$ and $\nu$ are Young’s modulus and Poisson’s ratio, respectively.

The total strain can be divided into a membrane strain ($\epsilon^m$) and a bending strain ($\epsilon^b$) and given by

$$\epsilon_x = \epsilon^m_x + \epsilon^b_x = \epsilon^m_x - zw_{xx}$$  
$$\epsilon_y = \epsilon^m_y + \epsilon^b_y = \epsilon^m_y - zw_{yy}$$  
$$\gamma_{xy} = \gamma^m_{xy} + \gamma^b_{xy} = \gamma^m_{xy} - 2zw_{xy}$$

where $w$ is the out-of-plane displacement in the z-direction (positive downwards in Fig. 2). The conventional notation $w_{xy}$ for $\partial^2 w/\partial x \partial y$, etc., is adopted. The bending strain distribution complies with Kirchhoff’s assumption [19]. In large deflection theory (large rotations, but small in-plane strains), the membrane strains in a plate with an initial imperfection $w_0$, that is additional to $w$, can be written as

$$\epsilon^m_x = u_x + \frac{1}{2}w^2_{xx} + w_0xw_x$$  
$$\epsilon^m_y = v_y + \frac{1}{2}w^2_{yy} + w_0yw_y$$  
$$\gamma^m_{xy} = u_y + v_x + w_{xx}w_y + w_0xw_y + w_0yw_x$$

where $u$ and $v$ are the displacements of the middle plane of the plate in x- and y-direction, respectively. These, with imperfections included, were given by Marguerre [20], and represent an extension of von Karman's plate theory [19, 21]. In a linear elastic buckling analysis, the imperfections $w_0$ is set to zero.

4 Elastic buckling stress limit (ESL)

The elastic buckling load (first eigenvalue) of a perfect, stiffened plate is computed using the well known Rayleigh-Ritz method. The assumed displacement field, which satisfies the boundary conditions of a simply supported plate, is given by

$$w(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} \sin(\pi ix/L) \sin(\pi jy/b)$$

where $a_{ij}$ are amplitudes, $L$ the plate length and $b$ the plate width. The first step is to establish the potential energy of the plate, $\Pi = U + T$, where $U$ is the strain energy and $T$ is the potential energy of the external loads. Equilibrium requires that $\Pi$ has
a stationary value, i.e., $\delta \Pi = 0$. This requirement leads to displacements that must satisfy the usual eigenvalue problem

$$(K_{ijkl}^M + \Lambda^e K_{ijkl}^G)\alpha_{kl}^e = 0$$

where

$$K_{ijkl}^M = \frac{\partial^2 U}{\partial \alpha_{ij} \partial \alpha_{kl}} \quad \text{and} \quad \Lambda^e K_{ijkl}^G = \frac{\partial^2 T}{\partial \alpha_{ij} \partial \alpha_{kl}}$$

Here, $\Lambda^e$ denote the eigenvalues and $\alpha_{kl}^e$ the eigenvectors. Superscripts $M$ and $G$ are used to designate the material stiffness matrix $K^M$ and the geometrical stiffness matrix $K^G$ (for the reference loading), respectively. In the common matrix notation, this eigenvalue problem can be written

$$(K^M + \Lambda^e K^G)\alpha^e = 0$$

In an analysis of a clamped plate, it would be more appropriate to assume a displacement field defined with a series of cosine functions. However, although each component in a series of sine functions represents a simply supported condition, added together they are nearly able to describe a clamped, or partially restrained, condition. The sine curve assumption is therefore able to handle plates with various boundary conditions along the edges.

The elastic strain energy contribution from bending of the plate is given by

$$U_{\text{plate}}^b = \frac{D}{2} \int_0^b \int_0^L \left( w_{,xx} + w_{,yy} \right)^2 dx dy - 2(1 - \nu)(w_{,xx} w_{,yy} - w_{,xy}^2) dx dy$$

where $D = EI t^3/12(1 - \nu^2)$ is the plate bending stiffness and $t$ is the plate thickness. By substituting the assumed displacement field, an analytical solution of this integral may be derived. The result is given in Appendix A. The membrane strain energy of the plate and the stiffeners (below) is not included as it does not affect computed eigenvalues.

The curvature of the stiffeners is equal to the curvature in the plate along the stiffeners. Thus, the bending strain energy due to an arbitrarily oriented stiffener, with length $L_s$ and cross-section area $A_s$, can be given by

$$U_{\text{stiff}}^b = \frac{EL_s^2}{2Et} \int_{L_s} \left( L_x^2 w_{,xx} + 2L_x L_y w_{,xy} + L_y^2 w_{,yy} \right)^2 dL_s$$

where $(x_1, y_1)$ and $(x_2, y_2)$ are the coordinates of the stiffener ends, $L_x = (x_2 - x_1)$, $L_y = (y_2 - y_1)$ and

$$I_c = \int_{A_s} (z - z_c)^2 dA_s + t b_e z_e^2$$

is an effective moment of inertia about the axis of bending. Here, $z_c$ is the distance from the middle plane of the plate to the centroidal axis (through the centre of area) of a cross-section consisting of the stiffener and an effective plate width $b_e$. The effective moment of inertia $I_c$ reflects the fact that eccentric stiffeners tend to "lift" the axis of bending. The strain energy integral in Eq. 15 may be solved analytically or by numerical integration. For a symmetric stiffener, $z_c = 0$. This value also represents a reasonable simplification in many cases also for eccentric stiffeners. This aspect is discussed in more detail in Section 8.

The potential energy of external loads due to plate bending is given by

$$T = -\Lambda \int_0^L \int_{L_s} \frac{1}{2} \left( S_{x0}(y) w_{,x}^2 + S_{y0}(x) w_{,y}^2 - 2S_{xy0} w_{,x} w_{,y} \right) dy dx$$

where $S_{x0}(y)$, $S_{y0}(x)$ and $S_{xy0}$ are the initial (reference) stresses and $\Lambda$ the load factor. An analytical solution of this integral is given in Appendix A.

In line with the snipped stiffener assumption, Eq. 17 does not include any contribution from stiffeners. It would be reasonable straightforward to extend Eq. 17 to also include end loaded (continuous) stiffeners. However, in cases with local plate buckling, which is of most practical interest, the stiffeners will remain nearly straight and only contribute negligibly to $T$. Then it makes little difference whether the stiffeners are sniped or continuous.

Both the displacements and the rotations along an arbitrary oriented line with length $S$ may be
restrained by applying translational and rotational springs, respectively. The strain energy due to these springs is

\[ U_{\text{spring}} = \frac{1}{2} \int \left( k_r w_n^2 + k_t w^2 \right) dS \quad (18) \]

Here, \( w_n \) is the derivative of \( w \) normal to the line, and \( k_t \) and \( k_r \) are the stiffness of the translational springs and the rotational springs, respectively. An analytical integration example for the case of rotational springs along the plate edge \( x = L \) is given in Appendix A.

5 Buckling strength limit (BSL)

5.1 Load incrementation and strength criterion

The ultimate strength limit of a plate obtained using full nonlinear analysis is here labelled USL. It is defined as the limit point (maximum point) of the load-displacement curve, i.e. when the curve starts to drop due to an instability. Unlike that strength, the present model does not account for the post-critical (or reserve) strength, beyond the elastic buckling stress limit, typical for thin (slender) plates. For the sake of distinction, the buckling strength limit predicted by the present model is labelled BSL. For thicker plates, the BSL and the USL results will converge and approach the “squash load”.

Due to redistribution of stresses caused by the formation of plastic regions in a plate, it will be able to carry loads beyond those causing yielding at the outer fibres. In order to allow in an approximate manner for some additional strength after yielding in outer fibres, and yet adopt a computationally simple criterion, criteria applied to stresses at various interior points, including the middle plane (membrane stress), have been considered.

First yield of the von Mises’ membrane stress

\[ \sigma_m^m = \sqrt{(\sigma_{x}^m)^2 + (\sigma_{y}^m)^2 - \sigma_{x}^m \sigma_{y}^m + 3(\tau_{xy}^m)^2} \quad (20) \]

is suggested as the buckling strength criterion \((\sigma_e^m = f_Y)\) in normal applications of the proposed model. This criterion is used in present computations. Here, \( \sigma_{x}^m \), \( \sigma_{y}^m \) and \( \tau_{xy}^m \) are the membrane stresses in the plate, following the redistribution due to the out-of-plane displacements, and \( f_Y \) is the yield strength. The critical points at which the von Mises’ membrane stress reaches first yield are typically located in the plate along the edges and along the stiffeners.

In normal applications, stiffeners are designed such as to prevent global buckling. In unusual situations, when this is not the case, the criterion above may still be acceptable. However, it has not been investigated in any detail and a more conservative criterion such as stress limitation at the outer fibre of the stiffener may be implemented for such cases.

5.2 Imperfection amplitude

Use of the displacement magnifier (Eq. 19) underestimates buckling strength predictions, and increasingly so for increasing slenderness. To partly compensate for this, a fictitious slenderness dependent maximum imperfection amplitude \( w_{0,\text{max}} \) that decreases with increasing slenderness, is proposed and adopted here. In terms of a chosen reduced slenderness \( \lambda \) defined as

\[ \lambda = \frac{\Lambda_Y}{\Lambda_{cr}} \quad (21) \]
it is necessary to satisfy in-plane strain compatibility. By differentiation and combination of Eqs. 7-9, the compatibility equation
\[
\epsilon_{x,y}^m + \epsilon_{y,x}^m - \gamma_{xy,xy}^m = w_{xy}^2 - w_{xx}w_{yy} + 2w_{0,xy}w_{xy} - w_{0,xx}w_{yy} - w_{0,yy}w_{xx}
\] (24)
can be obtained. Further, by substituting strains from Hooke’s law and Airy’s stress function into this equation, the following nonlinear plate compatibility equation (Marguerre [20]) results:
\[
\nabla^4 F = E(w_{xy}^2 - w_{xx}w_{yy} + 2w_{0,xy}w_{xy} - w_{0,xx}w_{yy} - w_{0,yy}w_{xx})
\] (25)
The shape of the imperfection displacement field \(w_0\) is taken equal to the first eigenmode and is defined by
\[
w_0(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{N} b_{ij} \sin\left(\frac{\pi ix}{L}\right) \sin\left(\frac{\pi jy}{b}\right)
\] (26)
where \(b_{ij}\) are the amplitudes.
A solution of Eq. 25, proposed by Levy [22] and given by
\[
F(x, y) = -\Lambda \left(\frac{1}{2}S_{0x}y^2 + \frac{1}{2}S_{y0}x^2 + S_{xy0}xy\right)
+ \sum_{i=0}^{2M} \sum_{j=0}^{2N} f_{ij} \cos\left(\frac{i\pi x}{L}\right) \cos\left(\frac{j\pi y}{b}\right)
\] (27)
is assumed. Here, \(\Lambda\) is the load factor at the current load step and the coefficients \(f_{ij}\) are functions of the amplitudes of \(w\) and \(w_0\). The coefficients \(f_{ij}\) are found by substituting \(F(x, y)\), \(w\) and \(w_0\) into the nonlinear plate compatibility equation, Eq. 25, and are given in Appendix A.

6 Validation

The present model was incorporated into a Fortran computer code and computed results have been compared with finite element analyses using ANSYS [23] for a variety of plate and stiffener dimensions. ESL results are verified by comparisons with ESL results by ANSYS, and the buckling strengths (BSL) are compared with ultimate strengths (USL) obtained from fully nonlinear finite element analyses. Results, presented
in subsequent sections, are limited to simply supported plates. Additional comparisons of both ESL and BSL results with ANSYS results are given by Brubak [18], where also clamped plates are considered.

The finite element model, based on Shell93 elements, is supported in the out-of-plane direction along the edges of the plate, and the edges are forced to remain straight during deformation. The plate is also supported in the in-plane directions, just enough to prevent rigid body motions. Further, the ends of the stiffeners are completely free and not loaded.

For validation purposes, the specified imperfection is taken as \(w_{0,\text{spec.}} = 5\) mm in all cases. In the BSL calculations, the maximum imperfection amplitude \(w_{0,\text{max}}\) is taken according to Eq. 22. In the fully nonlinear element analyses, the imperfection shape is taken equal to the first eigenmode of the plate with a maximum amplitude \(w_{0,\text{max}} = w_{0,\text{spec.}} = 5\) mm.

The adopted elastic material properties in each computation are Young's modulus \(E = 208000\) MPa, Poisson’s ratio \(\nu = 0.3\) and the yield strength \(f_Y = 235\) MPa. The fully nonlinear ANSYS analyses are performed with a bilinear stress-strain relationship having the same material properties \(E, \nu\) and \(f_Y\) as above, and additionally a hardening modulus \(E_T = 1000\) MPa.

In the present model, 225 degrees of freedom (15x15) are used in all the cases. Convergence studies have shown that this choice of degrees of freedom may overestimate the strength predictions by about 1-2 %. In comparisons with BSL analyses, the number of degrees of freedom used in ANSYS is typically about 20000, which is believed to be a sufficiently large number to ensure satisfactory results.

Typically, the present method is found to be from about 1000 to about 3000 times faster than a nonlinear FEM analysis (ANSYS) of the same problem on the same computer. This time factor difference, that is of several orders of magnitude, clearly demonstrates that the present method is comparatively very efficient computationally. These numbers are just given as an indication of the relative computational efficiency, and do not reflect the results of an in-depth study of the different factors affecting computer time consumption. The computer time on a medium fast computer (1.5 GHz Intel Pentium M processor and 512 MB memory), is about 1-2 seconds for a typical BSL result for a given loading.
7 Load-deflection and buckling strength-slenderness response

Fig. 5a shows a typical load-deflection response of an unstiffened, simply supported plate, subjected to a uniaxial stress $S_x$. The figure includes results calculated by the present BSL model (solid curve), by ANSYS using large deflection theory for an elastic material (empty, dotted curve) and by ANSYS for an elasto-plastic material (filled, dotted curve). The final edge stresses, redistributed due to the out-of-plane deflections, are illustrated schematically in Fig. 5b.

The BSL is reached for an external stress $S_x$ of 199 MPa. In comparison, the USL obtained using ANSYS is 203 MPa. Both the USL and BSL prediction are marked by an empty square in the figure. For the present model, the maximum additional displacement $w_{\text{max}}$ in the plate is 5.84 mm at the onset of membrane yielding. For displacements smaller than this value, the correspondence between the three response curves in Fig. 5a is relatively good. Generally, the accuracy of the approximate displacement magnifier in Eq. 19, decreases as the displacements increase.

The elastic buckling stress, also shown in the figure (ESL), is 364 MPa ($1.55 f_Y$). This results in a reduced slenderness $\lambda = 0.80$ and a maximum imperfection amplitude $w_{0,\text{max}} = 0.97w_{0,\text{spec}}$. Thus, the reduction of the imperfection amplitude in the BSL model is small at this slenderness value.

The effect of the reduction at a range of slenderness values can be seen in Fig 6 for an unstiffened plate with uniaxial, uniform loading on the shortest edges. The thick, solid curve represents BSL analyses with a reduced maximum imperfection amplitude according to Eq. 22 and the dashed curve is for BSL analyses with a constant maximum imperfection amplitude $w_{0,\text{max}} = w_{0,\text{spec}}$. The difference between these two curves is not significant in absolute terms, but significant for high slenderness values in relative terms (about 18% at about $\lambda = 1.8$). The thin, solid curve represents ESL results obtained by the present model. This curve will always be an upper bound on BSL predictions. By comparing with the USL results in the figure, obtained by fully nonlinear ANSYS analyses, it is seen that there is a considerable reserve strength beyond the predicted buckling strengths (BSL) at higher slenderness values.

8 Stiffener modelling

Local buckling of stiffeners is not a common type of failure in a practical situation and is less likely for sniped stiffeners than for continuous stiffeners. Indeed, practical constructional stiffener specifications in typical design rules are given to prevent local buckling of stiffeners. The neglect of local buckling of stiffeners in the present model consequently seems like a reasonable assumption.

In practical design, the stiffeners are also normally proportioned strong enough to prevent a global buckling mode. A stiffened plate with a global buckling load close to the local buckling load may exhibit a very unstable response. An important issue in design, then, is to choose a stiffener that is sufficient to prevent a global buckling mode. Closer examination shows that there exists a critical threshold value of the stiffener stiffness, which separates the global and the local buckling mode. A further increase in the stiffener stiffness above this value will not have a significant influence on the calculated ESL results. This aspect is illustrated in Fig. 7, showing typical results for a quadratic plate with one regular, eccentric stiffener. The stiffener is a flat bar of constant thickness $t_w$ and varying height $h_w$. The plate is sub-
Figure 7. Elastic buckling stress (ESL) in the global and local buckling range of a uniaxially loaded plate ($L = b = 2000$ mm, $t = 20$ mm) with an eccentric stiffener ($t_w = 12$ mm, height $h_w$).

The figure includes ANSYS results and ESL results for the present model for $I_e$ of the stiffener calculated about the middle plane of the plate, i.e. with $z_c = 0$ (solid curve), and with an effective plate width $b_e = 30t$ (dashed curve). In the local buckling range, the results for these two cases are identical, as expected in line with the discussion of a threshold value above. In the global buckling range, results for $I_e$ with $z_c = 0$ are somewhat non-conservative, while results for $I_e$ with $b_e = 30t$ compare well with the ANSYS results. However, the difference between the two curves is rather small, and use of $z_c = 0$ might be an acceptable simplification for the considered plate.

In other global buckling cases, use of $z_c = 0$ may give somewhat more non-conservative results. This may be so for plates stiffened with many flanged stiffeners. These will tend to “lift” the axis of bending more markedly above the middle plane of the plate than in the single stiffener case considered above. For such plates, it is appropriate to use a more conservative moment of inertia $I_e$. Specific $b_e$-values to be used for such cases have not been investigated in any detail, but $b_e$ should possible not be taken greater than about $20t$ in practical design work.

The effect of the torsional stiffness of the stiffener is accounted for in the finite element model, but not in the present model. This explains most of the marginal difference between these two curves.

As seen, there is not much to be gained by increasing the stiffener stiffness significantly beyond the threshold value at about $h_w/t_w = 12$.

9 ESL and BSL predictions

Results are presented for the four simply supported plates defined in Fig. 8. Each plate is provided with two inclined, eccentric stiffeners of the
The shape of the first buckling modes calculated by the present model and by ANSYS are quite similar in each case. Fig. 9 shows the first buckling mode of plate 2 subjected to a uniaxial constant stress $S_x$. The agreement between that mode calculated by the present model (a) and that by ANSYS (b) is seen to be good. The main difference is that the finite element model accounts for the sideways deflections of the stiffeners.

The buckling modes shown in the figure can be considered local buckling modes, as the out-of-plane displacements of the plate along the stiffeners are small. The same is found to be the case for the rest of the plates defined in Fig. 8. In such cases with local plate buckling, ESL and BSL predictions using the present model will not be significantly affected if the stiffeners’ $I_e$-value used here ($z_c = 0$ in Eq. 16) are replaced by the more conservative $I_e$ obtained for $z_c$ calculated with $b_e = 30 t$. Similarly, in such cases results would not be much affected if the sniped stiffeners used here had been replaced by continuous stiffeners (with end loads).

BSL results are obtained for each plate for various combinations of in-plane compression or tension. Results are shown by biaxial load interaction curves in Fig. 10. In the context of buckling behaviour, the most relevant loading situations are those with uniaxial or biaxial compressive stresses. However, for completeness, BSL results are also computed for plates subjected to tension stresses along one or all the edges.

Also included in the figure are the elastic buckling (eigenvalue) results (ESL) computed both by the present model and by ANSYS. In each case considered, the results are in close agreement in the region of interest. The squash load defined by the von Mises’ yield criterion is also shown for the sake of illustration. The slenderness ($\lambda$) of the plates in the figure increases with increasing plate number. The BSL results are confined between the ESL results calculated by the present model and the squash load.

In the figure, the BSL-curves can be compared with the USL-curves computed using ANSYS. These results are seen to be close to each other for plate 1, 2 and 3, which have relatively small to intermediate reduced slenderness values $\lambda$ (Eq. 21). For the more slender plate 4, the BSL results are very conservative relative to USL results. As mentioned before, this is because the method does not account for the post-critical (reserve) strength. Therefore, the present method is most feasible in practical design cases in which it is not accepted that structural elements buckle elastically.

The reduced slenderness $\lambda$ varies along the interaction curves. For plate 1, 2 and 3, it is smaller than 1.2 for all load combinations. The corresponding imperfection amplitude $w_{0,\text{max}}$ relative to the amplitude $w_{0,\text{spec.}}$ varies between 0.82 and 1. Thus, the reduction of the imperfection amplitude in the BSL procedure is small in these cases. For plate 4, the maximum slenderness is approximately $\lambda = 1.92$ (at $S_x = 1.43 S_y$), which gives an imperfection amplitude equal $w_{0,\text{max}} = 0.22 w_{0,\text{spec.}}$. According to Fig. 4, or $w_{0,\text{max}} = 1.1 \text{ mm with } w_{0,\text{spec.}} = 5 \text{ mm.}$ Physically, this (fictitious) imperfection amplitude is physically speaking, unreasonably small, but the calculated BSL results are still conservative compared to the fully nonlinear finite element analysis results. In this case, the USL predicted by ANSYS is almost twice as large as the BSL calculated by present model.
Figure 10. Interaction curves in the stress space $S_x$-$S_y$ for the plates no. 1, 2, 3 and 4, with two eccentric, inclined stiffeners.

10 A practical application

A stiffened girder like that in Fig. 1b is examined. The dimensions of the girder web are 5000x1500x15 mm, and the stiffener arrangement of the plating is shown in Fig. 11a, where $s = 1000$ mm. It is stiffened with five snipped, eccentric stiffeners with flat bar profiles of height $h_w = 150$ mm and thickness $t_w = 15$ mm. The moment of inertia $I_e$ is calculated about the middle plane of the plate ($z_c = 0$ in Eq. 16). The out-of-plane displacements of the girder web are restrained by four continuous flat bar stiffeners of height $h = 300$ mm.

To examine the effect of the restraints of the continuous stiffeners, analyses are performed for two cases, one with and one without these restraints. Fig. 11b presents ESL and BSL results for the two cases, for various combinations of the applied external uniaxial stress $S_y$ (along the girder length) and shear stress $S_{xy} (S_x = 0)$.

The thick, solid curve in the figure represents BSL results with the out-of-plane, continuous stiffener restraints included, and the thin, solid curve without these restraints. The corresponding ESL
11 Concluding remarks

An efficient computational model for buckling analysis of plates with arbitrary stiffener orientations is presented. Typically, it is more than 1000 times faster than nonlinear finite element analyses. Therefore, the model is ideally suited in design optimisation studies and also in reliability studies that normally require large number of case studies. A computer program based on this method is of a size that can easily be incorporated into a computerised design code (which has already been done [10, 11]). A minimal number of input parameters is required and the present model is therefore considerably more user friendly than commercial finite element programs, which requires experienced users to obtain reliable results.

The model predicts a buckling strength limit (BSL) by making use of a simplified displacement magnification method and using first yield of the von Mises’ membrane stress as a collapse criterion. The model captures both local and global plate buckling modes. The applicability of the model and
the presence of a reserve strength for slender plates are documented for a wide variety of parameters by comparing with more accurate ultimate strengths (USL) obtained from fully nonlinear finite element analysis.

The adopted displacement magnifier that involves the elastic buckling load and the first eigenmode, results in a buckling strength that never exceeds the elastic buckling load. In a design situation, when the structural elements are not accepted to buckle elastically, this may be a sound and conservative theoretical treatment. By using large deflection plate theory [8], the postbuckling reserve strength beyond the elastic buckling stress of slender plates can be investigated. To extend the present method to account for this reserve strength, further work is required.

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A Appendix

A.1 Coefficients in Airy’s stress function

The coefficients in Airy’s stress function

\[ f_{ij} = \frac{E}{4(i^2 \frac{b}{L} + j^2 \frac{L}{b})^2} \sum_{r=1}^{M} \sum_{s=1}^{N} \sum_{l=1}^{M} \sum_{p=1}^{N} c_{rsq} \left( a_{rs} a_{pq} + a_{rs} b_{pq} + a_{pq} b_{rs} \right) \]

(A.1)

where \( a_{rs} \) and \( b_{pq} \) are the amplitudes of \( w \) and \( w_0 \), respectively, \( f_{00} \) is zero, and \( c_{rsq} \) are integers given by

\[ c_{rsq} = r s p q + r^2 q^2 \]

(A.2)

if \( \pm(r - p) = i \) and \( s + q = j \), or \( r + p = i \) and \( \pm(s - q) = j \), or

\[ c_{rsq} = r s p q - r^2 q^2 \]

(A.3)

if \( r + p = i \) and \( s + q = j \), or \( \pm(r - p) = i \) and \( \pm(s - q) = j \), or

\[ c_{rsq} = 0 \]

(A.4)

for other cases.

More details of the derivation of the coefficients \( f_{ij} \) can be found in the literature [7].

A.2 Bending energy of the plate

By substitution of the assumed displacement field, Eq. 14 can be integrated analytically and the result written as

\[ U_{\text{plate}}^b = \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} a_{ij} \frac{D \pi^4 L b}{8} \left( \frac{i}{L} \right)^2 + \left( \frac{j}{b} \right)^2 \]

(A.5)

A.3 External energy

A.3.1 External energy of the stress in x-direction

By substituting the assumed displacement field, \( S_{yy}(x) = 0 \) and \( S_{xy} = 0 \), Eq. 17 can be integrated analytically and the result written as

\[ T_{S_x} = - \sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} a_{ij} \frac{S_{x0}^1 b (i \pi)^2}{8L} \]

\[ - \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{l=1}^{M} \sum_{p=1}^{N} a_{ij} a_{jl} (S_{x0}^1 - S_{x0}^1) A_{ijl} \]

(A.6)

where

\[ A_{ijl} = \begin{cases} \frac{b (i \pi)^2}{16L} & \text{if } i = k \text{ and } j = l \\ -\frac{b L}{(j^2 - i^2)^2} & \text{if } i = k \text{ and } j \pm l \text{ are odd} \\ 0 & \text{if } i = k \text{ and } j \pm l \text{ are even} \end{cases} \]

(A.7)
A.3.2 External energy of the stress in y-direction

By substituting the assumed displacement field, \( S_{x0}(y) = 0 \) and \( S_{y0} = 0 \), Eq. 17 can be integrated analytically and the result written as

\[
T^{S_y} = -\sum_{i=1}^{M} \sum_{j=1}^{N} a_{ij} a_{ij} \frac{S_{y0}^1 t L (j \pi)^2}{8 b} - \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{M} a_{ij} a_{kj} (S_{y0}^2 - S_{y0}^1) B_{ijk}
\]  

(A.8)

where

\[
B_{ijk} = \begin{cases} 
\frac{Lt (j \pi)^2}{16b} & \text{if } j = l \text{ and } i = k \\
-\frac{Lt j^2 i k}{b (i^2 - k^2)^2} & \text{if } j = l \text{ and } i \pm k \text{ are odd} \\
0 & \text{if } j = l \text{ and } i \pm k \text{ are even}
\end{cases}
\]

(A.9)

A.3.3 External energy of the shear stress

By substituting the assumed displacement field, \( S_{x0}(y) = 0 \) and \( S_{y0}(x) = 0 \), Eq. 17 can be integrated analytically and the result written as

\[
T^{S_{xy}} = \Lambda \sum_{i=1}^{M} \sum_{j=1}^{N} \sum_{k=1}^{M} \sum_{l=1}^{N} a_{ij} a_{kl} S_{x0} l \pi^2 t C_{ijkl}
\]  

(A.10)

where

\[
C_{ijkl} = 0 \quad \text{if } i = k \text{ or } j = l
\]

(A.11)

or,

\[
C_{ijkl} = \frac{1}{4 \pi^2} \left( \frac{\cos((k + i) \pi) - 1}{k + i} + \frac{\cos((k - i) \pi) - 1}{k - i} \right) \left( \frac{\cos((j + l) \pi) - 1}{j + l} + \frac{\cos((j - l) \pi) - 1}{j - l} \right)
\]

(A.12)

for other cases.

A.4 Rotational springs

The strain energy due to rotational springs with stiffness \( k_r \), along the edge \( x = L \) is

\[
U_{\text{spring}} = \frac{1}{2} \int_0^b k_r w_x^2 \bigg|_{x=L} dy
\]  

(A.13)