The Semantics and Complexity of Successor-free Nondeterministic Gödel's T

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Chapter 1

Introduction

1.1 Introduction

L. Kristiansen, P. Voda, N. Jones and M. Barra have in several papers studied the computational power of fragments of various successor-free computational models, such as Gödel's T in [1, 2], an imperative language in [1], PCF in [4] and function algebras in [3, 5]. In these papers they have demonstrated surprising computational power such models, and successfully captured well known complexity classes defined by explicit time and space bounds on Turing machines. In particular, L. Kristiansen and P. Voda showed in [1] that certain neat successor-free fragments of Gödel's T, perfectly match the well known alternating space-time hierarchy [1]

$$\text{SPACE } 2_0^{\text{LIN}} \subseteq \text{TIME } 2_1^{\text{LIN}} \subseteq \text{SPACE } 2_2^{\text{LIN}} \subseteq \text{TIME } 2_3^{\text{LIN}} \subseteq \text{SPACE } 2_4^{\text{LIN}} \subseteq \ldots$$

The three classes at the bottom of the hierarchy are called respectively LINSPACE, EXP and EXPSPACE in the literature. A natural direction of continued research is to study the nondeterminstic counterpart to this model, and that is the subject of this thesis. We develop a model for a successor-free nondeterministic flavour of Gödel's T, which is built with particular consideration made to keep it as convenient to compute as possible, while still being adequate[2]. We desire this convenience since we will use deterministic programs to compute the interpretation of nondeterministic programs, and from this establish a relationship between deterministic and nondeterminstic complexity classes. This was the motivating goal for this thesis.

[1]Let LIN denote $c|x|$ for some c, and let $2_0^x = x$ and $2_{i+1}^x = 2^{2_i^x}$. SPACE(f) and TIME(f) denote the set of problems decidable by a Turing machine working in $O(f)$ space and time respectively.

[2]A model is adequate when the interpretation of any closed term M of base type contains exactly those elements corresponding to canonical terms M reduces to.

1.2 Overview

In chapter 2 we define a deterministic and a nondeterministic successor-free flavour of Gödel's T, called T^- and $T^{\backsim}$ respectively. We establish that $T^{\backsim}$ is strongly normalizing, and define some concepts required later.

In chapter 3 we provide a denotational semantic for $T^{\backsim}$ and demonstrate its adequacy. This proof also gives a reduction strategy which preserves the interpretation of a running program[3] in a desirable way. The model itself is based on interpreting terms in domains with total functionals over power sets, and the domains are also equipped with an order relation and a binary operator. This operator is the key to understanding the calculus, since it precisely models the nondeterminstic term. We also embed the model of $T^{\backsim}$ into the natural numbers by way of a computable isomorphism, and this embedded model is later computed in chapter 5.

In chapter 4 we quickly present L. Kristiansens model for T^- embedded in natural numbers from [1]. We dramatically extend the computing machinery he develops, by demonstrating terms for input-bounded repetition and higher-level arithmetic. These are used in chapter 5 to compute the interpretation of terms.

In chapter 5 we finally compute the functions defining the model in chapter 2. We decide certain type requirements for doing this by relating the sizes of the deterministic and nondeterministic domains.

In chapter 6 we define complexity classes for T^- and $T^{\backsim}$, and use the main result from chapter 5 to relate them by computing the interpretation of nondeterministic programs with deterministic programs.

[3]We later define a progam to be any closed term of type $\iota \to \iota$, so a running program is understood to be such a term when given a closed term of type ι as input.

Chapter 2

Successor free Gödel's T

2.1 The Calculus

Definition 2.1. *We define the type space as the least set $\mathcal{T}$ satisfying (i) $\iota \in \mathcal{T}$ (ii) $\sigma \times \tau \in \mathcal{T}$ when $\sigma, \tau \in \mathcal{T}$ (iii) $\sigma \to \tau \in \mathcal{T}$ when $\sigma, \tau \in \mathcal{T}$. For convenience let the shorthands $\sigma_1 \times ... \times \sigma_n$ and $\sigma_1, ..., \sigma_{n-1} \to \sigma_n$ denote types $\sigma_1 \times (...(\sigma_{n-1} \times \sigma_n))$ and $\sigma_1 \to (\sigma_2 \to ...(\sigma_{n-1} \to \sigma_n))$ respectively.*

For each type $\sigma \in \mathcal{T}$ we have an infinite countable supply of symbols $\mathcal{V}^\sigma = \{x_0^\sigma, x_1^\sigma, x_2^\sigma, ...\}$ called the variables of type σ, and the set of all variables of all types is $\mathcal{V} = \bigcup_{\sigma \in \mathcal{T}} \mathcal{V}^\sigma$. Likewise we have $\mathcal{H}^\sigma = \{[]_0^\sigma, []_1^\sigma, []_2^\sigma, ...\}$ called the holes of type σ, and the set of all holes of all types is $\mathcal{H} = \bigcup_{\sigma \in \mathcal{T}} \mathcal{H}^\sigma$. We also have an infinite countable supply of symbols $\mathcal{K} = \{k_0, k_1, k_2, ...\}$ called numerals.

We inductively define C^- as the set of all deterministic successor-free contexts. All contexts $\mathcal{C} \in \mathrm{C}^-$ are said to be of some type $\sigma \in \mathcal{T}$, and we may clarify the type of $\mathcal{C}$ by writing $\mathcal{C}^\sigma$ or $\mathcal{C} : \sigma$.

(NUM). k_i is a context of type ι for all $k_i \in \mathcal{K}$

(HOLE). $[]_i^\sigma$ is a context of type σ for all $[]_i^\sigma \in \mathcal{H}$

(VAR). x_i^σ is a context of type σ for all $x_i^\sigma \in \mathcal{V}$

(APP). $(\mathcal{C}_1^{\sigma \to \tau} \mathcal{C}_2^\sigma)$ is a context of type τ for all contexts $\mathcal{C}_1^{\sigma \to \tau}$,$\mathcal{C}_2^\sigma$

(ABS). $\lambda x_i^\sigma.\mathcal{C}^\tau$ is a context of type $\sigma \to \tau$ for any $x_i^\sigma \in \mathcal{V}$ and all contexts $\mathcal{C}^\tau$

(PAIR). $\langle \mathcal{C}_1^\sigma, \mathcal{C}_2^\tau \rangle$ is a context of type $\sigma \times \tau$ for all contexts $\mathcal{C}_1^\sigma$,$\mathcal{C}_2^\tau$

(L-PRJ). $fst.\mathcal{C}^{\sigma \times \tau}$ is a term of type σ for all contexts $\mathcal{C}^{\sigma \times \tau}$

(R-PRJ). $snd.\mathcal{C}^{\sigma \times \tau}$ is a term of type τ for all contexts $\mathcal{C}^{\sigma \times \tau}$

(REC). $R_\sigma(\mathcal{C}_1^\iota, \mathcal{C}_2^{\iota, \sigma \to \sigma}, \mathcal{C}_3^\sigma)$ is a context of type σ for all contexts $\mathcal{C}_1^\iota, \mathcal{C}_2^{\iota, \sigma \to \sigma}, \mathcal{C}_3^\sigma$

We define $\mathrm{C}^\backsim$ as the set of nondeterministic successor-free contexts by naturally extending the induction schema above with

(NDT). $(\mathcal{C}_1^\sigma|\mathcal{C}_2^\sigma)$ *is a context of type* σ *for all contexts* $\mathcal{C}_1^\sigma, \mathcal{C}_2^\sigma$

We say that a context is simple when it has at most one hole in it. Let $\mathcal{C}$ *be a simple context, then for any simple context* $\mathcal{C}_*$ *which has the same type as any hole in* $\mathcal{C}$*, let* $\mathcal{C}[\mathcal{C}_*]$ *denote replacing* $\mathcal{C}_*$ *with any hole in* $\mathcal{C}$*.*[1]

We define the set of deterministic successor-free terms, denoted T^-*, as all contexts* $\mathcal{C} \in \mathrm{C}^-$ *which have no hole. We define* $\mathrm{T}^\backsim$ *the same way. Let* $\mathcal{C}^\sigma \in \mathrm{C}^\backsim$ *be a simple* $\mathrm{T}^\backsim$ *context, and let* s *be a* $\mathrm{T}^\backsim$*-term of appropriate type, observe then that* $\mathcal{C}[s]$ *is also a* $\mathrm{T}^\backsim$*-term. Given* $\mathrm{T}^\backsim$*-terms* $M : \sigma$ *and* $t : \tau$*, if there exists a context* $\mathcal{C} : \sigma$ *such that* $M = \mathcal{C}[t]$ *then we say that* t *is a subterm of* M*.*

For any $b \geq 0$ *let* T_b^- *denote the set of* T^-*-terms such that all numerals occurring in them are no greater then* k_b*, and we define* $\mathrm{T}_b^\backsim$ *the same way.*

When we simply refer to something as a term, then it may be either a T^- *or* $\mathrm{T}^\backsim$ *term. For convenience we allow some syntactic sugar; for a term of the form* (M) *we may write* M*, for a term of the form* $((((M)N_1)...)N_k)$ *we may write* $M(N_1, ..., N_k)$*, for a term of the form* $N(N(...(M)...))$ *where* N *occurs* k *times we may write* $(N^k(M))$*, for a term of the form* $\lambda x_1.(\lambda x_2.(...\lambda x_k.M))$ *we may write* $(\lambda x_1 x_2 ... x_k.M)$*, and in most situations concerning variables we will drop the subscript and sometimes also type.*

A variable is said to be bound in a term M *if each occurrence of it in* M *is within the scope of an abstraction for it, otherwise we say that each occurrence outside of the scope of an abstraction is free in* M*. A term is said to be closed when all variables are bound, otherwise it is said to be open.*

For terms M *and* $N : \tau$ *and variable* $x : \tau$*, we say that* N *is substitutable for* x *in* M *if substituting all free* x *in* M *with* N *does not result in binding free variables in* N*. Let* M_N^x *denote such a substitution whenever* N *is substitutable for* x *in* M*.*

We define the one step reduction relation $\to^1$ *for* T^-

(α). $\mathcal{C}[\lambda x.M] \to^1 \mathcal{C}[\lambda y.M_y^x]$ *when* y *is not free in* M

(β). $\mathcal{C}[(\lambda x.M)N] \to^1 \mathcal{C}[M_N^x]$

(LPRJ). $\mathcal{C}[fst.\langle M, N\rangle] \to^1 \mathcal{C}[M]$

(RPRJ). $\mathcal{C}[snd.\langle M, N\rangle] \to^1 \mathcal{C}[N]$

(0-REC). $\mathcal{C}[R_\sigma(k_0, M, N)] \to^1 \mathcal{C}[N]$

(REC). $\mathcal{C}[R_\sigma(k_{n+1}, M, N)] \to^1 \mathcal{C}[M(k_n, R_\sigma(k_n, M, N))]$

for any simple context and suitably typed M*,*N*. We define the one step reduction relation* $\rhd^1$ *for* $\mathrm{T}^\backsim$ *by naturally extending the list above with*

(γ). $\mathcal{C}[(M|N)] \rhd^1 \mathcal{C}[M]$ *and* $\mathcal{C}[(M|N)] \rhd^1 \mathcal{C}[N]$

[1] Notice that using $[\mathcal{C}*]$ respects the inductive structure of $\mathcal{C}$. By this we mean that for example given the simple $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2$ we have $\mathcal{C}[C_*] = \mathcal{C}_1[C_*]\mathcal{C}_2[C_*]$. This allows us to do convenient induction on the structure of $\mathcal{C}[\mathcal{C}_*]$.

We make the distinction between the T^- *and* $\mathrm{T}^{\backsim}$ *reduction relation by using the symbols* $\to^1$ *and* $\rhd^1$ *respectivley. Let the relations* $\to$ *and* $\rhd$ *be the symmetric,transitive closures of* $\to^1$ *and* $\rhd^1$*. A term is said to be on normal form when no non-*α*-reduction can be applied to it, and it is said to be on* γ*-normal form when no non-*α*-*γ*-reduction is available.*

Definition 2.2. *We define* $lv(\sigma)$ *as the level of* σ *by (i)* $lv(\iota) = 0$ *(i)* $lv(\sigma \times \tau) = \max\{lv(\sigma), lv(\tau)\}$ *(iii)* $lv(\sigma \to \tau) = \max\{lv(\sigma) + 1, lv(\tau)\}$*. Let* $R_{\sigma_1}, ..., R_{\sigma_n}$ *be an exact list of all recursors occurring in a term* M*, we define the recursor rank of* M *as* $Rk(M) = \max\{0, lv(\sigma_1), ..., lv(\sigma_n)\}$*. We also define* $Tr(M)$ *as the term rank of* M *by (i)* $Tr(k_n) = 0$ *(ii)* $Tr(x^\sigma) = lv(\sigma)$ *(iii)* $Tr(\lambda x^\sigma.P^\tau) = \max\{lv(\sigma \to \tau), Tr(P)\}$ *(iv)* $Tr(PQ) = \max\{Tr(P), Tr(Q)\}$ *(v)* $Tr(P|Q) = \max\{Tr(P), Tr(Q)\}$ *(vi)* $Tr(\langle P, Q\rangle) = \max\{Tr(P), Tr(Q)\}$ *(vii)* $Tr(fst.P) = Tr(P)$ *(viii)* $Tr(snd.P) = Tr(P)$ *(viiii)* $Tr(R_\sigma(P, Q, R)) = \max\{Tr(P), Tr(Q), Tr(R)\}$

Lemma 2.3. *For any term* $M : \sigma$ *we have*

(i) $Tr(M) \geq lv(\sigma)$

(ii) $Tr(M) \geq Tr(R)$ *for any subterm* R *of* M

(iii) $Tr(M) = lv(\tau) = Tr(R)$ *for some subterm* $R : \tau$ *of* M

Proof. All are proven by simple induction on structure of M. □

Definition 2.4. *We say that an infinite reduction of a term normalizes if it ends with an infinite use of only* α*-reductions, and a term is said to be strongly normalizable if all infinite reductions normalize.*

Theorem 2.5. *All* $\mathrm{T}^{\backsim}$*-terms are strongly normalizable*

Proof. In [7] Berger U. demonstrates that natural extensions of Gödels T are strongly normalizing by using W.W. Tait's method of strong computability for simply typed lambda calculus from [8]. However we cannot directly apply his result to our calculus because of a superficial discrepancy. In $\mathrm{T}^{\backsim}$ we do not consider R_σ, $fst.$, $snd.$ and $|_\sigma$ as independent constants and terms on their own, as is customary and done in [7]. To overcome this we simply construct a new calculus $\mathrm{T}_\mathcal{C}$ where for all types σ and τ we have constants

$$R_\sigma : \iota, (\sigma \to \sigma), \sigma \to \sigma, |_\sigma : \sigma, \sigma \to \sigma, fst_{\sigma\times\tau} : \sigma \times \tau \to \sigma, snd_{\sigma\times\tau} : \sigma \times \tau \to \tau$$

and the same reduction rules as before. It is obvious that $\mathrm{T}^{\backsim} \subset \mathrm{T}_\mathcal{C}$, and any reduction of a $\mathrm{T}^{\backsim}$-term is also a reduction of the same term in $\mathrm{T}_\mathcal{C}$. Therefor since the reduction sequence is normalizing by strong normalization of $\mathrm{T}_\mathcal{C}$, then $\mathrm{T}^{\backsim}$ is also strongly normalizing. □

Chapter 3

Denotational semantics of $\mathbf{T}^{\backsim}$

3.1 The domain D_b^σ

Definition 3.1. *For any type σ and $b > 0$ we define D_b^σ as the domain over type σ in base b and a binary relation $\sqsubseteq_b^\sigma$ over D_b^σ*

(i) $D_b^\iota = \mathcal{P}(\{0, ..., b-1\})$ *and* $d \sqsubseteq_b^\iota e \Leftrightarrow d \subseteq e$ *for all* $d, e \in D_b^\sigma$[1]

(ii) $D_b^{\sigma\times\tau} = D_b^\sigma \times D_b^\tau$ *and* $d \sqsubseteq_b^{\sigma\times\tau} e \Leftrightarrow fst(d) \sqsubseteq_b^\sigma fst(e) \wedge\ snd(d) \sqsubseteq_b^\tau snd(e)$ *for all* $d \in D_b^\sigma$, $e \in D_b^\tau$[2]

(iii) $D_b^{\sigma\to\tau}$ *is the set of all functions* $f : D_b^\sigma \to D_b^\tau$ *and* $f \sqsubseteq_b^{\sigma\to\tau} g \Leftrightarrow f(d) \sqsubseteq_b^\tau g(d)$ *for all* $d \in D_b^\sigma$

These domains are simple in structure and rich in elements not corresponding to any term, e.g. D_b^ι contains the empty set. The domains also contain non-monotonic functions which also do not correspond to any term, given certain natural conditions on open terms. As mentioned in the introduction, the reason no refinements have been made is because we want the domains to be of easily computable size, namely $||\sigma||_b$ as seen later.

We experience that our terms behave monotonically with respect to $\sqsubseteq_b^\sigma$, so one would expect that an accurate interpretation would be monotonic as well. Also, when $[\![N : \sigma \to \tau]\!]_\mathcal{A}$ is given input which is monotonic,e.g the interpretation of a term, we expect monotonic output. The complication is that monotonicity is usually defined for functions over domains which are them self's restricted to only monotonic elements. We overcome this by introducing a suitable extension of the notion of monotonicity.

[1] $\mathcal{P}(S)$ denotes the set of all subsets of S

[2] fst and snd refers to functions taking the first and second component of a 2-tuple respectively, not to be confused with $fst.$ and $snd.$ which are part of the term language.

Definition 3.2. *We define a unary relation for all domains, called deep monotonicity*

(i) $g \in D_b^\iota$ *is deeply monotonic*

(ii) $g \in D_b^{\sigma\times\tau}$ *is deeply monotonic when* $fst(g)$ *and* $snd(g)$ *are deeply monotonic*

(iii) $g \in D_b^{\sigma\to\tau}$ *is deeply monotonic when*

- $g(e)$ *is deeply monotonic for all deeply monotonic* $e \in D_b^\sigma$
- $d \sqsubseteq_b^\sigma e \Rightarrow g(d) \sqsubseteq_b^\tau g(e)$ *for all deeply monotonic* $d, e \in D_b^\sigma$

Proposition 3.3. *Let* $M : \sigma$ *be a* $\mathrm{T}_b^{\curlyvee}$*-term, and* $\mathcal{A}_{b+1}$ *be an assignment mapping all free variables in* M *to deeply monotonic elements, then* $[\![M]\!]_{\mathcal{A}}$ *is deeply monotonic.*

Proof. We omit a detailed proof since this proposition si no more then an observation. We prove this by induction on the structure of M. In each induction case we rely on the fact that the functions defining $[\![\cdot]\!]_{\mathcal{A}}$, such as $\vee_b^\sigma$ and Ψ_{b+1}^σ, preserve deep monotonicity. □

Proposition 3.4. $\langle D_b^\sigma, \sqsubseteq_b^\sigma\rangle$ *is a partial order.*

Proof. We prove this by induction on the structure of σ. □

Definition 3.5. *For any type* σ *and* $b > 0$ *we define the binary operator* $\vee_b^\sigma$ *over* D_b^σ *as the merge operator for type* σ *in base* b

(i) $d \vee_b^\iota e = d \cup e$ *for all* $d, e \in D_b^\iota$

(ii) $d \vee_b^{\sigma\times\tau} e = (fst(d) \vee_b^\sigma fst(e), snd(e) \vee_b^\tau snd(e))$ *for all* $d, e \in D_b^{\sigma\times\tau}$

(iii) $f \vee_b^{\sigma\to\tau} g = h$ *for all* $f, g \in D_b^{\sigma\to\tau}$*, where* $h(x) = f(x) \vee^\tau g(x)$ *for all* $x \in D_b^\sigma$

Lemma 3.6. *For any* $d, e, f \in D_b^\sigma$

(i) $d \vee_b^\sigma e \in D_b^\sigma$

(ii) $d \vee_b^\sigma e = e \vee_b^\sigma d$

(iii) $(d \vee_b^\sigma e) \vee_b^\sigma f = d \vee_b^\sigma (e \vee_b^\sigma f)$

(iv) $d \sqsubseteq_b^\sigma e \vee_b^\sigma d$

(v) $d \vee_b^\sigma d = d$

(vi) $d = e \Leftrightarrow e \sqsubseteq_b^\sigma d$ *and* $d \sqsubseteq_b^\sigma e$

(vii) $e \sqsubseteq_b^\sigma d$ *and* $f \sqsubseteq_b^\sigma d \Rightarrow (e \vee_b^\sigma f) \sqsubseteq_b^\sigma d$

Proof. All are easily proved by a standard induction on the structure σ, and some may also be derived from the others. $\square$

For convenience we define a shorthand for merging the elements of any finite set $S \subseteq D_b^\sigma$. Let $\bigvee_{d\in S}^{\sigma,b} d$ denote the merging of all $d \in S$ in some arbitrary order. This is well-defined, since merging is both commutative and associative by *(ii)* and *(iii)* in Lemma 3.6 respectively. Throughout the text we may use slight variations off this shorthand, but it will always be clear that we are merging over some finite subset of our domains.

3.2 Interpreting $\mathbf{T}_b^{\backsim}$ in D_{b+1}^σ

Definition 3.7. *For any* σ *and* $b > 0$ *we define* $\psi_b^\sigma : \mathbb{N}_b \times D_b^{\iota,\sigma\to\sigma} \times D_b^\sigma \to D_b^\sigma$ *and* $\Psi_b^\sigma : D_b^\iota \backslash \{\emptyset\} \times D_b^{\iota,\sigma\to\sigma} \times D_b^\sigma \to D_b^\sigma$ *by*

(i) $\psi_b^\sigma(0, f, g) = g$

(ii) $\psi_b^\sigma(i+1, f, g) = f(\{i\}, \psi_b^\sigma(i, f, g))$

(iii) $\Psi_b^\sigma(S, f, g) = \bigvee_{n\in S}^{\sigma,b} \psi_b^\sigma(n, f, g)$

Lemma 3.8. *For any* $S_1, S_2 \in D_b^\iota$*,*$f \in D_b^{\iota,\sigma\to\sigma}$ *and* $g \in D_b^\sigma$

$$\Psi_b^\sigma(S_1 \cup S_2, f, g) = \Psi_b^\sigma(S_1, f, g) \vee_b^\sigma \Psi_b^\sigma(S_1, f, g)$$

Proof.

$$\begin{aligned}
\Psi_b^\sigma(S_1 \cup S_2, f, g) &= \bigvee_{n\in S_1\cup S_2}^{\sigma,b} \psi_b^\sigma(n, f, g) && \Psi_b^\sigma \text{ def.} \\
&= \left(\bigvee_{n\in S_1}^{\sigma,b} \psi_b^\sigma(n, f, g)\right) \vee_b^\sigma \left(\bigvee_{n\in S_2}^{\sigma,b} \psi_b^\sigma(n, f, g)\right) \\
&= \Psi_b^\sigma(S_1, f, g) \vee_b^\sigma \Psi_b^\sigma(S_2, f, g) && \Psi_b^\sigma \text{ def.}
\end{aligned}$$

$\square$

Definition 3.9. *We define a domain assignment* $\mathcal{A}$ *in base* b *as a total map from* $\mathcal{V}$ *into* $\bigcup_\sigma D_b^\sigma$ *such that* $\mathcal{A}(x^\sigma) \in D_b^\sigma$*, occasionally we may write* $\mathcal{A}_b$ *to clarify the base, or only refer to it as an assignment if the context allows this. Let* $\mathcal{A}$ *be a domain assignment in base* $b+1$*, then we define* $[\![\cdot]\!]_\mathcal{A}$ *as the domain interpretation of* $\mathrm{T}^\backsim$*-terms under assignment* $\mathcal{A}$

$[\![k_n]\!]_\mathcal{A} = \{n\}$

$[\![x]\!]_\mathcal{A} = \mathcal{A}(x)$

$[\![MN]\!]_\mathcal{A} = [\![M]\!]_\mathcal{A}[\![N]\!]_\mathcal{A}$

$[\![\lambda x^\sigma . M^\tau]\!]_\mathcal{A} = f$ *where* $f(u) = [\![M]\!]_{\mathcal{A}^x_u}$[3] *for all* $u \in D^\sigma_{b+1}$

$[\![M^\sigma | N^\sigma]\!]_\mathcal{A} = [\![M]\!]_\mathcal{A} \vee^\sigma_b [\![N]\!]_\mathcal{A}$

$[\![\, \langle M, N \rangle \,]\!]_\mathcal{A} = ([\![M]\!]_\mathcal{A}, [\![N]\!]_\mathcal{A})$

$[\![fst.M^{\sigma\times\tau}]\!]_\mathcal{A} = fst([\![M]\!]_\mathcal{A})$

$[\![snd.M^{\sigma\times\tau}]\!]_\mathcal{A} = snd([\![M]\!]_\mathcal{A})$

$[\![R_\sigma(N, F, G)]\!]_\mathcal{A} = \Psi^\sigma_{b+1}([\![N]\!]_\mathcal{A}, [\![F]\!]_\mathcal{A}, [\![G]\!]_\mathcal{A})$

Lemma 3.10. *For any* $\mathrm{T}^{\frown}_b$*-term* $M : \sigma$ *and assignment* $\mathcal{A}_{b+1}$

(i) $[\![M]\!]_\mathcal{A} \in D^\sigma_{b+1}$

(ii) $[\![M]\!]_\mathcal{U} = [\![M]\!]_\mathcal{A}$ *for any assignment* $\mathcal{U}_{b+1}$ *if* M *is closed*

(iii) $[\![M]\!]_{\mathcal{A}^x_{[\![R]\!]_\mathcal{A}}} = [\![M^x_R]\!]_\mathcal{A}$ *for any* $\mathrm{T}^{\frown}_b$*-term* $R : \tau$ *and variable* $x : \tau$ *such that* R *is substitutable for* x *in* M.

Proof. All are easily proved by a standard induction on structure of M. □

We will for closed terms often omit the domain assignment and write $[\![M]\!]$ instead of $[\![M]\!]_\mathcal{A}$, this is possible since *(ii)* in the lemma above shows that assignments are irrelevant for closed terms.

Lemma 3.11. *For any context* $\mathcal{C} : \sigma$ *and terms* s^τ, t^τ *such that* $\mathcal{C}[s], \mathcal{C}[t]$ *are* $\mathrm{T}^{\frown}_b$*-terms and assignment* $\mathcal{A}_{b+1}$

(i) $[\![s]\!]_\mathcal{A} \sqsubseteq^\tau_{b+1} [\![t]\!]_\mathcal{A} \Rightarrow [\![\mathcal{C}[s]]\!]_\mathcal{A} \sqsubseteq^\sigma_{b+1} [\![\mathcal{C}[t]]\!]_\mathcal{A}$

(ii) $[\![s]\!]_\mathcal{A} = [\![t]\!]_\mathcal{A} \Rightarrow [\![\mathcal{C}[s]]\!]_\mathcal{A} = [\![\mathcal{C}[t]]\!]_\mathcal{A}$

Proof. *(i)* is easily proved by induction on the structure of $\mathcal{C}$,*(ii)* follows immediately. □

Lemma 3.12. *For any* $\mathrm{T}^{\frown}_b$*-term* $M : \sigma$ *where* $M \rhd^1 N$ *and assignment* $\mathcal{A}_{b+1}$*, then* $[\![N]\!]_\mathcal{A} \sqsubseteq^\sigma_{b+1} [\![M]\!]_\mathcal{A}$ *when the* γ*-reduction was used, otherwise* $[\![N]\!]_\mathcal{A} = [\![M]\!]_\mathcal{A}$.

Proof. Let $s : \tau$ be the subterm of M to which the reduction is directly applied, resulting in some term $t : \tau$, so there exists a context $\mathcal{C} : \sigma$ such that $M = \mathcal{C}[s]$ and $N = \mathcal{C}[t]$. We now consider each reduction rule and demonstrate that $[\![s]\!]_\mathcal{A} \sqsubseteq^\tau_{b+1} [\![t]\!]_\mathcal{A}$ if the γ-reduction was used, otherwise $[\![s]\!]_\mathcal{A} = [\![t]\!]_\mathcal{A}$.

[3] $\mathcal{A}^x_u$ is standard notation for the map sending x to u, and otherwise behaving like $\mathcal{A}$. It is used throughout this thesis.

(α). By definition of $[\![\cdot]\!]_A$ we have $[\![\lambda y^\rho.M^x_y]\!]_{\mathcal{A}} = g$ where $g(u) = [\![M^x_y]\!]_{\mathcal{A}^y_u}$, and $[\![\lambda x^\rho.M]\!]_{\mathcal{A}} = f$ where $f(u) = [\![M]\!]_{\mathcal{A}^x_u}$. So for all $u \in D^\rho_{b+1}$

$$\begin{aligned} g(u) = [\![M^x_y]\!]_{\mathcal{A}^y_u} &= [\![M]\!]_{\mathcal{A}^{y,x}_{u,\mathcal{A}^y_u(y)}} && \textit{(iii)} \text{ Lemma 3.10} \\ &= [\![M]\!]_{\mathcal{A}^{y,x}_{u,u}} && \mathcal{A}^y_u(y) = u \\ &= [\![M]\!]_{\mathcal{A}^x_u} && y \text{ is not free in } M \\ &= f(u) \end{aligned}$$

Therefor $f = g$, that is $[\![\lambda x^\rho.M]\!]_{\mathcal{A}} = [\![\lambda y^\rho.M^x_y]\!]_{\mathcal{A}}$.

(β).

$$[\![(\lambda x.R)S]\!]_{\mathcal{A}} = [\![\lambda x.R]\!]_{\mathcal{A}}[\![S]\!]_{\mathcal{A}} = [\![R]\!]_{\mathcal{A}^x_{[\![S]\!]_{\mathcal{A}}}} = [\![R^x_S]\!]_{\mathcal{A}}$$

The last equality holds by *(iii)* in Lemma 3.10

(γ). By definition of $[\![\cdot]\!]_{\mathcal{A}}$ we have

$$[\![R|S]\!]_{\mathcal{A}} = [\![R]\!]_{\mathcal{A}} \vee^\sigma_{b+1} [\![S]\!]_{\mathcal{A}}$$

and

$$[\![R]\!]_{\mathcal{A}} \sqsubseteq^\sigma_{b+1} [\![R]\!]_{\mathcal{A}} \vee^\sigma_{b+1} [\![S]\!]_{\mathcal{A}}$$

by *(iii)* in Lemma 3.6, hence

$$[\![R]\!]_{\mathcal{A}} \sqsubseteq^\sigma_{b+1} [\![R|S]\!]_{\mathcal{A}}$$

The argument is symmetric in S.

(PRJ).

$$[\![fst.\langle R,S\rangle]\!]_{\mathcal{A}} = fst([\![\langle R,S\rangle]\!]_{\mathcal{A}}) = fst([\![R]\!]_{\mathcal{A}},[\![S]\!]_{\mathcal{A}}) = [\![R]\!]_{\mathcal{A}}$$

$snd.\langle R,S\rangle$ is analogous.

(0-REC).

$$\begin{aligned} [\![R_\sigma(k_0,F,G)]\!]_{\mathcal{A}} &= \Psi^\sigma_{b+1}([\![k_0]\!]_{\mathcal{A}},[\![F]\!]_{\mathcal{A}},[\![G]\!]_{\mathcal{A}}) && [\![\cdot]\!]_A \text{ def.} \\ &= \bigvee_{n\in[\![k_0]\!]_{\mathcal{A}}}^{\sigma,b+1} \psi^\sigma_{b+1}(n,[\![F]\!]_{\mathcal{A}},[\![G]\!]_{\mathcal{A}}) && \Psi^\sigma_{b+1} \text{ def.} \\ &= \psi^\sigma_{b+1}(0,[\![F]\!]_{\mathcal{A}},[\![G]\!]_{\mathcal{A}}) && [\![k_0]\!]_{\mathcal{A}} = \{0\} \text{ def.} \\ &= [\![G]\!]_{\mathcal{A}} && \psi^\sigma_{b+1} \text{ def.} \end{aligned}$$

(REC).

$$
\begin{aligned}
&[\![R_\sigma(k_{n+1},F,G)]\!]_{\mathcal{A}} \\
&= \Psi^\sigma_{b+1}([\![k_{n+1}]\!]_{\mathcal{A}}, [\![F]\!]_{\mathcal{A}}, [\![G]\!]_{\mathcal{A}}) && [\![\cdot]\!]_{\mathcal{A}} \text{ def.} \\
&= \bigvee^{\sigma,b+1}_{m\in[\![k_{n+1}]\!]_{\mathcal{A}}} \psi^\sigma_{b+1}(m, [\![F]\!]_{\mathcal{A}}, [\![G]\!]_{\mathcal{A}}) && \Psi^\sigma_{b+1} \text{ def.} \\
&= \psi^\sigma_{b+1}(n+1, [\![F]\!]_{\mathcal{A}}, [\![G]\!]_{\mathcal{A}}) && [\![k_{n+1}]\!]_{\mathcal{A}} = \{n+1\} \\
&= [\![F]\!]_{\mathcal{A}}(\{n\}, \psi^\sigma_{b+1}(n, [\![F]\!]_{\mathcal{A}}, [\![G]\!]_{\mathcal{A}})) && \psi^\sigma_{b+1} \text{ def.} \\
&= [\![F]\!]_{\mathcal{A}} \left([\![k_n]\!]_{\mathcal{A}}, \bigvee^{\sigma,b+1}_{m\in[\![k_n]\!]_{\mathcal{A}}} \psi^\sigma_{b+1}(m, [\![F]\!]_{\mathcal{A}}, [\![G]\!]_{\mathcal{A}}) \right) \\
&= [\![F]\!]_{\mathcal{A}} \left([\![k_n]\!]_{\mathcal{A}}, \Psi^\sigma_{b+1}([\![k_n]\!]_{\mathcal{A}}, [\![F]\!]_{A}, [\![G]\!]_{\mathcal{A}}) \right) && \Psi^\sigma_{b+1} \text{ def.} \\
&= [\![F]\!]_{\mathcal{A}}([\![k_n]\!]_{\mathcal{A}}, [\![R_\sigma(k_n,F,G)]\!]_{\mathcal{A}}) && [\![\cdot]\!]_{\mathcal{A}} \text{ def.} \\
&= [\![Fk_n]\!]_{\mathcal{A}}[\![R_\sigma(k_n,F,G)]\!]_{\mathcal{A}} && [\![\cdot]\!]_{\mathcal{A}} \text{ def.} \\
&= [\![F(k_n, R_\sigma(k_n,F,G))]\!]_{\mathcal{A}} && [\![\cdot]\!]_{\mathcal{A}} \text{ def.}
\end{aligned}
$$

We see from the relations above, if a γ-reduction was applied then $[\![s]\!]_{\mathcal{A}} = [\![t]\!]_{\mathcal{A}}$, so by *(ii)* in Lemma 3.11 $[\![M]\!]_{\mathcal{A}} = [\![N]\!]_{\mathcal{A}}$ as desired. Otherwise $[\![s]\!]_{\mathcal{A}} \sqsubseteq^\tau_{b+1} [\![t]\!]_{\mathcal{A}}$, so by *(i)* in Lemma 3.11 $[\![N]\!]_{\mathcal{A}} \sqsubseteq^\sigma_b [\![M]\!]_{\mathcal{A}}$. ☐

Corollary 3.13. *For any $\mathrm{T}^{\frown}_b$-term $M : \sigma$ and assignment $\mathcal{A}_{b+1}$ where $M \rhd N$*

$$[\![N]\!]_{\mathcal{A}} \sqsubseteq^\sigma_b [\![M]\!]_{\mathcal{A}}$$

Proof. There is a sequence of $\mathrm{T}^{\frown}_b$-terms $R_1 : \sigma, ..., R_k : \sigma$ where $M = R_1$ and $N = R_k$ and $R_i \rhd^1 R_{i+1}$ for all $i < k$. By Lemma 3.12 we know that $[\![R_i]\!]_{\mathcal{A}} \sqsubseteq^\sigma_{b+1} [\![R_{i+1}]\!]_{\mathcal{A}}$, and combined with transitivity of $\sqsubseteq^\sigma_{b+1}$ from Proposition 3.4 we get that $[\![R_i]\!]_{\mathcal{A}} \sqsubseteq^\sigma_{b+1} [\![R_j]\!]_{\mathcal{A}}$ for all $i, j \leq k$. Hence $[\![R_1]\!]_{\mathcal{A}} \sqsubseteq^\sigma_{b+1} [\![R_k]\!]_{\mathcal{A}}$, that is $[\![N]\!]_{\mathcal{A}} \sqsubseteq^\sigma_{b+1} [\![M]\!]_{\mathcal{A}}$. ☐

Lemma 3.14. *Let M be a closed term on normal form of type σ*

(i) *$\sigma = \iota$ then M is a numeral*

(ii) *$\sigma = \tau \to \rho$ then M is a lambda abstraction*

(iii) *$\sigma = \tau \times \rho$ then M is a pair*

Proof. We do a simultaneous induction proof on the structure of M for all three claims, and keep in mind that any subterm of M must also be on normal form. In each claim we will argue that M cannot have certain forms, and some of these forms occur in all claims, therefor we argue these outside the claims to avoid repetition.

Since M is closed it cannot be a variable, and since it is on normal form it can obviously not be on form $(A|B)$. It cannot be on form $R(N, F, G)$ since $N : \iota$ by the induction hypothesis must be a numeral, and this allows us to apply the recursor reduction on M, contradicting it being on normal form. It cannot be on form $(A^{\tau\to\rho}B^{\tau})$,$fst.A^{\tau\times\rho}$ or $snd.A^{\tau\times\rho}$, since either one would by the induction hypothesis on A allow us to apply a reduction, β- and projection respectively, on M directly. Observe now that the only remaining possible forms are the numeral, lambda abstraction and pairing.

$M : \iota$: Since M is of type ι it cannot be on form $\lambda x.A$ or $\langle A, B\rangle$, so the only remaining form is numeral, so M is a numeral term.

$M : \tau \to \rho$: Since M is of type $\tau \to \rho$ it cannot be a numeral or on form $\langle A, B\rangle$, so the only remaining form is lambda abstraction, so M is a lambda abstraction term.

$M : \tau \times \rho$: Since M is of type $\tau \times \rho$ it cannot be a numeral or a lambda abstraction , so the only remaining form is pairing, so M is a pairing term.

□

Lemma 3.15. *A choice term is a closed $T^{\smile}$ term where the γ-reduction is available, but no other non-α-reduction.*[4] *Let $M : \sigma$ be a closed choice term not of the form $\lambda x.Q$ and $\langle P, Q\rangle$, then there exists a context $\mathcal{C} : \sigma$ such that $M = \mathcal{C}[p|q]$ and*

$$[\![\mathcal{C}[p|q]]\!] = [\![\mathcal{C}[p]]\!] \vee^{\sigma}_{b+1} [\![\mathcal{C}[q]]\!]$$

Proof. We demonstrate this by induction on the structure of M.

Case $M = k_n, x$. Neither of these cases are possible since M is a choice term.

Case $M = P^{\tau\to\sigma}Q^{\tau}$. P is a closed term since M is closed. P must be a choice term, otherwise it would have to be on normal form, which by *(ii)* in Lemma 3.14 puts it on form $\lambda x.u$, contradicting that M is a choice term. P cannot be on form $\langle u, v\rangle$ since it is of the arrow type, and not on form $\lambda x.u$ since this as mentioned contradicts that M is choice term. This allows the induction hypothesis to be applied on P, giving us a context $\mathcal{C}_1 : \sigma \to \tau$ such that $P = \mathcal{C}_1[p|q]$.

Let $\mathcal{C} = \mathcal{C}_1 Q$ and observe that $M = \mathcal{C}[p|q]$ and

$$\begin{aligned}
[\![\mathcal{C}[p|q]]\!] &= [\![\mathcal{C}_1[p|q]Q]\!] && \mathcal{C}[]\text{ def.}\\
&= [\![\mathcal{C}_1[p|q]]\!][\![Q]\!] && [\![\cdot]\!]\text{ def.}\\
&= \left\{[\![\mathcal{C}_1[p]]\!] \vee^{\tau\to\sigma}_{b+1} [\![\mathcal{C}_1[q]]\!]\right\}[\![Q]\!] && \text{IH.}\\
&= [\![\mathcal{C}_1[p]]\!]([\![Q]\!]) \vee^{\sigma}_{b+1} [\![\mathcal{C}_1[q]]\!]([\![Q]\!]) && \vee^{\tau\to\sigma}_{b+1}\text{ .def}\\
&= [\![\mathcal{C}_1[p]Q]\!] \vee^{\sigma}_{b+1} [\![\mathcal{C}_1[q]Q]\!] && [\![\cdot]\!]\text{ def.}\\
&= [\![\mathcal{C}[p]]\!] \vee^{\sigma}_{b+1} [\![\mathcal{C}[q]]\!] && \mathcal{C}
\end{aligned}$$

[4]Not to be confused with a term on γ-normal form, which does not guarantee having a nondeterministic reduction available. So a choice term is on γ-normal form, but the converse does not hold.

Case $M = \lambda x.P, \langle P, Q \rangle$. These cases are not possible by assumption.

Case $M = fst.P^{\sigma\times\tau}, snd.P^{\tau\times\sigma}$. We only consider the first case. P is clearly a choice term, and cannot be on form $\lambda x.u$ since it is of the product type, and not of form $\langle u, v\rangle$ since this as contradicts that M is choice term. This allows the induction hypothesis to be applied to P, giving us a context $\mathcal{C}_1$ such that $P = \mathcal{C}_1[p|q]$. Let $\mathcal{C} = fst.\mathcal{C}_1$ and observe that $M = \mathcal{C}[p|q]$ and

$$\begin{aligned}
[\![\mathcal{C}[p|q]]\!] &= [\![fst.\mathcal{C}_1[p|q]]\!] && \mathcal{C}[]\text{ def.}\\
&= fst([\![\mathcal{C}_1[p|q]]\!]) && [\![\cdot]\!]\text{ def.}\\
&= fst([\![\mathcal{C}_1[p]]\!] \vee^{\sigma\times\tau}_{b+1} [\![\mathcal{C}_1[q]]\!]) && \text{IH.}\\
&= fst([\![\mathcal{C}_1[p]]\!]) \vee^{\sigma}_{b+1} fst([\![\mathcal{C}_1[q]]\!]) && \vee^{\sigma\times\tau}_{b+1}\text{ def.}\\
&= [\![fst.\mathcal{C}_1[p]]\!] \vee^{\sigma}_{b+1} [\![fst.\mathcal{C}_1[q]]\!] && [\![\cdot]\!]\text{ def.}\\
&= [\![\mathcal{C}[p]]\!] \vee^{\sigma}_{b+1} [\![\mathcal{C}[q]]\!] && \mathcal{C}\text{ def.}
\end{aligned}$$

Case $M = (P|Q)$. Let $\mathcal{C} = []$ and observe that $M = \mathcal{C}[P|Q]$ and the desired result.

Case $M = R_\sigma(N^\iota, F^{\iota,\sigma\to\sigma}, G^\sigma)$. Terms N, F, G must all be closed. The term N must be a choice term, otherwise it would be on normal form and therefore by *(ii)* in Lemma 3.14 also on numeral form, contradicting that M is a choice term. N cannot be on form $\langle u, v\rangle$ or $\lambda x.u$ since it is of the ι type. This allows the induction hypothesis to be applied to N, giving us a context $\mathcal{C}_1$ such that $N = \mathcal{C}_1[p|q]$. Let $\mathcal{C} = R_\sigma(\mathcal{C}_1, F, G)$ and observe that $M = \mathcal{C}[p|q]$ and

$$\begin{aligned}
[\![\mathcal{C}^{\sigma,\rho}[p|q]]\!] &= [\![R_\sigma(\mathcal{C}_1[p|q], F, G)]\!] && \mathcal{C}[]\text{ def.}\\
&= \Psi^\sigma_{b+1}([\![\mathcal{C}_1[p|q]]\!], [\![F]\!], [\![G]\!]) && [\![\cdot]\!]\text{ def.}\\
&= \Psi^\sigma_{b+1}([\![\mathcal{C}_1[p]]\!] \vee^\iota_{b+1} [\![\mathcal{C}_1[q]]\!], [\![F]\!], [\![G]\!]) && \text{IH.}\\
&= \Psi^\sigma_{b+1}([\![\mathcal{C}_1[p]]\!] \cup [\![\mathcal{C}_1[q]]\!], [\![F]\!], [\![G]\!]) && \vee^\iota_{b+1}\text{ def.}\\
&= \Psi^\sigma_{b+1}([\![\mathcal{C}_1[p]]\!], [\![F]\!], [\![G]\!]) \vee^\sigma_{b+1} \Psi^\sigma_{b+1}([\![\mathcal{C}_1[q]]\!], [\![F]\!], [\![G]\!]) && \text{Lemma 3.8}\\
&= [\![R_\sigma(\mathcal{C}_1[p], F, G)]\!] \vee^\sigma_{b+1} [\![R_\sigma(\mathcal{C}_1[q], F, G)]\!] && [\![\cdot]\!]\text{ def.}\\
&= [\![\mathcal{C}[p]]\!] \vee^\sigma_{b+1} [\![\mathcal{C}[q]]\!] && \mathcal{C}\text{ def.}
\end{aligned}$$

□

This induction reveals that if a term has a nondeterministic choice as its only remaining non-α-reduction, then we will always find such a choice outside the scope of a recursor, meaning F or G for any $R_\sigma(N, F, G)$. We may however, find it completely outside a recursor, or alternatively in N. For example we may have $R([A|B], F, G)$ or even $R(...R([A|B], F_0, G_0)..., F_1, G_0)$, and so on. This is a fundamental ingredient in the adequacy proof, and it actually provides us with a guarantee of how we can suitably reduce a term to preserve its interpretation.

Corollary 3.16. *For any closed* $\mathrm{T}_b^{\frown}$*-term* $M : \iota$ *not on normal form with* $n \in [\![M]\!]$*, there exists a term* $N : \iota$ *such that* $M \rhd^1 N$ *by way of a non* α*-reduction and* $n \in [\![N]\!]$

Proof. If M is not a choice term, then it must have a deterministic reduction available, and applying any such reduction preserves the interpretation and thus $n \in [\![N]\!]$. Assume now that M is a choice term, and observe that it cannot be on form $\lambda x.P$ or $\langle P, Q\rangle$ since it it of type ι.

By Lemma 3.15 $M = \mathcal{C}[p|q]$ where

$$[\![\mathcal{C}[p|q]]\!] = [\![\mathcal{C}[p]]\!] \vee_{b+1}^{\iota} [\![\mathcal{C}[q]]\!]$$

alternatively

$$[\![M]\!] = [\![\mathcal{C}[p]]\!] \cup [\![\mathcal{C}[q]]\!]$$

which demonstrates the existence of an available nondeterministic reduction, moreover we must have $n \in [\![\mathcal{C}[p]]\!]$ or $n \in [\![\mathcal{C}[q]]\!]$. So picking the appropriate reduction will provide the desired result.

□

Theorem 3.17. *For any closed* $\mathrm{T}_b^{\frown}$*-term* $M : \iota$

$$n \in [\![M]\!] \Leftrightarrow M \rhd k_n$$

Proof. First observe that

$$\begin{aligned}
M \rhd k_n &\Rightarrow [\![k_n]\!] \sqsubseteq_{b+1}^{\iota} [\![M]\!] && \text{corollary 3.13}\\
&\Rightarrow \{n\} \sqsubseteq_{b+1}^{\iota} [\![M]\!] && [\![\cdot]\!]_A \text{ def.}\\
&\Rightarrow \{n\} \subseteq [\![M]\!] && \sqsubseteq_{b+1}^{\iota} \text{ def.}\\
&\Rightarrow n \in [\![M]\!]
\end{aligned}$$

For the converse implication let $n \in [\![M]\!]$. By Lemma 3.16 we can do a reduction $M \rhd^1 M_1 \rhd^1 M_2 \rhd^1 \ldots$ consisting of only non α-reductions such that

$$[\![M]\!] = [\![M_1]\!] = [\![M_2]\!] = \ldots$$

This reduction sequence must normalize since it otherwise would constitute a counterexample to Theorem 2.5. So there is a term M_i in the sequence on normal form. M_i must also be closed and of type ι since reductions preserves this, and so $M_i = k_m$ for some m by Lemma 3.14. Since and $n \in [\![M_i]\!] = [\![k_m]\!] = \{m\}$, so $M_i = k_n$ and thus $M \rhd k_n$ as desired. □

3.3 Mapping D_b^{σ} to $\mathcal{N}_b^{\sigma}$

In this section we show how to map D_b^{σ} into it's isomorphic counterpart and subset of the natural numbers $\mathcal{N}_b^{\sigma}$, by way of a bijection π_b^{σ}.

Definition 3.18. *For any type* σ *and* $b > 0$ *we define* $\|\sigma\|_b$ *as the nondeterministic cardinality of* σ *by (i)* $\|\iota\|_b = 2^b$ *(ii)* $\|\sigma \times \tau\|_b = \|\sigma\|_b \times \|\tau\|_b$ *(iii)* $\|\sigma \to \tau\|_b = \|\tau\|_b^{\|\sigma\|_b}$. *For any type* σ *and* $b > 0$ *we define* $\mathcal{N}_b^\sigma = \{0, ..., \|\sigma\|_b - 1\}$

This following lemma is necessary for following functions on $\mathcal{N}_b^\sigma$ to be well-defined

Lemma 3.19. *For any number* $a \in \mathcal{N}_b^\sigma$ *where*

(i) $\sigma = \iota$ *there are unique numbers* $d_0, ..., d_{b-1} \in \{0, 1\}$ *such that*

$$a = d_0 + d_1 2 + ... + d_{b-1} 2^{b-1}$$

(ii) $\sigma = \rho \times \tau$ *there are unique numbers* $d_0 \in \mathcal{N}_b^\tau$, $d_1 \in \mathcal{N}_b^\rho$ *such that*

$$a = d_0 + d_1 ||\tau||_b$$

(iii) $\sigma = \rho \to \tau$ *there are unique numbers* $d_0, ..., d_\ell \in \mathcal{N}_b^\tau$ *where* $\ell = ||\rho||_b - 1$ *such that*

$$a = d_0 + d_1 ||\tau||_b + ... + d_\ell {||\tau||_b}^\ell$$

Proof. *(i)* is proven the same way as *(iii)*, and both *(ii)* and *(iii)* are analogous to Lemma 4.2. □

Lemma 3.20. *For all* $b > 0$ *we define*

- $\xi_b : \mathcal{N}_b^\iota \to D_b^\iota$ *by* $\xi_b(d_0 + ... + d_{b-1}2^{b-1}) = \{i | d_i = 1\}$
- $\beta_b : \mathcal{N}_b^\iota \times \mathcal{N}_b \to \{0, 1\}$ *by* $\beta_b(d_0 + ... + d_{b-1}2^{b-1}, i) = d_i$
- $\mu_b : D_b^\iota \to \mathcal{N}_b^\iota$ *by* $\mu_b(S) = \sum_{i \in S} 2^i$

Then (i) ξ_b *is a bijection (ii)* μ_b *is the inverse of* ξ_b *(iii)* μ_b *is a bijection (iv)* $\beta(n, m) = 1 \Leftrightarrow \{m\} \subseteq \xi_b(n)$ *for any* $n \in \mathcal{N}_b^\iota$ *and* $m \in \mathcal{N}_b$.

Proof. For surjectivity of ξ_b fix $S \in D_b^\iota$ and let

$$d_i = \begin{cases} 1 & i \in S \\ 0 & \text{else} \end{cases} \quad \text{for all } i < b$$

then

$$\xi_b(d_0 + ... + d_{b-1}2^{b-1}) = \{i | d_i = 1\} = S$$

Injectivity of ξ_b follows from *(ii)* , so or any $a \in \mathcal{N}_b^\iota$

$$\begin{aligned}
\mu_b(\xi_b(a)) &= \mu_b(\xi_b(d_0^* + ... + d_{b-1}^* 2^{b-1})) \\
&= \mu_b(\{i | d_i^* = 1\}) \\
&= \sum_{i \in \{i | d_i^* = 1\}} 2^i \\
&= d_0^* + ... + d_{b-1}^* 2^{b-1} \\
&= a
\end{aligned}$$

Since μ_b is the inverse of a bijection, it is itself a bijection, and finally *(iv)* holds by straightforward computation. □

Definition 3.21. *For all types σ and $b > 0$ we define the binary relation $\preccurlyeq_b^\sigma$ on $\mathcal{N}_b^\sigma$ by*

(i) $n \preccurlyeq_b^\iota m \Leftrightarrow \xi_b(n) \subseteq \xi_b(m)$

(ii) $d_0 + d_1 \|\tau\|_b \preccurlyeq_b^{\sigma\times\tau} d_0^* + d_1^* \|\tau\|_b \Leftrightarrow d_1 \preccurlyeq_b^\sigma d_1^*$ *and* $d_0 \preccurlyeq_b^\tau d_0^*$

(iii) $d_0 + ... + d_\ell \|\tau\|_b^\ell \preccurlyeq_b^{\sigma\to\tau} d_0^* + ... + d_\ell^* \|\tau\|_b^\ell \Leftrightarrow d_0 \preccurlyeq^\tau d_0^*, ..., d_\ell \preccurlyeq_b^\tau d_\ell^*$ *where* $\ell = \|\sigma\|_b - 1$

Definition 3.22. *For any type σ and $b > 0$ we define $\pi_b^\sigma : D_b^\sigma \to \mathcal{N}_b^\sigma$ by*

(i) $\pi_b^\iota(S) = \mu_b(S)$

(ii) $\pi_b^{\sigma\times\tau}(d) = \pi_b^\tau(snd(d)) + \pi_b^\sigma(fst(d)) \|\tau\|_b$

(iii) $\pi_b^{\sigma\to\tau}(d) = \sum_{i<\|\sigma\|_b} \pi_b^\tau(d(\rho_b^\sigma(i))) \|\tau\|_b^i$, *where $\rho_b^\sigma : \mathcal{N}_b^\sigma \to D_b^\sigma$ is the inverse of π_b^σ*

We see that π_b^σ requires an inverse for $\pi_b^{\sigma\to\tau}$ to be well-defined, this is guaranteed by the bijectivity of π_b^σ.

Lemma 3.23.

(i) π_b^σ *is a bijection*

(ii) $d \sqsubseteq_b^\sigma e \Leftrightarrow \pi_b^\sigma(d) \preccurlyeq_b^\sigma \pi_b^\sigma(e)$ *for any* $d, e \in D_b^\sigma$

(iii) $\langle \mathcal{N}_b^\sigma, \preccurlyeq_b^\sigma \rangle$ *is a partial order isomorphic with* $\langle D_b^\sigma, \sqsubseteq_b^\sigma \rangle$

Proof. Both *(i)* and *(ii)* are proved straight forward by induction on the structure of σ. The partial ordering of $\langle \mathcal{N}_b^\sigma, \preccurlyeq_b^\sigma \rangle$ follows from it being isomorphic with the partial order $\langle D_b^\sigma, \sqsubseteq_b^\sigma \rangle$, and the isomorphism is established by *(i)* and *(ii)*. □

3.4 Interpreting $\mathbf{T}_b^\frown$ in $\mathcal{N}_{b+1}^\sigma$

In this section we show how to interpret $\mathbf{T}_b^\frown$ terms in $\mathcal{N}_{b+1}^\sigma$ and how the bijection π_{b+1}^σ preserves the interpretation of terms betweene D_{b+1}^σ and $\mathcal{N}_{b+1}^\sigma$. This shows that D_{b+1}^σ and $\mathcal{N}_{b+1}^\sigma$ are isomorphic models.

Definition 3.24. *For all $b > 0$ and all types σ,τ we define*

(i) $\theta_b^{\sigma\times\tau} : \mathcal{N}_b^\sigma \times \mathcal{N}_b^\tau \to \mathcal{N}_b^{\sigma\times\tau}$ *by* $\theta_b^{\sigma\times\tau}(d_1, d_0) = d_1 ||\tau||_b + d_0$

(ii) $\lambda_b^{\sigma\times\tau} : \mathcal{N}_b^{\sigma\times\tau} \to \mathcal{N}_b^\sigma$ *by* $\lambda_b^{\sigma\times\tau}(d_0 + d_1 \|\tau\|_b) = d_1$

(iii) $\rho_b^{\sigma\times\tau} : \mathcal{N}_b^{\sigma\times\tau} \to \mathcal{N}_b^{\tau}$ *by* $\rho_b^{\sigma\times\tau}(d_0 + d_1 \|\tau\|_b) = d_0$

(iv) $\delta_b^{\sigma\to\tau} : \mathcal{N}_b^{\sigma\to\tau} \times \mathcal{N}_b^{\sigma_1} \to \mathcal{N}_b^{\tau}$ *by* $\delta_b^{\sigma\to\tau}(d_0 + ... + d_k\|\tau\|_b{}^k, i) = d_i$*, where* $k = \|\sigma\|_b - 1$

For convenience we define a shorthand $a[b] = \delta_b^{\sigma\to\tau}(a, b)$ for any $a \in \mathcal{N}_b^{\sigma\to\tau}$ and $b \in \mathcal{N}_b^{\sigma}$, and let $a[b_1, ..., b_n] = a[b_1][b_2, ..., b_n]$ for appropriate $a, b_1, ..., b_n$.

Definition 3.25. *For all types* σ *and* $b > 0$ *we define* $Merge_b^{\sigma} : \mathcal{N}_b^{\sigma} \times \mathcal{N}_b^{\sigma} \to \mathcal{N}_b^{\sigma}$ *by induction on the structure of* σ

(i) $Merge_b^{\iota}(n, m) = \mu_b(\xi_b(n) \cup \xi_b(m))$

(ii) $Merge_b^{\sigma\times\tau}(d_1 + d_0 \|\tau\|_b, d_1^* + d_0^* \|\tau\|_b) = Merge_b^{\tau}(d_1, d_1^*) + Merge_b^{\sigma}(d_0, d_0^*) \|\tau\|_b$

(iii) $Merge_b^{\sigma\to\tau}(d_0 + ... + d_\ell \|\tau\|_b^{\ell}, d_0^* + ... + d_\ell^* \|\tau\|_b^{\ell}) = Merge_b^{\tau}(d_0, d_0^*) + ... + Merge_b^{\tau}(d_\ell, d_\ell^*) \|\tau\|_b^{\ell}$ *where* $\ell = \|\sigma\|_b - 1$

Lemma 3.26.

(i) $\pi_b^{\tau}(f(d)) = \pi_b^{\sigma\to\tau}(f)[\pi_b^{\sigma}(d)]$

(ii) $\pi_b^{\sigma}(d \vee_b^{\sigma} e) = Merge_b^{\sigma}(\pi_b^{\sigma}(d), \pi_b^{\sigma}(e))$

Proof. For *(i)* observe that

$$\begin{aligned}
\pi_b^{\sigma\to\tau}(f)[\pi_b^{\sigma}(d)] &= \left\{ \sum_{i<\|\sigma\|_b} \pi_b^{\tau}(f(\rho_b^{\sigma}(i))) \|\tau\|_b^{i} \right\} [\pi_b^{\sigma}(d)] && \pi_b^{\sigma\to\tau} \text{ def.} \\
&= \pi_b^{\tau}(f(\rho_b^{\sigma}(\pi_b^{\sigma}(d)))) && \delta_b^{\sigma\to\tau} \text{ def.} \\
&= \pi_b^{\tau}(f(d)) && \rho_b^{\sigma} \text{ is inverse of } \pi_b^{\sigma}
\end{aligned}$$

and *(ii)* is simply proven by induction on σ. □

We see that *(ii)* in Lemma 3.26 guarantees that $Merge_b^{\sigma}$ is indeed also commutative and associative by way of π_b^{σ}, so as before we define a shorthand $\bigsqcup_{d\in S}^{\sigma,b} d$ to denote the merging of all $d \in S$.

Definition 3.27. *For all* σ *and* $b > 0$ *we define* $\upsilon_b^{\sigma} : \mathbb{N}_b \times \mathcal{N}_b^{\iota,\sigma\to\sigma} \times \mathcal{N}_b^{\sigma} \to \mathcal{N}_b^{\sigma}$ *and* $\Upsilon_b^{\sigma} : \mathcal{P}(\mathbb{N}_b) \setminus \{\emptyset\} \times \mathcal{N}_b^{\iota,\sigma\to\sigma} \times \mathcal{N}_b^{\sigma} \to \mathcal{N}_b^{\sigma}$ *by*

(i) $\upsilon_b^{\sigma}(0, f, g) = g$

(ii) $\upsilon_b^{\sigma}(i + 1, f, g) = f[\mu_b(\{i\}), \upsilon_b^{\sigma}(i, f, g)]$[5]

(iii) $\Upsilon_b^{\sigma}(S, f, g) = \bigsqcup_{n\in S}^{\sigma,b} \upsilon_b^{\sigma}(n, f, g)$

Lemma 3.28.

[5] We use uncyrring for readability.

(i) $\pi_b^\sigma(\psi_b^\sigma(n,f,g)) = \upsilon_b^\sigma(n, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))$

(ii) $\pi_b^\sigma(\Psi_b^\sigma(S,f,g)) = \Upsilon_b^\sigma(S, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))$

Proof. We prove *(i)* by induction on n, so first

$$\pi_b^\sigma(\psi_b^\sigma(0,f,g)) = \pi_b^\sigma(g) = \upsilon_b^\sigma(0, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))$$

then for $n = i+1 > 0$

$$\begin{aligned}
\pi_b^\sigma(\psi_b^\sigma(n,f,g)) &= \pi_b^\sigma(\psi_b^\sigma(i+1,f,g)) && n = i+1 \\
&= \pi_b^\sigma(f(\{i\}, \psi_b^\sigma(i,f,g))) && \psi_b^\sigma \text{ def.} \\
&= \pi_b^{\sigma\to\sigma}(f(\{i\}))[\pi_b^\sigma(\psi_b^\sigma(i,f,g))] && \textit{(i)} \text{ in Lemma 3.26} \\
&= \pi_b^{\sigma\to\sigma}(f(\{i\}))[\upsilon_b^\sigma(i, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))] && \text{IH.} \\
&= \{\pi_b^{\iota,\sigma\to\sigma}(f)[\pi_b^\iota(\{i\})]\}\,[\upsilon_b^\sigma(i, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))] && \textit{(i)} \text{ in Lemma 3.26} \\
&= \{\pi_b^{\iota,\sigma\to\sigma}(f)[\mu_b(\{i\})]\}\,[\upsilon_b^\sigma(i, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))] && \pi_b \text{ def.} \\
&= \pi_b^{\iota,\sigma\to\sigma}(f)[\mu_b(\{i\}), \upsilon_b^\sigma(i, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g))] && [] \text{ def.} \\
&= \upsilon_b^\sigma(i+1, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g)) && \upsilon_b^\sigma \text{ def.}
\end{aligned}$$

For *(ii)* we then have

$$\begin{aligned}
\pi_b^\sigma(\Psi_b^\sigma(S,f,g)) &= \pi_b^\sigma(\bigvee_{n\in S}^{\sigma,b} \psi_b^\sigma(n,f,g)) && \Psi_b^\sigma \text{ def.} \\
&= \bigsqcup_{n\in S}^{\sigma,b} \pi_b^\sigma \psi_b^\sigma(n,f,g)) && \textit{(ii)} \text{ in Lemma 3.26} \\
&= \bigsqcup_{n\in S}^{\sigma,b} \upsilon_b^\sigma(n, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g)) && \textit{(i)} \\
&= \Upsilon_b^\sigma(S, \pi_b^{\iota,\sigma\to\sigma}(f), \pi_b^\sigma(g)) && \text{by } \Upsilon_b^\sigma \text{ def.}
\end{aligned}$$

□

Definition 3.29. *We define a value assignment $\mathcal{A}$ in base b as a total map from $\mathcal{V}$ into $\bigcup_\sigma \mathcal{N}_b^\sigma$ such that $\mathcal{A}(x^\sigma) \in \mathcal{N}_b^\sigma$, occasionally we may simply write $\mathcal{A}_b$ for shorthand, or only refer to it as an assignment if the context allows this. Furthermore, for some value assignment $\mathcal{A}_b$, let the derived value assignment of $\mathcal{A}_b$ be the value assignment $\widehat{\mathcal{A}}_b$, given by $\widehat{\mathcal{A}}_b(x^\sigma) = \pi_b^\sigma(\mathcal{A}_b(x^\sigma))$. Let $\mathcal{A}$ be a value assignment in base $b+1$, then we inductively define $\boldsymbol{nval}_{b+1}^{\mathcal{A}}(\cdot)$ as the value interpretation of $\mathrm{T}_b^{\frown}$-terms under assignment $\mathcal{A}$.*

$\boldsymbol{nval}_{b+1}^{\mathcal{A}}(k_n) = \mu_{b+1}(\{n\})$

$\boldsymbol{nval}_{b+1}^{\mathcal{A}}(x^\sigma) = \mathcal{A}(x)$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(M^{\sigma\to\tau}N^{\sigma}) = \boldsymbol{nval}^{\mathcal{A}}_{b+1}(M)[\boldsymbol{nval}^{\mathcal{A}}_{b+1}(N)]$$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(\lambda x^{\sigma}.M^{\tau}) = \textstyle\sum_{i<\|\sigma\|_{b+1}} \boldsymbol{nval}^{\mathcal{A}^x_i}_{b+1}(M)\,\|\tau\|^i_{b+1}$$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(M^{\sigma}|N^{\sigma}) = Merge^{\sigma}_{b+1}(\boldsymbol{nval}^{\mathcal{A}}_{b+1}(M), \boldsymbol{nval}^{\mathcal{A}}_{b+1}(N))$$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(\langle M^{\sigma}, N^{\tau}\rangle) = \theta^{\sigma\times\tau}_{b+1}(\boldsymbol{nval}^{\mathcal{A}}_{b+1}(M), \boldsymbol{nval}^{\mathcal{A}}_{b+1}(N))$$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(fst.M^{\sigma\times\tau}) = \lambda^{\sigma\times\tau}_{b+1}(\boldsymbol{nval}^{\mathcal{A}}_{b+1}(M))$$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(snd.M^{\sigma\times\tau}) = \rho^{\sigma\times\tau}_{b+1}(\boldsymbol{nval}^{\mathcal{A}}_{b+1}(M))$$

$$\boldsymbol{nval}^{\mathcal{A}}_{b+1}(R_{\sigma}(N^{\iota}, F^{\iota,\sigma\to\sigma}, G^{\sigma})) = \Upsilon^{\sigma}_{b+1}(\xi_{b+1}(\boldsymbol{nval}^{\mathcal{A}}_{b+1}(N)), \boldsymbol{nval}^{\mathcal{A}}_{b+1}(F), \boldsymbol{nval}^{\mathcal{A}}_{b+1}(G))$$

Lemma 3.30. *For any* $\mathrm{T}^{\frown}_b$ *term* M^{σ} *and domain assignment* $\mathcal{A}_{b+1}$

$$\pi^{\sigma}_{b+1}([\![M]\!]_{\mathcal{A}}) = \boldsymbol{nval}^{\widehat{\mathcal{A}}}_{b+1}(M)$$

Proof. We prove this by induction on the structure of M.

Case $M = k_n$.

$$\pi^{\iota}_{b+1}([\![k_n]\!]_{\mathcal{A}}) = \pi^{\iota}_{b+1}(\{k_n\}) = \mu_{b+1}(\{k_n\}) = \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(k_n)$$

Case $M = x^{\sigma}$.

$$\pi^{\iota}_{b+1}([\![x]\!]_{\mathcal{A}}) = \pi^{\iota}_{b+1}(\mathcal{A}(x)) = \widehat{\mathcal{A}}(x) = \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(x)$$

Case $M = U^{\sigma\to\tau}V^{\sigma}$.

$$\begin{aligned}\pi^{\tau}_{b+1}([\![UV]\!]_{\mathcal{A}}) &= \pi^{\tau}_{b+1}([\![U]\!]_{\mathcal{A}}[\![V]\!]_{\mathcal{A}}) = \pi^{\sigma\to\tau}_{b+1}([\![U]\!]_{\mathcal{A}})[\pi^{\sigma}_{b+1}([\![V]\!]_{\mathcal{A}})] && \textit{(i)} \text{ in Lemma 3.26}\\ &= \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(U)[\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(V)] = \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(UV) && \text{IH.}\end{aligned}$$

Case $M = \lambda x^{\sigma}.N^{\tau}$. Let $z = \rho^{\sigma}_{b+1}$ and observe first

$$\widehat{\mathcal{A}^x_{z(i)}}(y) = \pi^{\sigma}_{b+1}(\mathcal{A}^x_{z(i)}(y)) = \begin{cases}\pi^{\sigma}_{b+1}(\rho^{\sigma}_{b+1}(i)) & \text{if } x = y\\ \pi^{\sigma}_{b+1}(\mathcal{A}(y)) & \text{else}\end{cases} = \begin{cases} i & \text{if } x = y\\ \widehat{\mathcal{A}}(y) & \text{else}\end{cases} = \widehat{\mathcal{A}}^x_i\,(\dagger)$$

and so

$$\begin{aligned}
\pi^{\sigma\to\tau}_{b+1}(\llbracket \lambda x.N\rrbracket_{\mathcal{A}}) &= \pi^{\sigma\to\tau}_{b+1}(f) && f(j)=\llbracket M\rrbracket_{\mathcal{A}^x_j}\\
&= \sum_{i<\|\sigma\|_{b+1}} \pi^{\tau}_{b+1}(f(\rho^{\sigma}_{b+1}(i)))\,\|\tau\|^i_{b+1} && \pi^{\sigma}_{b+1}\text{ def.}\\
&= \sum_{i<\|\sigma\|_{b+1}} \pi^{\tau}_{b+1}(\llbracket N\rrbracket_{\mathcal{A}^x_{z(i)}})\,\|\tau\|^i_{b+1}\\
&= \sum_{i<\|\sigma\|_{b+1}} \mathbf{nval}^{\widehat{\mathcal{A}^x_{z(i)}}}_{b+1}(N)\,\|\tau\|^i_{b+1} && \text{IH.}\\
&= \sum_{i<\|\sigma\|_{b+1}} \mathbf{nval}^{\widehat{\mathcal{A}}^x_i}_{b+1}(N)\,\|\tau\|^i_{b+1} && (\dagger)\\
&= \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(\lambda x.N) && \mathbf{nval}^{\hat{\mathcal{A}}}_{b+1}(\cdot)\text{ def.}
\end{aligned}$$

Case $M=(U^\sigma|V^\sigma)$.

$$\begin{aligned}
\pi^{\sigma}_{b+1}(\llbracket U|V\rrbracket_{\mathcal{A}}) &= \pi^{\sigma}_{b+1}(\llbracket U\rrbracket_{\mathcal{A}} \vee^{\sigma}_{b+1} \llbracket V\rrbracket_{\mathcal{A}}) && \llbracket\cdot\rrbracket_{\mathcal{A}}\text{ def.}\\
&= Merge^{\sigma}_{b+1}(\pi^{\sigma}_{b+1}(\llbracket U\rrbracket_{\mathcal{A}}),\pi^{\sigma}_{b+1}(\llbracket V\rrbracket_{\mathcal{A}})) && \textit{(ii)}\text{ in Lemma 3.26}\\
&= Merge^{\sigma}_{b+1}(\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(U),\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(V)) && \text{IH.}\\
&= \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(U|V) && \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(\cdot)\text{ def.}
\end{aligned}$$

Case $M=R_\sigma(N^\iota,F^{\iota,\sigma\to\sigma},G^\sigma)$. By induction hypothesis

$$\pi^{\iota}_{b+1}(\llbracket N\rrbracket_{\mathcal{A}}) = \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(N)$$

and by π_{b+1} definition

$$\mu_{b+1}(\llbracket N\rrbracket_{\mathcal{A}}) = \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(N)$$

and by *(ii)* in Lemma 3.20

$$\llbracket N\rrbracket_{\mathcal{A}} = \xi_{b+1}(\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(N)) \qquad (\ddagger)$$

so finally

$$\begin{aligned}
&\pi^{\sigma}_{b+1}(\llbracket R_\sigma(N,F,G)\rrbracket_{\mathcal{A}})\\
&= \pi^{\sigma}_{b+1}(\Psi^{\sigma}_{b+1}(\llbracket N\rrbracket_{\mathcal{A}},\llbracket F\rrbracket_{\mathcal{A}},\llbracket G\rrbracket_{\mathcal{A}})) && \llbracket\cdot\rrbracket_{A}\text{ def.}\\
&= \Upsilon^{\sigma}_{b+1}(\llbracket N\rrbracket_{\mathcal{A}},\pi^{\iota,\sigma\to\sigma}_{b+1}(\llbracket F\rrbracket_{\mathcal{A}}),\pi^{\sigma}_{b+1}(\llbracket G\rrbracket_{\mathcal{A}})) && \textit{(ii)}\text{ in Lemma 3.28}\\
&= \Upsilon^{\sigma}_{b+1}(\llbracket N\rrbracket_{\mathcal{A}},\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(F),\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(G)) && \text{IH.}\\
&= \Upsilon^{\sigma}_{b+1}(\xi_{b+1}(\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(N)),\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(F),\mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(G)) && (\ddagger)\\
&= \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(R_\sigma(N,F,G)) && \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(\cdot)\text{ def.}
\end{aligned}$$

□

Corollary 3.31. *For any* $T_b^{\frown}$*-terms* $M : \sigma$*,* $N : \sigma$ *and assignment* $\mathcal{A}_{b+1}$

(i) $[\![N]\!]_{\mathcal{A}} \sqsubseteq^{\sigma}_{b+1} [\![M]\!]_{\mathcal{A}} \Leftrightarrow \ \boldsymbol{nval}^{\widehat{\mathcal{A}}}_{b+1}(N) \preccurlyeq^{\sigma}_{b+1} \boldsymbol{nval}^{\widehat{\mathcal{A}}}_{b+1}(M)$

(ii) $[\![N]\!]_{\mathcal{A}} = [\![M]\!]_{\mathcal{A}} \Leftrightarrow \ \boldsymbol{nval}^{\widehat{\mathcal{A}}}_{b+1}(N) = \boldsymbol{nval}^{\widehat{\mathcal{A}}}_{b+1}(M)$

(iii) $M \rhd N \Rightarrow \boldsymbol{nval}^{\mathcal{A}}_{b+1}(N) \preccurlyeq^{\sigma}_{b} \boldsymbol{nval}^{\mathcal{A}}_{b+1}(M)$

(iv) $\mu_{b+1}(\{n\}) \preccurlyeq^{\iota}_{b+1} \boldsymbol{nval}_{b+1}(M) \Leftrightarrow \ M \rhd k_n$ *where* $M : \iota$ *is closed*

Proof. *(ii)* follows immediately from *(i)*. We show *(i)* and *(iii)* at once

$$
\begin{aligned}
M \rhd N \Rightarrow\ & [\![N]\!]_{\mathcal{A}} \sqsubseteq^{\sigma}_{b+1} [\![M]\!]_{\mathcal{A}} && \text{(ii) in Lemma 3.12}\\
\Leftrightarrow\ & \pi^{\sigma}_{b+1}([\![N]\!]_{\mathcal{A}}) \preccurlyeq^{\sigma}_{b+1} \pi^{\sigma}_{b+1}([\![M]\!]_{\mathcal{A}}) && \text{(ii) in Lemma 3.23}\\
\Leftrightarrow\ & \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(N) \preccurlyeq^{\sigma}_{b+1} \mathbf{nval}^{\widehat{\mathcal{A}}}_{b+1}(M) && \text{lemma 3.30}
\end{aligned}
$$

For *(iv)*

$$
\begin{aligned}
M \rhd k_n \Leftrightarrow\ & n \in [\![M]\!] && \text{(ii) in Lemma 3.12}\\
\Leftrightarrow\ & \{n\} \subseteq [\![M]\!] && \\
\Leftrightarrow\ & \{n\} \sqsubseteq^{\iota}_{b+1} [\![M]\!] && \sqsubseteq^{\iota}_{b+1} \text{ def.}\\
\Leftrightarrow\ & \pi^{\iota}_{b+1}(\{n\}) \preccurlyeq^{\iota}_{b+1} \pi^{\iota}_{b+1}([\![M]\!]) && \text{(ii) in Lemma 3.23}\\
\Leftrightarrow\ & \mu_{b+1}(\{n\}) \preccurlyeq^{\iota}_{b+1} \pi^{\iota}_{b+1}([\![M]\!]) && \pi_{b+1} \text{ def.}\\
\Leftrightarrow\ & \mu_{b+1}(\{n\}) \preccurlyeq^{\iota}_{b+1} \mathbf{nval}_{b+1}(M) && \text{Lemma 3.30}
\end{aligned}
$$

□

Chapter 4

Successor free computation in $\mathbf{T}^-$

4.1 Interpreting $\mathbf{T}_b^-$ in $\mathbb{N}_{b+1}^\sigma$

Definition 4.1. *For all types σ and $b > 0$ we define $|\sigma|_b$ as the deterministic cardinality in base b by (i) $|\iota|_b = b$ (ii) $|\sigma\times\tau|_b = |\sigma|_b\times|\tau|_b$ (iii) $|\sigma \to \tau|_b = |\tau|_b^{|\sigma|_b}$. For all types σ and $b > 0$ we define $\mathbb{N}_b = \{0, ..., b-1\}$ and $\mathbb{N}_b^\sigma = \mathbb{N}_{|\sigma|_b}$.*

Lemma 4.2. *For any number $a \in \mathbb{N}_b^\sigma$ where*

(i) $\sigma = \rho \times \tau$, there are unique numbers $d_1 \in \mathbb{N}_b^\rho, d_0 \in \mathbb{N}_b^\tau$ such that

$$a = d_0 + d_1|\tau|_b$$

(ii) $\sigma = \rho \to \tau$, there are unique numbers $d_0, ..., d_\ell \in \mathbb{N}_b^\tau$ where $\ell = |\rho|_b - 1$ such that

$$a = d_0 + d_1|\tau|_b + ... + d_\ell|\tau|_b^\ell$$

Proof. We omit *(i)* since the proof is analogous to *(ii)*.

Given any $\sigma = \rho \to \tau$ and $b > 0$, we define we define the map

$$\phi_b^\sigma : \underbrace{\mathbb{N}_b^\tau \times .. \times \mathbb{N}_b^\tau}_{\ell+1} \to \mathbb{N}_b^\sigma$$

such that

$$\phi_b^\sigma(d_0, ..., d_\ell) = d_0 + d_1|\tau|_b + ... + d_\ell|\tau|_b^\ell$$

If ϕ_b^σ is surjective, then there are indeed such numbers $d_0, ..., d_\ell \in \mathbb{N}_b^\tau$ for any $a \in \mathbb{N}_b^\sigma$, and if it is injective then these numbers are unique for each $a \in \mathbb{N}_b^\sigma$, so demonstrating both is sufficient to prove *(ii)*.

We first show injectivity, so let $d_0, ..., d_\ell \in \mathbb{N}_b^\tau$ and $d_0^*, ..., d_\ell^* \in \mathbb{N}_b^\tau$ be two distinct sequences, and let $i = \max\{j | d_j \neq d_j^*\}$ be the greatest coordinate for which they disagree. Assume without loss off generality that $d_i > d_i^*$[1]. When $i = 0$

$$\phi_b^\sigma(d_0, ..., d_j) - \phi_b^\sigma(d_0^*, ..., d_j^*) = d_0 - d_0^* > 0$$

so we are done in this case. When $i > 0$

$$\begin{aligned}\phi_b^\sigma(d_0, ..., d_j) - \phi_b^\sigma(d_0^*, ..., d_j^*) &= (d_0 - d_0^*) + ... + (d_i - d_i^*)|\tau|_b^i \\ &= X + Y|\tau|_b^i\end{aligned}$$

where $X = (d_0 - d_0^*) + ... + (d_{i-1} - d_{i-1}^*)$ and $Y = (d_i - d_i^*)$. We require

$$|X| < |\tau|_b^i \qquad \textbf{(Claim)}$$

and under this claim assume first that $X < 0$, then

$$|X| < |\tau|_b^i \Rightarrow -X < |\tau|_b^i \quad \Rightarrow -X < Y|\tau|_b^i \Rightarrow X > -Y|\tau|_b^i \quad \Rightarrow X + Y|\tau|_b^i > 0$$

and when $X \geq 0$ we also obviously have $X+Y|\tau|_b^i > 0$ as well, so $\phi_b^\sigma(d_0, ..., d_j) \neq \phi_b^\sigma(d_0^*, ..., d_j^*)$ as desired. Finally we prove the required claim[2]

$$\begin{aligned}|X| &= |(d_0 - d_0^*) + ... + (d_{i-1} - d_{i-1}^*)|\tau|_b^{i-1} \\ &\leq |(d_0 - d_0^*)| + ... + |(d_{i-1} - d_{i-1}^*)||\tau|_b^{i-1} && \text{triangle inequality} \\ &< |\tau|_b + ... + |\tau|_b^i && d_j, d_0^* < |\tau|_b \\ &= \frac{|\tau|_b^{i+1} - |\tau|_b}{|\tau|_b - 1} && \text{sum of geometric series} \\ &= \frac{|\tau|_b^i(|\tau|_b - 1)}{|\tau|_b - 1} = |\tau|_b^i\end{aligned}$$

Moving on to surjectivity. Given any $a < |\sigma|_b - 1$ (†) where

$$\phi_b^\sigma(d_0, ..., d_\ell) = a$$

we exhibit $d_0^*, ..., d_\ell^* \in \mathbb{N}_b^\tau$ such that

$$\phi_b^\sigma(d_0^*, ..., d_\ell^*) = a + 1(\ddagger)$$

which together with $\phi_b^\sigma(0, ..., 0) = 0 \in \mathbb{N}_b^\sigma$ demonstrates that ϕ_b^σ is onto $\mathbb{N}_b^\sigma$.

So let $d_0, ..., d_\ell \in \mathbb{N}_b^\tau$ be such that $\phi_b^\sigma(d_0^*, ..., d_\ell^*) = a$. Consider first when $d_0 < |\tau|_b - 1$. It should be easy to convince oneself that the following sequence satisfies (‡)

$$d_j^* = \begin{cases} d_0 + 1 & \text{if } j = 0 \\ d_j & \text{otherwise} \end{cases}$$

[1] The opposite subtraction could considered if $d_i < d_i^*$

[2] Recall that $r + ... + r^i = \frac{r^{i+1} - r}{r-1}$

Suppose now $d_0 = |\tau|_b - 1$, and Let $i = \max\{j | d_0, ..., d_j = |\tau|_b - 1\}$. Notice that $i < \ell$, otherwise $a = |\sigma|_b - 1$ which contradicts (†). Observe the sequence

$$d_j^* = \begin{cases} 0 & \text{if } j < i+1 \\ d_j + 1 & \text{if } j = i+1 \\ d_j & \text{if } j > i+1 \end{cases}$$

and by sum of a geometric series

$$|\tau|_b^{i+1} = (|\tau|_b - 1)(1 + ... + |\tau|_b^i) + 1 = (|\tau|_b - 1) + ... + (|\tau|_b - 1)|\tau|_b^i + 1 = d_0 + ... + d_i|\tau|_b^i + 1$$

which we use in confirming

$$\begin{aligned} \phi_b^\sigma(d_0^*, ..., d_\ell^*) &= \phi_b^\sigma(0, ..., 0, d_{i+1} + 1, ..., d_\ell) \\ &= (d_{i+1} + 1)|\tau|_b^{i+1} + ... + d_\ell|\tau|_b^\ell \\ &= |\tau|_b^{i+1} + d_{i+1}|\tau|_b^{i+1} + ... + d_\ell|\tau|_b^\ell \\ &= \left\{d_0 + ... + d_i|\tau|_b^i + 1\right\} + d_{i+1}|\tau|_b{}^{i+1} + ... + d_\ell|\tau|_b{}^\ell \\ &= \left\{d_0 + ... + d_i|\tau|_b{}^i + d_{i+1}|\tau|_b{}^{i+1} + ... + d_\ell|\tau|_b{}^\ell\right\} + 1 \\ &= \phi_b^\sigma(d_0, ..., d_\ell) + 1 \\ &= a + 1 \end{aligned}$$

□

For any $a = d_0 + d_1|\tau|_b + ... + d_\ell|\tau|_b^\ell \in \mathbb{N}_b^{\sigma\to\tau}$ and $b \in \mathbb{N}_b^\sigma$, let $a[b]$ denote $d_b \in \mathbb{N}_b^\tau$ as before and $a[b_1, ..., b_n] = a[b_1][b_2, ..., b_n]$ for appropriate $b_1, ..., b_n$. It will always be clear from the context whether we are referring to T^- or $\mathrm{T}^\backsim$ when using this shorthand.

Definition 4.3. *We define a deterministic value assignment $\mathcal{A}$ in base b as a total map from $\mathcal{V}$ into $\bigcup_\sigma \mathbb{N}_b^\sigma$ such that $\mathcal{A}(x^\sigma) \in \mathbb{N}_b^\sigma$, occasionally we may simply write $\mathcal{A}_b$ for shorthand, or only refer to it as an assignment if the context allows this. Let $\mathcal{A}$ be a value assignment in base $b+1$, then we inductively define $\boldsymbol{val}_{b+1}^{\mathcal{A}}(\cdot)$ as the value interpretation of $\mathrm{T}_b^\backsim$-terms under value assignment $\mathcal{A}$.*

$\boldsymbol{val}_{b+1}^{\mathcal{A}}(k_n) = n$

$\boldsymbol{val}_{b+1}^{\mathcal{A}}(x^\sigma) = \mathcal{A}(x)$

$\boldsymbol{val}_{b+1}^{\mathcal{A}}(M^{\sigma\to\tau}N^\sigma) = \boldsymbol{val}_{b+1}^{\mathcal{A}}(M)[\boldsymbol{val}_{b+1}^{\mathcal{A}}(N)]$

$\boldsymbol{val}_{b+1}^{\mathcal{A}}(\lambda x^\sigma.M^\tau) = \sum_{i<|\sigma|_{b+1}} \boldsymbol{val}_{b+1}^{\mathcal{A}_i^x}(M)|\tau|_{b+1}^i$

$\boldsymbol{val}_{b+1}^{\mathcal{A}}(\langle M^\sigma, N^\tau\rangle) = \boldsymbol{val}_{b+1}^{\mathcal{A}}(M)|\tau|_{b+1} + \boldsymbol{val}_{b+1}^{\mathcal{A}}(N)$

$\boldsymbol{val}_{b+1}^{\mathcal{A}}(fst.M^{\sigma\times\tau}) = \boldsymbol{val}_{b+1}^{\mathcal{A}}(M)\ div\ |\tau|_{b+1}$

$\boldsymbol{val}^{\mathcal{A}}_{b+1}(snd.M^{\sigma\times\tau}) = \boldsymbol{val}^{\mathcal{A}}_{b+1}(M) \mod |\tau|_{b+1}$

$\boldsymbol{val}^{\mathcal{A}}_{b+1}(R_\sigma(N^\iota, F^{\iota,\sigma\to\sigma}, G^\sigma)) = f(\boldsymbol{val}^{\mathcal{A}}_{b+1}(N))$ *where* $f(0) = \boldsymbol{val}^{\mathcal{A}}_{b+1}(G)$ *and* $f(i+1) = \boldsymbol{val}^{\mathcal{A}}_{b+1}(F)[i, f(i)]$

Lemma 4.4. *Let $M : \sigma$ be a T^-_b term and $\mathcal{A}_{b+1}$ be an assignment*

(i) $\boldsymbol{val}^{\mathcal{A}}_{b+1}(M) \in \mathbb{N}^\sigma_{b+1}$

(ii) $M \rhd N \Rightarrow \boldsymbol{val}^{\mathcal{A}}_{b+1}(M) = \boldsymbol{val}^{\mathcal{A}}_{b+1}(N)$

(iii) $\boldsymbol{val}^{\mathcal{A}}_{b+1}(M) = n \Leftrightarrow M \rhd k_n$

Proof. See [1] Lemma 9. □

4.2 Computation with $\mathbf{val}^{\mathcal{A}}_{b+1}$

Lemma 4.5. *For all types σ there exists a closed T^--terms*

(i) $\boldsymbol{Cond}_\sigma : \iota, \sigma, \sigma \to \sigma$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Cond}_\sigma(C_1, F, G)) = \begin{cases} \boldsymbol{val}_{b+1}(F) & \boldsymbol{val}_{b+1}(C_1) = 0 \\ \boldsymbol{val}_{b+1}(G) & \textit{else} \end{cases}$$

(ii) $\boldsymbol{Or}_\sigma : \iota, \iota, \sigma, \sigma \to \sigma$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Or}_\sigma(C_1, C_2, F, G)) = \begin{cases} \boldsymbol{val}_{b+1}(F) & \boldsymbol{val}_{b+1}(C_1) = 0 \textit{ or } \boldsymbol{val}_{b+1}(C_2) = 0 \\ \boldsymbol{val}_{b+1}(G) & \textit{else} \end{cases}$$

(iii) $\boldsymbol{And}_\sigma : \iota, \iota, \sigma, \sigma \to \sigma$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{And}_\sigma(C_1, C_2, F, G)) = \begin{cases} \boldsymbol{val}_{b+1}(F) & \boldsymbol{val}_{b+1}(C_1) = 0 \textit{ and } \boldsymbol{val}_{b+1}(C_2) = 0 \\ \boldsymbol{val}_{b+1}(G) & \textit{else} \end{cases}$$

for all T^-_b terms F,G,C_1 and C_2, more over all have have recursor rank of zero.

Proof. See [1] Lemma 7 for $\mathbf{Cond}_\sigma$ term. For all σ let

$$\mathbf{Or} = \lambda C_1^\iota C_2^\iota F^\sigma G^\sigma.\mathbf{Cond}_\sigma(C_1, F, \mathbf{Cond}_\sigma(C_2, F, G))$$

and

$$\mathbf{And} = \lambda C_1^\iota C_2^\iota F^\sigma G^\sigma.\mathbf{Cond}_\sigma(C_1, \mathbf{Cond}_\sigma(C_2, F, G), G)$$

which are obviously correct and of the required rank. □

Lemma 4.6. *For all types σ,τ there exists a T^- term $\boldsymbol{It}^\tau_\sigma : (\iota, \sigma \to \sigma, \sigma) \to \sigma$ such that*

$$\boldsymbol{val}_{b+1}(\boldsymbol{It}^\tau_\sigma(k_b, F, G)) = g^{|\sigma|_{b+1}}(\boldsymbol{val}_{b+1}(G))$$

where $g(x) = \boldsymbol{val}_{b+1}(F)[x]$, for all appropriately typed closed T^- terms F and G, moreover $Rk(\boldsymbol{It}^\tau_\sigma) = lv(\sigma) + lv(\tau)$.

Proof. By [1] Lemma 8 we know that for all σ there exists a term $\mathbf{It}_\sigma^\tau : (\iota, \sigma \to \sigma, \sigma) \to \sigma$ of the mentioned rank such that

$$\mathbf{It}_\sigma^\tau(k_b, F, G) \to F^{|\sigma|_{b+1}} G$$

this combined with *(ii)* in Lemma 4.4 gives that

$$\mathbf{val}_{b+1}(\mathbf{It}_\sigma^\tau(k_b, F, G)) = \mathbf{val}_{b+1}(F^{|\sigma|_{b+1}} G)$$

which immediately give the desired result by simple iterated evaluation following $\mathbf{val}_{b+1}$. ☐

Lemma 4.7. *For all types σ there exists* T^- *terms*

(i) $\boldsymbol{0}_\sigma : \iota$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{0}_\sigma) = 0$

(ii) $\boldsymbol{Leq}_\sigma : \iota, \sigma, \sigma \to \iota$ *where* $\boldsymbol{Leq}_\sigma(k_b, F, G) \rhd k_0$ *iff* $\boldsymbol{val}_{b+1}(F) \le \boldsymbol{val}_{b+1}(G)$

(iii) $\boldsymbol{Eq}_\sigma : \iota, \sigma, \sigma \to \iota$ *where* $\boldsymbol{Eq}_\sigma(k_b, F, G) \rhd k_0$ *iff* $\boldsymbol{val}_{b+1}(F) = \boldsymbol{val}_{b+1}(G)$

(iv) $\boldsymbol{Less}_\sigma : \iota, \sigma, \sigma \to \iota$ *where* $\boldsymbol{Less}_\sigma(k_b, F, G) \rhd k_0$ *iff* $\boldsymbol{val}_{b+1}(F) < \boldsymbol{val}_{b+1}(G)$

(v) $\boldsymbol{Succ}_\sigma : \iota, \sigma \to \sigma$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Succ}_\sigma(k_b, F)) = \boldsymbol{val}_{b+1}(F) + 1 \mod |\sigma|_{b+1}$

(vi) $\boldsymbol{Pred}_\sigma : \iota, \sigma \to \sigma$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Pred}_\sigma(k_b, F)) = \boldsymbol{val}_{b+1}(F) \dot{-} 1$

for all appropriately typed T_b^- *terms F and G. Moreover, all terms have recursor rank less then* $2lv(\sigma) \dot{-} 2$.

Proof. See [1] Lemma 10 for *(i)-(iii)* and *(v)*. Also it is obvious that *(iv)* can be implemented using the others, and *(vi)* has a construction completely analogous to *(v)*. ☐

This next term is analogous to for-loops in imperative languages, in the sense that the you can specify the precise number of iterations $\mathbf{val}_{b+1}(V)$ you want of a supplied procedure U, and in each iteration the procedure U will be given an iteration number.

Lemma 4.8. *For all types σ and τ there exists a closed* T^- *term* $\boldsymbol{Rep}_\sigma^\tau : (\iota, \sigma \times \tau \to \tau, \sigma, \tau) \to \tau$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Rep}_\sigma^\tau(k_b, U, V, W)) = f(\boldsymbol{val}_{b+1}(V))$$

for any appropriately typed closed T_b^-*-terms U, V, W where* $f(0) = \boldsymbol{val}_{b+1}(W)$ *and* $f(i+1) = \boldsymbol{val}_{b+1}(U)[i|\tau|_{b+1} + f(i)]$*, moreover* $Rk(\boldsymbol{Rep}_\sigma^\tau) = \max(lv(\tau) + lv(\sigma), 2lv(\sigma))$.

Proof. Let

$$\mathbf{Rep}_\sigma^\tau = \lambda b^\iota U^{\sigma\times\tau\to\tau} V^\sigma W^\tau . snd.\mathbf{It}_\sigma^{\sigma\times\tau}(b, D(b, U, V), \langle \mathbf{0}_\sigma, W\rangle)$$

where

$$D = \lambda b^{\iota} U^{\sigma\times\tau\to\tau} V^{\sigma} T^{\sigma\times\tau}.\mathbf{Cond}_{\sigma\times\tau}(\mathbf{Less}_{\sigma}(b, fst.T, V), \langle \mathbf{Succ}_{\sigma}(b, fst.T), U(T)\rangle, T)$$

Fix closed T_b^--terms U, V, W to be terms of appropriate type, and let $u = \mathbf{val}_{b+1}(U)$, $v = \mathbf{val}_{b+1}(V)$ and $w = \mathbf{val}_{b+1}(W)$. Then

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{Rep}^{\tau}_{\sigma}(k_b, U, V, W)) &= \mathbf{val}_{b+1}(snd.\mathbf{It}^{\sigma\times\tau}_{\sigma}(k_b, D(k_b, U, V), \langle \mathbf{0}_{\sigma}, W\rangle))\\ &= \mathbf{val}_{b+1}(\mathbf{It}^{\sigma\times\tau}_{\sigma}(k_b, D(k_b, U, V), \langle \mathbf{0}_{\sigma}, W\rangle)) \bmod |\tau|_{b+1}\\ &= g^{|\sigma|_{b+1}}(\mathbf{val}_{b+1}(\langle \mathbf{0}_{\sigma}, W\rangle)) \bmod |\tau|_{b+1} \ (\flat)\end{aligned}$$

where g is the function from lemma 4.6, and

$$\begin{aligned}g(x) &= \mathbf{val}_{b+1}(D(k_b, U, V))[x]\\ &= \begin{cases}\{(x \text{ div } |\tau|_{b+1}) + 1\} \times |\tau|_{b+1} + u[x] & \text{if } x \text{ div } |\tau|_{b+1} < v\\ x & \text{else}\end{cases}\end{aligned}$$

by applying the definition of $\mathbf{val}_{b+1}$. By iteration we see

$$g^{i+1}(w) = \begin{cases}(i+1)|\tau|_{b+1} + u[g^i(w)] & \text{if } i+1 < v\\ g^v(w) & \text{else}\end{cases}$$

By induction on i we show that

$$g^{i}(w) = \begin{cases} i|\tau|_{b+1} + f(i) & \text{if } i < v\\ v|\tau|_{b+1} + f(v) & \text{else}\end{cases} \quad (\dagger)$$

so finally

$$\begin{aligned}\mathbf{val}_{b}(\mathbf{Rep}^{\tau}_{\sigma}(k_b, U, V, W)) &= g^{|\sigma|_{b+1}}(\mathbf{val}_{b+1}(\langle \mathbf{0}_{\sigma}, W\rangle)) \bmod |\tau|_{b+1}\\ &= g^{|\sigma|_{b+1}}(w) \quad \bmod |\tau|_{b+1}\\ &= (v|\tau|_{b+1} + f(v)) \quad \bmod |\tau|_{b+1}\\ &= f(v)\\ &= f(\mathbf{val}_{b+1}(V))\end{aligned}$$

For the rank

$$\begin{aligned}Rk(\mathbf{Rep}^{\tau}_{\sigma}) &= \max(Rk(\mathbf{It}^{\sigma\times\tau}_{\sigma}), Rk(\mathbf{Less}_{\sigma}), Rk(\mathbf{Succ}_{\sigma}))\\ &\le \max(lv(\sigma\times\tau) + lv(\sigma), 2lv(\sigma) \dot{-} 2)\\ &= \max(lv(\tau) + lv(\sigma), 2lv(\sigma), 2lv(\sigma) \dot{-} 2)\\ &= \max(lv(\tau) + lv(\sigma), 2lv(\sigma))\end{aligned}$$

□

This next term is used to convert quantities between different types, it can be thought of as rewriting a number into a number system with a different base. It is also the last proof where we compute $\mathbf{val}_b^{\mathcal{A}}(\cdot)$ and compute recursor rank at this level of detail.

Lemma 4.9. *For any* σ, τ *and* $b > 0$ *there exists a closed* T^- *term* $\boldsymbol{V}_\sigma^\tau : \iota, \sigma \to \tau$ *where*

(i) $\boldsymbol{val}_{b+1}(\boldsymbol{V}_\sigma^\tau(k_b, F)) = \boldsymbol{val}_{b+1}(F) \ mod\ |\tau|_b$

(ii) $\boldsymbol{val}_{b+1}(\boldsymbol{V}_\sigma^\tau(k_b, F)) = \boldsymbol{val}_{b+1}(F)$ *when* $|\sigma|_x \leq |\tau|_x$ *for all* x

for all closed T_b^-*-terms* F *of appropriate type, more over* $Rk(\boldsymbol{V}_\sigma^\tau) \leq \max(lv(\tau) + lv(\sigma), 2lv(\sigma), 2lv(\tau) \dot{-} 2)$.

Proof. For case *(i)* let

$$\mathbf{V}_\sigma^\tau = \lambda b^\iota F^\sigma.\mathbf{Rep}_\sigma^\tau(b, \lambda x^{\sigma\times\tau}.\mathbf{Succ}_\tau(b, snd.x), F, \mathbf{0}_\tau)$$

Fix a closed T_b^- term F of appropriate type, we get

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{V}_\sigma^\tau(k_b, F)) &= \mathbf{val}_{b+1}(\mathbf{Rep}_\sigma^\tau(k_b, \lambda x^{\sigma\times\tau}.\mathbf{Succ}_\tau(b, snd.x), F, \mathbf{0}_\tau))\\ &= f(\mathbf{val}_{b+1}(F))(\clubsuit)\end{aligned}$$

where f is from lemma 4.8, so $f(0) = \mathbf{val}_{b+1}(\mathbf{0}_\tau) = 0$ and

$$\begin{aligned}
f(i+1) &= \mathbf{val}_{b+1}(\lambda x^{\sigma\times\tau}.\mathbf{Succ}_\tau(k_b, snd.x))[r] && r = i|\tau|_{b+1} + f(i)\\
&= \left\{\sum_{j<|\sigma\times\tau|_{b+1}} \mathbf{val}_{b+1}^{A_j^x}(\mathbf{Succ}_\tau(k_b, snd.x))|\tau|_{b+1}^j\right\}[r] && \mathbf{val}_{b+1}(\cdot)\\
&= \mathbf{val}_{b+1}^{A_r^x}(\mathbf{Succ}_\tau(k_b, snd.x)) && \delta_{b+1}^{\sigma\times\tau\to\tau}\\
&= \left\{\mathbf{val}_{b+1}^{A_r^x}(snd.x) + 1\right\} \mod |\tau|_{b+1} && \textit{(ii)} \text{ in lemma 4.7}\\
&= \{(r \mod |\tau|_{b+1}) + 1\} \mod |\tau|_{b+1}\\
&= \{(\{i|\tau|_{b+1} + f(i)\} \mod |\tau|_{b+1}) + 1\} \mod |\tau|_{b+1}\\
&= \{f(i) + 1\} \mod |\tau|_{b+1} && f(i) < |\tau|_{b+1}
\end{aligned}$$

We can iterate this to get $f(i) = i \bmod |\tau|_{b+1}$, so

$$f(\mathbf{val}_{b+1}(F)) = \mathbf{val}_{b+1}(F) \bmod |\tau|_b$$

which extends ($\clubsuit$) to give us the desired result. When $|\sigma|_b \leq |\tau|_b$ for all b then mod $|\tau|_b$ falls away since $\mathbf{val}_{b+1}(F) < |\sigma|_b$, giving us *(ii)*. For the recursor rank

$$\begin{aligned}Rk(\mathbf{V}_\sigma^\tau) &= \max(Rk(\mathbf{Rep}_\sigma^\tau), Rk(\mathbf{Succ}_\tau))\\ &\leq \max(\max(lv(\tau) + lv(\sigma), 2lv(\sigma)), 2lv(\tau) \dot{-} 2)\\ &= \max(lv(\tau) + lv(\sigma), 2lv(\sigma), 2lv(\tau) \dot{-} 2)\end{aligned}$$

□

Lemma 4.10. *For all types σ there exists closed T^- terms*

(i) $\boldsymbol{Add}_\sigma : \iota, \sigma, \sigma \to (\sigma \times \sigma)$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Add}_\sigma(k_b, F, G)) = \boldsymbol{val}_{b+1}(F) + \boldsymbol{val}_{b+1}(G)$

(ii) $\boldsymbol{Sub}_\sigma : \iota, \sigma, \sigma \to \sigma$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Sub}_\sigma(k_b, F, G)) = \boldsymbol{val}_{b+1}(F) \dot{-} \boldsymbol{val}_{b+1}(G)$

(iii) $\boldsymbol{Prod}^\tau_\sigma : \iota, \sigma, \tau \to (\sigma \times \tau)$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Prod}_\sigma(k_b, F, G)) = \boldsymbol{val}_{b+1}(F) \times \boldsymbol{val}_{b+1}(G)$

(iv) $\boldsymbol{Exp}^\sigma_\tau : \iota, \tau, \sigma \to (\sigma \to \tau)$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Exp}_\sigma(k_b, F, G)) = \boldsymbol{val}_{b+1}(F)^{\boldsymbol{val}_{b+1}(G)}$

(v) $\boldsymbol{Div}_\sigma : \iota, \sigma, \sigma \to \sigma$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Div}_\sigma(k_b, F, G)) = \boldsymbol{val}_{b+1}(F)$ *div* $\boldsymbol{val}_{b+1}(G)$

(vi) $\boldsymbol{Mod}_\sigma : \iota, \sigma, \sigma \to \sigma$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Mod}_\sigma(k_b, F, G)) = \boldsymbol{val}_{b+1}(F) \mod \boldsymbol{val}_{b+1}(G)$

for all closed T^-_b-terms F and G of appropriate type, more over they are of recursor rank no greater then $2lv(\sigma)$, except (iii) and (iv) which have recursor rank no greater then $2lv(\sigma \times \tau)$ and $2lv(\sigma \to \tau)$ respectively.

Proof. For natural numbers we can do addition by repeated incrementing, multiplication by repeated addition, and exponentiation by repeated multiplication.

So for any σ let

$$\mathbf{Add}_\sigma = \lambda b^\iota F^\sigma G^\sigma.\mathbf{Rep}^{\sigma\times\sigma}_\sigma(b, \lambda x^{\sigma\times(\sigma\times\sigma)}.\mathbf{Succ}_{\sigma\times\sigma}(b, snd.x), G, \langle \mathbf{0}_\sigma, F\rangle)$$
$$\mathbf{Sub}_\sigma = \lambda b^\iota F^\sigma G^\sigma.\mathbf{Rep}^{\sigma}_\sigma(b, \lambda x^{\sigma\times\sigma}.\mathbf{Pred}_{\sigma}(b, snd.x), G, F)$$

Recognize that both $\mathbf{Add}_\sigma, \mathbf{Sub}_\sigma$ are very similar to $\mathbf{V}^\tau_\sigma$, and their correctness can indeed be demonstrated the same way, meanwhile observe that

$$\begin{aligned} Rk(\mathbf{Add}_\sigma) &= \max(Rk(\mathbf{Rep}^{\sigma\times\sigma}_\sigma), Rk(\mathbf{Succ}_{\sigma\times\sigma})) \\ &\leq \max(\max(lv(\sigma\times\sigma) + lv(\sigma), 2lv(\sigma)), 2lv(\sigma\times\sigma)\dot{-}2) \\ &= \max(lv(\sigma\times\sigma) + lv(\sigma), 2lv(\sigma), 2lv(\sigma\times\sigma)\dot{-}2) \\ &= \max(lv(\sigma) + lv(\sigma), 2lv(\sigma)) \\ &= 2lv(\sigma) \end{aligned}$$

and a similar calculation also yields $Rk(\mathbf{Sub}_\sigma) \leq 2lv(\sigma)$.

For any σ, τ let $\rho = \sigma \times \tau$ and

$$\mathbf{Prod}^\tau_\sigma = \lambda b^\iota F^\sigma G^\tau.\mathbf{Rep}^\rho_\sigma(b, \lambda x^{\sigma\times\rho}.\mathbf{V}^\rho_{\rho\times\rho}(b, \mathbf{Add}_\rho(b, \langle \mathbf{0}_\sigma, G\rangle, snd.x)), F, \mathbf{0}_\rho)$$

We omit correctness proof since $\mathbf{Exp}^\sigma_\tau$ is similar but more complicated, and a quick calculation shows that $Rk(\mathbf{Prod}^\tau_\sigma) \leq 2lv(\sigma \times \tau)$.

For any σ, τ let $\rho = \sigma \to \tau$ and

$$\begin{aligned} \mathbf{Exp}^\sigma_\tau = &\lambda b^\iota F^\tau G^\sigma. \\ &\mathbf{Rep}^\rho_\sigma(b, \lambda x^{\sigma\times\rho}.\mathbf{V}^\rho_{\rho\times\rho}(b, \mathbf{Prod}^\rho_\rho(b, \mathbf{V}^\rho_\tau(F), snd.x)), G, \mathbf{Succ}_\rho(b, \mathbf{0}_\rho)) \end{aligned}$$

Fix closed T_b^- terms F and G to be terms of appropriate type, then as before for

$$\mathbf{val}_{b+1}(\mathbf{Exp}^\sigma_\tau(k_b, F, G)) = f(\mathbf{val}_{b+1}(G))(\clubsuit)$$

where $f(0) = \mathbf{val}_{b+1}(\mathbf{Succ}_\rho(k_b, \mathbf{0}_\rho)) = 1$ and

$$f(i+1) = \mathbf{val}_{b+1}(F) \times f(i) \mod |\rho|_{b+1}$$

Observe that $f(i) < \{|\tau|_{b+1}\}^i$, so

$$f(i) < |\tau|_{b+1}^{|\sigma|_{b+1}} = |\rho|_{b+1} \text{ for all } i < |\sigma|_{b+1} (\dagger)$$

therefore

$$f(i+1) = \mathbf{val}_{b+1}(F) \times f(i) \text{ for all } i+1 < |\sigma|_{b+1}$$

iterated to

$$f(i) = \mathbf{val}_{b+1}(F)^i \text{ for all } i < |\sigma|_{b+1}$$

giving $f(\mathbf{val}_{b+1}(G)) = \mathbf{val}_{b+1}(F)^{\mathbf{val}_{b+1}(G)}$ which extends ($\clubsuit$) as desired.

For the recursor rank

$$\begin{aligned}
Rk(\mathbf{Exp}^\tau_\sigma) &= \max(Rk(\mathbf{Prod}^\rho_\rho), Rk(\mathbf{Rep}^\rho_\sigma), Rk(\mathbf{V}^\rho_{\rho\times\rho}), Rk(\mathbf{V}^\rho_\tau), Rk(\mathbf{Succ}_\rho)) \\
&\leq \max(2lv(\rho \times \rho), lv(\rho) + lv(\sigma), 2lv(\sigma), \\
&\quad lv(\rho) + lv(\rho \times \rho), 2lv(\rho \times \rho), 2lv(\rho) \dot{-} 2, lv(\rho) + lv(\tau), 2lv(\tau) \dot{-} 2) \\
&= \max(2lv(\rho), lv(\rho) + lv(\sigma), 2lv(\sigma), 2lv(\rho) \dot{-} 2, lv(\rho) + lv(\tau), 2lv(\tau) \dot{-} 2) \\
&= 2lv(\rho) \\
&= 2lv(\sigma \to \tau)
\end{aligned}$$

Integer division can be implemented as how many times the divisor can be subtracted from the dividend. So for any σ let

$$\begin{aligned}
&\mathbf{Div}_\sigma = \lambda b^\iota F^\sigma G^\sigma . \mathbf{Rep}^\sigma_\sigma(b, \lambda x^{\sigma\times\sigma}. \\
&\mathbf{Cond}_{\sigma\times\sigma}(\mathbf{Leq}_{\sigma\times\sigma}(b, \mathbf{Prod}^\sigma_\sigma(b, G, \mathbf{Succ}_\sigma(k_b, fst.x)), \mathbf{V}^\sigma_{\sigma\times\sigma}(F))), fst.x, snd.x), F, \mathbf{0}_\sigma)
\end{aligned}$$

As before $\mathbf{val}_{b+1}(\mathbf{Div}_\sigma(k_b, F, G)) = f(\mathbf{val}_{b+1}(F))$ where $f(0) = \mathbf{val}_{b+1}(\mathbf{0}_\sigma) = 0$ and

$$f(i+1) = \begin{cases} (i+1) & \mathbf{val}_{b+1}(G) \times (i+1) \leq \mathbf{val}_{b+1}(F) \\ f(i) & \text{else} \end{cases}$$

We see that $f(i+1)$ is the greatest $j \leq i+1$ such that $\mathbf{val}_{b+1}(G) \times j \leq \mathbf{val}_{b+1}(F)$, and so for $f(\mathbf{val}_{b+1}(F))$ the j becomes the number of times $\mathbf{val}_{b+1}(G)$ goes in $\mathbf{val}_{b+1}(F)$ which is indeed $\mathbf{val}_{b+1}(F)$ div $\mathbf{val}_{b+1}(G)$. We also see

$$\begin{aligned}
Rk(\mathbf{Div}_\sigma) &= \max(Rk(\mathbf{Prod}^\sigma_\sigma), Rk(\mathbf{Rep}^\sigma_\sigma), Rk(\mathbf{Leq}_{\sigma\times\sigma}), Rk(\mathbf{V}^\sigma_{\sigma\times\sigma}), Rk(\mathbf{Succ}_\sigma)) \\
&\leq \max(2lv(\sigma \times \sigma), 2lv(\sigma), lv(\sigma \times \sigma) + lv(\sigma), 2lv(\sigma \times \sigma) \dot{-} 2, 2lv(\sigma) \dot{-} 2) \\
&= 2lv(\sigma)
\end{aligned}$$

From elementary number theory we know that any $n \geq m$ can be written as $n = q \times m + r$ for some $r < m$, where $q = n$ div m and $r = n$ mod m. We use this to construct *(vi)* by observing

$$n \text{ mod } m = r = n - q \times m = n - (n \text{ div } m) \times m$$

so for any σ let

$$\mathbf{Mod}_\sigma = \lambda b^\iota F^\sigma G^\sigma . \mathbf{Sub}_\sigma(b, F, \mathbf{V}^\sigma_{\sigma\times\sigma}(\mathbf{Prod}^\sigma_\sigma(b, G, \mathbf{Div}_\sigma(b, F, G))))$$

which is a direct implementation of the identity.

For the recursor rank

$$\begin{aligned}
Rk(\mathbf{Mod}_\sigma) &= \max(Rk(\mathbf{Sub}_\sigma), Rk(\mathbf{Prod}^\sigma_\sigma), Rk(\mathbf{Div}_\sigma), \mathbf{V}^\sigma_{\sigma\times\sigma}) \\
&\leq \max(2lv(\sigma), 2lv(\sigma\times\sigma), 2lv(\sigma), lv(\sigma) + lv(\sigma\times\sigma), lv(\sigma\times\sigma), lv(\sigma) \dot{-} 2) \\
&= 2lv(\sigma)
\end{aligned}$$

□

Chapter 5

Computing $\mathbf{nval}_b^{\mathcal{A}}$ in $\mathbf{T}^-$

5.1 Relating $|\sigma|_b$ and $||\sigma||_b$

Lemma 5.1. *For any polynomial $p(x)$ let $2_0^{p(x)} = p(x)$ and $2_{i+1}^{p(x)} = 2^{2_i^{p(x)}}$*

(i) for any type σ of rank n there is a polynomial $Q_\sigma(x) > 0$ such that $|\sigma|_x < 2_n^{Q_\sigma(x)}$ for all x

(ii) for any polynomial $p(x)$ and $n \geq 0$ there exists a type σ_p^n of rank n such that $2_n^{p(x)} < |\sigma_p^n|_x$ for all x

(iii) for any $n \geq 0$ and polynomials $p(x), r(x) > 0$ we have $2_n^{p(x)} \times 2_n^{r(x)} \leq 2_n^{p(x)\times r(x)}$

Proof. *(i)* and *(ii)* are proved in [1], and we prove *(iii)* by induction on n.

So first let $n = 0$, then $2_0^{p(x)} \times 2_0^{r(x)} = p(x) \times r(x) = 2_0^{p(x)\times r(x)}$.

Now let $n = i + 1$, then

$$
\begin{aligned}
2_{i+1}^{p(x)} \times 2_{i+1}^{r(x)} &= 2^{2_i^{p(x)}} \times 2^{2_i^{r(x)}} && 2_{i+1}^c = 2^{2_i^c} \\
&= 2^{2_i^{p(x)} + 2_i^{r(x)}} && \\
&\leq 2^{2_i^{p(x)} \times 2_i^{r(x)}} && p(x), r(x) > 0 \\
&= 2^{2_i^{p(x)\times r(x)}} && \text{IH.} \\
&= 2_{i+1}^{p(x)\times r(x)} && 2_{i+1}^c = 2^{2_i^c}
\end{aligned}
$$

□

Lemma 5.2. *For each type σ of rank n there is a type $\bar{\sigma}$ of rank $n+1$ such that $||\sigma||_{b+1} < |\bar{\sigma}|_{b+1}$ for all $b \geq 2$. Moreover, $lv(\overline{\sigma \times \tau}) = lv(\bar{\sigma} \times \bar{\tau})$ and $lv(\overline{\sigma \to \tau}) = lv(\bar{\sigma} \to \bar{\tau})$ for all types σ and τ*

Proof. We prove the first part by induction on the structure of σ.

Assume $\sigma = \iota$, let $\bar{\sigma} = \iota \to \iota$ then

$$\begin{aligned}
||\iota||_{b+1} &= 2^{b+1} && ||\iota||_{b+1} \text{ def.}\\
&< (b+1)^{(b+1)} && b \geq 2\\
&= |\iota \to \iota|_{b+1} && |\iota \to \iota|_{b+1} \text{ def.}\\
&= |\bar{\iota}|_{b+1} && \bar{\iota} = \iota \to \iota
\end{aligned}$$

Assume $\sigma = \rho \times \tau$ and observe that ρ and τ are of rank no greater then n, then by the induction hypothesis $\bar{\rho}$ and $\bar{\tau}$ are of rank no greater then $n+1$. Let $z(x) = Q_{\bar{\rho}}(x) \times Q_{\bar{\tau}}(x)$ and let $\bar{\sigma} = \sigma_z^{n+1}$, then for all $b \geq 2$

$$\begin{aligned}
||\rho \times \tau||_{b+1} &= ||\rho||_{b+1} \times ||\tau||_{b+1} && ||\rho \times \tau||_{b+1} \text{ def.}\\
&< |\bar{\rho}|_{b+1} \times |\bar{\tau}|_{b+1} && \text{IH.}\\
&< 2_{n+1}^{Q_{\bar{\rho}}(b+1)} \times 2_{n+1}^{Q_{\bar{\tau}}(b+1)} && \textit{(i)} \text{ in Lemma 5.1}\\
&\leq 2_{n+1}^{Q_{\bar{\rho}}(b+1) \times Q_{\bar{\tau}}(b+1)} && \textit{(iii)} \text{ in Lemma 5.1}\\
&= 2_{n+1}^{z(b+1)} && z(x) = Q_{\bar{\rho}}(x) \times Q_{\bar{\tau}}(x)\\
&< |\sigma_z^{n+1}|_{b+1} && \textit{(ii)} \text{ in Lemma 5.1}\\
&= |\bar{\sigma}|_{b+1} && \bar{\sigma} = \sigma_z^{n+1}
\end{aligned}$$

Assume $\sigma = \rho \to \tau$ and observe that ρ and τ are of rank no greater then $n-1$ and n respectively, then by the induction hypothesis $\bar{\rho}$ and $\bar{\tau}$ are of rank no greater then n and $n+1$ respectively. Let $z(x) = Q_{\bar{\rho}}(x) \times Q_{\bar{\tau}}(x)$ and let $\bar{\sigma} = \sigma_z^{n+1}$, then for all $b \geq 2$

$$\begin{aligned}
||\rho \to \tau||_{b+1} &= ||\tau||_{b+1}^{||\rho||_{b+1}} && ||\rho \to \tau||_{b+1} \text{ def.}\\
&< |\bar{\tau}|_{b+1}^{|\bar{\rho}|_{b+1}} && \text{IH.}\\
&< \left\{2_{n+1}^{Q_{\bar{\tau}}(b+1)}\right\}^{\left\{2_n^{Q_{\bar{\rho}}(b+1)}\right\}} && \textit{(i)} \text{ in Lemma 5.1}\\
&= \left\{2^{2_n^{Q_{\bar{\tau}}(b+1)}}\right\}^{\left\{2_n^{Q_{\bar{\rho}}(b+1)}\right\}} && 2_{i+1}^c = 2^{2_i^c}\\
&= 2^{2_n^{Q_{\bar{\tau}}(b+1)} \times 2_n^{Q_{\bar{\rho}}(b+1)}}\\
&\leq 2^{2_n^{Q_{\bar{\tau}}(b+1) \times Q_{\bar{\rho}}(b+1)}} && \textit{(iii)} \text{ in Lemma 5.1}\\
&= 2_{n+1}^{Q_{\bar{\tau}}(b+1) \times Q_{\bar{\rho}}(b+1)} && 2_{i+1}^c = 2^{2_i^c}\\
&= 2_{n+1}^{z(b+1)} && z(x) = Q_{\bar{\rho}}(x) \times Q_{\bar{\tau}}(x)\\
&< |\sigma_z^{n+1}|_{b+1} && \textit{(ii)} \text{ in Lemma 5.1}\\
&= |\bar{\sigma}|_{b+1} && \bar{\sigma} = \sigma_z^{n+1}
\end{aligned}$$

The two identities are computed straight forward. □

5.2 Computing $||\sigma||_b, \xi_b, \beta_b, \mu_b, \theta_b^\sigma, \lambda_b^\sigma, \rho_b^\sigma, \delta_b^\sigma, \Upsilon_b^\sigma, Merge_b^\sigma$

Lemma 5.3. *For all types σ, τ there exists closed T^- terms*

(i) $\boldsymbol{Cardinality}_\sigma : \iota \to \bar{\sigma}$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Cardinality}_\sigma(k_b)) = ||\sigma||_{b+1}$

(ii) $\boldsymbol{Radix}_\sigma^\tau : \iota, \sigma \to (\sigma \to \bar{\tau})$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Radix}_\sigma^\tau(k_b, F)) = ||\tau||_{b+1}^{\boldsymbol{val}_{b+1}(F)}$

for $b \geq 2$ and appropriately typed closed T_b^- term F.
Moreover $Rk(\boldsymbol{Cardinality}_\sigma) \leq 2lv(\bar{\sigma})$ and $Rk(\boldsymbol{Radix}_\sigma^\tau) \leq 2lv(\overline{\sigma \times \tau})$.

Proof. We construct $\mathbf{Cardinality}_\sigma$ terms by induction on σ.

When $\sigma = \iota$ let

$$\mathbf{Cardinality}_\iota = \lambda b^\iota.\mathbf{V}_{\iota\times\bar{\iota}}^{\bar{\iota}}(b, \mathbf{Prod}_\iota^{\bar{\iota}}(b, k_2, \mathbf{Exp}_\iota^\iota(b, k_2, b)))$$

observe

$$\begin{aligned}
\mathbf{val}_{b+1}(\mathbf{Cardinality}_\iota(k_b)) &= \mathbf{val}_{b+1}(\mathbf{V}_{\iota\times\bar{\iota}}^{\bar{\iota}}(b, \mathbf{Prod}_\iota^{\bar{\iota}}(k_b, k_2, \mathbf{Exp}_\iota^\iota(k_b, k_2, k_b)))) \\
&= \mathbf{val}_{b+1}(\mathbf{Prod}_\iota^{\bar{\iota}}(k_b, k_2, \mathbf{Exp}_\iota^\iota(k_b, k_2, k_b))) \mod |\bar{\iota}|_{b+1} \\
&= 2 \times \mathbf{val}_{b+1}(\mathbf{Exp}_\iota^\iota(k_b, k_2, k_b)) \mod |\bar{\iota}|_{b+1} \\
&= 2 \times 2^b \mod |\bar{\iota}|_{b+1} = 2^{b+1} \mod |\bar{\iota}|_{b+1} \\
&= ||\iota||_{b+1} \mod |\bar{\iota}|_{b+1} = ||\iota||_{b+1} && \text{Lemma 5.2}
\end{aligned}$$

also

$$\begin{aligned}
Rk(\mathbf{Cardinality}_\iota) &= \max(Rk(\mathbf{Prod}_\iota^{\bar{\iota}}), Rk(\mathbf{Exp}_\iota^\iota), Rk(\mathbf{V}_{\iota\times\bar{\iota}}^{\bar{\iota}})) \\
&\leq \max(2lv(\bar{\iota} \times \iota), 2lv(\iota \to \iota), 2lv(\bar{\iota})) \\
&= 2lv(\bar{\iota})
\end{aligned}$$

When $\sigma = \rho \times \tau$ let

$$\mathbf{Cardinality}_{\rho\times\tau} = \lambda b^\iota.\mathbf{V}_{\bar{\rho}\times\bar{\tau}}^{\overline{\rho\times\tau}}(b, \mathbf{Prod}_{\bar{\rho}}^{\bar{\tau}}(b, \mathbf{Cardinality}_\rho(b), \mathbf{Cardinality}_\tau(b)))$$

observe

$$\begin{aligned}
&\mathbf{val}_{b+1}(\mathbf{Cardinality}_{\rho\times\tau}(k_b)) \\
&\quad = \mathbf{val}_{b+1}(\mathbf{V}_{\bar{\rho}\times\bar{\tau}}^{\overline{\rho\times\tau}}(b, \mathbf{Prod}_{\bar{\rho}}^{\bar{\tau}}(k_b, \mathbf{Cardinality}_\rho(k_b), \mathbf{Cardinality}_\tau(k_b)))) \\
&\quad = \mathbf{val}_{b+1}(\mathbf{Prod}_{\bar{\rho}}^{\bar{\tau}}(k_b, \mathbf{Cardinality}_\rho(k_b), \mathbf{Cardinality}_\tau(k_b))) \mod |\overline{\rho \times \tau}|_{b+1} \\
&\quad = \mathbf{val}_{b+1}(\mathbf{Cardinality}_\rho(k_b)) \times \mathbf{val}_{b+1}(\mathbf{Cardinality}_\tau(k_b))) \mod |\overline{\rho \times \tau}|_{b+1} \\
&\quad = ||\rho||_{b+1} \times ||\tau||_{b+1} \mod |\overline{\rho \times \tau}|_{b+1} \\
&\quad = ||\rho \times \tau||_{b+1} \mod |\overline{\rho \times \tau}|_{b+1} \\
&\quad = ||\rho \times \tau||_{b+1} && \text{Lemma 5.2}
\end{aligned}$$

also

$$
\begin{aligned}
Rk(\mathbf{Cardinality}_{\rho\times\tau}) &= \max(Rk(\mathbf{Prod}^{\bar\tau}_{\bar\rho}), Rk(\mathbf{Cardinality}_{\rho}), Rk(\mathbf{Cardinality}_{\tau}), Rk(\mathbf{V}^{\overline{\rho\times\tau}}_{\bar\rho\times\bar\tau}))\\
&\leq \max(2lv(\bar\tau\times\bar\rho), 2lv(\bar\rho), 2lv(\bar\tau), 2lv(\overline{\rho\times\tau}))\\
&= 2lv(\overline{\rho\times\tau})
\end{aligned}
$$

When $\sigma = \rho \to \tau$ let

$$
\mathbf{Cardinality}_{\rho\to\tau} = \lambda b^{\iota}.\mathbf{V}^{\overline{\rho\to\tau}}_{\bar\rho\to\bar\tau}(b, \mathbf{Exp}^{\bar\tau}_{\bar\rho}(b, \mathbf{Cardinality}_{\tau}(b), \mathbf{Cardinality}_{\rho}(b)))
$$

observe

$$
\begin{aligned}
&\mathbf{val}_{b+1}(\mathbf{Cardinality}_{\rho\to\tau}(k_b))\\
&\quad= \mathbf{val}_{b+1}(\mathbf{V}^{\overline{\rho\to\tau}}_{\bar\rho\to\bar\tau}(b, \mathbf{Exp}^{\bar\tau}_{\bar\rho}(k_b, \mathbf{Cardinality}_{\tau}(k_b), \mathbf{Cardinality}_{\rho}(k_b))))\\
&\quad= \mathbf{val}_{b+1}(\mathbf{Exp}^{\bar\tau}_{\bar\rho}(k_b, \mathbf{Cardinality}_{\tau}(k_b), \mathbf{Cardinality}_{\rho}(k_b))) \mod |\overline{\rho\to\tau}|_{b+1}\\
&\quad= \mathbf{val}_{b+1}(\mathbf{Cardinality}_{\tau}(k_b))^{\mathbf{val}_{b+1}(\mathbf{Cardinality}_{\rho}(k_b))} \mod |\overline{\rho\to\tau}|_{b+1}\\
&\quad= ||\tau||_{b+1}^{||\rho||_{b+1}} \mod |\overline{\rho\to\tau}|_{b+1} = ||\rho\to\tau||_{b+1} \mod |\overline{\rho\to\tau}|_{b+1} = ||\rho\to\tau||_{b+1} \quad \text{Lemma 5.2}
\end{aligned}
$$

also

$$
\begin{aligned}
Rk(\mathbf{Cardinality}_{\rho\times\tau}) &= \max(Rk(\mathbf{Exp}^{\bar\tau}_{\bar\rho}), Rk(\mathbf{Cardinality}_{\rho}), Rk(\mathbf{Cardinality}_{\tau}), \mathbf{V}^{\overline{\rho\to\tau}}_{\bar\rho\to\bar\tau})\\
&\leq \max(2lv(\bar\rho\to\bar\tau), 2lv(\bar\rho), 2lv(\bar\tau), 2lv(\overline{\rho\to\tau}))\\
&= 2lv(\overline{\rho\to\tau})
\end{aligned}
$$

For all types σ, τ let

$$
\mathbf{Radix}^{\tau}_{\sigma} = \lambda b^{\iota} F^{\sigma}.\mathbf{Exp}^{\bar\tau}_{\sigma}(b, \mathbf{Cardinality}_{\tau}(b), F)
$$

for any T^{-}_{b} term $F : \sigma$ observe that

$$
\begin{aligned}
\mathbf{val}_{b'}(\mathbf{Radix}^{\tau}_{\sigma}(k_b, F)) &= \mathbf{val}_{b+1}(\mathbf{Exp}^{\bar\tau}_{\sigma}(b, \mathbf{Cardinality}_{\tau}(k_b), F))\\
&= \mathbf{val}_{b+1}(\mathbf{Exp}^{\bar\tau}_{\sigma}(k_b, \mathbf{Cardinality}_{\tau}(k_b), F)\\
&= \mathbf{val}_{b+1}(\mathbf{Cardinality}_{\tau}(k_b))^{\mathbf{val}_{b+1}(F)}\\
&= ||\tau||_{b+1}^{\mathbf{val}_{b+1}(F)}
\end{aligned}
$$

also

$$
\begin{aligned}
Rk(\mathbf{Radix}^{\tau}_{\sigma}) &= \max(Rk(\mathbf{Exp}^{\bar\tau}_{\sigma}), Rk(\mathbf{Cardinality}_{\tau}))\\
&\leq \max(2lv(\sigma\to\bar\tau), 2lv(\bar\tau)) = 2\max(lv(\sigma)+1, lv(\bar\tau))\\
&= 2\max(lv(\bar\sigma), lv(\bar\tau)) = 2lv(\bar\sigma\times\bar\tau)\\
&= 2lv(\overline{\sigma\times\tau})
\end{aligned}
$$

□

Lemma 5.4. *For all types σ, τ there exists closed* T^- *terms*

(i) $\boldsymbol{Left}^\tau_\sigma : \iota, \overline{\sigma \times \tau} \to \bar{\sigma}$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Left}^\tau_\sigma(k_b, F)) = \lambda^{\sigma\times\tau}_{b+1}(\boldsymbol{val}_{b+1}(F))$

(ii) $\boldsymbol{Right}^\tau_\sigma : \iota, \overline{\sigma \times \tau} \to \bar{\tau}$ *where* $\boldsymbol{val}_{b+1}(\boldsymbol{Right}^\tau_\sigma(k_b, F)) = \rho^{\sigma\times\tau}_{b+1}(\boldsymbol{val}_{b+1}(F))$

(iii) $\boldsymbol{Pair}^\tau_\sigma : \iota, \bar{\sigma}, \bar{\tau} \to \overline{\sigma \times \tau}$ *where*
$\boldsymbol{val}_{b+1}(\boldsymbol{Pair}^\tau_\sigma(k_b, G, H)) = \theta^{\sigma\times\tau}_{b+1}(\boldsymbol{val}_{b+1}(G), \boldsymbol{val}_{b+1}(H))$

for $b \geq 2$ and all appropriately typed closed T^-_b *terms F,G and H where* $\boldsymbol{val}_{b+1}(F) < ||\sigma \times \tau||_{b+1}$,$\boldsymbol{val}_{b+1}(G) < ||\sigma||_{b+1}$ *and* $\boldsymbol{val}_{b+1}(H) < ||\tau||_{b+1}$.

Moreover $Rk(\boldsymbol{Left}^\tau_\sigma), Rk(\boldsymbol{Right}^\tau_\sigma), \boldsymbol{Pair}^\tau_\sigma \leq 2lv(\overline{\sigma \times \tau})$

Proof. We only show *(i)* since the approach is completely analogous for *(ii)* and *(iii)*.

For any types σ, τ let $\phi = \overline{\sigma \times \tau}$ and let

$$\mathbf{Left}^\tau_\sigma = \lambda b^\iota F^\phi.\mathbf{V}^{\bar{\sigma}}_\phi(b, \mathbf{Div}_\phi(b, F, \mathbf{V}^\phi_{\bar{\tau}}(b, \mathbf{Cardinality}_\tau(b))))$$

Observe for any term $F : \phi$ as mentioned where $\mathbf{val}_{b+1}(F) = f_0 + f_1||\tau||_{b+1}$

$$\mathbf{val}_{b+1}(\mathbf{Left}^\tau_\sigma(k_b, F)) = \{f_0 + f_1||\tau||_{b+1}\} \text{ div } ||\tau||_{b+1} = f_1 = \lambda^{\sigma\times\tau}_{b+1}(\mathbf{val}_{b+1}(F))$$

also

$$\begin{aligned} Rk(\mathbf{Left}^\tau_\sigma) &= \max(Rk(\mathbf{Div}_\phi), Rk(\mathbf{V}^\phi_{\bar{\tau}}), Rk(\mathbf{V}^{\bar{\sigma}}_\phi) Rk(\mathbf{Cardinality}_\tau)) \\ &\leq \max(2lv(\phi), 2lv(\phi), 2lv(\bar{\tau})) \\ &= 2lv(\overline{\sigma \times \tau}) \end{aligned}$$

□

Lemma 5.5. *For all types σ, τ there exists closed* T^- *terms* $\boldsymbol{Digit}^\tau_\sigma : \iota, \overline{\sigma \to \tau}, \bar{\sigma} \to \bar{\tau}$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Digit}^\tau_\sigma(k_b, F, G)) = \delta^{\sigma\to\tau}_{b+1}(\boldsymbol{val}_{b+1}(F), \boldsymbol{val}_{b+1}(G))$$

for $b \geq 2$ and all appropriately typed closed T^-_b *terms F,G where* $\boldsymbol{val}_{b+1}(F) < ||\sigma \to \tau||_{b+1}$ *and* $\boldsymbol{val}_{b+1}(G) < ||\sigma||_{b+1}$.

Moreover $Rk(\boldsymbol{Digit}^\tau_\sigma) \leq 2lv(\overline{\sigma \to \tau})$.

Proof. For any types σ, τ let $\phi = \overline{\sigma \to \tau}$ and

$$\mathbf{Digit}^\tau_\sigma = \lambda b^\iota F^\phi G^{\bar{\sigma}}.\mathbf{V}^{\bar{\tau}}_\phi(b, \mathbf{Mod}_\phi(b, D(b, F, G), \mathbf{V}^\phi_{\bar{\tau}}(b, \mathbf{Cardinality}_\tau(b))))$$

where

$$D = \lambda b^\iota F^\phi G^{\bar{\sigma}}.\mathbf{Div}_\phi(b, F, \mathbf{V}^\phi_{\bar{\sigma}\to\bar{\tau}}(b, \mathbf{Radix}^\tau_{\bar{\sigma}}(b, G)))$$

For any terms $F : \phi$ and $G : \bar{\sigma}$ as mentioned, let $z = \mathbf{val}_{b+1}(F)$ and $r = \mathbf{val}_{b+1}(G)$. Since $z < ||\sigma \to \tau||_{b+1}$, then by *(iii)* in lemma 3.19 we write

$$z = d_0 + d_1||\tau||_{b+1} + ... + d_k||\tau||^k_{b+1}$$

where $k = ||\sigma||_{b+1} - 1$. We omit the calculation showing

$$\mathbf{val}_{b+1}(D(k_b, F, G)) = d_r + d_{r+1}||\tau||_{b+1} + ... + d_k||\tau||_{b+1}^{k-r}$$

which is combined to give

$$\begin{aligned}
&\mathbf{val}_{b+1}(\mathbf{Digit}_\sigma^\tau(k_b, F, G)) \\
&= \{\mathbf{val}_{b+1}(D(k_b, F, G)) \mod \mathbf{val}_{b+1}(\mathbf{Cardinality}_\tau(k_b))\} \mod |\overline{\tau}|_{b+1} \\
&= \left\{\left\{d_r + d_{r+1}||\tau||_{b+1} + ... + d_k||\tau||_{b+1}^{k-r}\right\} \mod ||\tau||_{b+1}\right\} \mod |\overline{\tau}|_{b+1} \\
&= d_r \mod |\overline{\tau}|_{b+1} \\
&= d_r \\
&= \delta_{b+1}^{\sigma\to\tau}(d_0 + d_1||\tau||_{b+1} + ... + d_k||\tau||_{b+1}, r) \\
&= \delta_{b+1}^{\sigma\to\tau}(z, r) = \delta_{b+1}^{\sigma\to\tau}(\mathbf{val}_{b+1}(F), \mathbf{val}_{b+1}(G))
\end{aligned}$$

We find $Rk(D) \leq 2(\phi)$ and also $Rk(\mathbf{Digit}_\sigma^\tau) \leq 2(\phi) = 2lv(\overline{\sigma \to \tau})$. □

Lemma 5.6. *There exists a closed* T^- *terms*

(i) $\boldsymbol{IsMember} : \iota, \overline{\iota}, \iota \to \iota$ *where*
$\boldsymbol{val}_{b+1}(\boldsymbol{IsMember}(k_b, F, H)) = \beta_{b+1}(\boldsymbol{val}_{b+1}(F), \boldsymbol{val}_{b+1}(H))$

(ii) $\boldsymbol{Union} : \iota, \overline{\iota}, \overline{\iota} \to \overline{\iota}$ *where*
$\boldsymbol{val}_{b+1}(\boldsymbol{Union}(k_b, F, G)) = \mu_{b+1}(\xi_{b+1}(\boldsymbol{val}_{b+1}(F)) \cup \xi_{b+1}(\boldsymbol{val}_{b+1}(G)))$

for $b \geq 2$ *and appropriately typed closed* T_b^- *terms* F,G *and* H *where* $\boldsymbol{val}_{b+1}(F), \boldsymbol{val}_{b+1}(G) < ||\iota||_{b+1}$.

Moreover $Rk(\boldsymbol{IsMember}) = Rk(\boldsymbol{Union}) = 2$.

Proof. Let

$$\mathbf{IsMember} = \lambda b^\iota F^{\overline{\iota}} G^\iota.\mathbf{V}_{\overline{\iota}}^\iota(b, \mathbf{Mod}_{\overline{\iota}}(b, \mathbf{Div}_{\overline{\iota}}(b, F, \mathbf{Exp}_\iota^\iota(b, k_2, G)), \mathbf{V}_\iota^{\overline{\iota}}(b, k_2)))$$

and

$$\mathbf{Union} = \lambda b^\iota F^{\overline{\iota}} G^{\overline{\iota}}.\mathbf{Rep}_\iota^{\overline{\iota}}(b, \lambda x^{\iota\times\overline{\iota}}.D(b, F, G), b, \mathbf{0}_{\overline{\iota}})$$

where
$D = \lambda b^\iota F^{\overline{\iota}} G^{\overline{\iota}}.\mathbf{Or}_\iota(\mathbf{IsMember}(b, F, fst.x), \mathbf{IsMember}(b, G, fst.x),$
$\mathbf{V}_{\overline{\iota}\times\overline{\iota}}^{\overline{\iota}}(b, \mathbf{Add}_{\overline{\iota}}(b, snd.x, \mathbf{Exp}_\iota^\iota(b, k_2, fst.x))), snd.x)$

We omit correctness and rank calculation for **IsMember** since it is very similar to $\mathbf{Digit}_\sigma^\tau$.

Moving on to *(ii)*, for any terms $F : \overline{\iota}$ and $G : \overline{\iota}$, we can by *(iii)* in lemma 3.19 write

$$\mathbf{val}_{b+1}(F) = f_0 + ... + f_k 2^b$$
$$\mathbf{val}_{b+1}(G) = g_0 + ... + g_k 2^b$$

Observe that

$$\mu_{b+1}(\xi_{b+1}(\mathbf{val}_{b+1}(F)) \cup \xi_{b+1}(\mathbf{val}_{b+1}(G))) = z_0 + ... + z_b 2^b$$

where $z_i = 1$ when $f_i = 1$ or $g_i = 1$, otherwise $z_i = 0$. We then compute

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{Union}(k_b, F, G)) &= \mathbf{val}_{b+1}(b, \mathbf{Rep}_\iota^{\bar{\iota}}(k_b, \lambda x^{\iota \times \bar{\iota}}.D(k_b, F, G), k_b, \mathbf{0}_{\bar{\iota}})) \\ &= f(\mathbf{val}_{b+1}(k_b))(\clubsuit)\end{aligned}$$

where $f(0) = z_0$ and $f(i+1) = f(i) + z_{i+1}2^{i+1}$. We iterate this and get $f(i) = z_0 + ... + z_i 2^i$, so we have that $f(\mathbf{val}_{b+1}(k_b)) = z_0 + ... + z_b 2^b$, which extends ($\clubsuit$) to give us the desired result. We also see $Rk(D) = 2$, which finally gives $Rk(\mathbf{Union}_\sigma) = 2$ as well. □

Lemma 5.7. *For all types σ, τ there exists a T^- term $\boldsymbol{Table}_\sigma^\tau : (\iota, \bar{\sigma} \to \bar{\tau}) \to \overline{\sigma \to \tau}$ where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Table}_\sigma^\tau(k_b, F)) = \sum_{i < \|\sigma\|_{b+1}} \delta_{b+1}^{\bar{\sigma} \to \bar{\tau}}(\boldsymbol{val}_{b+1}(F), i) \times \|\tau\|_{b+1}^i$$

for $b \geq 2$ and all appropriately typed closed T_b^- terms F.

Moreover $Rk(\boldsymbol{Table}_\sigma^\tau) \leq 2lv(\overline{\sigma \to \tau})$.

Proof. For any types σ, τ let $\phi = \bar{\sigma} \to \bar{\tau}$ and let

$$\mathbf{Table}_\sigma^\tau = \lambda b^\iota F^\phi.\mathbf{Rep}_{\bar{\sigma}}^\phi(b, \lambda x^{\bar{\sigma} \times \phi}.D(b, F, x), \mathbf{Cardinality}_\sigma(b), \mathbf{0}_\phi)$$

where

$$\begin{aligned}D = \lambda b^\iota F^\phi x^{\bar{\sigma} \times \phi}.\mathbf{V}_{\phi \times \phi}^{\overline{\sigma \to \tau}}(&\mathbf{Add}_\phi(b, snd.x, \\ &\mathbf{V}_{\bar{\tau} \times \bar{\sigma} \to \bar{\tau}}^\phi(\mathbf{Prod}_{\bar{\tau}}^{\bar{\sigma} \to \bar{\tau}}(b, F(fst.x), \mathbf{Radix}_{\bar{\sigma}}^\tau(b, fst.x)))))\end{aligned}$$

Observe for any term $F : \bar{\sigma} \to \bar{\tau}$

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{Table}_\sigma^\tau(k_b, F)) &= \mathbf{val}_{b+1}(\mathbf{Rep}_{\bar{\sigma}}^\phi(k_b, \lambda x^{\bar{\sigma} \times \phi}.D(k_b, F, x), \mathbf{Cardinality}_\sigma(k_b), \mathbf{0}_\phi)) \\ &= f(\mathbf{val}_{b+1}(\mathbf{Cardinality}_\sigma(k_b))) \\ &= f(||\sigma||_{b+1})(\clubsuit)\end{aligned}$$

where $f(0) = \mathbf{val}_{b+1}(\mathbf{0}_\sigma) = 0$ and

$$f(i+1) = f(i) + \delta_{b+1}^{\bar{\sigma} \to \bar{\tau}}(\mathbf{val}_{b+1}(F), i) \times \|\tau\|_{b+1}^i$$

which is iterated to

$$f(j) = \sum_{i<j} \delta_{b+1}^{\bar{\sigma} \to \bar{\tau}}(\mathbf{val}_{b+1}(F), i) \times \|\tau\|_{b+1}^i$$

for all $||\sigma||_{b+1} \geq j > 0$, which extends ($\clubsuit$) to give us the desired result.

We also find that $Rk(D) \leq 2lv(\phi)$ and $Rk(\mathbf{Table}_\sigma^\tau) \leq 2lv(\phi) = 2lv(\overline{\sigma \to \tau})$ □

Lemma 5.8. *For all types σ there exists a closed* T^- *term* $\boldsymbol{Merge}_\sigma : \iota, \bar{\sigma}, \bar{\sigma} \to \bar{\sigma}$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Merge}_\sigma(k_b, F, G)) = Merge^\sigma_{b+1}(\boldsymbol{val}_{b+1}(F), \boldsymbol{val}_{b+1}(G))$$

for $b \geq 2$ *and all appropriately typed closed* T^-_b *terms* F *and* G *where* $\boldsymbol{val}_{b+1}(F)$, $\boldsymbol{val}_{b+1}(G) < ||\sigma||_{b+1}$.

Moreover $Rk(\boldsymbol{Merge}^\sigma_\tau) \leq 2lv(\bar{\sigma})$.

Proof. We construct $\mathbf{Merge}_\sigma$ terms by induction on σ, but omit the lengthy recursor rank and value computation.

In each case, let $\phi = \bar{\sigma}$.When $\sigma = \iota$ let

$$\mathbf{Merge}_\iota = \lambda b^\iota F^\phi G^\phi.\mathbf{Union}(b, F, G)$$

When $\sigma = \rho \times \pi$ let

$$\mathbf{Merge}_{\rho\times\pi} = \lambda b^\iota F^\phi G^\phi.\mathbf{Pair}^\pi_\rho(b, \mathbf{Merge}_\rho(b, \mathbf{Left}^\pi_\rho(F, b), \mathbf{Left}^\pi_\rho(G, b)), \mathbf{Merge}_\pi(b, \mathbf{Right}^\pi_\rho(F, b), \mathbf{Right}^\pi_\rho(G, b)))$$

When $\sigma = \rho \to \pi$ let

$$\mathbf{Merge}_{\rho\to\pi} = \lambda b^\iota F^\phi G^\phi.\mathbf{Table}^\pi_\rho(b, \lambda x^{\bar{\rho}}.\mathbf{Merge}_\pi(b, \mathbf{Digit}^\pi_\rho(F, x), \mathbf{Digit}^\pi_\rho(G, x)))$$

□

Lemma 5.9. *For all types there exists a closed* T^- *term* $\boldsymbol{Rec}_\sigma : \iota, \bar{\iota}, \overline{\iota, \sigma \to \sigma}, \bar{\sigma} \to \bar{\sigma}$ *where*

$$\boldsymbol{val}_{b+1}(\boldsymbol{Rec}_\sigma(k_b, N, F, G)) = \Upsilon^\sigma_{b+1}(\xi_{b+1}(\boldsymbol{val}_{b+1}(N)), \boldsymbol{val}_{b+1}(F), \boldsymbol{val}_{b+1}(G))$$

for $b \geq 2$ *and all appropriately typed closed* T^-_b *terms* N,F,G *where* $\boldsymbol{val}_{b+1}(N) < ||\iota||_{b+1}$,$\boldsymbol{val}_{b+1}(F) < ||\iota, \sigma \to \sigma||_{b+1}$ *and* $\boldsymbol{val}_{b+1}(G) < ||\sigma||_{b+1}$.

Moreover $Rk(\boldsymbol{Rec}_\sigma) \leq 2lv(\bar{\sigma})$.

Proof. For any σ let $\phi = \overline{\iota, \sigma \to \sigma}$ and let

$$\mathbf{Rec}_\sigma = \lambda b^\iota N^{\bar{\iota}} F^\phi G^{\bar{\sigma}}.\mathbf{Rep}^{\bar{\sigma}}_\iota(b, \lambda x^{\iota\times\bar{\sigma}}.D(b, N, F, G), b, \mathbf{0}_{\bar{\sigma}})$$

where

$$D = \lambda b^\iota N^{\bar{\iota}} F^\phi G^{\bar{\sigma}}.\mathbf{Cond}_{\bar{\sigma}}(\mathbf{IsMember}(b, N, fst.x), \mathbf{Merge}_\sigma(b, R_{\bar{\sigma}}(fst.x, \mathbf{V}^{\iota,\bar{\sigma}\to\bar{\sigma}}_\phi(F), G), snd.x), snd.x)$$

For any terms N, F, G as mentioned, we can by *(iii)* in lemma 3.19 write

$$\mathbf{val}_{b+1}(F) = n_0 + ... + n_b 2^b$$

Let

$$X = \xi_{b+1}(\mathbf{val}_{b+1}(F)) = \{i | n_i = 1\}$$

and let $X_j = \{i \in X | i \leq j\}$ so $X_b = X$. As before we find

$$\mathbf{val}_{b+1}(\mathbf{Rec}_\sigma(k_b, N, F, G)) = f(\mathbf{val}_{b+1}(k_b))(\clubsuit)$$

where $f(0) = 0$ and

$$f(i+1) = \begin{cases} Merge^\sigma_{b+1}(f(i), \upsilon^\sigma_{b+1}(i+1, \mathbf{val}_{b+1}(F), \mathbf{val}_{b+1}(G))) & \text{if } n_{i+1} = 1 \\ f(i) & \text{else} \end{cases}$$

which is iterated to yield

$$f(i+1) = \Upsilon^\sigma_{b+1}(X_{i+1}, \mathbf{val}_{b+1}(F), \mathbf{val}_{b+1}(G))$$

giving

$$\begin{aligned} f(\mathbf{val}_{b+1}(k_b)) &= f(b) = \Upsilon^\sigma_{b+1}(X_b, \mathbf{val}_{b+1}(F), \mathbf{val}_{b+1}(G)) \\ &= \Upsilon^\sigma_{b+1}(X, \mathbf{val}_{b+1}(F), \mathbf{val}_{b+1}(G)) \\ &= \Upsilon^\sigma_{b+1}(\xi_{b+1}(\mathbf{val}_{b+1}(F)), \mathbf{val}_{b+1}(F), \mathbf{val}_{b+1}(G)) \end{aligned}$$

which extends ($\clubsuit$) to give us the desired result. □

5.3 Modelling $\mathcal{A}_b$

Definition 5.10. *Given a term $\mathrm{T}_b^{\frown}$-term M, let $\boldsymbol{vars}(M)$ denote the set of distinct variables in M, also counting variables which only occur as $\lambda x.P$. A map $f : \mathcal{V} \to \mathbb{N}$ is called a variable listing for M when it is injective on $\boldsymbol{vars}(M)$. Let $M_f = \{f(x) \mid x \in \boldsymbol{vars}(M)\}$, and we call $\max(M_f \cup \{0\})$[1] the length of f on M. Given a variable listing f for M, then any type of the form $\overline{\pi_0} \times \ ... \times \overline{\pi_\ell}$ is called an assignment type of M given f when*

- *ℓ is no smaller then the length of f on M*
- *π_i is the type of the variable $f^{-1}(i)$ for all $i \in M_f$*

When ω is an assignment type, let $\omega[i]$ denote $\overline{\pi_i}$. Given some variable listing f for M, let ω be an assignment type for M given f. Any closed T_b^--term $V : \omega$ is called a term assignment for M. Let $\mathcal{A}_{b+1}$ be a value assignment, then V is also a derived term assignment of $\mathcal{A}$ for M when

$$\boldsymbol{val}_{b+1}(fst.snd^i.V) = \mathcal{A}(x) \text{ where } x = f^{-1}(i)$$

for all $i \in M_f$.

[1]The set $\{0\}$ is added to make length of f well defined on variable free terms, i.e $M_f = \emptyset$

The following lemma is necessary to apply the induction hypothesis to subterms in in Theorem 5.13 by claim *(i)*, and allows the theorem to initially be applied to any term by claim *(ii)*.

Lemma 5.11. *For any* $\mathrm{T}^{\frown}$*-term* M

(i) given a variable listing f *for* M *with corresponding assignment type* ω *and term assignment* $V : \omega$*, then they all apply to any subterm of* M*.*

(ii) there exists variable listings, and corresponding assignment types and term assignments for M

Proof. For *(i)* consider a subterm N of M, and observe that $\mathbf{vars}(N) \subseteq \mathbf{vars}(M)$. Obviously f is still injective on $\mathbf{vars}(N)$, even if $\mathbf{vars}(N) = \emptyset$, so f is a variable listing for N. M_f and the length of f on N is also well defined, since even if $\mathbf{vars}(N) = \emptyset$, then the length is zero. Let $\omega = \overline{\pi_0} \times \ \ldots \times \overline{\pi_\ell}$. Since ℓ is no smaller then the length f on M by definition, and since the length of f on M is no smaller then the length of f on N, ℓ is of adequate length. Moreover $N_f \subseteq M_f$, so the condition on π_i still holds for N_f, hence ω is also an assignment type for N. For the same reason V is also a term assignment for N.

For *(ii)*, simply let M be the subterm of some term P, then by *(i)* applied to P we are done. □

Lemma 5.12. *Let* $\mathcal{A}_b$ *be a value assignment. Let* f *be a variable listing for* $\mathrm{T}_b^{\frown}$*-term* $M : \sigma$ *with* $V : \omega$ *as a corresponding derived term assignment of* $\mathcal{A}$ *for* M*. Then for each* $i \in M_f$ *there exists closed* T^- *terms*

$$\boldsymbol{GetVar}^i_\omega : \omega \to \omega[i], \boldsymbol{SetVar}^i_\omega : \omega, \omega[i] \to \omega$$

of recursor rank zero, such that when $f^{-1}(i) = x$

(i) $\boldsymbol{val}_{b+1}(\boldsymbol{GetVar}^i_\omega(V)) = \mathcal{A}(x)$

(ii) $\boldsymbol{SetVar}^i_\omega(V, t)$ *is a derived term assignment of* $\mathcal{A}^x_v$ *for* M*, for any closed* T_b^-*-term* $t : \omega[i]$ *with* $v = \boldsymbol{val}_{b+1}(t)$

Proof. For *(i)* let

$$\mathbf{GetVar}^i_\omega = \lambda V^\omega.fst.snd^i.V$$

and observe

$$\begin{aligned}
\mathbf{val}_{b+1}(\mathbf{GetVar}^i_\omega(V)) &= \mathbf{val}_{b+1}(fst.snd^i.V) && \mathbf{GetVar}^i_\omega \text{ def.}\\
&= \mathcal{A}(f^{-1}(i)) && V \text{ def.}\\
&= \mathcal{A}(x) && f^{-1}(i) = x
\end{aligned}$$

For *(ii)* let

$$\mathbf{SetVar}^i_\omega = \lambda V^\omega.\left\langle \mathbf{GetVar}^0_\omega(V), ..., t, ..., \mathbf{GetVar}^\ell_\omega(V)\right\rangle$$

where t is in the i'th position and ℓ is the length of f on M. We show that $\mathbf{SetVar}^i_\omega$ is a derived term assignment of $\mathcal{A}$ for M by first letting $j \in M_f$ be such that $j \neq i$. Let $y = f^{-1}(j)$ and observe that $x \neq y$ since $i \neq j$ and f is injective. We see

$$\begin{aligned}\mathbf{val}_{b+1}(fst.snd^j.\mathbf{SetVar}^i_\omega(V,t)) &= \mathbf{val}_{b+1}(\mathbf{GetVar}^j_\omega(V)) && j \neq i\\ &= \mathcal{A}(y) && (i)\\ &= \mathcal{A}^x_v(y) && x \neq y\end{aligned}$$

Now let $i = j$, and so

$$\begin{aligned}\mathbf{val}_{b+1}(fst.snd^j.\mathbf{SetVar}^i_\omega(V,t)) &= \mathbf{val}_{b+1}(t) && j = i\\ &= v\\ &= A^x_v(x) && \mathcal{A}^x_v \text{ def.}\end{aligned}$$

and so $\mathbf{SetVar}^i_\omega(V,t)$ fulfills the requirements of Definition 5.10 and is indeed a derived term assignment of $\mathcal{A}$ for M.

Lastly it should be clear that $\mathbf{GetVar}^i_\omega$ and $\mathbf{SetVar}^i_\omega$ both are recursor free, and hence of recursor rank zero. □

5.4 Computing $\mathbf{nval}^{\mathcal{A}}_{b+1}$

Theorem 5.13. *Let $\mathcal{A}_{b+1}$ be a value assignment. Let $M : \sigma$ be a $\mathrm{T}^{\smile}_b$-term with variable listing f and a corresponding derived term assignment $V : \omega$ of $\mathcal{A}$ with recursor rank zero. Then there exists a closed T^- term $\boldsymbol{M} : \iota, \omega \to \bar{\sigma}$ such that*

$$\boldsymbol{val}_{b+1}(\boldsymbol{M}(k_b, V)) = \boldsymbol{nval}^{\mathcal{A}}_{b+1}(M)$$

for all $b \geq 2$.

Moreover $Rk(\boldsymbol{M}) \leq 2(Tr(M) + 1)$.

Proof. We prove this by induction on the structure of M. Observe that lemma 5.11 guarantees that V is also a term assignments for subterms in the induction, so we can apply the induction hypothesis.

Case $M = k_n$. Let

$$\mathbf{M} = \lambda b^\iota V^\omega.\mathbf{Exp}^\iota_\iota(b, k_2, k_n)$$

and observe

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) = \mathbf{val}_{b+1}(\mathbf{Exp}^\iota_\iota(k_b, k_2, k_n)) &= \mathbf{val}_{b+1}(k_2)^{\mathbf{val}_{b+1}(k_n)}\\ = 2^n = \mu_{b+1}(n) &= \mathbf{nval}^{\mathcal{A}}_{b+1}(k_n)\end{aligned}$$

also

$$Rk(\mathbf{M}) = Rk(\mathbf{Exp}^\iota_\iota) = 1 < 2 = 2(Tr(k_n) + 1)$$

Case $M = x^\sigma$. Let $f(x) = j$ and let

$$\mathbf{M} = \lambda b^\iota V^\omega.\mathbf{GetVar}^j_\omega(V)$$

and observe

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) &= \mathbf{val}_{b+1}(\mathbf{GetVar}^j_\omega(V)) \\ &= \mathcal{A}(x) && \textit{(i)} \text{ in Lemma 5.12} \\ &= \mathbf{nval}^{\mathcal{A}}_{b+1}(x)\end{aligned}$$

also

$$Rk(\mathbf{M}) = Rk(\mathbf{GetVar}^j_\omega) = 0 < 2(Tr(x) + 1)$$

Case $M = P^{\sigma\to\tau}Q^\sigma$. Let

$$\mathbf{M} = \lambda b^\iota V^\omega.\mathbf{Digit}^\tau_\sigma(b, \mathbf{P}(b, V), \mathbf{Q}(b, V))$$

and observe

$$\begin{aligned}\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) &= \mathbf{val}_{b+1}(\mathbf{Digit}^\tau_\sigma(k_b, \mathbf{P}(k_b, V), \mathbf{Q}(k_b, V))) \\ &= \delta^{\sigma\to\tau}_{b+1}(\mathbf{val}_{b+1}(\mathbf{P}(k_b, V)), \mathbf{val}_{b+1}(\mathbf{Q}(k_b, V))) \\ &= \delta^{\sigma\to\tau}_{b+1}(\mathbf{nval}^{\mathcal{A}}_{b+1}(P), \mathbf{nval}^{\mathcal{A}}_{b+1}(Q)) && \text{IH} \\ &= \mathbf{nval}^{\mathcal{A}}_{b+1}(PQ)\end{aligned}$$

also

$$\begin{aligned}Rk(\mathbf{M}) &= \max\{Rk(\mathbf{Digit}^\tau_\sigma), Rk(\mathbf{P}), Rk(\mathbf{Q})\} \\ &\leq \max\{2(lv(\sigma\to\tau) + 1), 2(Tr(P) + 1), 2(Tr(Q) + 1)\} && \text{IH} \\ &= 2(\max\{lv(\sigma\to\tau), Tr(P), Tr(Q)\} + 1) \\ &= 2(\max\{Tr(P), Tr(Q)\} + 1) && \text{Lemma 2.3} \\ &= 2(Tr(PQ) + 1)\end{aligned}$$

Case $M = \lambda x^\sigma.P^\tau$. Let $f(x) = j$ and let

$$\mathbf{M} = \lambda b^\iota V^\omega.\mathbf{Table}^\tau_\sigma(b, \lambda x^{\bar\sigma}.\mathbf{P}(b, \mathbf{SetVar}^j_\omega(V, x)))$$

and observe

$$\begin{aligned}
&\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) \\
&= \mathbf{val}_{b+1}(\mathbf{Table}^{\tau}_{\sigma}(k_b, \lambda x^{\bar{\sigma}}.\mathbf{P}(k_b, \mathbf{SetVar}^{j}_{\omega}(V, x)))) \\
&= \sum_{i<\|\sigma\|_{b+1}} \delta^{\bar{\sigma}\to\bar{\tau}}_{b+1}(\mathbf{val}_{b+1}(\lambda x^{\bar{\sigma}}.\mathbf{P}(k_b, \mathbf{SetVar}^{j}_{\omega}(V, x))), i) \times \|\tau\|^{i}_{b+1} && \text{Lemma 5.7} \\
&= \sum_{i<\|\sigma\|_{b+1}} \delta^{\bar{\sigma}\to\bar{\tau}}_{b+1}\left(\sum_{z<|\bar{\sigma}|_{b+1}} \mathbf{val}^{\mathcal{A}^{x}_{z}}_{b+1}(\mathbf{P}(k_b, \mathbf{SetVar}^{j}_{\omega}(V, x))) \times |\bar{\tau}|^{z}_{b+1}, i\right) \times \|\tau\|^{i}_{b+1} && \mathbf{val}(\cdot) \\
&= \sum_{i<\|\sigma\|_{b+1}} \mathbf{val}^{\mathcal{A}^{x}_{i}}_{b+1}(\mathbf{P}(k_b, \mathbf{SetVar}^{j}_{\omega}(V, x))) \times \|\tau\|^{i}_{b+1} && \delta^{\bar{\sigma}\to\bar{\tau}}_{b+1} \\
&= \sum_{i<\|\sigma\|_{b+1}} \mathbf{nval}^{\mathcal{A}^{x}_{\mathcal{A}^{x}_{i}(i)}}_{b+1}(P) \times \|\tau\|^{i}_{b+1} && \text{IH} \\
&= \sum_{i<\|\sigma\|_{b+1}} \mathbf{nval}^{\mathcal{A}^{x}_{i}}_{b+1}(P) \times \|\tau\|^{i}_{b+1} \\
&= \mathbf{nval}^{\mathcal{A}}_{b+1}(\lambda x.P)
\end{aligned}$$

also

$$\begin{aligned}
Rk(\mathbf{M}) &= \max\{Rk(\mathbf{Table}^{\tau}_{\sigma}), Rk(\mathbf{P})\} \\
&\leq \max\{2(lv(\sigma \to \tau) + 1), 2(Tr(P) + 1))\} && \text{IH} \\
&= 2(\max\{lv(\sigma \to \tau), Tr(P)\} + 1) \\
&= 2(Tr(\lambda x^{\sigma}.P^{\tau}) + 1)
\end{aligned}$$

Case $M = P^{\sigma}|Q^{\sigma}, \langle P^{\sigma}, Q^{\tau}\rangle$. We only consider the first case,since the second is completely analogous. Let

$$\mathbf{M} = \lambda b^{\iota} V^{\omega}.\mathbf{Merge}_{\sigma}(\mathbf{P}(b, V), \mathbf{Q}(b, V))$$

and observe

$$\begin{aligned}
\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) &= \mathbf{val}_{b+1}(\mathbf{Merge}^{\sigma}_{\sigma}(\mathbf{P}(k_b, V), \mathbf{Q}(k_b, V))) \\
&= Merge^{\sigma}_{b+1}(\mathbf{val}_{b+1}(\mathbf{P}(k_b, V)), \mathbf{val}_{b+1}(\mathbf{Q}(k_b, V))) \\
&= Merge^{\sigma}_{b+1}(\mathbf{nval}^{\mathcal{A}}_{b+1}(P), \mathbf{nval}^{\mathcal{A}}_{b+1}(Q)) && \text{IH} \\
&= \mathbf{nval}_{b+1}(P|Q)
\end{aligned}$$

also

$$\begin{aligned}
Rk(\mathbf{M}) &= \max\{Rk(\mathbf{Merge}_{\sigma}), Rk(\mathbf{P}), Rk(\mathbf{Q})\} \\
&\leq \max\{2(lv(\sigma) + 1), 2(Tr(P) + 1), 2(Tr(Q) + 1)\} && \text{IH} \\
&= \max\{2(Tr(P) + 1), 2(Tr(Q) + 1)\} && 2.3 \\
&= 2(\max\{Tr(P), Tr(Q)\} + 1) \\
&= 2(Tr(P|Q) + 1)
\end{aligned}$$

Case $M = fst.P^{\sigma\times\tau}, snd.P^{\sigma\times\tau}$. We only consider the first case,since the secondly is completly analogus. Let

$$\mathbf{M} = \lambda b^\iota V^\omega.\mathbf{Left}^\tau_\sigma(b, \mathbf{P}(b, V))$$

and observe

$$\begin{aligned}
\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) &= \mathbf{val}_{b+1}(\mathbf{Left}^\tau_\sigma(k_b, \mathbf{P}(k_b, V))) \\
&= \lambda^{\sigma\times\tau}_{b+1}(\mathbf{val}_{b+1}(\mathbf{P}(b, V))) \\
&= \lambda^{\sigma\times\tau}_{b+1}(\mathbf{nval}^{\mathcal{A}}_{b+1}(P)) && \text{IH} \\
&= \mathbf{nval}^{\mathcal{A}}_{b+1}(fst.P)
\end{aligned}$$

also

$$\begin{aligned}
Rk(\mathbf{M}) &= \max\{Rk(\mathbf{Left}^\tau_\sigma), Rk(\mathbf{P})\} \\
&\leq \max\{2(lv(\sigma\times\tau) + 1), 2(Tr(P) + 1))\} && \text{IH} \\
&= 2(\max\{lv(\sigma\times\tau), Tr(P)\} + 1) \\
&= 2(Tr(P) + 1) && 2.3 \\
&= 2(Tr(fst.P) + 1)
\end{aligned}$$

Case $M = R_\sigma(N^\iota, F^{\iota,\sigma\to\sigma}, G^\sigma)$. Let

$$\mathbf{M} = \lambda b^\iota V^\omega.\mathbf{Rec}_\sigma(b, \mathbf{N}(b, V), \mathbf{F}(b, V), \mathbf{G}(b, V))$$

and observe

$$\begin{aligned}
&\mathbf{val}_{b+1}(\mathbf{M}(k_b, V)) \\
&= \mathbf{val}_{b+1}(\mathbf{Rec}_\sigma(k_b, \mathbf{N}(k_b, V), \mathbf{F}(k_b, V), \mathbf{G}(k_b, V))) \\
&= \Upsilon^\sigma_{b+1}(\xi_{b+1}(\mathbf{val}_{b+1}(\mathbf{N}(k_b, V))), \mathbf{val}_{b+1}(\mathbf{F}(k_b, V)), \mathbf{val}_{b+1}(\mathbf{G}(k_b, V)))) \\
&= \Upsilon^\sigma_{b+1}(\xi_{b+1}(\mathbf{nval}^{\mathcal{A}}_{b+1}(N)), \mathbf{nval}^{\mathcal{A}}_{b+1}(F), \mathbf{nval}^{\mathcal{A}}_{b+1}(G))) && \text{IH} \\
&= \mathbf{nval}^{\mathcal{A}}_{b+1}(R_\sigma(N, F, G))
\end{aligned}$$

also

$$\begin{aligned}
Rk(\mathbf{M}) &= \max\{Rk(\mathbf{Rec}_\sigma), Rk(\mathbf{N}), Rk(\mathbf{F}), Rk(\mathbf{G})\} \\
&\leq \max\{2(lv(\sigma) + 1), 2(Tr(N) + 1), 2(Tr(F) + 1), 2(Tr(G) + 1)\} && \text{IH} \\
&= 2(\max\{lv(\sigma), Tr(N), Tr(F), Tr(G)\} + 1) \\
&= 2(\max\{Tr(N), Tr(F), Tr(G)\} + 1) && 2.3 \\
&= 2(Tr(R_\sigma(N, F, G)) + 1)
\end{aligned}$$

□

Chapter 6

Complexity classes $\mathcal{F}_i^-$ and $\mathcal{F}_i^{\backsim}$

Definition 6.1. *We define a problem as any $S \subseteq \mathbb{N}$. We say that a closed* T^- *term $N : \iota \to \iota$ is a* T^- *program, and likewise for* $\mathrm{T}^{\backsim}$*. Let N be a* T^- *or* $\mathrm{T}^{\backsim}$ *program, we say that N decides a problem S if $N(k_n) \rhd k_1 \Leftrightarrow n \in S$. Two programs are said to be equivalent when they decide the same problem. Let $\mathcal{F}_i^-$ be the family of all problems decided by a* T^- *program with recursor rank no greater then i, and likewise for $\mathcal{F}_i^{\backsim}$.*

Definition 6.2. *We define the desirable terms, denoted $\mathcal{D}$, as all* $\mathrm{T}^{\backsim}$*-terms on γ-normal form and have no subterm of the forms $fst.(P|Q)$, $snd.(P|Q)$ or $(P|Q)R$.*

Lemma 6.3. *For any* $\mathrm{T}_b^{\backsim}$*-term $M : \sigma$ there is a desirable term $N : \sigma$ such that $Rk(N) \leq Rk(M)$ and N and for any assignment $\mathcal{A}_{b+1}$*

$$[\![M]\!]_{\mathcal{A}} = [\![N]\!]_{\mathcal{A}}$$

Proof. Consider the following procedure.

REDUCE($\mathrm{T}^{\backsim}$-term X)

1. Let X_1 be the result of reducing X to γ-normal form using no γ reduction.

2. Let X_2 be the result of replacing any $fst.(P|Q)$ with $(fst.P|fst.Q)$, any $snd.(P|Q)$ with $(snd.P|snd.Q)$ and any $(P|Q)R$ with $(PR|QR)$ in X_1.

3. No replacements where made in *2.* then return X_2, otherwise return **REDUCE**(X_2)

First observe that the procedure terminates for any $\mathrm{T}^{\backsim}$-term as input,the reason is that we could extend our calculus to have the reduction rules;

(i) $fst.(P|Q) \rhd^1 (fst.P|fst.Q)$

(ii) $snd.(P|Q) \rhd^1 (snd.P|snd.Q)$

(iii) $(P|Q)R \rhd^1 (PR|QR)$

and that calculus would be strongly normalizing as well. Also observe that

$$[\![\mathbf{REDUCE}(M)]\!]_{\mathcal{A}} = [\![M]\!]_{\mathcal{A}}$$

since we only use non γ-reductions and *(i)*,*(ii)*,*(iii)* which all preserve $[\![\ \cdot\ ]\!]_{\mathcal{A}}$. $\mathbf{REDUCE}(M)$ will also be desirable because of the halting condition, and since no new recursors are introduced

$$Rk(\mathbf{REDUCE}(M)) \leq Rk(M)$$

□

Lemma 6.4. *Let N be a* T$^{\frown}$*-program, there exist a* T$^{\frown}$*-program M such that*

$$[\![M]\!] = [\![N]\!]$$

and $Tr(N) \leq Rk(M) + 1$.

Proof. Let N be the desirable program from Lemma 6.3 when given M, such that $[\![M]\!] = [\![N]\!]$ and $Rk(N) \leq Rk(M)$. By *(iii)* in Lemma 2.3 we can select a subterm $R : \phi$ of N such that $Tr(N) = lv(\phi) = Tr(R)$, and which is not of the form (PQ), $\langle P, Q\rangle$,$(P|Q)$,$fst.P$ or $snd.P$. The latter requirement will always be satisfied since either P or Q will qualify in all of the above cases.

Lets consider the remaining possible forms of R. It cannot be a numeral or variable because of the type of N, and N being closed respectively. It cannot be of the form $R_\phi(A, B, C)$ since then $Tr(B) \geq lv(\iota, \phi \to \phi) = lv(\phi) + 1 > lv(\phi) = Tr(N)$ contradicting *(ii)* in Lemma 2.3. Therefor it must be that

$$R = \lambda x^\sigma.X^\tau \text{ and } \phi = \sigma \to \tau$$

Now we define $\mathcal{W}$[1] as the least set such that

(i) $R \in \mathcal{W}$

(ii) $\lambda x^\pi.Q \in \mathcal{W}$ where $Q \in \mathcal{W}$ and $lv(\pi) < lv(\phi)$

(iii) $(Q|P), (P|Q) \in \mathcal{W}$ where $Q \in \mathcal{W}$ and $Tr(P) \leq lv(\phi)$

(iv) $\langle Q|P\rangle, \langle P|Q\rangle \in \mathcal{W}$ where $Q \in \mathcal{W}$ and $Tr(P) \leq lv(\phi)$

[1] $\mathcal{W}$ is meant to capture a substructure of N.

and we can by induction on the structure of $\mathcal{W}$-terms easily show that for any term $U : \rho \in \mathcal{W}$ we have

$$Tr(U) = lv(\rho) = lv(\phi) \qquad (\clubsuit)$$

Let U be the longest subterm of N which is also in $\mathcal{W}$, this will always be possible since at least $R \in \mathcal{W}$.

Assume first that $U = N$, then by ($\clubsuit$) we have $Tr(N) = lv(\iota \to \iota) = 1$. For any recursor $R_\pi(A, B, C)$ in N we see that $lv(\pi) = 0$, otherwise $Tr(B) \geq lv(\iota, \pi \to \pi) = lv(\pi) + 1 \geq 2 > Tr(N)$ contradicting *(ii)* in Lemma 2.3. So we have $Tr(N) = Rk(N) + 1$

Assume now that U is not N itself. We examine in what context U must occur in N. We can conclude that U cannot occur in an abstraction, a pair or a nondeterministic choice since no longer term then N is in $\mathcal{W}$. It cannot occur on the right hand side of an application, that is as (VU) since then $Tr(V) > lv(\phi) = Tr(N)$ contradicting *(ii)* in Lemma 2.3. It cannot occur on the left hand side of the application either, since N is desirable. It cannot occur in a projection since N is desirable. The only remaining case is in a recursor $R_\pi(A, B, C)$. Obviously U cannot be A because it is a program. It cannot be C either, since then $Tr(B) \geq lv(\phi) + 1 > Tr(N)$ contradicting *(ii)* in Lemma 2.3. So $B = U$ and $R_\pi(A, U, C)$ must be a subterm of N, and ρ is of the form $\iota, \pi \to \pi$. We now see $lv(\pi) + 1 = lv(\rho) = lv(\phi)$, and there can be no greater recursor type by the same argument applied several times before, so $Rk(N) = lv(\pi) = lv(\rho) - 1$, that is $Rk(N) + 1 = lv(\rho) = lv(\phi) = Tr(N)$ as desired. So under both assumptions $Tr(N) = Rk(N) + 1 \leq Rk(M) + 1$ as desired. □

Theorem 6.5. *For any nondeterministic program with recursor rank i there is an equivalent deterministic program with recursor rank no more then $2(i + 2)$, and therefore $\mathcal{F}_i^{\backsim} \subseteq \mathcal{F}_{2(i+2)}^{-}$.*

Proof. Let Z be a $\mathrm{T}^{\backsim}$ program such that $Rk(Z) \leq i$ and Z decides the problem $\mathcal{L}$. By lemma 6.4 there is a $\mathrm{T}^{\backsim}$-program N such that $[\![N]\!] = [\![Z]\!]$ and $Tr(N) \leq Rk(Z) + 1$. By lemma 5.11 there exists a variable listing f for N, and corresponding assignment type ω for N. Let $r_0 = 1$ if $N(k_0) \rhd k_1$, otherwise let $r_0 = 0$. Likewise let $r_1 = 1$ if $N(k_1) \rhd k_1$, otherwise $r_1 = 0$. We define the following program

$$\begin{aligned} M = \lambda x^\iota.\mathbf{Cond}_\iota(&\mathbf{Eq}_\iota(x, k_0), k_{r_1}, \\ &\mathbf{Cond}_\iota(\mathbf{Eq}_\iota(x, k_1), k_{r_2}, \\ &\mathbf{IsMember}(x, \mathbf{Digit}_\iota^\iota(x, \mathbf{N}(x, \mathbf{0}_\omega), \mathbf{Exp}_\iota^\iota(x, k_2, x)), k_1))) \end{aligned}$$

where $\mathbf{N} : \iota, \omega, \to \overline{\iota \to \iota}$ is the term guaranteed by Theorem 5.13. Now if $b = 0$ or $b = 1$, then M will decide k_b correctly with respect to $\mathcal{L}$, since it has been statically programmed to do so.

Assume now that $b \geq 2$ for the rest of the proof. Let $\mathcal{A}$ be the value assignment that maps all variables to 0, and recognize that $\mathbf{0}_\omega$ is indeed a derived value assignment of $\mathcal{A}$ for M, we then see

$$\begin{aligned}
\mathbf{val}_{b+1}(M(k_b)) &= \beta_{b+1}(\delta^{\iota\to\iota}_{b+1}(\mathbf{val}_{b+1}(\mathbf{N}(k_b,\mathbf{0}_\omega)),2^b),1) \\
&= \beta_{b+1}(\delta^{\iota\to\iota}_{b+1}(\mathbf{val}_{b+1}(\mathbf{N}(k_b,\mathbf{0}_\omega)),2^b),1) \\
&= \beta_{b+1}(\delta^{\iota\to\iota}_{b+1}(\mathbf{nval}^{\mathcal{A}}_{b+1}(N),2^b),1) && \text{Theorem 5.13} \\
&= \beta_{b+1}(\delta^{\iota\to\iota}_{b+1}(\mathbf{nval}_{b+1}(N),2^b),1) && N \text{ is closed} \\
&= \beta_{b+1}(\delta^{\iota\to\iota}_{b+1}(\mathbf{nval}_{b+1}(N),\mu_{b+1}(b)),1) \\
&= \beta_{b+1}(\delta^{\iota\to\iota}_{b+1}(\mathbf{nval}_{b+1}(N),\mathbf{nval}_{b+1}(k_b),1)) \\
&= \beta_{b+1}(\mathbf{nval}_{b+1}(N(k_b)),1)
\end{aligned}$$

and so

$$\begin{aligned}
b \in \mathcal{L} &\Leftrightarrow Z(k_b) \rhd k_1 && \text{Z decides } \mathcal{L} \\
&\Leftrightarrow 1 \in [\![Z(k_b)]\!] && \text{Theorem 3.17} \\
&\Leftrightarrow 1 \in [\![Z]\!][\![k_b]\!] \Leftrightarrow 1 \in [\![N]\!][\![k_b]\!] && [\![N]\!] = [\![Z]\!] \\
&\Leftrightarrow 1 \in [\![N(k_b)]\!] \Leftrightarrow N(k_b) \rhd k_1 && \text{Theorem 3.17} \\
&\Leftrightarrow \mu_{b+1}(\{1\}) \preccurlyeq^{\iota}_{b+1} \mathbf{nval}_{b+1}(N(k_b)) && \text{Corollary 3.31} \\
&\Leftrightarrow \{1\} \subseteq \xi_{b+1}(\mathbf{nval}_{b+1}(N(k_b))) && \preccurlyeq^{\iota}_{b+1} \text{ def.} \\
&\Leftrightarrow \beta_{b+1}(\mathbf{nval}_{b+1}(N(k_b)),1) = 1 && \beta_{b+1}(n,1)=1 \text{ iif } \{1\} \subseteq \xi_{b+1}(n) \\
&\Leftrightarrow \mathbf{val}_{b+1}(M(k_b)) = 1 && \text{above} \\
&\Leftrightarrow M(k_b) \rhd k_1 && \text{Lemma 4.4}
\end{aligned}$$

and so M decides $\mathcal{L}$ as well, so M is equivalent to Z, moreover

$$\begin{aligned}
Rk(M) &= \max\{Rk(\mathbf{N}), Rk(\mathbf{Digit}^{\iota}_{\iota}), Rk(\mathbf{Exp}^{\iota}_{\iota})\} \\
&= \max\{2(Tr(N)+1),4,4)\} \\
&\leq \max\{2((Rk(Z)+1)+1),4)\} \\
&= \max\{2(i+2),4\} \\
&= 2(i+2)
\end{aligned}$$

□

We may supplement our result in Theorem 6.5 and say that the equivalent deterministic term not only exists, but it is computable from the nondeterministic term itself. The computation is simply to take the initial nondeterministic term M, apply the procedure in Lemma 6.3, then take the result and build a new term N as in Theorem 5.13, which then computes the interpretation of M.

Based on the proof of

$$\mathcal{F}^{-}_{2i} = \text{SPACE } 2^{\text{LIN}}_{i} \qquad (\clubsuit)$$

in [1] we have reason to conjecture

Conjecture 6.6. $\mathcal{F}^{\smile}_{2i} = \mathit{NSPACE}\ 2^{\mathit{LIN}}_{i}$

The proof of this would be completely analogous to (♣). We would first develop terms analogous to $\mathbf{Succ}_\sigma$ and $\mathbf{Leq}_\sigma$ for $\mathbf{nval}(\cdot)$, and then use them to simulate a nondeterministic space bound Turing machine, proving NSPACE $2_i^{\text{LIN}} \subseteq \mathcal{F}_{2i}^{\asymp}$. Then we would use a nondeterministic space bound Turing machine to rewrite the running program with recursor rank $2i$ into a term with recursor rank i, and then compute $\mathbf{nval}(\cdot)$ for this term, thereby proving $\mathcal{F}_{2i}^{\asymp} \subseteq$ NSPACE 2_i^{LIN}. This conjecture combined with the wellknown result from complexity theory that SPACE $2_i^{\text{LIN}} =$ NSPACE 2_i^{LIN} for $i > 0$ would give us

Conjecture 6.7. $\mathcal{F}_{2i}^{\asymp} = \mathcal{F}_{2i}^{-}$ *for* $i > 0$

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