**Universitetet i Oslo
Institutt for informatikk**

Only Knowing with Confidence Levels: Reductions and Complexity

Espen Hallenstvedt
Lian

Hovedoppgave

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Chapter 1

Introduction

An important problem in nonmonotonic reasoning is deciding what the precise beliefs of an agent might be, given an incomplete specification. The notion of a stable expansion of an autoepistemic theory is a way of capturing this. As autoepistemic logic is not a logic per se, to actually find the stable expansions, one cannot do this on the object level, but has to use an algorithm involving meta-concepts like set inclusion. Levesque's "Only knowing" logic can be used to represent autoepistemic theories, with the benefit of finding the expansions within the logic, on the object level. Waaler's logic $\mathcal{A}\mathcal{E}$ generalizes this logic, adding confidence levels. We will show how a generalization of stable expansions can be found, strictly using equivalences in $\mathcal{A}\mathcal{E}$, thus providing a rewriting procedure.

In Ch. 2, we introduce $\mathcal{A}\mathcal{E}$ and the related $\mathcal{A}\mathcal{E}_\rho$. In Ch. 3, we give three rewriting procedures for the case when there is only one confidence level. In Ch. 4 we give a rewriting procedure for the general case with multiple confidence levels. In Ch. 5 we examine the complexity of the problem of deciding whether expansions exist, and give an algorithm for generating them.

Chapter 2

Logic

2.1 Introduction

In this chapter we introduce the logic $\mathcal{A}\mathcal{E}$, which is the logic of our main interest. Since it is a propositional modal logic, propositional logic and modal logic are presented first.

2.2 Propositional logic

Propositional logic (PL) is the basis for the other logics we will see in this thesis. Sentences in the propositional language are on the following form.

$p_1, \dots, p_n$	propositional variables
$\top$	verum, the truth constant
$\perp$	falsum, the falsity constant
$\neg\varphi$	negation of φ
$\varphi \vee \psi$	disjunction of φ and ψ
$\varphi \wedge \psi$	conjunction of φ and ψ
$\varphi \supset \psi$	φ materially implies ψ
$\varphi \equiv \psi$	φ and ψ are equivalent

Our *propositional language* $\mathcal{L}_{\text{PL}}$ is a set of well-formed formulae, of which the simplest are called *atomic*: $\top$, $\perp$ and elements from a countable set of propositional variables $\Phi = \{p_1, \dots, p_n\}$. *Non-atomic* formulae are formed using the atomic formulae and the symbols $\neg$, $\vee$, $\wedge$, $\supset$ and $\equiv$. Let $\mathcal{L}_{\text{PL}}$ be the least set that satisfies the following properties: $\Phi \cup \{\top, \perp\} \subseteq \mathcal{L}_{\text{PL}}$, and

if $\varphi, \psi \in \mathcal{L}_{\text{PL}}$ then so are the following: $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \supset \psi)$ and $(\varphi \equiv \psi)$. The parentheses are usually skipped.

The *semantics* of PL is given by a *valuation* function $v : \mathcal{L}_{\text{PL}} \rightarrow \{0, 1\}$ that assigns a truth value to each atomic formula and is constrained by

$$\begin{aligned} v(\top) &= 1 \\ v(\perp) &= 0 \\ v(\neg\varphi) &= 1 \text{ iff } v(\varphi) = 0 \\ v(\varphi \vee \psi) &= 1 \text{ iff } v(\varphi) = 1 \text{ or } v(\psi) = 1 \\ v(\varphi \wedge \psi) &= 1 \text{ iff } v(\varphi) = 1 \text{ and } v(\psi) = 1 \\ v(\varphi \supset \psi) &= 1 \text{ iff } v(\varphi) = 0 \text{ or } v(\psi) = 1 \\ v(\varphi \equiv \psi) &= 1 \text{ iff } v(\varphi) = v(\psi) \end{aligned}$$

Since each atomic formula $p_1, \dots, p_n$ evaluates to either 0 or 1, there are 2^n different valuations. How v evaluates can be summarized in the *truth table*.

φ	ψ	$\varphi \vee \psi$	$\varphi \wedge \psi$	$\varphi \supset \psi$	$\varphi \equiv \psi$
0	0	0	0	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	1	1	1	1

φ	$\neg\varphi$
0	1
1	0

$\top$	$\perp$
1	0

We write $v \models \varphi$ iff $v(\varphi) = 1$ and $v \not\models \varphi$ iff $v(\varphi) = 0$. If $v \models \varphi$ holds for all v , we write $\models \varphi$, and say that φ is *valid*, and that φ is a *tautology*. If $v \models \varphi$ holds for some v , we say that φ is *satisfiable*.

In addition to v , we define a vector-valued (see Appendix) valuation function π . We do this in the standard inductive way (like for v), initially specifying the values of the vectors corresponding to propositional variables $\pi(p_1), \dots, \pi(p_n)$. These are set such that each bit corresponds to a different valuation. Assume $n = |\Phi|$. Define $\pi : \mathcal{L}_{\text{PL}} \rightarrow \mathbf{2}^{2^n}$ as

$$\pi(p_1)_i = 1 \text{ iff } i > 2^{n-1} \quad (2.1)$$

$$\pi(p_k) = \pi(p_{k-1}) \oplus (\pi(p_{k-1}) \ll 2^{n-k}) \text{ where } 2 \leq k \leq n \quad (2.2)$$

$$\pi(\top)_i = 1 \quad (2.3)$$

$$\pi(\perp)_i = 0 \quad (2.4)$$

$$\pi(\neg\varphi) = \neg\pi(\varphi) \quad (2.5)$$

$$\pi(\varphi \vee \psi) = \pi(\varphi) \vee \pi(\psi) \quad (2.6)$$

$$\pi(\varphi \wedge \psi) = \pi(\varphi) \wedge \pi(\psi) \quad (2.7)$$

$$\pi(\varphi \equiv \psi) = \pi(\varphi) \equiv \pi(\psi) \quad (2.8)$$

$$\pi(\varphi \supset \psi) = \neg\pi(\varphi) \vee \pi(\psi) \quad (2.9)$$

Example 2.1. For $n = 3$ we need 8 bits.

$$\begin{aligned}\pi(p) &= [0\ 0\ 0\ 0\ 1\ 1\ 1\ 1] \\ \pi(q) &= [0\ 0\ 1\ 1\ 0\ 0\ 1\ 1] \\ \pi(r) &= [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1] \\ &\quad 0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\end{aligned}$$

From looking at the vectors corresponding to the atomic formulae we see that they span all 8 valuations. Let $\varphi = (p \wedge q) \supset r$. Then

$$\pi(\varphi) = \pi((p \wedge q) \supset r) = \neg(\pi(p) \wedge \pi(q)) \vee \pi(r) = [1\ 1\ 1\ 1\ 1\ 1\ 0\ 1]$$

φ is false only in valuation 6, where p and q are false and r is true.

We need a notation for establishing a correspondence between our vector valuation function and the standard propositional valuation function. Assume $n = |\Phi|$, and let $v_0, \dots, v_{2^n-1}$ be defined by: $v_i \models p_j$ iff bit j in vec i is 1. For example, if $n = 3$, $v_5 = v_{101_2}$ is the valuation where p_1 is true, p_2 is false and p_3 is true. Note that $v_0, \dots, v_{2^n-1}$ are all different valuations.

Theorem 2.2. The i th bit in π corresponds to the i th valuation v_i .

$$\pi(\varphi)_i = v_i(\varphi)$$

Proof. Eq. 2.1 and 2.2 are proven by induction over k . The k th most significant bit is first set in the valuation immediately after the 2^{n-k} th. It is then set the next 2^{n-k} valuations. These 2^{n-k+1} bits amounts to a *block*. We will refer to 2^{n-k+1} as the *block size*. There are 2^{k-1} blocks (as there are $2^{n-k+1} \cdot 2^{k-1} = 2^n$ different valuations).

Base step: The most significant bit is first set in the valuation immediately after the 2^{n-1} th, and is set the next 2^{n-1} valuations, which amounts to all 2^n . So $v_i(p_1) = 1$ iff $i > 2^{n-1}$, and from Eq. 2.1, iff $\pi(p_1)_i = 1$.

Induction step: The induction hypothesis is that $\pi(p_{k-1})_i = v_i(p_{k-1})$. The block size is 2^{n-k+2} . Let $x = 2^{n-k}$. Shifting $\pi(p_{k-1})$ left x times amounts to a vector where in each block the x first bits are not set, the next $2x$ are, and the last x are not. $0_{[m]}$ and $1_{[m]}$ are vectors of length m where all elements are 0 and 1 respectively.

$$(0_{[2x]} \otimes 1_{[2x]}) \oplus (0_{[x]} \otimes 1_{[2x]} \otimes 0_{[x]}) = (0_{[x]} \otimes 1_{[x]} \otimes 0_{[x]} \otimes 1_{[x]})$$

XORing $\pi(p_{k-1})$ with the left-shifted vector gives us a vector with block size $2x = 2^{n-k+1}$, thus $\pi(p_k)_i = v_i(p_k)$. Eq. 2.3 through 2.9 can proven by induction over the length of formulae. $\square$

Our model becomes a pair (π, i) , such that i picks out the value of the i th valuation of φ from the vector $\pi(\varphi)$. We define the propositional satisfiability relation $\models$ as

$$(\pi, i) \models \varphi \text{ iff } \pi(\varphi)_i = 1$$

We write $(\pi, i) \not\models \varphi$ iff $\pi(\varphi)_i = 0$, and $\models \varphi$ iff $(\pi, i) \models \varphi$ holds for all i .

It follows immediately from Theorem 2.2 that the two propositional satisfiability relations are equivalent.

Corollary 2.3. $\models \varphi$ iff $\models \varphi$.

For $n = 2$, the following table summarizes how all non-equivalent formulae are valued with the standard valuation function v , our vector-valued valuation function π and their integer representation.

φ	$v_{00_2}(\varphi)$	$v_{01_2}(\varphi)$	$v_{10_2}(\varphi)$	$v_{11_2}(\varphi)$	$\pi(\varphi)$	int $\pi(\varphi)$
$\perp$	0	0	0	0	[0 0 0 0]	0
$p \wedge q$	0	0	0	1	[0 0 0 1]	1
$p \wedge \neg q$	0	0	1	0	[0 0 1 0]	2
p	0	0	1	1	[0 0 1 1]	3
$\neg p \wedge q$	0	1	0	0	[0 1 0 0]	4
q	0	1	0	1	[0 1 0 1]	5
$p \equiv \neg q$	0	1	1	0	[0 1 1 0]	6
$p \vee q$	0	1	1	1	[0 1 1 1]	7
$\neg p \wedge \neg q$	1	0	0	0	[1 0 0 0]	8
$p \equiv q$	1	0	0	1	[1 0 0 1]	9
$\neg q$	1	0	1	0	[1 0 1 0]	10
$q \supset p$	1	0	1	1	[1 0 1 1]	11
$\neg p$	1	1	0	0	[1 1 0 0]	12
$p \supset q$	1	1	0	1	[1 1 0 1]	13
$\neg p \vee \neg q$	1	1	1	0	[1 1 1 0]	14
$\top$	1	1	1	1	[1 1 1 1]	15

Table 2.1: Valuations for $n = 2$.

2.3 Modal logic

Sentences in the modal language include those of the propositional language. In addition, *modal* formulae contain subformulae on the following form.

$\Box\varphi$ φ is necessary
 $\Diamond\varphi$ φ is possible ($\Diamond$ is an abbreviation for $\neg\Box\neg$)

Let the language of modal logic, $\mathcal{L}_M$ be the least set that satisfies the following properties: $\Phi \cup \{\top, \perp\} \subseteq \mathcal{L}_{PL}$, and if $\varphi, \psi \in \mathcal{L}_M$ then so are the following: $\Box\varphi$, $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \supset \psi)$ and $(\varphi \equiv \psi)$.

By allowing axioms, we can define different modal logics. Some axiom schemata are

K	$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$	Distribution
T	$\Box\varphi \supset \varphi$	Truth
D	$\Box\perp \supset \perp$	The Deontic principle
4	$\Box\varphi \supset \Box\Box\varphi$	Positive introspection
5	$\neg\Box\varphi \supset \Box\neg\Box\varphi$	Negative introspection
Triv	$\varphi \supset \Box\varphi$	Triviality

Two common rules of inference are

MP	If $\vdash \varphi$ and $\vdash \varphi \supset \psi$ then $\vdash \psi$	Modus ponens
RN	If $\vdash \varphi$ then $\vdash \Box\varphi$	Rule of necessitation

All *normal* logics include the K axiom and the Rule of necessitation. The *logic* K is the weakest normal logic, and includes only these. Below are some common logics.

K	PL + MP + RN + axiom K
T	logic K + axiom T
S4	logic T + axiom 4
S5	logic S4 + axiom 5
K45	logic K + axiom 4 and 5
KD45	logic K + axiom D, 4 and 5

Systems containing both D and Triv collapses to PL because together they yield $\vdash \varphi \equiv \Box\varphi$ and $\vdash \varphi \equiv \Diamond\varphi$. Refer to [3] and [7] for details.

2.4 $\mathcal{AE}$

$\mathcal{AE}$ (from autoepistemic) is a sound and complete [11, 18] multimodal propositional logic used for describing the preferred beliefs of an agent.

2.4.1 Syntax

Sentences in the language of $\mathcal{AE}$ include those of the modal language. In addition, $\mathcal{AE}$ -formulae contain subformulae on the following form (the last three are abbreviations), where the subscript denotes the confidence level.

$B_k\varphi$	belief of at least φ with confidence k
$C_k\varphi$	belief of at most $\neg\varphi$ with confidence k
$b_k\varphi$	φ is consistent with the agent's beliefs ($\neg B_k\neg\varphi$)
$c_k\varphi$	φ is not the agent's strongest belief ($\neg C_k\neg\varphi$)
$O_k\varphi$	"all I know" is φ ($B_k\varphi \wedge C_k\neg\varphi$)

$(I, \prec)$ is a strict poset (see Appendix) where I is a finite non-empty set indexing confidence levels. Let $\mathcal{L}$ be the least set such that $\Phi \cup \{\top, \perp\} \subseteq \mathcal{L}$, and if $\varphi, \psi \in \mathcal{L}$ then so are the following: $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$, $(\varphi \supset \psi)$, $(\varphi \equiv \psi)$, $B_k\varphi$ if $k \in I$, $C_k\varphi$ if $k \in I$ and $\Box\varphi$.

$\mathcal{AE}$ contains every substitution instance of tautologies (PL). The additional axiom schemata of $\mathcal{AE}$ are

$\Box$	$\vdash \Box\varphi \equiv B_k\varphi \wedge C_k\varphi$	Definition of $\Box$
K_B	$\vdash B_k(\varphi \supset \psi) \supset (B_k\varphi \supset B_k\psi)$	The K axiom for B
K_C	$\vdash C_k(\varphi \supset \psi) \supset (C_k\varphi \supset C_k\psi)$	The K axiom for C
$B_\Box$	$\vdash B_k\varphi \supset \Box B_k\varphi$	The 4 axiom for B (cross)
$C_\Box$	$\vdash C_k\varphi \supset \Box C_k\varphi$	The 4 axiom for C (cross)
$\overline{B}_\Box$	$\vdash \neg B_k\varphi \supset \Box \neg B_k\varphi$	The 5 axiom for B (cross)
$\overline{C}_\Box$	$\vdash \neg C_k\varphi \supset \Box \neg C_k\varphi$	The 5 axiom for C (cross)
T	$\vdash \Box\varphi \supset \varphi$	The T axiom for $\Box$
P_B	$\vdash B_i\varphi \supset B_k\varphi$ for all $i \prec k$	Persistence for B
P_C	$\vdash C_i\varphi \supset C_k\varphi$ for all $i \prec k$	Persistence for C

The inference rules of $\mathcal{AE}$ are

MP	If $\vdash \varphi$ and $\vdash \varphi \supset \psi$ then $\vdash \psi$	Modus ponens
RN	If $\vdash \varphi$ then $\vdash \Box\varphi$	Rule of necessitation

If $\vdash (\varphi_1 \wedge \dots \wedge \varphi_n) \supset \psi$, we sometimes write $\varphi, \dots, \varphi_n \vdash \psi$. We define (if β then φ else ψ) as $(\varphi \wedge \beta) \vee (\psi \wedge \neg\beta)$, equivalently $(\beta \supset \varphi) \wedge (\neg\beta \supset \psi)$. $\vdash_{PL}$, the provability relation of PL is a subset of $\mathcal{A}\mathcal{E}$'s provability relation $\vdash$. For non-propositional φ , $\vdash_{PL}$ treats φ as atomic, so any substitution instance of a formula valid in PL is a tautology, for instance $\vdash_{PL} \Box p \supset \Box p$. Observe that $\Box$ is S5, while B_k and C_k are K45 [18].

A *modal atom* is formula on the form $B_k\varphi$, $C_k\varphi$ or $O_k\varphi$. A modal atom is *prime* if φ is propositional. A *modal literal* is a modal atom or its negation. A *completely modalized* formula is a formula where all propositional variables occur within the scope of a modal operator. $\text{mod}(\varphi)$ is the set of all modal atoms occurring as subformulae of φ . Define $\text{mod} : \mathcal{L} \rightarrow 2^{\mathcal{L}}$ as

$$\begin{aligned} \text{mod}(\varphi) &= \emptyset \text{ if } \varphi \in \mathcal{L}_{PL} \\ \text{mod}(\varphi) &= \{\varphi\} \cup \text{mod}(\psi) \text{ if } \varphi \text{ is a modal atom } (B_k\psi, C_k\psi, O_k\psi) \\ \text{mod}(\neg\varphi) &= \text{mod}(\varphi) \\ \text{mod}(\varphi \vee \psi) &= \text{mod}(\varphi \wedge \psi) = \text{mod}(\varphi \supset \psi) \\ &= \text{mod}(\varphi \equiv \psi) = \text{mod}(\varphi) \cup \text{mod}(\psi) \end{aligned}$$

An *extensional* occurrence of a subformula in φ is an occurrence not within the scope of any modal operator. $\text{ext}(\varphi)$ is the set of all modal atoms occurring extensionally as subformulae of φ . $\text{ext} : \mathcal{L} \rightarrow 2^{\mathcal{L}}$ is defined as mod except if φ is a modal atom.

$$\begin{aligned} \text{ext}(\varphi) &= \emptyset \text{ if } \varphi \in \mathcal{L}_{PL} \\ \text{ext}(\varphi) &= \{\varphi\} \text{ if } \varphi \text{ is a modal atom} \\ \text{ext}(\neg\varphi) &= \text{ext}(\varphi) \\ \text{ext}(\varphi \vee \psi) &= \text{ext}(\varphi \wedge \psi) = \text{ext}(\varphi \supset \psi) \\ &= \text{ext}(\varphi \equiv \psi) = \text{ext}(\varphi) \cup \text{ext}(\psi) \end{aligned}$$

This means that a modal atom β has an extensional occurrence in φ iff $\beta \in \text{ext}(\varphi)$. Note that $\text{ext}(\varphi) \subseteq \text{mod}(\varphi)$.

Example 2.4. Let $\varphi = B_1p \supset B_2\neg C_1q$. Then $\text{ext}(\varphi) = \{B_1p, B_2\neg C_1q\}$ and $\text{mod}(\varphi) = \text{ext}(\varphi) \cup \{C_1q\}$.

The *depth* of φ , $d(\varphi)$ is the maximal nesting of modal subformulae plus one. Define $d : \mathcal{L} \rightarrow \mathbb{N}$ as

$$\begin{aligned} d(\varphi) &= 0 \text{ if } \varphi \in \mathcal{L}_{PL} \\ d(\varphi) &= 1 + d(\psi) \text{ if } \varphi \text{ is a modal atom } (B\psi, C\psi \text{ or } O\psi) \\ d(\neg\varphi) &= d(\varphi) \\ d(\varphi \vee \psi) &= d(\varphi \wedge \psi) = d(\varphi \supset \psi) = d(\varphi \equiv \psi) = \max(d(\varphi), d(\psi)) \end{aligned}$$

Example 2.5. $d(p \supset q) = 0$. $d(p \vee Oq) = 1$. $d(Bp \wedge B\neg Cq) = 2$.

If $X = \{\varphi_1, \dots, \varphi_n\}$ and $\phi : \mathcal{L} \rightarrow \mathcal{L}$,

$$\bigvee_{\psi \in X} \phi(\psi) \quad \text{and} \quad \bigwedge_{\psi \in X} \phi(\psi)$$

denote $\phi(\varphi_1) \vee \dots \vee \phi(\varphi_n) \vee \perp$ and $\phi(\varphi_1) \wedge \dots \wedge \phi(\varphi_n) \wedge \top$ respectively. Note that $\bigvee_{\psi \in \emptyset} \phi(\psi) = \perp$ and $\bigwedge_{\psi \in \emptyset} \phi(\psi) = \top$.

Let $\pm p_i$ mean either p_i or $\neg p_i$, and let $n = |\Phi|$. An *atom* is a conjunction $\pm p_1 \wedge \dots \wedge \pm p_n$. Let $\alpha_1, \dots, \alpha_{2^n}$ be every atom. Observe that for every atom α , either $\alpha \vdash \varphi$ or $\alpha \vdash \neg \varphi$.

A logical *logical space*, denoted λ , is a formula on the form

$$\Diamond \alpha_1 \wedge \dots \wedge \Diamond \alpha_k \wedge \Box(\alpha_1 \wedge \dots \wedge \alpha_k),$$

equivalently $\Diamond \alpha_1 \wedge \dots \wedge \Diamond \alpha_k \wedge \neg \Diamond \alpha_{k+1} \wedge \dots \wedge \neg \Diamond \alpha_{2^n}$. $\alpha_1 \wedge \dots \wedge \alpha_k$ is called the *characteristic formula* of λ . The *maximal* logical space, denoted $\lambda_\top$, is $\Diamond \alpha_1 \wedge \dots \wedge \Diamond \alpha_{2^n}$. The *size* of λ is denoted $|\lambda|$, and is defined as the number of conjuncts in its characteristic formula, in particular $|\lambda_\top| = 2^n$. Every propositional formula is equivalent to a disjunction of atoms.

By $\varphi[\psi_1/\psi_2]$ we mean the formula which differs from φ only in that all subformula occurrences of ψ_1 are substituted with ψ_2 .

$$\varphi[\psi_1/\psi_2] = \begin{cases} \varphi & \text{if } \varphi \in \Phi \text{ and } \varphi \neq \psi_1 \\ \psi_2 & \text{if } \varphi \in \Phi \text{ and } \varphi = \psi_1 \end{cases}$$

$$(\Box \varphi)[\psi_1/\psi_2] = \Box(\varphi[\psi_1/\psi_2])$$

$$(B_k \varphi)[\psi_1/\psi_2] = B_k(\varphi[\psi_1/\psi_2])$$

$$(C_k \varphi)[\psi_1/\psi_2] = C_k(\varphi[\psi_1/\psi_2])$$

$$(\neg \varphi)[\psi_1/\psi_2] = \neg(\varphi[\psi_1/\psi_2])$$

$$(\varphi_1 \vee \varphi_2)[\psi_1/\psi_2] = \varphi_1[\psi_1/\psi_2] \vee \varphi_2[\psi_1/\psi_2]$$

$$(\varphi_1 \wedge \varphi_2)[\psi_1/\psi_2] = \varphi_1[\psi_1/\psi_2] \wedge \varphi_2[\psi_1/\psi_2]$$

$$(\varphi_1 \supset \varphi_2)[\psi_1/\psi_2] = \varphi_1[\psi_1/\psi_2] \supset \varphi_2[\psi_1/\psi_2]$$

$$(\varphi_1 \equiv \varphi_2)[\psi_1/\psi_2] = \varphi_1[\psi_1/\psi_2] \equiv \varphi_2[\psi_1/\psi_2]$$

2.4.2 Semantics

The semantics presented here differs slightly from the semantics as specified in [18]. We view points as valuations instead of having a valuation function picking out points. Given a signature $(\Phi, I, \prec)$, we define models for the language. Φ is assumed to be finite for the rest of this thesis.

A *universe* is a non-empty set of points, where each point is a different propositional valuation function. A *model* based on a universe U is a structure $M = (U, U^+, U^-)$, where $U^+ : I \rightarrow 2^U$ and $U^- : I \rightarrow 2^U$ such that $U = U_k^+ \cup U_k^-$ for all $k \in I$ and $U_k^+ \subseteq U_i^+$ if $i \prec k$, where U_k^+ and U_k^- denotes $U^+(k)$ and $U^-(k)$ respectively. The model is said to be *k-bisected* if $U_k^+ \cap U_k^- = \emptyset$. The model is simply *bisected* if it is *k-bisected* for all $k \in I$. $M \models_v \varphi$ denotes that φ is true at the point v in the model M , $M \not\models_v \varphi$ that not $M \models_v \varphi$, and $M \models \varphi$ that $M \models_v \varphi$ for all $v \in U$. Let $v \in U$. Then

$$M \models_v p_k \text{ iff } v \models p_k \quad (2.10)$$

$$M \models_v \top \quad (2.11)$$

$$M \not\models_v \perp \quad (2.12)$$

$$M \models_v B_k \varphi \text{ iff } M \models_w \varphi \text{ for all } w \in U_k^+ \quad (2.13)$$

$$M \models_v C_k \varphi \text{ iff } M \models_w \varphi \text{ for all } w \in U_k^- \quad (2.14)$$

$$M \models_v \Box \varphi \text{ iff } M \models_w \varphi \text{ for all } w \in U \quad (2.15)$$

$$M \models_v \neg \varphi \text{ iff } M \not\models_v \varphi \quad (2.16)$$

$$M \models_v \varphi \vee \psi \text{ iff } M \models_v \varphi \text{ or } M \models_v \psi \quad (2.17)$$

$$M \models_v \varphi \wedge \psi \text{ iff } M \models_v \varphi \text{ and } M \models_v \psi \quad (2.18)$$

$$M \models_v \varphi \supset \psi \text{ iff } M \not\models_v \varphi \text{ or } M \models_v \psi \quad (2.19)$$

$$M \models_v \varphi \equiv \psi \text{ iff } M \models_v \varphi \supset \psi \text{ and } M \models_v \psi \supset \varphi \quad (2.20)$$

The semantics of the dual modalities and some other properties follow.

$$M \models_v b\varphi \text{ iff } M \models_w \varphi \text{ for a } w \in U^+ \quad (2.21)$$

$$M \models_v c\varphi \text{ iff } M \models_w \varphi \text{ for a } w \in U^- \quad (2.22)$$

$$M \models_v \Diamond \varphi \text{ iff } M \models_w \varphi \text{ for a } w \in U \quad (2.23)$$

$$M \models_v \varphi \text{ iff } v \models \varphi \text{ for } \varphi \in \mathcal{L}_{\text{PL}} \quad (2.24)$$

$$M \models_v \Box \varphi \text{ iff } M \models \varphi \quad (2.25)$$

The truth value of propositional formulae and formulae where the only modalities are $\Box$ and $\Diamond$ do not rely on U_k^+ or U_k^- . If φ is such a formula, we may write $U \models_v \varphi$ instead of $M \models_v \varphi$. We may always write $U \models \varphi$ instead of $M \models \varphi$. If $U \models \varphi$ for all universes U , we write $\models \varphi$, and say that φ is *valid*. If there exists a model M and a valuation v such that $M \models_v \varphi$, we say that φ is *satisfiable*.

$\|\varphi\|$ is the subset of U where φ holds: $\|\varphi\| = \{v \mid U \models_v \varphi\}$. It is worth noticing that $\|\varphi\| \cap \|\neg\varphi\| = \emptyset$, $\|\varphi\| \cup \|\neg\varphi\| = U$, $M \models B_k \psi$ iff $U_k^+ \subseteq \|\psi\|$, and $M \models C_k \psi$ iff $U_k^- \subseteq \|\psi\|$. O_k -formulae have *k-bisected* models.

Lemma 2.6. If $M \models O_k \varphi$ then $U_k^+ = \|\varphi\|$ and $U_k^- = \|\neg\varphi\|$.

Proof. The semantics of $O_k\varphi$ tells us that $U_k^+ \subseteq \|\varphi\|$ and $U_k^- \subseteq \|\neg\varphi\|$. U_k^+ and U_k^- are subsets of disjoint sets, thus they are disjoint themselves. As $U_k^+ \cup U_k^- = U = \|\varphi\| \cup \|\neg\varphi\|$, $U_k^+ = \|\varphi\|$ and $U_k^- = \|\neg\varphi\|$. $\square$

When φ contains B_k or C_k , the expressions $U_k^+ = \|\varphi\|$ and $U_k^- = \|\neg\varphi\|$ should be regarded as fixpoint equations, as U_k^+ (U_k^-) appears on both sides. Thus, U_k^+ and U_k^- are not necessarily uniquely specified by $O_k\varphi$. When φ is propositional, the situation is different.

Theorem 2.7. If $M \models O_k\varphi$ then M is k -bisected.

Proof. From Lemma 2.6 it follows that $U_k^+ = \|\varphi\|$ and $U_k^- = \|\neg\varphi\|$, and since $\|\varphi\| \cap \|\neg\varphi\| = \emptyset$, M is k -bisected. $\square$

In bisected models, determining whether prime modal atoms hold, can be done in propositional logic.

Theorem 2.8. Let φ and ψ be propositional and $M \models O_k\varphi$. Then

$$\begin{aligned} M \models B_k\psi &\text{ iff } M \models \varphi \supset \psi \\ M \models C_k\psi &\text{ iff } M \models \neg\varphi \supset \psi \\ M \models O_k\psi &\text{ iff } M \models \varphi \equiv \psi \end{aligned}$$

Proof. From Lemma 2.6 we know that $U_k^+ = \|\varphi\|$ and $U_k^- = \|\neg\varphi\|$. Observe that $\|\varphi\| \subseteq \|\psi\|$ iff $M \models \varphi \supset \psi$. $M \models B_k\psi$ iff $U_k^+ \subseteq \|\psi\|$ iff $\|\varphi\| \subseteq \|\psi\|$ iff $M \models \varphi \supset \psi$. $M \models C_k\psi$ iff $U_k^- \subseteq \|\psi\|$ iff $\|\neg\varphi\| \subseteq \|\psi\|$ iff $M \models \neg\varphi \supset \psi$. The last equivalence follows from the first two, as $M \models C_k\neg\psi$ iff $M \models \psi \supset \varphi$. $\square$

2.4.3 Uniqueness

There is a correspondence between atoms in a logical space and points in a universe.

Lemma 2.9. Let $U \models \lambda$, and assume $v_i \models \alpha_i$ for all i . $\lambda \vdash \Diamond\alpha_i$ iff $v_i \in U$.

Proof. *If:* As $v_i \in U$ and $v_i \models \alpha_i$, it follows that $U \models \Diamond\alpha_i$ for some i . As $U \models \lambda$ and either $\lambda \vdash \Diamond\alpha_i$ or $\lambda \not\vdash \Diamond\alpha_i$, it must be the case that $\lambda \vdash \Diamond\alpha_i$. *Only if:* It follows that $U \models \Diamond\alpha_i$. So there exists a $v \in U$ such that $U \models_v \alpha_i$, which means $v \models \alpha_i$, thus $v = v_i$. $\square$

Let $K \subseteq I$, and let φ^K be a formula on the form $\bigwedge_{i \in K} O_i\varphi_i$. We will refer to φ^K as an O_K -atom, and in the case where φ_i is propositional for every $i \in K$, a *prime* O_K -atom.

A logical space in conjunction with a prime O_I -atom corresponds to a bisected model.

Theorem 2.10 (Uniqueness of models). Let φ^I be a prime O_I -atom, and let λ be a logical space. There is a unique universe U such that $U \models \lambda$, and there is a unique model M based on U such that $M \models \varphi^I$.

Proof. It follows from Lemma 2.9 that U is unique. Lemma 2.6 tells us that U_k^+ and U_k^- must be uniquely specified. $\square$

Thus, for any completely modalized formula β , either

$$\lambda \wedge \varphi^I \vdash \beta \text{ or } \lambda \wedge \varphi^I \vdash \neg\beta.$$

2.4.4 Autoepistemic logic

Autoepistemic logic or AEL was introduced by Moore [12]. The language of AEL is the same as the language of modal logic, $\mathcal{L}_M$, but using our notation, we must use B instead of $\Box$ (and b instead of $\Diamond$). AEL doesn't need its own provability relation. We can use propositional provability, where modal atoms (in this case formulae on the form $B\varphi$) are treated atomic. Thus AEL isn't a modal logic.

Define $\text{Th}(T) = \{\varphi \mid T \vdash_{\text{PL}} \varphi\}$. A set of autoepistemic formulae is called an *autoepistemic theory*. An autoepistemic theory T is *stable* iff

- AE1. $T = \text{Th}(T)$
- AE2. $B\varphi \in T$ if $\varphi \in T$
- AE3. $\neg B\varphi \in T$ if $\varphi \notin T$

AE1 states that T is closed under propositional consequence, AE2 that T is closed under necessitation. AE3 provides a completeness property. This means that for every φ , either $B\varphi$ or $\neg B\varphi$ is element of T , or both if T is inconsistent. For consistent theories, AE2 and AE3 can be stated as

- AE2. $B\varphi \in T$ iff $\varphi \in T$
- AE3. $\neg B\varphi \in T$ iff $\varphi \notin T$

Given an autoepistemic theory, what is reasonable to believe? E is a *stable expansion* of an autoepistemic theory T iff

$$E = \{\varphi \mid T \cup \{B\varphi \mid \varphi \in E\} \cup \{\neg B\varphi \mid \varphi \notin E\} \vdash_{\text{PL}} \varphi\}$$

An autoepistemic theory can have several stable expansions. A propositional theory can only have one, its stable closure.

The following method for generating stable expansions is found in [1]. To find the stable expansions of T , partition the set of modal atoms occurring in T into two sets, a set of formulae you believe in, and a set you don't believe in, and generate the *kernel* (without modal formulae) E_0 of a potential stable expansion E . E_0 is the kernel of a potential stable expansion iff the set of formulae you believe in are included, and the set you don't believe in is disjoint with E_0 .

Example 2.11. Let $T = \{(Bp \supset r), (Bq \equiv q)\}$.

E^+	E^-	E_0	$E^+ \subseteq E_0$	$E^- \cap E_0 = \emptyset$	expansion
$\emptyset$	$\{p, q\}$	$\text{Th}(\{\neg q\})$	yes	yes	yes
$\{q\}$	$\{p\}$	$\text{Th}(\{q\})$	yes	yes	yes
$\{p\}$	$\{q\}$	$\text{Th}(\{r, \neg q\})$	no	yes	no
$\{p, q\}$	$\emptyset$	$\text{Th}(\{r, q\})$	no	no	no

Thus the kernels of the stable expansions of T are $\text{Th}(\{q\})$ and $\text{Th}(\{\neg q\})$.

2.4.5 Levesque's "Only knowing" logic

$\mathcal{AE}$ is based on the sound and complete propositional subset of Levesque's "Only knowing" logic $\mathcal{ONL}$ [9, 16]. It has a deep relationship with AEL, in the way that models of $O\varphi$ corresponds to stable expansions of φ . The generation of a generalization of stable expansions is the topic of the next chapter. The axiom schemata of this logic are (using $\mathcal{AE}$ notation) PL, K_B , K_C and

$\vdash B\varphi$	if φ is valid
$\vdash \varphi \supset B\varphi$	if φ is completely modalized
$\vdash \varphi \supset C\varphi$	if φ is completely modalized
$\vdash C\varphi \supset \neg B\varphi$	if φ is propositional and not valid
$\vdash O\varphi \equiv (B\varphi \wedge C\neg\varphi)$	

$C\varphi \supset \neg B\varphi$ is known as the " C vs. B " axiom. This is the only axiom that does not also hold in $\mathcal{AE}$.

2.4.6 $\mathcal{AE}_\rho$

As the logical space is exponential in the number of atomic formulae, having to specify it might not be feasible. Axiomatizing the logical space in a manner similar to Levesque solves this problem. As an atom corresponds to a propositional valuation, the logical space λ corresponds to a *satisfiable*

propositional formula ρ : the characteristic formula of λ . By specifying ρ instead of λ and adding the following axioms, we give an implicit, non-exponential specification of the logical space.

Define $\mathcal{A}_\rho$ as containing every axiom schemata and inference rule of $\mathcal{A}$ in addition to the following axiom and inference rule:

$$\vdash_\rho \Box \rho \quad (\text{RI}) \qquad \frac{\rho, \varphi \not\vdash_{\text{PL}} \perp \text{ and } \varphi \in \mathcal{L}_{\text{PL}}}{\vdash_\rho \Diamond \varphi} \quad (\text{RC})$$

From RI and RC follow *conceivable* and *inconceivable* formulae respectively. For $\rho = \top$, RC coincides with Levesque's "*C* vs. *B*" axiom, as it instantiates to " $\Diamond \varphi$ is a theorem if φ is PL-consistent". Since RI instantiates to $\Box \top$, $\mathcal{A}_\top$ is equivalent with $\mathcal{O}\mathcal{N}\mathcal{L}$.

We need to establish the correspondence between a logical space and its characteristic formula. Since both have countably infinite number of equivalents, the correspondence is between equivalence classes of equivalent formulae. We also need a way to generate a characteristic formula from a logical space and vice versa.

Let λ be on the form $\Diamond \alpha_1 \wedge \dots \wedge \Diamond \alpha_k \wedge \Box(\alpha_1 \vee \dots \vee \alpha_k)$. Then

$$\rho(\lambda) = \alpha_1 \vee \dots \vee \alpha_k$$

Let S be the subset of the set of every atom, such that $S = \{\alpha \mid \alpha \vdash \rho\}$. Then

$$\lambda(\rho) = \bigwedge_{\alpha \in S} \Diamond \alpha \wedge \Box \bigvee_{\alpha \in S} \alpha$$

Lemma 2.12. $\vdash \lambda(\rho(\lambda)) \equiv \lambda$ and $\vdash \rho(\lambda(\rho)) \equiv \rho$.

Proof. $\lambda(\rho(\lambda)) = \lambda(\alpha_1 \vee \dots \vee \alpha_k)$, and $\vdash \lambda(\alpha_1 \vee \dots \vee \alpha_k) \equiv \lambda$. ρ is equivalent to a formula ρ' on the form $\alpha_1 \vee \dots \vee \alpha_k$ such that $\rho(\lambda(\rho)) = \rho(\Diamond \alpha_1 \wedge \dots \wedge \Diamond \alpha_k \wedge \Box(\alpha_1 \vee \dots \vee \alpha_k)) = \rho'$. $\square$

Theorem 2.13. $\lambda \vdash \varphi$ iff $\vdash_{\rho(\lambda)} \varphi$.

Proof. λ is on the form $\Diamond \alpha_1 \wedge \dots \wedge \Diamond \alpha_k \wedge \Box(\alpha_1 \vee \dots \vee \alpha_k)$, so $\rho = \rho(\lambda) = \alpha_1 \vee \dots \vee \alpha_k$.

Only if: Assume $\lambda \vdash \varphi$. We need to show that $\vdash_\rho \varphi$. Instantiating RC with $\varphi = \alpha_i$ for $1 \leq i \leq k$ gives $\vdash_\rho \Diamond \alpha_i$. $\vdash_\rho \Box(\alpha_1 \vee \dots \vee \alpha_k)$ is the RI axiom.

If: We need to show that the consequences of λ are the *only* additional theorems of $\mathcal{A}_\rho$. This can be shown by induction over proofs, by showing

that RI and RC are deducible from λ in $\mathcal{A}$. For RI this is trivial. For any $\Diamond\varphi$ stemming from RC, it must be the case that $\rho \not\vdash_{\text{PL}} \neg\varphi$. Assume that for all $1 \leq i \leq k$, $\alpha_i \not\vdash_{\text{PL}} \varphi$. As either $\alpha \vdash_{\text{PL}} \varphi$ or $\alpha \vdash_{\text{PL}} \neg\varphi$, $\alpha_i \vdash_{\text{PL}} \neg\varphi$, thus $\rho \vdash_{\text{PL}} \neg\varphi$, a contradiction. There must exist an α_i such that $\alpha_i \vdash_{\text{PL}} \varphi$, thus $\Diamond\alpha_i \vdash \Diamond\varphi$ and $\vdash (\Diamond\alpha_1 \wedge \dots \wedge \Diamond\alpha_k \wedge \Box(\alpha_1 \vee \dots \vee \alpha_k)) \supset \Diamond\varphi$. $\square$

An immediate consequence of the preceding Lemma and Theorem is the fact that it doesn't matter whether we start with a λ or a ρ , as we can always convert between the two.

Corollary 2.14. $\lambda(\rho) \vdash \varphi$ iff $\vdash_\rho \varphi$.

Example 2.15. Let $\rho = (p \equiv q)$.

Conceivable points (from inference rule RC):

$$\frac{\rho, \neg p \wedge \neg q \not\vdash_{\text{PL}} \perp}{\vdash_\rho \Diamond(\neg p \wedge \neg q)} \quad \frac{\rho, p \wedge q \not\vdash_{\text{PL}} \perp}{\vdash_\rho \Diamond(p \wedge q)}$$

Inconceivable points (from the RI axiom, as $\vdash \rho \equiv ((\neg p \wedge \neg q) \vee (p \wedge q))$):

$$\vdash_\rho \Box((\neg p \wedge \neg q) \vee (p \wedge q))$$

Since $\Box((p \wedge q) \vee (\neg p \wedge \neg q))$ is equivalent to $\neg\Diamond(p \wedge \neg q) \wedge \neg\Diamond(\neg p \wedge q)$, this exhausts all points. In terms of a logical space,

$$\lambda(\rho) = \Diamond(\neg p \wedge \neg q) \wedge \Diamond(p \wedge q) \wedge \Box((\neg p \wedge \neg q) \vee (p \wedge q))$$

What about the semantics of $\mathcal{A}_\rho$? Since the logical space already is a theorem of $\mathcal{A}_\rho$, a universe is implicitly specified by the axiomatization.

Theorem 2.16. (1) Every model for a theorem of $\mathcal{A}_\rho$ is based on the same unique universe. (2) There is a unique model M based on this universe such that $M \models \varphi^I$ if φ^I is a prime O_I -atom.

Proof. (1) Let $\vdash_\rho \psi_1$ and $\vdash_\rho \psi_2$, and let $M_1 \models \psi_1$ and $M_2 \models \psi_2$. Let U_1 and U_2 be the universes M_1 and M_2 are based on respectively. Since $U_1 \models \lambda(\rho)$ and $U_2 \models \lambda(\rho)$, it follows from Theorem 2.10 that $U_1 = U_2$. (2) M must be based on this universe, thus $M \models \lambda$. It follows from Theorem 2.10 that M is unique. $\square$

As it is unique, we can let U_ρ denote the universe such that $U_\rho \models \lambda(\rho)$.

Example 2.17. Let $|I| = 1$, let $\Phi = \{p, q\}$, and let U be the universe corresponding to $\lambda_{\top}$. All bisected models based on U are listed below¹. Every possible propositional valuation is a point, and all $O\varphi$ for non-equivalent propositional φ have different models. Note that the number preceding each $O\varphi$ is $\text{int } \pi(\varphi)$ (see Table 2.2), as for instance $\pi(p \supset q) = [1 \ 1 \ 0 \ 1]$ (the truth table, only horizontally) and $\text{int } [1 \ 1 \ 0 \ 1] = 13$.

Figure 2.1: Every model for every $O\varphi$ with $\rho = \top$.

Example 2.18. If U' is the universe corresponding to a logical space

$$\lambda = \Diamond(\neg p \wedge \neg q) \wedge \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \neg\Diamond(p \wedge q)$$

however, there are only half as many models based on U' as on U (of the previous example). Now two and two $O\varphi$ have the same model, as for instance $\lambda \vdash O\perp \equiv O(p \wedge q)$ and $\lambda \vdash O(p \wedge \neg q) \equiv Op$. This is also evident from the vector valued valuation corresponding to each point, as ρ is an axiom of $\mathcal{A}\mathcal{E}_\rho$. As $\rho(\lambda) = (\neg p \vee \neg q)$, $\pi(\rho) \wedge \pi(\perp) = \pi(\rho) \wedge \pi(p \wedge q) = [0 \ 0 \ 0 \ 0]$, and $\text{int } [0 \ 0 \ 0 \ 0] = 0$, and $\pi(\rho) \wedge \pi(\neg p) = \pi(\rho) \wedge \pi(p \supset q) = [1 \ 1 \ 0 \ 0]$, and $\text{int } [1 \ 1 \ 0 \ 0] = 12$.

¹The intended reading of the figure is that points enclosed by a thick line are in U^+ , points enclosed by a thin line are in U^- and all other points are not in U .

Figure 2.2: Every model for every $O\varphi$ with $\rho = (\neg p \vee \neg q)$.

2.4.7 Propositionality

Lemma 2.19. $v \models \rho$ iff $v \in U_\rho$.

Proof. *If:* Holds since $U_\rho \models \rho$. *Only if:* Assume $v_i \models \alpha_i$, and assume that there exists a v_i such that $v_i \models \rho$. Then ρ is a disjunction of α_i , such that for each i , $1 \leq i \leq 2^n$. For all such α_i , $\lambda(\rho) \vdash \Diamond \alpha_i$, and by Lemma 2.9, $v_i \in U_\rho$. $\square$

This means that the propositional formulae true on U_ρ are exactly the propositional consequences of ρ .

Corollary 2.20. $U_\rho \models \varphi$ iff $\rho \vdash \varphi$, for propositional φ .

Theorem 2.21. Let φ and ψ be propositional.

$$\vdash_\rho O_k \varphi \supset B_k \psi \text{ iff } \rho \vdash \varphi \supset \psi \quad (2.26)$$

$$\vdash_\rho O_k \varphi \supset C_k \psi \text{ iff } \rho \vdash \neg \varphi \supset \psi \quad (2.27)$$

$$\vdash_\rho O_k \varphi \supset O_k \psi \text{ iff } \rho \vdash \varphi \equiv \psi \quad (2.28)$$

Proof. $O_k \varphi$ has a unique model in $\mathcal{A}_{\rho}$. This model must be M , so $\vdash_\rho O_k \varphi \supset B_k \psi$ iff $M \models B_k \psi$. It follows from Theorem 2.8 that $M \models B_k \psi$ iff $M \models \varphi \supset \psi$, and from Corollary 2.20 iff $\rho \vdash \varphi \supset \psi$. Similarly for C_k and O_k . $\square$

The significance of all this is that if we want to check if a prime modal atom is implied by a prime O_k -formulae in $\mathcal{A}_\rho$, this can be done in PL, and without having to go via any λ . This will be useful when *collapsing* formulae in the next chapter. To summarize; the following are equivalent.

$$\begin{aligned} U_\rho \models \varphi & \quad \varphi \text{ holds in the universe where the logical space } \lambda(\rho) \text{ holds} \\ \lambda(\rho) \vdash \varphi & \quad \varphi \text{ follows from the logical space } \lambda(\rho) \\ \vdash_\rho \varphi & \quad \varphi \text{ is a theorem of } \mathcal{A}_\rho \end{aligned}$$

2.4.8 Normal form

A property of $\mathcal{AE}$ is that every formula is equivalent to another formula without nested modalities.

A formula is said to be on *negation normal form* (NNF) if every negation is in front of a propositional variable or a modal atom. A formula is said to be on *conjunctive normal form* (CNF) if it is a conjunction of disjunctions of literals and modal literals. A formula is said to be on *disjunctive normal form* (DNF) if it is a disjunction of conjunctions of literals and modal literals. A formula is said to be on *modal normal form* (MNF) if it is on CNF such that for each modal literal (on the form $B\psi$, $C\psi$ or $O\psi$), ψ is a disjunction of literals. A formula on MNF is without nested modalities.

Lemma 2.22. Let β be a completely modalized formula. Then

$$\begin{aligned} \vdash \beta \supset B_i\beta \\ \vdash \beta \supset C_i\beta \end{aligned}$$

Proof. Induction on β , which we assume is on NNF.

Base step: β is on the form $B_k\psi$, $\neg B_k\psi$, $C_k\psi$ or $\neg C_k\psi$. Then both $\beta^k \supset B_i\beta^k$ and $\beta^k \supset C_i\beta^k$ follow from $B_\square$, $C_\square$, $\overline{B}_\square$ and $\overline{C}_\square$ respectively, no matter how i and k are related.

Induction step: We only prove $\beta \supset B_i\beta$, as $\beta \supset C_i\beta$ is proven similarly. We prove the cases for $\wedge$ and $\vee$.

β is on the form $\beta_1 \wedge \beta_2$. The induction hypothesis is that $\beta_1 \supset B_i\beta_1$ and $\beta_2 \supset B_i\beta_2$. Thus $(\beta_1 \wedge \beta_2) \supset B_i(\beta_1 \wedge \beta_2)$, as B_i distributes over conjunctions.

β is on the form $\beta_1 \vee \beta_2$. The induction hypothesis is that $\beta_1 \supset B_i\beta_1$ and $\beta_2 \supset B_i\beta_2$. Thus $(\beta_1 \vee \beta_2) \supset B_i(\beta_1 \vee \beta_2)$ as $\beta_1 \supset B_i(\beta_1 \vee \beta_2)$ and $\beta_2 \supset B_i(\beta_1 \vee \beta_2)$. $\square$

Lemma 2.23. Let β be a completely modalized formula. Then

$$\begin{aligned} \vdash B_i(\varphi \vee \beta) &\equiv (B_i\varphi \vee \beta) \\ \vdash C_i(\varphi \vee \beta) &\equiv (C_i\varphi \vee \beta) \end{aligned}$$

Proof. If: From Lemma 2.22 it follows that $\beta \supset B_i(\varphi \vee \beta)$, and in conjunction with $B_i\varphi \supset B_i(\varphi \vee \beta)$ it follows that $(B_i\varphi \vee \beta) \supset B_i(\varphi \vee \beta)$.

Only if: $B_i(\beta \vee \varphi) \supset (B_i\varphi \vee \neg B_i\neg\beta)$ (equivalently $B_i(\neg\beta \supset \varphi) \supset (B_i\neg\beta \supset B_i\varphi)$) is an instance of KB. From this and $\neg B_i\neg\beta \supset \beta$ (the contraposition of $\neg\beta \supset B_i\neg\beta$ (Lemma 2.22)) it follows that $B_i(\varphi \vee \beta) \supset (B_i\varphi \vee \beta)$. $\square$

Corollary 2.24. Let β be a completely modalized formula. Then

$$\begin{aligned} \vdash B_i\beta &\equiv (B_i\perp \vee \beta) \\ \vdash C_i\beta &\equiv (C_i\perp \vee \beta) \end{aligned}$$

Proof. Follows from the fact that $\vdash B_i\beta \equiv (\perp \vee B_i\beta)$ (and similarly for C_i) and Lemma 2.23. $\square$

Lemma 2.25. Any modal atom β is equivalent to a formula β' without nested modalities.

Proof. Induction over $d(\beta)$.

Base step: If $d(\beta) = 1$, let $\beta' = \beta$, and the theorem trivially holds.

Induction step: Let β be on the form $B\psi$ such that $d(\psi) = n$. Then $d(\beta) = n + 1$. The induction hypothesis is that ψ is equivalent to a formula $\psi^C = \psi_1 \wedge \cdots \wedge \psi_m$ on CNF where each conjunct ψ_i is on the form

$$\varphi_1 \vee \cdots \vee \varphi_n \vee \beta_1 \vee \cdots \vee \beta_k$$

where $\varphi_1, \dots, \varphi_n$ are propositional and $\beta_1, \dots, \beta_k$ are modal atoms of depth 1. If $n = 0$, $\varphi_1 \vee \cdots \vee \varphi_n$ is replaced with $\perp$. So $d(\psi^C) = 1$. Let $\beta' = B\psi^C = B(\psi_1 \wedge \cdots \wedge \psi_m)$, which is equivalent to $B\psi_1 \wedge \cdots \wedge B\psi_m$, and so by Lemma 2.23 each disjunct is equivalent to a formula on the form

$$B(\varphi_1 \vee \cdots \vee \varphi_n) \vee \beta_1 \vee \cdots \vee \beta_k$$

So $d(\beta') = 1$.

The case when β is on the form $C\psi$ is symmetrical. The case when β is on the form $O\psi$ trivially follows from these two cases, since $d(O\psi) = d(B\psi \wedge C\neg\psi) = d(B\psi) = d(C\neg\psi)$. $\square$

Theorem 2.26. Any formula β is equivalent to a formula β' on MNF.

Proof. Let β^C be β on CNF. By Lemma 2.25, all modal atoms are equivalent to a formula on MNF. Obtain β' by replacing all modal atoms occurring as disjuncts in β^C with their MNF counterparts, then put every conjunct on CNF. β' is on MNF. $\square$

Chapter 3

The Modal reduction theorem

3.1 Introduction

Given a logical space λ , if Φ is finite¹, any $O\varphi$ ² is equivalent to a disjunction of prime O -formulae, whose disjuncts or λ -*expansions* can be viewed as generalization of the stable expansions of AEL and $\mathcal{ONL}$ in the propositional case. The *Modal reduction theorem* for a singleton I is stated as follows.

Theorem 3.1. For each φ and logical space λ , there exists propositional $\psi_1, \dots, \psi_k$ such that

$$\lambda \vdash O\varphi \equiv (O\psi_1 \vee \dots \vee O\psi_k)$$

Proof. This was proven semantically by Waaler [17] and syntactically by Langholm [8]. We will prove the Theorem later in this section. $\square$

In this chapter we are concerned with generating λ -expansions. We give three sets of equivalence preserving rewriting rules, which are used to rewrite a formula $O\varphi$ to a disjunction of λ -expansions, $O\psi_1 \vee \dots \vee O\psi_k$.

The idea is that if a modal atom β is true in a model of $\lambda \wedge O\varphi$, we can substitute it with $\top$ ($\varphi_\top$) and if not, we can substitute it with $\perp$ ($\varphi_\perp$). In the former case $O\varphi_\top$ must imply β , in the latter $O\varphi_\perp$ must imply $\neg\beta$ for our assumption to hold. The algorithms we know of for computing stable expansions in AEL [1] and $\mathcal{ONL}$ [10] adhere to this scheme.

Given a formula $O\varphi$, let ψ be φ with every member of $\text{ext}(\varphi)$ substituted with $\top$ or $\perp$. Then $O\psi$ is a *model candidate* for $O\varphi$.

¹This we have assumed since Sect. 2.4.2.

²Throughout this chapter, we will assume that $|I| = 1$ and therefore omit subscripts on modalities.

To find the $\lambda_{\top}$ -expansions (recall that $\lambda_{\top}$ is the maximal logical space) of $O\varphi$, we make a truth table over the modal atoms occurring in φ and see if φ with modal atoms substituted imply all of those (possibly negated) modal atoms.

Example 3.2. Let $\varphi = (\kappa \wedge (\neg p \supset B\neg p) \wedge (\neg q \supset B\neg q))$. The $\pm$ in the fourth column is $\neg$ iff the value in the first column is $\perp$. Columns five and two have the same relationship.

$B\neg p$	$B\neg q$	model candidate	$\supset \pm B\neg p$	$\supset \pm B\neg q$	$\lambda_{\top}$ -exp
$\perp$	$\perp$	$O(\kappa \wedge p \wedge q)$	yes	yes	yes
$\perp$	$\top$	$O(\kappa \wedge p)$	yes	no	no
$\top$	$\perp$	$O(\kappa \wedge q)$	no	yes	no
$\top$	$\top$	$O\kappa$	no	no	no

This of course is a very simple example, as there are no O - or C -formulae, no nested formulae and the logical space is maximal.

3.1.1 Preliminaries

The rewriting process has two phases. When rewriting, the formula is *expanded*, then *collapsed*. Every expand and collapse step is done by substituting a formula with an equivalent one. To show which equivalence is used, and which way it is used, we define a number of rewriting rules. The *collapse rule* relies on a logical space, as how a formula collapses depends on whether it is consistent, while the *expand rule* takes a subformula as a parameter.

Each set of rules has its pros and cons. The *simple* rule set chooses a bare minimum of subformulae for rewriting, generating a smaller fully expanded formula than the others. This does, however, require a more powerful theorem prover when testing for consistency, as it has to deal with modal formulae. *Langholm's* rule set recursively rewrites every modal atom into an O -formula, creating a bigger fully expanded formula than the others. As every modal atom is prime, only propositional formulae need to be tested for consistency when collapsing. The *prime* rule set rewrites every modal subformula, the advantage of this being that, as in the case of Langholm's rules, every modal atom is prime. This rule set is the basis of the rule set of the next chapter, when confidence levels are taken into consideration.

An operation which isn't used in the proof, but which is useful nonetheless, is the minimization of $\top$ and $\perp$ occurrences in a formula. Since each expand step introduces a $\top$ or $\perp$, this can make the generated formulae much simpler. The following function does just this. It reduces a formula to another one without any occurrences of $\top$ and $\perp$, or to $\top$ or $\perp$ themselves if that is the case.

Define $\varsigma : \mathcal{L} \rightarrow \mathcal{L}$ as

$$\begin{aligned}
& \varsigma(\varphi) = \varphi \text{ if } \varphi \in \Phi \cup \{\top, \perp\} \\
& \varsigma(B\varphi) = B\varsigma(\varphi) \\
& \varsigma(C\varphi) = C\varsigma(\varphi) \\
& \varsigma(O\varphi) = O\varsigma(\varphi) \\
& \varsigma(\neg\varphi) = \begin{cases} \perp & \text{if } \varsigma(\varphi) = \top \\ \top & \text{if } \varsigma(\varphi) = \perp \\ \psi & \text{if } \varsigma(\varphi) \text{ is on the form } \neg\psi \\ \neg\varsigma(\varphi) & \text{else} \end{cases} \\
& \varsigma(\varphi \vee \psi) = \begin{cases} \top & \text{if } \varsigma(\varphi) = \top \text{ or } \varsigma(\psi) = \top \\ \varsigma(\psi) & \text{if } \varsigma(\varphi) = \perp \\ \varsigma(\varphi) & \text{if } \varsigma(\psi) = \perp \\ \varsigma(\varphi) \vee \varsigma(\psi) & \text{else} \end{cases} \\
& \varsigma(\varphi \wedge \psi) = \begin{cases} \varsigma(\psi) & \text{if } \varsigma(\varphi) = \top \\ \perp & \text{if } \varsigma(\varphi) = \perp \text{ or } \varsigma(\psi) = \perp \\ \varsigma(\varphi) & \text{if } \varsigma(\psi) = \top \\ \varsigma(\varphi) \wedge \varsigma(\psi) & \text{else} \end{cases} \\
& \varsigma(\varphi \supset \psi) = \begin{cases} \varsigma(\psi) & \text{if } \varsigma(\varphi) = \top \\ \top & \text{if } \varsigma(\varphi) = \perp \text{ or } \varsigma(\psi) = \top \\ \varsigma(\neg\varphi) & \text{if } \varsigma(\psi) = \perp \\ \varsigma(\varphi) \supset \varsigma(\psi) & \text{else} \end{cases} \\
& \varsigma(\varphi \equiv \psi) = \begin{cases} \varsigma(\psi) & \text{if } \varsigma(\varphi) = \top \\ \varsigma(\neg\psi) & \text{if } \varsigma(\varphi) = \perp \\ \varsigma(\varphi) & \text{if } \varsigma(\psi) = \top \\ \varsigma(\neg\varphi) & \text{if } \varsigma(\psi) = \perp \\ \varsigma(\varphi) \equiv \varsigma(\psi) & \text{else} \end{cases}
\end{aligned}$$

Theorem 3.3. $\vdash \varphi \equiv \varsigma(\varphi)$, and either (1) $\varsigma(\varphi) = \top$, (2) $\varsigma(\varphi) = \perp$ or (3) $\top$ and $\perp$ do not occur as subformulae of $\varsigma(\varphi)$.

Proof. Induction over φ . □

3.2 The simple rule set

This rule set has the simplest definition. Only extensional occurrences of modal atoms are substituted with the expand rule. For formulae containing nested modalities, fewer instances of the expansion rule have to be applied than if all occurrences were substituted. This means that we may have to collapse modal atoms containing nested modalities.

3.2.1 The rewriting rules

Let $\beta \in \text{ext}(\varphi)$. The *simple expand* rule is defined as

$$O\varphi \xrightarrow[E]{\beta} (O\varphi[\beta/\top] \wedge \beta) \vee (O\varphi[\beta/\perp] \wedge \neg\beta)$$

The *simple collapse* rule is defined for a propositional φ and a modal atom β as

$$\begin{aligned} O\varphi \wedge \beta &\xrightarrow[C]{\lambda} O\varphi \text{ if } \lambda \vdash O\varphi \supset \beta \\ O\varphi \wedge \neg\beta &\xrightarrow[C]{\lambda} \perp \text{ if } \lambda \vdash O\varphi \supset \beta \\ O\varphi \wedge \beta &\xrightarrow[C]{\lambda} \perp \text{ if } \lambda \not\vdash O\varphi \supset \beta \\ O\varphi \wedge \neg\beta &\xrightarrow[C]{\lambda} O\varphi \text{ if } \lambda \not\vdash O\varphi \supset \beta \end{aligned}$$

The *distributive* rule is defined as

$$(\psi_1 \vee \dots \vee \psi_n) \wedge \beta \xrightarrow[D]{} (\psi_1 \wedge \beta) \vee \dots \vee (\psi_n \wedge \beta)$$

The *constant simplification* rule is defined as

$$\varphi \xrightarrow[S]{} \psi \text{ if } \psi = \varsigma(\varphi)$$

After each collapse we are left with a disjunction where some disjuncts may be $\perp$. These $\perp$ are superfluous, and we need to get rid of them. Whereas the constant simplification rule could be used to do this, only a subset of its functionality is needed.

The *disjunct simplification* rule is defined as

$$\psi_1 \vee \dots \vee \psi_n \xrightarrow[\perp]{} \bigvee_{i \in \Gamma} \psi_i \text{ such that } \psi_i \in \Gamma \text{ iff } \psi_i \text{ is PL-consistent}$$

Example 3.4. $O\varphi \vee \perp \vee O\psi \vee O\varphi \xrightarrow[\perp]{} O\varphi \vee O\psi$, while $O(\neg p \supset \perp) \xrightarrow[S]{} Op$.

3.2.2 Soundness of the rules

The expand rule is based on an equivalence in propositional logic that lets us substitute subformulae, and the distribution of O 's over the (if $\cdot$ then $\cdot$ else $\cdot$) construct.

Lemma 3.5 (Substitution). $\vdash_{\text{PL}} \varphi \equiv (\text{if } \beta \text{ then } \varphi[\beta/\top] \text{ else } \varphi[\beta/\perp])$.

Proof. Induction over φ . The PL subscript is dropped for readability.

Base step: If β is not a subformula of φ , $\vdash \varphi \equiv (\varphi \wedge \beta) \vee (\varphi \wedge \neg\beta)$. If $\beta = \varphi$, $\vdash \varphi \equiv (\varphi \wedge \varphi) \vee (\varphi \wedge \neg\varphi)$.

Induction step: We prove the cases for $\wedge$, $\vee$ and $\neg$.

$$(i) \vdash \varphi_1 \equiv (\varphi_1[\beta/\top] \wedge \beta) \vee (\varphi_1[\beta/\perp] \wedge \neg\beta)$$

$$(ii) \vdash \varphi_2 \equiv (\varphi_2[\beta/\top] \wedge \beta) \vee (\varphi_2[\beta/\perp] \wedge \neg\beta)$$

1. $\varphi = \varphi_1 \wedge \varphi_2$. The induction hypothesis is (i) and (ii). From this it follows that $\varphi_1 \wedge \varphi_2$ is equivalent to

$$((\varphi_1[\beta/\top] \wedge \beta) \vee (\varphi_1[\beta/\perp] \wedge \neg\beta)) \wedge ((\varphi_2[\beta/\top] \wedge \beta) \vee (\varphi_2[\beta/\perp] \wedge \neg\beta)).$$

On disjunctive normal form this becomes

$$(\varphi_1[\beta/\top] \wedge \varphi_2[\beta/\top] \wedge \beta) \vee (\varphi_1[\beta/\perp] \wedge \varphi_2[\beta/\perp] \wedge \neg\beta).$$

As the substitution operator is distributive,

$$((\varphi_1 \wedge \varphi_2)[\beta/\top] \wedge \beta) \vee ((\varphi_1 \wedge \varphi_2)[\beta/\perp] \wedge \neg\beta).$$

2. $\varphi = \varphi_1 \vee \varphi_2$. The induction hypothesis is (i) and (ii). From this it follows that $\varphi_1 \vee \varphi_2$ is equivalent to

$$(\varphi_1[\beta/\top] \wedge \beta) \vee (\varphi_2[\beta/\top] \wedge \beta) \vee (\varphi_1[\beta/\perp] \wedge \neg\beta) \vee (\varphi_2[\beta/\perp] \wedge \neg\beta),$$

equivalently

$$((\varphi_1[\beta/\top] \vee \varphi_2[\beta/\top]) \wedge \beta) \vee ((\varphi_1[\beta/\perp] \vee \varphi_2[\beta/\perp]) \wedge \neg\beta).$$

As the substitution operator is distributive,

$$((\varphi_1 \vee \varphi_2)[\beta/\top] \wedge \beta) \vee ((\varphi_1 \vee \varphi_2)[\beta/\perp] \wedge \neg\beta).$$

3. $\varphi = \neg\varphi_1$. The induction hypothesis is (i). From this it follows that $\neg\varphi_1$ is equivalent to

$$(\neg(\varphi_1[\beta/\top]) \vee \neg\beta) \wedge (\neg(\varphi_1[\beta/\perp]) \vee \beta),$$

equivalently

$$(\neg(\varphi_1[\beta/\top]) \wedge \beta) \vee (\neg(\varphi_1[\beta/\perp]) \wedge \neg\beta).$$

As the substitution operator is distributive,

$$((\neg\varphi_1)[\beta/\top] \wedge \beta) \vee ((\neg\varphi_1)[\beta/\perp] \wedge \neg\beta).$$

The cases when $\varphi_1 \supset \varphi_2$ and $\varphi_1 \equiv \varphi_2$ are proven similarly. $\square$

Lemma 3.6 (*O* distribution). Let β be any completely modalized formula. Then

$$\vdash O(\text{if } \beta \text{ then } \varphi \text{ else } \psi) \equiv (\text{if } \beta \text{ then } O\varphi \text{ else } O\psi)$$

Proof. By definition $O(\text{if } \beta \text{ then } \varphi \text{ else } \psi)$ is equivalent to

$$B((\beta \supset \varphi) \wedge (\neg\beta \supset \psi)) \wedge C\neg((\beta \wedge \varphi) \vee (\neg\beta \wedge \psi)).$$

B and C distribute over conjunctions:

$$B(\beta \supset \varphi) \wedge B(\neg\beta \supset \psi) \wedge C\neg(\beta \wedge \varphi) \wedge C\neg(\neg\beta \wedge \psi),$$

equivalently

$$B(\neg\beta \vee \varphi) \wedge B(\beta \vee \psi) \wedge C(\neg\beta \vee \neg\varphi) \wedge C(\beta \vee \neg\psi).$$

Using Lemma 2.23 we can put the formula on normal form:

$$(\neg\beta \vee B\varphi) \wedge (\beta \vee B\psi) \wedge (\neg\beta \vee C\neg\varphi) \wedge (\beta \vee C\neg\psi),$$

which is equivalent to

$$(\neg\beta \vee (B\varphi \wedge C\neg\varphi)) \wedge (\beta \vee (B\psi \wedge C\neg\psi)),$$

which can be abbreviated $(\beta \supset O\varphi) \wedge (\neg\beta \supset O\psi)$. This is definition is equivalent to $(\text{if } \beta \text{ then } O\varphi \text{ else } O\psi)$. $\square$

The soundness of the expand rule follows from the following Theorem.

Theorem 3.7. Let $\beta \in \text{ext}(\varphi)$. Then

$$\vdash O\varphi \equiv ((O\varphi[\beta/\top] \wedge \beta) \vee (O\varphi[\beta/\perp] \wedge \neg\beta)) \quad (3.1)$$

Proof. From Lemma 3.5 it follows that $O\varphi$ is equivalent to

$$O(\text{if } \beta \text{ then } \varphi[\beta/\top] \text{ else } \varphi[\beta/\perp]),$$

which from Lemma 3.6 is equivalent to

$$(\text{if } \beta \text{ then } O\varphi[\beta/\top] \text{ else } O\varphi[\beta/\perp]),$$

which by definition is equivalent to $((O\varphi[\beta/\top] \wedge \beta) \vee (O\varphi[\beta/\perp] \wedge \neg\beta))$. $\square$

The soundness of the collapse rule follows from the following two Theorems.

Theorem 3.8.

$$\begin{aligned} \lambda \vdash (O\varphi \wedge \beta) &\equiv O\varphi \text{ iff } \lambda \vdash O\varphi \supset \beta \\ \lambda \vdash (O\varphi \wedge \neg\beta) &\equiv \perp \text{ iff } \lambda \vdash O\varphi \supset \beta \end{aligned}$$

Proof. Both are equivalent and follow from propositional logic. $\square$

Theorem 3.9. Let φ be propositional, and let β be a modal atom.

$$\begin{aligned}\lambda \vdash (O\varphi \wedge \beta) &\equiv \perp \text{ iff } \lambda \not\vdash O\varphi \supset \beta \\ \lambda \vdash (O\varphi \wedge \neg\beta) &\equiv O\varphi \text{ iff } \lambda \not\vdash O\varphi \supset \beta\end{aligned}$$

Proof. Both are equivalent to “ $\lambda \not\vdash O\varphi \supset \beta$ iff $\lambda \vdash O\varphi \supset \neg\beta$ ”, which holds because of Theorem 2.10. $\square$

3.2.3 The proof

We are now ready to prove the Modal reduction theorem.

Proof. Induction over $|\text{ext}(\varphi)|$. If $|\text{ext}(\varphi)| > 0$, a modal atom β occurs extensionally as a subformula of φ . Let $\varphi_{\top} = \varphi[\beta/\top]$ and $\varphi_{\perp} = \varphi[\beta/\perp]$. By Theorem 3.7, $O\varphi$ expands to $(O\varphi_{\top} \wedge \beta) \vee (O\varphi_{\perp} \wedge \neg\beta)$.

Base step: $|\text{ext}(\varphi)| = 1$. The collapse rule, possibly in combination with the disjunct simplification rule, collapses this formula to one of the following prime O -formulae: $O\varphi_{\top} \vee O\varphi_{\perp}$, $O\varphi_{\top}$, $O\varphi_{\perp}$ or $\perp$.

Induction step: $|\text{ext}(\varphi)| = n$. $|\text{ext}(\varphi_{\top})| = |\text{ext}(\varphi_{\perp})| = n - 1$. The induction hypothesis is that $O\varphi_{\top}$ and $O\varphi_{\perp}$ are equivalent to disjunctions of prime O -formulae. By applying the distributive rule, and then the collapse rule, this also holds for $O\varphi_{\top} \wedge \beta$ and $O\varphi_{\perp} \wedge \neg\beta$. $O\varphi$ is the disjunction of these two formulae, which means this also holds for $O\varphi$, possibly by applying the disjunct simplification rule. $\square$

3.2.4 Examples

To see how these rules can be used, we give an example with a propositional κ and two supernormal default rules [15]. We then give an example to illustrate the use of the distributive rule. In both examples we assume the logical space λ to be maximal. Examples with other logical spaces are found in later sections.

Example 3.10. Let $\delta_p = (\neg p \supset B\neg p)$ and $\delta_q = (\neg q \supset B\neg q)$, and let $\varphi = \kappa \wedge \delta_p \wedge \delta_q$. We choose to first substitute $B\neg p$.

$$\begin{aligned}O\varphi &\xrightarrow[B]{B\neg p} [O(\kappa \wedge (\neg p \supset \top) \wedge \delta_q) \wedge B\neg p] \vee \\ &\quad [O(\kappa \wedge (\neg p \supset \perp) \wedge \delta_q) \wedge \neg B\neg p] \\ &\xrightarrow{S} [O(\kappa \wedge \delta_q) \wedge B\neg p] \vee [O(\kappa \wedge p \wedge \delta_q) \wedge \neg B\neg p]\end{aligned}$$

If we let $\varphi_1 = (\kappa \wedge \delta_q)$ and $\varphi_2 = (\kappa \wedge p \wedge \delta_q)$, we can write $O\varphi$ as $(O\varphi_1 \wedge B\neg p) \vee (O\varphi_2 \wedge \neg B\neg p)$.

Figure 3.1: $O\varphi$ after expanding $B\neg p$.

Two new O -formulae have been generated. Substitute $B\neg q$ in $O\varphi_1$.

$$\begin{aligned} O\varphi_1 &\xrightarrow[B]{B\neg q} [O(\kappa \wedge (\neg q \supset \top)) \wedge B\neg q] \vee [O(\kappa \wedge (\neg q \supset \perp)) \wedge \neg B\neg q] \\ &\xrightarrow{S} [O\kappa \wedge B\neg q] \vee [O(\kappa \wedge q) \wedge \neg B\neg q] \end{aligned}$$

Then substitute $B\neg q$ in $O\varphi_2$.

$$\begin{aligned} O\varphi_2 &\xrightarrow[B]{B\neg q} [O(\kappa \wedge p \wedge (\neg q \supset \top)) \wedge B\neg q] \vee \\ &\quad [O(\kappa \wedge p \wedge (\neg q \supset \perp)) \wedge \neg B\neg q] \\ &\xrightarrow{S} [O(\kappa \wedge p) \wedge B\neg q] \vee [O(\kappa \wedge p \wedge q) \wedge \neg B\neg q] \end{aligned}$$

Figure 3.2: $O\varphi$ after expanding $B\neg q$.

As the formula is fully expanded, we identify the model candidates as $O\kappa$, $O(\kappa \wedge q)$, $O(\kappa \wedge p)$ and $O(\kappa \wedge p \wedge q)$. Collapsing the leaf nodes, we can use Theorem 2.21 because all our modal atoms are prime. $\rho(\lambda) = \top$, so $\vdash O\varphi \supset B\psi$ iff $\vdash \varphi \supset \psi$, and $\vdash O\varphi \supset \neg B\psi$ iff $\not\vdash \varphi \supset \psi$.

$$\begin{aligned} O\kappa \wedge B\neg q &\xrightarrow[C]{\lambda} \perp & O(\kappa \wedge q) \wedge \neg B\neg q &\xrightarrow[C]{\lambda} O(\kappa \wedge q) \\ O(\kappa \wedge p) \wedge B\neg q &\xrightarrow[C]{\lambda} \perp & O(\kappa \wedge p \wedge q) \wedge \neg B\neg q &\xrightarrow[C]{\lambda} O(\kappa \wedge p \wedge q) \\ \perp \vee O(\kappa \wedge q) &\xrightarrow[\perp]{} O(\kappa \wedge q) & \perp \vee O(\kappa \wedge p \wedge q) &\xrightarrow[\perp]{} O(\kappa \wedge p \wedge q) \end{aligned}$$

Figure 3.3: $O\varphi$ expanded and partially collapsed.

The new leaf nodes are collapsed as follows.

$$\begin{aligned}
O(\kappa \wedge q) \wedge B\neg p &\xrightarrow[\text{C}]{\lambda} \perp \\
O(\kappa \wedge p \wedge q) \wedge \neg B\neg p &\xrightarrow[\text{C}]{\lambda} O(\kappa \wedge p \wedge q) \\
\perp \vee O(\kappa \wedge p \wedge q) &\xrightarrow[\perp]{} O(\kappa \wedge p \wedge q)
\end{aligned}$$

This means that $\lambda \vdash O(\kappa \wedge \delta_p \wedge \delta_q) \equiv O(\kappa \wedge p \wedge q)$.

Example 3.11. Let $\varphi = (Bp \supset r) \wedge (Bq \equiv q)$. We use a φ corresponding to the autoepistemic theory in Ex. 2.11. First we substitute Bp .

$$\begin{aligned}
O\varphi &\xrightarrow[\text{E}]{Bp} [O((\top \supset r \wedge (Bq \equiv q)) \wedge Bp) \vee \\
&\quad [O((\perp \supset r) \wedge (Bq \equiv q)) \wedge \neg Bp] \\
&\xrightarrow[\text{S}]{} [O(r \wedge (Bq \equiv q) \wedge Bp) \vee [O(Bq \equiv q) \wedge \neg Bp]
\end{aligned}$$

Two new formulae in need of expanding have been generated. If we let $\varphi_1 = (r \wedge (Bq \equiv q))$ and $\varphi_2 = (Bq \equiv q)$, we can write $O\varphi$ as $(O\varphi_1 \wedge Bp) \vee (O\varphi_2 \wedge \neg Bp)$. First we expand $O\varphi_1$.

$$\begin{aligned}
O\varphi_1 &\xrightarrow[\text{E}]{Bq} [O(r \wedge (\top \equiv q)) \wedge Bq] \vee [O(r \wedge (\perp \equiv q)) \wedge \neg Bq] \\
&\xrightarrow[\text{S}]{} [O(r \wedge q) \wedge Bq] \vee [O(r \wedge \neg q) \wedge \neg Bq]
\end{aligned}$$

Collapse $O(r \wedge q) \wedge Bq \xrightarrow[\text{C}]{\lambda} O(r \wedge q)$ and $O(r \wedge \neg q) \wedge \neg Bq \xrightarrow[\text{C}]{\lambda} O(r \wedge \neg q)$.

$$\begin{aligned}
O\varphi_2 &\xrightarrow[\text{E}]{Bq} [Bq \wedge O(\top \equiv q)] \vee [\neg Bq \wedge O(\perp \equiv q)] \\
&\xrightarrow[\text{S}]{} [Bq \wedge Oq] \vee [\neg Bq \wedge O\neg q]
\end{aligned}$$

Collapse $Oq \wedge Bq \xrightarrow[\text{C}]{\lambda} Oq$ and $O\neg q \wedge \neg Bq \xrightarrow[\text{C}]{\lambda} O\neg q$.

Figure 3.4: $O\varphi$ expanded and partially collapsed.

Now we have two situations where the distributive rule applies.

$$\begin{aligned} (O(r \wedge q) \vee O(r \wedge \neg q)) \wedge Bp &\xrightarrow{D} (O(r \wedge q) \wedge Bp) \vee (O(r \wedge \neg q) \wedge Bp) \\ (Oq \vee O\neg q) \wedge \neg Bp &\xrightarrow{D} (Oq \wedge \neg Bp) \vee (O\neg q \wedge \neg Bp) \end{aligned}$$

We collapse every disjunct.

$$\begin{aligned} O(r \wedge q) \wedge Bp &\xrightarrow{C} \perp & Oq \wedge \neg Bp &\xrightarrow{C} Oq \\ O(r \wedge \neg q) \wedge Bp &\xrightarrow{C} \perp & O\neg q \wedge \neg Bp &\xrightarrow{C} O\neg q \end{aligned}$$

We then collapse the left subtree, then the entire tree.

$$\begin{aligned} \perp \vee \perp &\xrightarrow{\perp} \perp \\ \perp \vee (Oq \vee O\neg q) &\xrightarrow{\perp} Oq \vee O\neg q \end{aligned}$$

In conclusion $\lambda \vdash O[(Bp \supset r) \wedge (Bq \equiv q)] \equiv (Oq \vee O\neg q)$.

3.3 Langholm's rule set

The expand rule given in the previous section is not identical to the one originally given in [8]. There all modal subformulae are rewritten to prime O -formulae. This means that only a subset of the the collapse rule is needed.

3.3.1 The rewriting rules

Let $\beta \in \text{ext}(\varphi)$. The *Langholm expand* rule is defined as

$$\begin{aligned} O\varphi &\xrightarrow{E^L} (O\varphi[B\psi/\top] \wedge O(\varphi[B\psi/\top] \wedge \psi)) \vee \\ &\quad (O\varphi[B\psi/\perp] \wedge \neg O(\varphi[B\psi/\perp] \wedge \psi)) \\ O\varphi &\xrightarrow{E^L} (O\varphi[C\psi/\top] \wedge O(\psi \supset \varphi[C\psi/\top])) \vee \\ &\quad (O\varphi[C\psi/\perp] \wedge \neg O(\psi \supset \varphi[C\psi/\perp])) \\ O\varphi &\xrightarrow{E^L} (O\varphi[O\psi/\top] \wedge O\psi) \vee (O\varphi[O\psi/\perp] \wedge \neg O\psi) \end{aligned}$$

The *Langholm collapse rule* is defined for propositional φ and ψ as

$$\begin{aligned} O\varphi \wedge O\psi &\xrightarrow[C^L]{\lambda} O\varphi \text{ if } \lambda \vdash O\varphi \supset O\psi \\ O\varphi \wedge \neg O\psi &\xrightarrow[C^L]{\lambda} \perp \text{ if } \lambda \vdash O\varphi \supset O\psi \\ O\varphi \wedge O\psi &\xrightarrow[C^L]{\lambda} \perp \text{ if } \lambda \not\vdash O\varphi \supset O\psi \\ O\varphi \wedge \neg O\psi &\xrightarrow[C^L]{\lambda} O\varphi \text{ if } \lambda \not\vdash O\varphi \supset O\psi \end{aligned}$$

With these rules, the distributive rule isn't strong enough, so we introduce a more general rule for putting formulae on disjunctive normal form.

The *DNF rule* is defined as

$$\varphi \xrightarrow[\text{DNF}]{} \psi \text{ if } \vdash \varphi \equiv \psi \text{ and } \psi \text{ is on DNF}$$

3.3.2 Soundness of the rules

Lemma 3.12.

$$\vdash O\varphi \supset (B\psi \equiv O(\varphi \wedge \psi)) \quad (3.2)$$

$$\vdash O\varphi \supset (C\psi \equiv O(\psi \supset \varphi)) \quad (3.3)$$

Proof. To prove (3.2), we need to prove (i) $O\varphi \wedge B\psi \vdash O(\varphi \wedge \psi)$ and (ii) $O\varphi \wedge O(\varphi \wedge \psi) \vdash B\psi$. (i) holds since $\vdash C\neg\varphi \supset C\neg(\varphi \wedge \psi)$. (ii) obviously holds. (3.3) can be proven similarly. $\square$

With this expand rule, all modal atoms are substituted with O -formulae, which results in an expression without nested modal operators.

The soundness of the expand rule follows from the following Theorem.

Theorem 3.13. Let $\beta \in \text{ext}(\varphi)$, and let $\varphi_{\top} = \varphi[\beta/\top]$ and $\varphi_{\perp} = \varphi[\beta/\perp]$.

$$\begin{aligned} \vdash O\varphi &\equiv [(O\varphi_{\top} \wedge O(\varphi_{\top} \wedge \psi)) \vee (O\varphi_{\perp} \wedge \neg O(\varphi_{\perp} \wedge \psi))] \text{ if } \beta = B\psi \\ \vdash O\varphi &\equiv [(O\varphi_{\top} \wedge O(\psi \supset \varphi_{\top})) \vee (O\varphi_{\perp} \wedge \neg O(\psi \supset \varphi_{\perp}))] \text{ if } \beta = C\psi \\ \vdash O\varphi &\equiv [(O\varphi_{\top} \wedge O\psi) \vee (O\varphi_{\perp} \wedge \neg O\psi)] \text{ if } \beta = O\psi \end{aligned}$$

Proof. This follows from Theorem 3.7 and Lemma 3.12. $\square$

The soundness of the collapse rule follows from the following two Theorems.

Theorem 3.14.

$$\begin{aligned}\lambda \vdash (O\varphi \wedge O\psi) &\equiv O\varphi \text{ iff } \lambda \vdash O\varphi \supset O\psi \\ \lambda \vdash (O\varphi \wedge \neg O\psi) &\equiv \perp \text{ iff } \lambda \vdash O\varphi \supset O\psi\end{aligned}$$

Proof. Both are equivalent and follow from propositional logic. $\square$

Theorem 3.15. Let φ be propositional.

$$\begin{aligned}\lambda \vdash (O\varphi \wedge O\psi) &\equiv \perp \text{ iff } \lambda \not\vdash O\varphi \supset O\psi \\ \lambda \vdash (O\varphi \wedge \neg O\psi) &\equiv O\varphi \text{ iff } \lambda \not\vdash O\varphi \supset O\psi\end{aligned}$$

Proof. Both are equivalent to “ $\lambda \not\vdash O\varphi \supset O\psi$ iff $\lambda \vdash O\varphi \supset \neg O\psi$ ”, which holds because of Theorem 2.10. $\square$

3.3.3 The proof

We are ready to prove the Modal reduction theorem again. Notice that although we induce over $|\text{mod}(\varphi)|$, during expansion only extensional formulae are substituted. But because of Lemma 3.12, *all* modal atoms occurring as subformulae of φ are eventually substituted.

Proof. Induction over $|\text{mod}(\varphi)|$. If $|\text{mod}(\varphi)| > 0$, a modal atom β occurs as a subformula of φ . Let $\varphi_{\top} = \varphi[\beta/\top]$ and $\varphi_{\perp} = \varphi[\beta/\perp]$. Depending on β , by Theorem 3.13, $O\varphi$ expands to one of the following three formulae: $(O\varphi_{\top} \wedge O(\varphi_{\top} \wedge \psi)) \vee (O\varphi_{\perp} \wedge \neg O(\varphi_{\perp} \wedge \psi))$ if $\beta = B\psi$, $(O\varphi_{\top} \wedge O(\psi \supset \varphi_{\top})) \vee (O\varphi_{\perp} \wedge \neg O(\psi \supset \varphi_{\perp}))$ if $\beta = C\psi$, and $(O\varphi_{\top} \wedge O\psi) \vee (O\varphi_{\perp} \wedge \neg O\psi)$ if $\beta = O\psi$.

Base step: $|\text{mod}(\varphi)| = 1$. The expanded formula is collapsed to one of the following prime O -formulae: $O\varphi_{\top} \vee O\varphi_{\perp}$, $O\varphi_{\top}$, $O\varphi_{\perp}$ or $\perp$.

Induction step: $|\text{mod}(\varphi)| = n$. $|\text{mod}(\varphi_{\top})| = |\text{mod}(\varphi_{\perp})| = |\text{mod}(\psi)| = n - 1$. The induction hypothesis is that $O\varphi_{\top}$ and $O\varphi_{\perp}$ are equivalent to disjunctions of prime O -formulae and that, depending on β , so are $O\psi$, $O(\varphi_{\top} \wedge \psi)$, $O(\varphi_{\perp} \wedge \psi)$, $O(\psi \supset \varphi_{\top})$ and $O(\psi \supset \varphi_{\perp})$. By applying the DNF rule, then the collapse rule and finally the disjunct simplification rule, this also holds for $O\varphi_{\top} \wedge O(\varphi_{\top} \wedge \psi)$, $O\varphi_{\top} \wedge O(\psi \supset \varphi_{\top})$, $O\varphi_{\top} \wedge O\psi$, $O\varphi_{\perp} \wedge \neg O(\varphi_{\perp} \wedge \psi)$, $O\varphi_{\perp} \wedge \neg O(\psi \supset \varphi_{\perp})$ and $O\varphi_{\perp} \wedge \neg O\psi$. $O\varphi$ is the disjunction of two of these formulae, which means that this also holds for $O\varphi$. $\square$

3.3.4 Examples

These are similar to Examples 3.10 and 3.11, except that the we use the Langholm rewriting rules. The logical space λ is maximal.

Example 3.16. Let $\delta_1 = (\neg p \supset B\neg p)$, and $\delta_2 = (\neg q \supset B\neg q)$, and let $\varphi = \kappa \wedge \delta_1 \wedge \delta_2$. First we substitute $B\neg p$.

$$\begin{aligned} O\varphi &\xrightarrow[E^L]{B\neg p} [O(\kappa \wedge (\neg p \supset \top) \wedge \delta_2) \wedge O(\kappa \wedge (\neg p \supset \top) \wedge \delta_2 \wedge \neg p)] \vee \\ &\quad [O(\kappa \wedge (\neg p \supset \perp) \wedge \delta_2) \wedge \neg O(\kappa \wedge (\neg p \supset \perp) \wedge \delta_2 \wedge \neg p)] \\ &\xrightarrow[S]{} [O(\kappa \wedge \delta_2) \wedge O(\kappa \wedge \delta_2 \wedge \neg p)] \vee [O(\kappa \wedge p \wedge \delta_2) \wedge \neg O\perp] \end{aligned}$$

If we let $\psi_1 = (\kappa \wedge \delta_2 \wedge \neg p)$, $\varphi_1 = (\kappa \wedge \delta_2)$ and $\varphi_2 = (\kappa \wedge p \wedge \delta_2)$, we can write $O\varphi$ as $(O\varphi_1 \wedge O\psi_1) \vee (O\varphi_2 \wedge \neg O\perp)$.

Figure 3.5: $O\varphi$ after expanding $B\neg p$.

We have generated four new O -formulae (as opposed to two in Example 3.10), one of which is prime. First we substitute $B\neg q$ in $O\psi_1$.

$$\begin{aligned} O\psi_1 &\xrightarrow[E^L]{B\neg q} [O(\kappa \wedge (\neg q \supset \top) \wedge \neg p) \wedge O(\kappa \wedge (\neg q \supset \top) \wedge \neg p \wedge \neg q)] \vee \\ &\quad [O(\kappa \wedge (\neg q \supset \perp) \wedge \neg p) \wedge \neg O(\kappa \wedge (\neg q \supset \perp) \wedge \neg p \wedge \neg q)] \\ &\xrightarrow[S]{} [O(\kappa \wedge \neg p) \wedge O(\kappa \wedge \neg p \wedge \neg q)] \vee [O(\kappa \wedge \neg p \wedge q) \wedge \neg O\perp] \end{aligned}$$

Then we substitute $B\neg q$ in $O\varphi_1$.

$$\begin{aligned} O\varphi_1 &\xrightarrow[E^L]{B\neg q} [O(\kappa \wedge (\neg q \supset \top)) \wedge O(\kappa \wedge (\neg q \supset \top) \wedge \neg q)] \vee \\ &\quad [O(\kappa \wedge (\neg q \supset \perp)) \wedge \neg O(\kappa \wedge (\neg q \supset \perp) \wedge \neg q)] \\ &\xrightarrow[S]{} [O\kappa \wedge O(\kappa \wedge \neg q)] \vee [O(\kappa \wedge q) \wedge \neg O\perp] \end{aligned}$$

Figure 3.6: $O\psi_1$ after expanding $B\neg q$.

Figure 3.7: $O\varphi_1$ after expanding $B\neg q$.

The expanded formula is bigger than in Example 3.10 since the formulae in $\text{mod}(\varphi)$ are rewritten to O -formulae, which are then expanded. All modal subformulae are prime after expansion, which was the case in the previous example as well but there it was so because φ didn't contain nested modalities. The leaf nodes in the left subtree ($O\psi_1$) are collapsed as follows.

$$\begin{aligned}
O(\kappa \wedge \neg p) \wedge O(\kappa \wedge \neg p \wedge \neg q) &\xrightarrow[\text{CL}]{\lambda} \perp \\
O(\kappa \wedge \neg p \wedge q) \wedge \neg O\perp &\xrightarrow[\text{CL}]{\lambda} O(\kappa \wedge \neg p \wedge q) \\
\perp \vee O(\kappa \wedge \neg p \wedge q) &\xrightarrow[\perp]{} O(\kappa \wedge \neg p \wedge q)
\end{aligned}$$

The leaf nodes in the right subtree ($O\varphi_1$) are collapsed as follows.

$$\begin{aligned}
O\kappa \wedge O(\kappa \wedge \neg q) &\xrightarrow[\text{CL}]{\lambda} \perp \\
O(\kappa \wedge q) \wedge \neg O\perp &\xrightarrow[\text{CL}]{\lambda} O(\kappa \wedge q) \\
\perp \vee O(\kappa \wedge q) &\xrightarrow[\perp]{} O(\kappa \wedge q)
\end{aligned}$$

We substitute $B\neg q$ in $O\varphi_2$.

$$\begin{aligned}
O\varphi_2 &\xrightarrow[\text{EL}]{B\neg q} [O(\kappa \wedge p \wedge (\neg q \supset \top)) \wedge O(\kappa \wedge p \wedge (\neg q \supset \top) \wedge \neg q)] \vee \\
&\quad [O(\kappa \wedge p \wedge (\neg q \supset \perp)) \wedge \neg O(\kappa \wedge p \wedge (\neg q \supset \perp) \wedge \neg q)] \\
&\xrightarrow[S]{} [O(\kappa \wedge p) \wedge O(\kappa \wedge p \wedge \neg q)] \vee [O(\kappa \wedge p \wedge q) \wedge \neg O\perp]
\end{aligned}$$

Figure 3.8: $O\varphi_2 \wedge \neg O\perp$ after expanding $B\neg q$.

The leaf nodes are collapsed as follows.

$$\begin{aligned}
O(\kappa \wedge p) \wedge O(\kappa \wedge p \wedge \neg q) &\xrightarrow[C^L]{\lambda} \perp \\
O(\kappa \wedge p \wedge q) \wedge \neg O\perp &\xrightarrow[C^L]{\lambda} O(\kappa \wedge p \wedge q) \\
\perp \vee O(\kappa \wedge p \wedge q) &\xrightarrow[\perp]{} O(\kappa \wedge p \wedge q)
\end{aligned}$$

Figure 3.9: $O\varphi$ expanded and partially collapsed.

The new leaf nodes are collapsed as follows.

$$\begin{aligned}
O(\kappa \wedge q) \wedge O(\kappa \wedge \neg p \wedge q) &\xrightarrow[C^L]{\lambda} \perp \\
O(\kappa \wedge p \wedge q) \wedge \neg O\perp &\xrightarrow[C^L]{\lambda} O(\kappa \wedge p \wedge q)
\end{aligned}$$

As in Example 3.10, $\lambda \vdash O(\kappa \wedge \delta_p \wedge \delta_q) \equiv O(\kappa \wedge p \wedge q)$.

Example 3.17. Let $\varphi = (Bp \supset r) \wedge (Bq \equiv q)$. First we substitute Bp .

$$\begin{aligned}
O\varphi &\xrightarrow[E^L]{Bp} [O((\top \supset r \wedge (Bq \equiv q)) \wedge O((\top \supset r) \wedge (Bq \equiv q) \wedge p)) \vee \\
&\quad [O((\perp \supset r) \wedge (Bq \equiv q)) \wedge \neg O((\perp \supset r) \wedge (Bq \equiv q) \wedge p)] \\
&\xrightarrow[S]{} [O(r \wedge (Bq \equiv q)) \wedge O(p \wedge r \wedge (Bq \equiv q))] \vee \\
&\quad [O(Bq \equiv q) \wedge \neg O(p \wedge (Bq \equiv q))]
\end{aligned}$$

Four new formulae in need of further expanding have been generated. If we let $\varphi_1 = (p \wedge (Bq \equiv q))$, $\varphi_2 = (p \wedge r \wedge (Bq \equiv q))$, $\varphi_3 = (Bq \equiv q)$ and $\varphi_4 = (r \wedge (Bq \equiv q))$, we can write $O\varphi$ as $(O\varphi_1 \wedge O\varphi_2) \vee (O\varphi_3 \wedge \neg O\varphi_4)$.

$$\begin{aligned}
O\varphi_1 &\xrightarrow[E^L]{Bq} [O(p \wedge (\top \equiv q)) \wedge O(p \wedge (\top \equiv q) \wedge q)] \vee \\
&\quad [O(p \wedge (\perp \equiv q)) \wedge \neg O(p \wedge (\perp \equiv q) \wedge q)] \\
&\xrightarrow[S]{} O(p \wedge q) \vee [O(p \wedge \neg q) \wedge \neg O\perp]
\end{aligned}$$

After expanding $O\varphi_1$, we collapse $O(p \wedge \neg q) \wedge \neg O\perp \xrightarrow{C^L} O(p \wedge \neg q)$.

$$\begin{aligned} O\varphi_2 &\xrightarrow{Bq} [O(p \wedge r \wedge (\top \equiv q)) \wedge O(p \wedge r \wedge (\top \equiv q) \wedge q)] \vee \\ &\quad [O(p \wedge r \wedge (\perp \equiv q)) \wedge \neg O(p \wedge r \wedge (\perp \equiv q) \wedge q)] \\ &\xrightarrow{S} O(p \wedge r \wedge q) \vee [O(p \wedge r \wedge \neg q) \wedge \neg O\perp] \end{aligned}$$

After expanding $O\varphi_2$, we collapse $O(p \wedge r \wedge \neg q) \wedge \neg O\perp \xrightarrow{C^L} O(p \wedge r \wedge \neg q)$.

$$\begin{aligned} O\varphi_3 &\xrightarrow{Bq} [O((\top \equiv q)) \wedge O((\top \equiv q) \wedge q)] \vee \\ &\quad [O(\perp \equiv q) \wedge \neg O((\perp \equiv q) \wedge q)] \\ &\xrightarrow{S} Oq \vee [O\neg q \wedge \neg O\perp] \end{aligned}$$

After expanding $O\varphi_3$, we collapse $O\neg q \wedge \neg O\perp \xrightarrow{C^L} O\neg q$.

$$\begin{aligned} O\varphi_4 &\xrightarrow{Bq} [O(r \wedge (\top \equiv q)) \wedge O(r \wedge (\top \equiv q) \wedge q)] \vee \\ &\quad [O(r \wedge (\perp \equiv q)) \wedge \neg O(r \wedge (\perp \equiv q) \wedge q)] \\ &\xrightarrow{S} O(r \wedge q) \vee [O(r \wedge \neg q) \wedge \neg O\perp] \end{aligned}$$

After expanding $O\varphi_4$, we collapse $O(r \wedge \neg q) \wedge \neg O\perp \xrightarrow{C^L} O(r \wedge \neg q)$.

Figure 3.10: $O\varphi$ expanded and partially collapsed.

Now we have two situations where the DNF rule applies. First we apply it to the left subtree.

$$\begin{aligned} &(O(p \wedge q) \vee O(p \wedge \neg q)) \wedge (O(p \wedge r \wedge q) \vee O(p \wedge r \wedge \neg q)) \xrightarrow{\text{DNF}} \\ &((O(p \wedge q) \wedge O(p \wedge r \wedge q)) \vee (O(p \wedge \neg q) \wedge O(p \wedge r \wedge q)) \vee \\ &\quad (O(p \wedge q) \wedge O(p \wedge r \wedge \neg q)) \vee (O(p \wedge \neg q) \wedge O(p \wedge r \wedge \neg q))) \end{aligned}$$

Then to the right subtree.

$$\begin{aligned} &(Oq \vee O\neg q) \wedge \neg(O(r \wedge q) \vee O(r \wedge \neg q)) \xrightarrow{\text{DNF}} \\ &((Oq \wedge \neg O(r \wedge q) \wedge \neg O(r \wedge \neg q)) \vee (O\neg q \wedge \neg O(r \wedge q) \wedge \neg O(r \wedge \neg q))) \end{aligned}$$

We see that we get a conjunction with more than one modal atom. When this is the case, we simply apply the collapse rule on each modal atom separately, then apply the disjunct simplification rule (skipped here).

Collapse the left subtree.

$$\begin{aligned} O(p \wedge q) \wedge O(p \wedge r \wedge q) &\xrightarrow{C^L} \perp & O(p \wedge \neg q) \wedge O(p \wedge r \wedge q) &\xrightarrow{C^L} \perp \\ O(p \wedge q) \wedge O(p \wedge r \wedge \neg q) &\xrightarrow{C^L} \perp & O(p \wedge \neg q) \wedge O(p \wedge r \wedge \neg q) &\xrightarrow{C^L} \perp \end{aligned}$$

Collapse the right subtree.

$$\begin{aligned} Oq \wedge \neg O(r \wedge q) &\xrightarrow{C^L} Oq & O\neg q \wedge \neg O(r \wedge q) &\xrightarrow{C^L} O\neg q \\ Oq \wedge \neg O(r \wedge \neg q) &\xrightarrow{C^L} Oq & O\neg q \wedge \neg O(r \wedge \neg q) &\xrightarrow{C^L} O\neg q \end{aligned}$$

What remains of the tree is collapsed as follows.

$$\perp \vee (Oq \vee O\neg q) \xrightarrow{\perp} Oq \vee O\neg q$$

As in Example 3.11, $\lambda \vdash O[(Bp \supset r) \wedge (Bq \equiv q)] \equiv (Oq \vee O\neg q)$.

3.4 The prime rule set

The expand rule of this rule set guarantees that only prime modal atoms occur when collapsing.

Let $\beta \in \text{mod}(\varphi)$ such that $d(\beta) = 1$. The *prime expand rule* is defined as

$$O\varphi \xrightarrow{E'}^{(\beta)} (O\varphi[\beta/\top] \wedge \beta) \vee (O\varphi[\beta/\perp] \wedge \neg\beta)$$

Substitution Lemma 3.5 cannot be generalized directly into $\mathcal{AE}$ as the following three equivalent statements demonstrate.

$$\begin{aligned} \vdash \Box p &\equiv (\text{if } p \text{ then } \Box p[p/\top] \text{ else } \Box p[p/\perp]) \\ \vdash \Box p &\equiv (\Box \top \wedge p) \vee (\Box \perp \wedge \neg p) \\ \vdash \Box p &\equiv p \end{aligned}$$

$\Box p \equiv p$ cannot be valid as it would yield the Triv system. However, as mentioned in [8], the following holds.

Lemma 3.18 (Substitution). Let $\beta \in \text{mod}(\varphi)$. Then

$$\vdash \varphi \equiv \text{if } \beta \text{ then } \varphi[\beta/\top] \text{ else } \varphi[\beta/\perp].$$

Proof. As in Lemma 3.5, induction over φ . The base step is similar.

Induction step: $\varphi = B\psi$. The induction hypothesis is

$$\vdash \psi \equiv (\psi[\beta/\top] \wedge \beta) \vee (\psi[\beta/\perp] \wedge \neg\beta).$$

From this it follows that $B\psi$ is equivalent to

$$B((\psi[\beta/\top] \vee \neg\beta) \wedge (\psi[\beta/\perp] \vee \beta)).$$

B distributes over conjunctions, thus

$$B(\psi[\beta/\top] \vee \neg\beta) \wedge B(\psi[\beta/\perp] \vee \beta).$$

Using Lemma 2.23 we can put the formula on normal form:

$$(B(\psi[\beta/\top]) \vee \neg\beta) \wedge (B(\psi[\beta/\perp]) \vee \beta),$$

which is equivalent to

$$(B(\psi[\beta/\top]) \wedge \beta) \vee (B(\psi[\beta/\perp]) \wedge \neg\beta).$$

The case when $\varphi = C\psi$ is similar. The other cases are similar to the ones in the proof of Lemma 3.5. $\square$

Theorem 3.19. Let $\beta \in \text{mod}(\varphi)$ such that $d(\beta) = 1$. Then

$$\vdash O\varphi \equiv ((\beta \wedge O\varphi[\beta/\top]) \vee (\neg\beta \wedge O\varphi[\beta/\perp])).$$

Proof. The proof is similar to that of Theorem 3.7, only with Substitution Lemma 3.18 instead of 3.5. $\square$

The proof of the Modal reduction theorem using this strengthened expand rule is similar to the proof using the simple expand rule. The only difference is that the induction is over $|\text{mod}(\varphi)|$. We can still use the simple collapse rule.

To illustrate the strength of the prime expand rule, we need an example containing nested modalities.

3.5 The rule sets compared

We give three examples, each using a different rule set, on a formula containing nested modalities, to illustrate the differences between the approaches. In all three examples, let $\varphi = \kappa \wedge (p \supset BCp)$, and let the logical space λ be maximal. We will show that $\lambda \vdash O\varphi \equiv O(\kappa \wedge \neg p)$.

Example 3.20. The simple rules.

$$\begin{aligned} O\varphi &\xrightarrow[BCp]{E} [O(\kappa \wedge (p \supset \top)) \wedge BCp] \vee \\ &\quad [O(\kappa \wedge (p \supset \perp)) \wedge \neg BCp] \\ &\xrightarrow[S]{} [O\kappa \wedge BCp] \vee [O(\kappa \wedge \neg p) \wedge \neg BCp] \end{aligned}$$

Collapse $O\kappa \wedge BCp \xrightarrow[C]{\lambda} \perp$ and $O(\kappa \wedge \neg p) \wedge \neg BCp \xrightarrow[C]{\lambda} O(\kappa \wedge \neg p)$.

These collapses have no equivalent in PL, so one has to use some sort of K45 theorem prover. Then collapse $\perp \vee O(\kappa \wedge \neg p) \xrightarrow[\perp]{} O(\kappa \wedge \neg p)$.

Example 3.21. The Langholm rules.

$$\begin{aligned} O\varphi &\xrightarrow[BCp]{EL} [O(\kappa \wedge (p \supset \top)) \wedge O(Cp \supset (\kappa \wedge (p \supset \top)))] \vee \\ &\quad [O(\kappa \wedge (p \supset \perp)) \wedge \neg O(Cp \supset (\kappa \wedge (p \supset \perp)))] \\ &\xrightarrow[S]{} [O\kappa \wedge O(Cp \supset \kappa)] \vee [O(\kappa \wedge \neg p) \wedge \neg O(Cp \supset (\kappa \wedge \neg p))] \end{aligned}$$

Two non-prime O -formulae generated.

$$\begin{aligned} O(Cp \supset \kappa) &\xrightarrow[EL]{Cp} [O(\top \supset \kappa) \wedge O(p \supset (\top \supset \kappa))] \vee \\ &\quad [O(\perp \supset \kappa) \wedge \neg O(p \supset (\perp \supset \kappa))] \\ &\xrightarrow[S]{} [O\kappa \wedge O(p \supset \kappa)] \vee [O\top \wedge \neg O\top] \end{aligned}$$

Collapse $O\kappa \wedge O(p \supset \kappa) \xrightarrow[C^L]{\lambda} \perp$ and $O\top \wedge \neg O\top \xrightarrow[C^L]{\lambda} \perp$, then

$\perp \vee \perp \xrightarrow[\perp]{} \perp$ and $\perp \wedge O\kappa \xrightarrow[DNF]{} \perp$.

$$\begin{aligned} O(Cp \supset (\kappa \wedge \neg p)) &\xrightarrow[EL]{Cp} [O(\top \supset (\kappa \wedge \neg p)) \wedge O(p \supset (\top \supset (\kappa \wedge \neg p)))] \vee \\ &\quad [O(\perp \supset (\kappa \wedge \neg p)) \wedge \neg O(p \supset (\perp \supset (\kappa \wedge \neg p)))] \\ &\xrightarrow[S]{} [O(\kappa \wedge \neg p) \wedge O\neg p] \vee [O\top \wedge \neg O\top] \end{aligned}$$

Collapse $O(\kappa \wedge \neg p) \wedge O\neg p \xrightarrow[C^L]{\lambda} \perp$ and $O\top \wedge \neg O\top \xrightarrow[C^L]{\lambda} \perp$, then

$$\perp \vee \perp \xrightarrow[\perp]{} \perp \text{ and } O(\kappa \wedge \neg p) \wedge \neg \perp \xrightarrow[\text{DNF}]{} O(\kappa \wedge \neg p).$$

$$\text{Thus } \perp \vee O(\kappa \wedge \neg p) \xrightarrow[\perp]{} O(\kappa \wedge \neg p).$$

Example 3.22. The prime rules.

$$O\varphi \xrightarrow[E']{Cp} [O(\kappa \wedge (p \supset B\top)) \wedge Cp] \vee [O(\kappa \wedge (p \supset B\perp)) \wedge \neg Cp]$$

Two non-prime O -formulae generated.

$$\begin{aligned} O(\kappa \wedge (p \supset B\top)) &\xrightarrow[E']{B\top} [O(\kappa \wedge (p \supset \top)) \wedge B\top] \vee \\ &\quad [O(\kappa \wedge (p \supset \perp)) \wedge \neg B\top] \\ &\xrightarrow[S]{} [O\kappa \wedge B\top] \vee [O(\kappa \wedge \neg p)) \wedge \neg B\top] \end{aligned}$$

Collapse $O\kappa \wedge B\top \xrightarrow[C]{\lambda} O\kappa$ and $O(\kappa \wedge \neg p) \wedge \neg B\top \xrightarrow[C]{\lambda} \perp$, then

$O\kappa \vee \perp \xrightarrow[\perp]{} O\kappa$ and $O\kappa \wedge Cp \xrightarrow[C]{\lambda} \perp$. Expand

$$\begin{aligned} O(\kappa \wedge (p \supset B\perp)) &\xrightarrow[E']{B\perp} [O(\kappa \wedge (p \supset \top)) \wedge B\perp] \vee \\ &\quad [O(\kappa \wedge (p \supset \perp)) \wedge \neg B\perp] \\ &\xrightarrow[S]{} [O\kappa \wedge B\perp] \vee [O(\kappa \wedge \neg p)) \wedge \neg B\perp] \end{aligned}$$

Collapse $O\kappa \wedge B\perp \xrightarrow[C]{\lambda} \perp$ and $O(\kappa \wedge \neg p) \wedge \neg B\perp \xrightarrow[C]{\lambda} O(\kappa \wedge \neg p)$, then

$$\perp \vee O(\kappa \wedge \neg p) \xrightarrow[\perp]{} O(\kappa \wedge \neg p) \text{ and } O(\kappa \wedge \neg p) \wedge \neg Cp \xrightarrow[C]{\lambda} O(\kappa \wedge \neg p).$$

$$\text{Thus } \perp \vee O(\kappa \wedge \neg p) \xrightarrow[\perp]{} O(\kappa \wedge \neg p).$$

We see that using the simple rules we only need to expand once and collapse twice, at the expense of not being able to do the collapses in PL. Using Langholm's rules there is much more expanding and collapsing going on. However, every collapse can be done in PL. With the prime rules we get the benefits of both. Collapses can be done in PL, and we only expand once per modal atom occurring as subformula.

In the next chapter, when generalizing the rewriting process to handle modalities of different confidence levels, we base the rules on the prime rule set (whose collapse rules are the same as for the simple rule set).

Chapter 4

Confidence levels

4.1 Introduction

In this chapter we generalize the results from the previous to the full logic with confidence levels. We will give rewriting rules to generate the λ -expansions of formulae on the form $\bigwedge_{i \in I} O_i \varphi_i$ (O_I -atoms). The Modal reduction theorem for a general I is stated as follows.

Theorem 4.1. For each O_I -atom φ^I , for some $m \geq 0$, there are prime modal O_I -atoms $\psi_1^I, \dots, \psi_m^I$ such that

$$\lambda \vdash \varphi^I \equiv (\psi_1^I \vee \dots \vee \psi_m^I)$$

Proof. The proof is given in Sect. 4.4. □

Every such ψ_k^I that is *consistent* with λ , is a λ -*expansion* of φ^I . In Ch. 3, when I was a singleton, ψ_k^I was on the form $O\psi_k$, a formula which cannot be inconsistent with λ . Here, however, if $O_i\psi_i$ and $O_k\psi_k$ are conjuncts of ψ_k^I , and $i \prec k$, ψ_k^I is inconsistent if $\lambda \not\vdash \psi_k \supset \psi_i$, because of persistence. In the case when φ^I has no λ -expansion, $\lambda \vdash \neg\varphi^I$.

Let V be a function from a set of modal atoms of depth 1, to $\{\perp, \top\}$, represented by a binary relation in the usual way. Let φ^K be an O_K -atom, and let $v = (\beta, V(\beta))$ such that β occurs as a subformula of φ^K and $d(\beta) = 1$. Define

$$\varphi^K[V] = \varphi^K[\beta/V(\beta)][V \setminus \{v\}]$$

If $|V| = |\text{mod}(\varphi^K)|$ and $\varphi^K[V]$ is prime, we say that V is a *modal valuation* of φ^K , and $\varphi^I[V]$ is said to be a *model candidate* for φ^I .

Example 4.2. If $\varphi = C_1(p \wedge B_2q)$, and V is a modal valuation of φ such that $V(B_2q) = \top$ and $V(C_1(p \wedge \top)) = \perp$, then $O_1\varphi[V] = O_1\perp$.

4.2 The general rule set

Although different from the simple rules (Sect. 3.2), we use E and C as subscript.

Assume $\beta \in \text{mod}(\varphi)$ such that $d(\beta) = 1$. The *expand rule* is defined as

$$O_i\varphi \xrightarrow[\beta]{E} (\beta \wedge O_i\varphi[\beta/\top]) \vee (\neg\beta \wedge O_i\varphi[\beta/\perp])$$

Not all conjunctions containing only completely modalized formulae, including at least one O_i -formula are equivalent to an O_i -formula, as was the case for a singleton I . $O_ip \wedge B_kq$, where $i \prec k$ is consistent but cannot be simplified.

Let $i \preceq k$. The *collapse rule* is defined for propositional φ and ψ as

$$O_i\varphi \wedge O_i\psi \xrightarrow{\lambda}{C} O_i\varphi \text{ if } \lambda \vdash O_i\varphi \supset O_i\psi \quad (O^1)$$

$$O_i\varphi \wedge \neg O_i\psi \xrightarrow{\lambda}{C} \perp \text{ if } \lambda \vdash O_i\varphi \supset O_i\psi \quad (O^2)$$

$$O_i\varphi \wedge O_k\psi \xrightarrow{\lambda}{C} \perp \text{ if } \lambda \not\vdash O_k\psi \supset B_k\varphi \quad (O^3)$$

$$O_i\varphi \wedge \neg O_k\psi \xrightarrow{\lambda}{C} O_i\varphi \text{ if } \lambda \not\vdash O_k\psi \supset B_k\varphi \quad (O^4)$$

and

$$O_i\varphi \wedge B_k\psi \xrightarrow{\lambda}{C} O_i\varphi \text{ if } \lambda \vdash O_i\varphi \supset B_i\psi \quad (B^1)$$

$$O_i\varphi \wedge \neg B_k\psi \xrightarrow{\lambda}{C} \perp \text{ if } \lambda \vdash O_i\varphi \supset B_i\psi \quad (B^2)$$

$$O_k\varphi \wedge B_i\psi \xrightarrow{\lambda}{C} \perp \text{ if } \lambda \not\vdash O_k\varphi \supset B_k\psi \quad (B^3)$$

$$O_k\varphi \wedge \neg B_i\psi \xrightarrow{\lambda}{C} O_k\varphi \text{ if } \lambda \not\vdash O_k\varphi \supset B_k\psi \quad (B^4)$$

and

$$O_k\varphi \wedge C_i\psi \xrightarrow{\lambda}{C} O_k\varphi \text{ if } \lambda \vdash O_k\varphi \supset C_k\psi \quad (C^1)$$

$$O_k\varphi \wedge \neg C_i\psi \xrightarrow{\lambda}{C} \perp \text{ if } \lambda \vdash O_k\varphi \supset C_k\psi \quad (C^2)$$

$$O_i\varphi \wedge C_k\psi \xrightarrow{\lambda}{C} \perp \text{ if } \lambda \not\vdash O_i\varphi \supset C_i\psi \quad (C^3)$$

$$O_i\varphi \wedge \neg C_k\psi \xrightarrow{\lambda}{C} O_i\varphi \text{ if } \lambda \not\vdash O_i\varphi \supset C_i\psi \quad (C^4)$$

4.3 Soundness of the rules

Theorem 4.3. Let $\beta \in \text{mod}(\varphi)$ such that $d(\beta) = 1$.

$$\vdash O_i\varphi \equiv ((\beta \wedge O_i\varphi[\beta/\top]) \vee (\neg\beta \wedge O_i\varphi[\beta/\perp])) \quad (4.1)$$

Proof. The proof is similar to that of Theorem 3.19. $\square$

Theorem 4.4. O^1 and O^2 .

$$\lambda \vdash (O_i\varphi \wedge O_i\psi) \equiv O_i\varphi \text{ iff } \lambda \vdash O_i\varphi \supset O_i\psi \quad (4.2)$$

$$\lambda \vdash (O_i\varphi \wedge \neg O_i\psi) \equiv \perp \text{ iff } \lambda \vdash O_i\varphi \supset O_i\psi \quad (4.3)$$

Proof. Eq. 4.2 and 4.3 are both equivalent and follow from propositional logic. $\square$

Theorem 4.5. O^3 and O^4 . Let ψ be propositional, and let $i \preceq k$.

$$\lambda \vdash (O_i\varphi \wedge O_k\psi) \equiv \perp \text{ if } \lambda \not\vdash O_k\psi \supset B_k\varphi \quad (4.4)$$

$$\lambda \vdash (O_i\varphi \wedge \neg O_k\psi) \equiv O_i\varphi \text{ if } \lambda \not\vdash O_k\psi \supset B_k\varphi \quad (4.5)$$

Proof. Eq. 4.4 and 4.5 are both equivalent, and because of Theorem 2.10, the contraposition is “ $\lambda \wedge O_k\psi \vdash B_k\varphi$ if $\lambda \wedge O_k\psi \vdash O_i\varphi$ ”, which follows from persistence. $\square$

Theorem 4.6. B^1 and B^2 . Let $i \preceq k$.

$$\lambda \vdash (O_i\varphi \wedge B_k\psi) \equiv O_i\varphi \text{ if } \lambda \vdash O_i\varphi \supset B_i\psi \quad (4.6)$$

$$\lambda \vdash (O_i\varphi \wedge \neg B_k\psi) \equiv \perp \text{ if } \lambda \vdash O_i\varphi \supset B_i\psi \quad (4.7)$$

Proof. Eq. 4.6 and 4.7 are both equivalent to “ $\lambda \wedge O_i\varphi \vdash B_k\psi$ if $\lambda \wedge O_i\varphi \vdash B_i\psi$ ”, which follows from persistence. $\square$

Theorem 4.7. B^3 and B^4 . Let φ be propositional, and let $i \preceq k$.

$$\lambda \vdash (O_k\varphi \wedge B_i\psi) \equiv \perp \text{ if } \lambda \not\vdash O_k\varphi \supset B_k\psi \quad (4.8)$$

$$\lambda \vdash (O_k\varphi \wedge \neg B_i\psi) \equiv O_k\varphi \text{ if } \lambda \not\vdash O_k\varphi \supset B_k\psi \quad (4.9)$$

Proof. Eq. 4.4 and 4.5 are both equivalent, and because of Theorem 2.10, the contraposition is “ $\lambda \wedge O_k\varphi \vdash B_k\psi$ if $\lambda \wedge O_k\varphi \vdash B_i\psi$ ”, which follows from persistence. $\square$

Theorem 4.8. C^1 and C^2 . Let $i \preceq k$.

$$\lambda \vdash (O_k \varphi \wedge C_i \psi) \equiv O_k \varphi \text{ if } \lambda \vdash O_k \varphi \supset C_k \psi \quad (4.10)$$

$$\lambda \vdash (O_k \varphi \wedge \neg C_i \psi) \equiv \perp \text{ if } \lambda \vdash O_k \varphi \supset C_k \psi \quad (4.11)$$

Proof. Eq. 4.6 and 4.7 are both equivalent to “ $\lambda \wedge O_k \varphi \vdash C_i \psi$ if $\lambda \wedge O_k \varphi \vdash C_k \psi$ ”, which follows from persistence. $\square$

Theorem 4.9. C^3 and C^4 . Let φ be propositional, and let $i \preceq k$.

$$\lambda \vdash (O_i \varphi \wedge C_k \psi) \equiv \perp \text{ if } \lambda \not\vdash O_i \varphi \supset C_i \psi \quad (4.12)$$

$$\lambda \vdash (O_i \varphi \wedge \neg C_k \psi) \equiv O_i \varphi \text{ if } \lambda \not\vdash O_i \varphi \supset C_i \psi \quad (4.13)$$

Proof. Eq. 4.12 and 4.13 are both equivalent, and because of Theorem 2.10, the contraposition is “ $\lambda \wedge O_i \varphi \vdash C_i \psi$ if $\lambda \wedge O_i \varphi \vdash C_k \psi$ ”, which follows from persistence. $\square$

4.4 The proof

For simplicity we introduce a function that given a modal valuation and a formula, gives a formula equivalent to a conjunction of modal literals (the modal atoms that are substituted when valuating $\varphi[V]$, negated iff substituted with $\perp$). Let V be a modal valuation, and let D be the subset of the domain of V that applies to φ . Define

$$M(V, \varphi) = \bigwedge_{\beta \in D} (\beta \equiv V(\beta))$$

Example 4.10. Let $\varphi = C_1(p \wedge B_2 q)$, and let $V = \{(B_2 q, \top), (C_1(p \wedge \top), \perp)\}$ (as in Ex. 4.2). Then $D = \{B_2 q, C_1(p \wedge \top)\}$ and

$$M(V, \varphi) = (\top \equiv B_2 q \wedge) \wedge (\perp \equiv C_1(p \wedge \top)),$$

which is equivalent to $B_2 q \wedge \neg C_1 p$.

Lemma 4.11. Let $m = |\text{mod}(\varphi)|$, and Let $V_1, \dots, V_{2^m}$ be every different modal valuation of φ . Then

$$\lambda \vdash O_i \varphi \equiv ((O_i \varphi[V_1] \wedge M(V_1, \varphi)) \vee \dots \vee (O_i \varphi[V_{2^m}] \wedge M(V_{2^m}, \varphi)))$$

Proof. Induction over $|\text{mod}(\varphi)|$. If $|\text{mod}(\varphi)| > 0$, a modal atom β of depth 1 occurs as a subformula of φ . Let $\varphi_\top = \varphi[\beta/\top]$ and $\varphi_\perp = \varphi[\beta/\perp]$. By Lemma 4.3, $O_i \varphi$ expands to $(O_i \varphi_\top \wedge \beta) \vee (O_i \varphi_\perp \wedge \neg \beta)$.

Base step: Assume $\text{mod}(\varphi) = \{\beta\}$. Let $V_1(\beta) = \top$ and $V_2(\beta) = \perp$. Then

$$\lambda \vdash O_i \varphi \equiv ((O_i \varphi[V_1] \wedge \beta) \vee (O_i \varphi[V_2] \wedge \neg \beta))$$

Induction step: $|\text{mod}(\varphi_\top)| = |\text{mod}(\varphi_\perp)| = k - 1$, thus there is some $\beta \in \text{mod}(\varphi)$ such that $\beta \notin \text{mod}(\varphi_\top) \cup \text{mod}(\varphi_\perp)$. For all $1 \leq i \leq 2^{k-1}$, let $V_i(\beta) = \top$ and $V_{i+2^{k-1}}(\beta) = \perp$, but otherwise equal. The induction hypothesis is that $O_i \varphi_\top$ and $O_i \varphi_\perp$ are equivalent to

$$\begin{aligned} & ((O_i \varphi_\top[V_1] \wedge M(V_1, \varphi_\top)) \vee \cdots \vee (O_i \varphi_\top[V_{2^{k-1}}] \wedge M(V_{2^{k-1}}, \varphi_\top))) \text{ and} \\ & ((O_i \varphi_\perp[V_1] \wedge M(V_1, \varphi_\perp)) \vee \cdots \vee (O_i \varphi_\perp[V_{2^{k-1}}] \wedge M(V_{2^{k-1}}, \varphi_\perp))), \end{aligned}$$

respectively. Thus, by distributing β and $\neg \beta$, $O_i \varphi$ is equivalent to

$$\begin{aligned} & (O_i \varphi_\top[V_1] \wedge M(V_1, \varphi_\top) \wedge \beta) \vee \\ & \cdots \vee (O_i \varphi_\top[V_{2^{k-1}}] \wedge M(V_{2^{k-1}}, \varphi_\top) \wedge \beta) \vee \\ & (O_i \varphi_\perp[V_1] \wedge M(V_1, \varphi_\perp) \wedge \neg \beta) \vee \\ & \cdots \vee (O_i \varphi_\perp[V_{2^{k-1}}] \wedge M(V_{2^{k-1}}, \varphi_\perp) \wedge \neg \beta) \end{aligned}$$

which is equivalent to

$$\begin{aligned} & (O_i \varphi[V_1] \wedge M(V_1, \varphi)) \vee \cdots \vee (O_i \varphi[V_{2^{k-1}}] \wedge M(V_{2^{k-1}}, \varphi)) \vee \\ & (O_i \varphi[V_{1+2^{k-1}}] \wedge M(V_{1+2^{k-1}}, \varphi)) \vee \cdots \vee (O_i \varphi[V_{2^k}] \wedge M(V_{2^k}, \varphi)) \end{aligned}$$

Thus the Lemma holds. $\square$

Lemma 4.12. Let $\varphi = O_1 \varphi_1 \wedge \cdots \wedge O_n \varphi_n$, let $m = |\text{mod}(\varphi)|$, and let $V_1, \dots, V_{2^m}$ be every different modal valuation of φ . Then

$$\lambda \vdash \varphi \equiv ((\varphi[V_1] \wedge M(V_1, \varphi)) \vee \cdots \vee (\varphi[V_{2^m}] \wedge M(V_{2^m}, \varphi)))$$

Proof. Induction over n .

Base step: By Lemma 4.11,

$$\lambda \vdash O_1 \varphi_1 \equiv ((O_1 \varphi_1[V_1] \wedge M(V_1, \varphi)) \vee \cdots \vee (O_1 \varphi_1[V_{2^m}] \wedge M(V_{2^m}, \varphi)))$$

Induction step: Let $\varphi' = O_1 \varphi_1 \wedge \cdots \wedge O_{k-1} \varphi_{k-1}$. The induction hypothesis is that

$$\lambda \vdash \varphi' \equiv ((\varphi'[V_1] \wedge M(V_1, \varphi)) \vee \cdots \vee (\varphi'[V_{2^m}] \wedge M(V_{2^m}, \varphi)))$$

By Lemma 4.11,

$$\lambda \vdash O_k \varphi_k \equiv ((O_k \varphi_k[V_1] \wedge M(V_1, \varphi)) \vee \cdots \vee (O_k \varphi_k[V_{2^m}] \wedge M(V_{2^m}, \varphi)))$$

Thus $O_1\varphi_1 \wedge \cdots \wedge O_{k-1}\varphi_{k-1} \wedge O_k\varphi_k$ is equivalent to

$$\begin{aligned} & (\varphi'[V_1] \wedge M(V_1, \varphi') \wedge O_k\varphi_k[V_1] \wedge M(V_1, \varphi_k)) \vee \\ & \cdots \vee (\varphi'[V_{2^m}] \wedge M(V_{2^m}, \varphi') \wedge O_k\varphi_k[V_1] \wedge M(V_1, \varphi_k)) \vee \\ & (\varphi'[V_1] \wedge M(V_1, \varphi') \wedge O_k\varphi_k[V_{2^m}] \wedge M(V_{2^m}, \varphi_k)) \vee \\ & \cdots \vee (\varphi'[V_{2^m}] \wedge M(V_{2^m}, \varphi') \wedge O_k\varphi_k[V_{2^m}] \wedge M(V_{2^m}, \varphi_k)) \end{aligned}$$

which is equivalent to $(\varphi[V_1] \wedge M(V_1, \varphi)) \vee \cdots \vee (\varphi[V_{2^m}] \wedge M(V_{2^m}, \varphi))$, as every disjunct where $M(V_i, \varphi)$ and $M(V_j, \varphi)$ are conjuncts and $j \neq i$, is inconsistent. $\square$

The Modal reduction theorem follows trivially.

Proof of MRT. Let φ^I be an O_I -atom. Given a logical space, φ^I is equivalent to a formula on disjunctive normal form where each conjunction consists entirely of modal literals including prime O_i -formula for all $i \in I$. This follows directly from Lemma 4.12. Because every conjunction has formulae on the form $O\psi_i$ for all $i \in I$ as conjuncts, the *simple* collapse rule always applies such that the conjunction either is inconsistent, or every other modal atom is removed by a collapse. $\square$

Notice that you only need the simple collapse rule (which is a subset of the general one), as for every modal atom with modality i , there is always a prime O_i -formula in the same conjunction. The richer rules are only needed for providing the possibility of collapsing earlier.

4.5 Examples

As in Ch. 3, we examine an example with two supernormal defaults. Here, however, we have introduced confidence levels, and we examine four different logical spaces, collapsing the formula using the characteristic formula.

Example 4.13. Let $\delta_1 = (\neg p \supset B_1\neg p)$ and $\delta_2 = (\neg q \supset B_2\neg q)$ and

$$\varphi = O_0\kappa \wedge O_1(\kappa \wedge \delta_1) \wedge O_2(\kappa \wedge \delta_1 \wedge \delta_2)$$

We expand $O_1(\kappa \wedge \delta_1)$ and $O_2(\kappa \wedge \delta_1 \wedge \delta_2)$ by substituting $B_1\neg p$.

$$\begin{aligned} O_1(\kappa \wedge \delta_1) & \xrightarrow[E]{B_1\neg p} (O_1\kappa \wedge B_1\neg p) \vee (O_1(\kappa \wedge p) \wedge \neg B_1\neg p) \\ O_2(\kappa \wedge \delta_1 \wedge \delta_2) & \xrightarrow[E]{B_1\neg p} (O_2(\kappa \wedge \delta_2) \wedge B_1\neg p) \vee \\ & (O_2(\kappa \wedge p \wedge \delta_2) \wedge \neg B_1\neg p) \end{aligned}$$

This provides us with two more formulae we need to expand.

$$\begin{aligned} O_2(\kappa \wedge \delta_2) &\xrightarrow[E]{B_2 \neg q} (O_2\kappa \wedge B_2 \neg q) \vee (O_2(\kappa \wedge q) \wedge \neg B_2 \neg q) \\ O_2(\kappa \wedge p \wedge \delta_2) &\xrightarrow[E]{B_2 \neg q} (O_2(\kappa \wedge p) \wedge B_2 \neg q) \vee (O_2(\kappa \wedge p \wedge q) \wedge \neg B_2 \neg q) \end{aligned}$$

We examine four cases with different logical spaces, represented by their characteristic formulae.

1. $\kappa \supset (\neg p \wedge \neg q)$
2. $\kappa \supset (\neg p \wedge q)$
3. $\kappa \supset (\neg p \vee \neg q)$
4. $\top$

Note that the following axioms of the logic $\mathcal{AE}_\rho$, for each ρ .

1. $\text{RI} \Rightarrow \neg \Diamond(\kappa \wedge p), \text{RI} \Rightarrow \neg \Diamond(\kappa \wedge q)$
2. $\text{RI} \Rightarrow \neg \Diamond(\kappa \wedge p), \text{RC} \Rightarrow \Diamond(\kappa \wedge q)$
3. $\text{RI} \Rightarrow \neg \Diamond(\kappa \wedge p \wedge q), \text{RC} \Rightarrow \Diamond(\kappa \wedge p \wedge q) \Rightarrow \Diamond(\kappa \wedge p)$
4. $\text{RC} \Rightarrow \Diamond(\kappa \wedge p \wedge q)$

In each case we need to collapse (from $O_1(\kappa \wedge \delta_1)$):

$$\psi_1 = O_1\kappa \wedge B_1 \neg p \quad \psi_2 = O_1(\kappa \wedge p) \wedge \neg B_1 \neg p$$

And from $O_2(\kappa \wedge \delta_1 \wedge \delta_2)$:

$$\begin{aligned} \psi_3 &= O_2\kappa \wedge B_2 \neg q & \psi_4 &= O_2(\kappa \wedge q) \wedge \neg B_2 \neg q \\ \psi_5 &= O_2(\kappa \wedge p) \wedge B_2 \neg q & \psi_6 &= O_2(\kappa \wedge p \wedge q) \wedge \neg B_2 \neg q \end{aligned}$$

φ fully expanded is

$$O_0\kappa \wedge (\psi_1 \vee \psi_2) \wedge (((\psi_3 \vee \psi_4) \wedge B_1 \neg p) \vee ((\psi_5 \vee \psi_6) \wedge \neg B_1 \neg p)).$$

Case 1: $\rho = \kappa \supset (\neg p \wedge \neg q)$

$$\begin{aligned} O_1\kappa \wedge B_1 \neg p &\xrightarrow[C]{\lambda} O_1\kappa & O_1(\kappa \wedge p) \wedge \neg B_1 \neg p &\xrightarrow[C]{\lambda} \perp \\ O_2\kappa \wedge B_2 \neg q &\xrightarrow[C]{\lambda} O_2\kappa & O_2(\kappa \wedge q) \wedge \neg B_2 \neg q &\xrightarrow[C]{\lambda} \perp \\ O_2(\kappa \wedge p) \wedge B_2 \neg q &\xrightarrow[C]{\lambda} O_2(\kappa \wedge p) & O_2(\kappa \wedge p \wedge q) \wedge \neg B_2 \neg q &\xrightarrow[C]{\lambda} \perp \end{aligned}$$

φ fully expanded is $O_0\kappa \wedge O_1\kappa \wedge ((O_2\kappa \wedge B_1 \neg p) \vee (O_2(\kappa \wedge p) \wedge \neg B_1 \neg p))$. In order to further collapse the formula, we need to put it on disjunctive normal

form: $(O_0\kappa \wedge O_1\kappa \wedge O_2\kappa \wedge B_1\neg p) \vee (O_0\kappa \wedge O_1\kappa \wedge O_2(\kappa \wedge p) \wedge \neg B_1\neg p)$. Since $\rho \vdash \kappa \supset \neg p$,

$$O_1\kappa \wedge B_1\neg p \xrightarrow[C]{\lambda} O_1\kappa \quad \text{and} \quad O_1\kappa \wedge \neg B_1\neg p \xrightarrow[C]{\lambda} \perp$$

Accordingly $\lambda_\rho \vdash \varphi \equiv O_0\kappa \wedge O_1\kappa \wedge O_2\kappa$.

Case 2: $\rho = \kappa \supset (\neg p \wedge q)$

$$\begin{aligned} O_1\kappa \wedge B_1\neg p &\xrightarrow[C]{\lambda} O_1\kappa & O_1(\kappa \wedge p) \wedge \neg B_1\neg p &\xrightarrow[C]{\lambda} \perp \\ O_2\kappa \wedge B_2\neg q &\xrightarrow[C]{\lambda} \perp & O_2(\kappa \wedge q) \wedge \neg B_2\neg q &\xrightarrow[C]{\lambda} O_2(\kappa \wedge q) \\ O_2(\kappa \wedge p) \wedge B_2\neg q &\xrightarrow[C]{\lambda} O_2(\kappa \wedge p) & O_2(\kappa \wedge p \wedge q) \wedge \neg B_2\neg q &\xrightarrow[C]{\lambda} \perp \end{aligned}$$

φ fully expanded and on DNF is $(O_0\kappa \wedge O_1\kappa \wedge O_2(\kappa \wedge q) \wedge B_1\neg p) \vee (O_0\kappa \wedge O_1\kappa \wedge O_2(\kappa \wedge p) \wedge \neg B_1\neg p)$. Since $\rho \vdash \kappa \supset \neg p$,

$$O_1\kappa \wedge B_1\neg p \xrightarrow[C]{\lambda} O_1\kappa \quad \text{and} \quad O_1\kappa \wedge \neg B_1\neg p \xrightarrow[C]{\lambda} \perp$$

Accordingly $\lambda_\rho \vdash \varphi \equiv O_0\kappa \wedge O_1\kappa \wedge O_2(\kappa \wedge q)$.

Case 3: $\rho = \kappa \supset (\neg p \vee \neg q)$, thus for each confidence level i , the possible O_i -formula constraining U_i^+ is equivalent to one of the following (if κ is assumed to be true):

Figure 4.1: Same as Fig. 2.2.

$$\begin{aligned} O_1\kappa \wedge B_1\neg p &\xrightarrow[C]{\lambda} \perp & O_1(\kappa \wedge p) \wedge \neg B_1\neg p &\xrightarrow[C]{\lambda} O_1(\kappa \wedge p) \\ O_2\kappa \wedge B_2\neg q &\xrightarrow[C]{\lambda} \perp & O_2(\kappa \wedge q) \wedge \neg B_2\neg q &\xrightarrow[C]{\lambda} O_2(\kappa \wedge q) \\ O_2(\kappa \wedge p) \wedge B_2\neg q &\xrightarrow[C]{\lambda} O_2(\kappa \wedge p) & O_2(\kappa \wedge p \wedge q) \wedge \neg B_2\neg q &\xrightarrow[C]{\lambda} \perp \end{aligned}$$

φ fully expanded is $O_0\kappa \wedge O_1(\kappa \wedge p) \wedge ((O_2(\kappa \wedge q) \wedge B_1 \neg p) \vee (O_2(\kappa \wedge p) \wedge \neg B_1 \neg p))$.
 Since $\rho \not\models (\kappa \wedge q) \supset \neg p$ and $\rho \not\models (\kappa \wedge p) \supset \neg p$,

$$O_2(\kappa \wedge q) \wedge B_1 \neg p \xrightarrow[\mathcal{C}]{\lambda} \perp \quad \text{and} \quad O_2(\kappa \wedge p) \wedge \neg B_1 \neg p \xrightarrow[\mathcal{C}]{\lambda} O_2(\kappa \wedge p)$$

Accordingly $\lambda_\rho \vdash \varphi \equiv O_0\kappa \wedge O_1(\kappa \wedge p) \wedge O_2(\kappa \wedge p)$. $U_0^\pm$, $U_1^\pm$ and $U_2^\pm$ are characterized by 14, 2 and 2 respectively (Fig. 4.1), as $\lambda_\rho \wedge \kappa \vdash O_i(p \wedge \neg q) \equiv O_i p$.

Case 4: $\rho = \top$ thus for each confidence level i , the possible O_i -formula constraining U_i^+ is equivalent to one of the following (if κ is assumed to be true):

Figure 4.2: Same as Fig. 2.1.

$$\begin{array}{ll} O_1\kappa \wedge B_1 \neg p \xrightarrow[\mathcal{C}]{\lambda} \perp & O_1(\kappa \wedge p) \wedge \neg B_1 \neg p \xrightarrow[\mathcal{C}]{\lambda} O_1(\kappa \wedge p) \\ O_2\kappa \wedge B_2 \neg q \xrightarrow[\mathcal{C}]{\lambda} \perp & O_2(\kappa \wedge q) \wedge \neg B_2 \neg q \xrightarrow[\mathcal{C}]{\lambda} O_2(\kappa \wedge q) \\ O_2(\kappa \wedge p) \wedge B_2 \neg q \xrightarrow[\mathcal{C}]{\lambda} \perp & O_2(\kappa \wedge p \wedge q) \wedge \neg B_2 \neg q \xrightarrow[\mathcal{C}]{\lambda} O_2(\kappa \wedge p) \end{array}$$

φ fully expanded is $O_0\kappa \wedge O_1(\kappa \wedge p) \wedge ((O_2(\kappa \wedge q) \wedge B_1\neg p) \vee (O_2(\kappa \wedge p) \wedge \neg B_1\neg p))$.
Since $\rho \not\models (\kappa \wedge q) \supset \neg p$ and $\rho \not\models (\kappa \wedge p \wedge q) \supset \neg p$,

$$O_2(\kappa \wedge q) \wedge B_1\neg p \xrightarrow[C]{\lambda} \perp \quad \text{and} \quad O_2(\kappa \wedge p \wedge q) \wedge \neg B_1\neg p \xrightarrow[C]{\lambda} O_2(\kappa \wedge p \wedge q)$$

$U_0^\pm$, $U_1^\pm$ and $U_2^\pm$ are characterized by 15, 3 and 1 respectively (Fig. 4.2), as $\lambda_\rho \vdash \varphi \equiv O_0\kappa \wedge O_1(\kappa \wedge p) \wedge O_2(\kappa \wedge p \wedge q)$.

Example 4.14. Let the logical space λ be maximal and

$$\varphi = O_0\kappa \wedge O_1(\kappa \wedge (p \equiv B_0p)) \wedge O_2(\kappa \wedge (p \equiv B_1p))$$

Only modalities of lower preference than i inside the scope of each O_i . Let $\varphi_1 = \kappa \wedge (p \equiv B_0p)$.

$$\begin{aligned} O_1\varphi_1 &\xrightarrow[E]{B_0p} [B_0p \wedge O_1(\kappa \wedge (p \equiv \top))] \vee [\neg B_0p \wedge O_1(\kappa \wedge (p \equiv \perp))] \\ &\xrightarrow[S]{} [B_0p \wedge O_1(\kappa \wedge p)] \vee [\neg B_0p \wedge O_1(\kappa \wedge \neg p)] = \psi_1 \end{aligned}$$

Let $\varphi_2 = \kappa \wedge (p \equiv B_1p)$.

$$\begin{aligned} O_2\varphi_2 &\xrightarrow[E]{B_1p} [B_1p \wedge O_2(\kappa \wedge (p \equiv \top))] \vee [\neg B_1p \wedge O_2(\kappa \wedge (p \equiv \perp))] \\ &\xrightarrow[S]{} [B_1p \wedge O_2(\kappa \wedge p)] \vee [\neg B_1p \wedge O_2(\kappa \wedge \neg p)] = \psi_2 \end{aligned}$$

No collapse is applicable. In order to perform a collapse, the entire expanded formula $O_0\kappa \wedge \psi_1 \wedge \psi_2$ must be put on DNF:

$$\begin{aligned} &(O_0\kappa \wedge B_0p \wedge O_1(\kappa \wedge p) \wedge B_1p \wedge O_2(\kappa \wedge p)) \vee \\ &(O_0\kappa \wedge B_0p \wedge O_1(\kappa \wedge p) \wedge \neg B_1p \wedge O_2(\kappa \wedge \neg p)) \vee \\ &(O_0\kappa \wedge \neg B_0p \wedge O_1(\kappa \wedge \neg p) \wedge B_1p \wedge O_2(\kappa \wedge p)) \vee \\ &(O_0\kappa \wedge \neg B_0p \wedge O_1(\kappa \wedge \neg p) \wedge \neg B_1p \wedge O_2(\kappa \wedge \neg p)) \end{aligned}$$

Since

$$O_0\kappa \wedge B_0p \xrightarrow[C]{\lambda} \perp \quad \text{and} \quad O_1(\kappa \wedge \neg p) \wedge B_1p \xrightarrow[C]{\lambda} \perp,$$

the first three disjuncts are inconsistent. Only the last is consistent, as

$$O_0\kappa \wedge \neg B_0p \xrightarrow[C]{\lambda} O_0\kappa \quad \text{and} \quad O_1(\kappa \wedge \neg p) \wedge \neg B_1p \xrightarrow[C]{\lambda} O_1(\kappa \wedge \neg p).$$

Accordingly $\lambda \vdash \varphi \equiv O_0\kappa \wedge O_1(\kappa \wedge \neg p) \wedge O_2(\kappa \wedge \neg p)$. If κ is assumed to be true, $U_0^\pm$, $U_1^\pm$ and $U_2^\pm$ are characterized by 15, 12 and 12 respectively (Fig. 4.2).

Chapter 5

Algorithms and complexity

5.1 Introduction

Satisfiability of $\mathcal{AE}$ is NP-complete, as B_i and C_i are K45, and $\Box$ is S5. As the Modal reduction theorem relies on a logical space, whose size is exponential in the number of propositional variables, the problem of deciding whether a formula $O\varphi$ has any $\lambda_{\top}$ -expansions is different from deciding whether φ has any stable expansions.

The logic $\mathcal{AE}_\rho$ was introduced as a means to not having to specify a logical space. Because of the RC axiom, as in Levesque's case, satisfiability of this logic is harder than it is for $\mathcal{AE}$.

5.2 Preliminaries on complexity.

Recall that NP is the class of problems decided by nondeterministic Turing machines in polynomial time. Problems in NP have the property that, if an instance of the problem is *accepted*, this can be verified in polynomial time by a *succinct certificate*. For propositional satisfiability a certificate is a valuation that satisfies the formula in question. Where NP has succinct certificates, coNP, the class of problems complementary to those in NP, has *succinct disqualifications*. The *rejection* of an instance can be verified in polynomial time. Accordingly, for propositional validity a disqualification is a valuation that does not satisfy the formula.

The hardest problems in a complexity class C are called C-*hard*. All harder problems are also considered C-hard, so hardness is a lower bound on complexity, whereas membership is an upper bound. A problem A is at least as hard as a problem B, if B *reduces* to A, that is: we can use an algorithm

for solving A to solve B. To prove that A is at least as hard as B, we give a *reduction* from B to A.

If a problem is in C and is C-hard, it is called *C-complete*. Propositional satisfiability and validity are prototypical NP-complete and coNP-complete problems respectively. Refer to [14] for details.

The *polynomial hierarchy* or PH is defined as follows.

$$\Delta_0^p = \Sigma_0^p = \Pi_0^p = P$$

For all $i \geq 0$

$$\Delta_{i+1}^p = P^{\Sigma_i^p} \quad \Sigma_{i+1}^p = NP^{\Sigma_i^p} \quad \Pi_{i+1}^p = coNP^{\Sigma_i^p}$$

On the first level of the polynomial hierarchy we find the familiar classes $\Delta_1^p = P$, $\Sigma_1^p = NP$ and $\Pi_1^p = coNP$. On level 2 we find the versions of the level 1 classes that have access to an NP oracle: $\Delta_2^p = P^{NP}$, $\Sigma_2^p = NP^{NP}$ and $\Pi_2^p = coNP^{NP}$. The certificates/disqualifications of the first level is a general property of PH. The problem of verifying that an instance of a problem lies Σ_i^p , lies in Π_{i-1}^p . The problem of verifying that an instance of a problem lies Π_i^p , lies in Σ_{i-1}^p .

A *quantified boolean formula with k quantifiers* or *kQBF* [2] has the form

$$Q_1 X_1 \cdots Q_k X_k \varphi$$

φ is a propositional formula, $Q_1, \dots, Q_k$ are alternating quantifiers and $X_1, \dots, X_k$ are disjoint sets whose union is the set of propositional variables occurring in φ . $\exists X \varphi$ in this context means “there exist valuations of the members in X such that φ is true” and $\forall X \varphi$ means “for all valuations of the members in X , φ is true”. When Q_1 is $\exists$ we write $kQBF_{\exists}$, and when Q_1 is $\forall$ we write $kQBF_{\forall}$. Evaluating a $kQBF_{\exists}$ is Σ_k^p -complete, and evaluating a $kQBF_{\forall}$ is Π_k^p -complete.

Evaluating a $1QBF_{\exists}$ and a $1QBF_{\forall}$ are the problems of determining the satisfiability and validity of a propositional formula respectively. An example of $2QBF_{\forall}$ is

$$\forall\{p\}\exists\{q\}(p \vee q) \wedge (\neg p \vee \neg q)$$

which means: for all valuations of p , does there exist a valuation of q such that $(p \vee q) \wedge (\neg p \vee \neg q)$ is true? There does: if $v(p) = 1$, let $v(q) = 0$, and if $v(p) = 0$, let $v(q) = 1$. The order of the quantifiers is significant, as the $2QBF_{\exists}$

$$\exists\{q\}\forall\{p\}(p \vee q) \wedge (\neg p \vee \neg q)$$

is false, as if $v(q) = 0$, then $v(p) = 0$ makes φ false, and if $v(q) = 1$, then $v(p) = 1$ makes φ false. A general 2QBF $_{\exists}$ has the form

$$\exists\{p_1, \dots, p_n\} \forall\{q_1, \dots, q_m\} \varphi$$

A syntactical formulation of 2QBF $_{\exists}$ is: does there exist an $\alpha \in \{\pm p_1 \wedge \dots \wedge \pm p_n\}$ such that $\alpha \supset \varphi$ is valid?

5.3 Satisfiability

Every λ -expansion of an O_I -atom φ^I has a unique model, and these models are exactly the models of φ^I . The “if”-part of the equivalence in the Modal reduction theorem states that a model of an λ -expansion is a model of φ^I , while the “only if”-part states that no other model is a model of φ^I .

Lemma 5.1. Let $z = 2^{|\lambda_{\top}|}$. If $M \models O_k \varphi$, there exist non-equivalent propositional formulae $\psi_1, \dots, \psi_z$ such that $M \models O_k \psi_1 \vee \dots \vee O_k \psi_z$.

Proof. From Lemma 2.6 it follows that $U_k^+ = \|\varphi\|$ and $U_k^- = \|\neg\varphi\|$. Thus there exists a ψ_i ($1 \leq i \leq z$) such that $U_k^+ = \|\psi\|$ and $U_k^- = \|\neg\psi\|$, and $M \models O_k \psi_i$. $\square$

In the rest of this section, let λ be some logical space, let φ^I be an O_I -atom, and let $\psi_1^I, \dots, \psi_m^I$ be every consistent prime O_I -atom, no two of which are equivalent under λ .

Lemma 5.2. Let $M_i \models \lambda \wedge \psi_i^I$ for all $1 \leq i \leq k$. Then $M_i \models \varphi^I$ for all $1 \leq i \leq k$ iff $\lambda \vdash (\psi_1^I \vee \dots \vee \psi_k^I) \supset \varphi^I$.

Proof. $\lambda \vdash (\psi_1^I \vee \dots \vee \psi_k^I) \supset \varphi^I$ is equivalent to $\lambda \wedge \psi_i^I \vdash \varphi^I$ for all $1 \leq i \leq k$. As every $\lambda \wedge \psi_i^I$ has a unique model, $M_i \models \varphi^I$. $\square$

Lemma 5.3. Let $M_i \models \lambda \wedge \psi_i^I$ for all $k+1 \leq i \leq m$. Then $M_i \not\models \varphi^I$ for all $1 \leq i \leq k$ iff $\lambda \vdash \varphi^I \supset (\psi_1^I \vee \dots \vee \psi_k^I)$.

Proof. If: Assume the opposite, that $\lambda \vdash \varphi^I \supset (\psi_1^I \vee \dots \vee \psi_k^I)$ and there exists an M_i for $k+1 \leq i \leq m$ such that $M_i \models \varphi^I$. This means that $M_i \models \psi_1^I \vee \dots \vee \psi_k^I$. However, since $M_i \models \psi_j^I$ iff $\lambda \vdash \psi_i^I \equiv \psi_j^I$, and all ψ_j^I are non-equivalent under λ , this cannot be the case.

Only if: For $k+1 \leq i \leq m$, $\lambda \wedge \psi_i^I \not\models \varphi^I$, equivalently (Theorem 2.10) that $\lambda \vdash \varphi^I \supset \neg\psi_i^I$. Let $M \models \lambda \wedge \varphi^I$. It follows from Lemma 5.1 that $M \models \psi_1^I \vee \dots \vee \psi_m^I$, thus $\lambda \wedge \varphi^I \vdash \psi_1^I \vee \dots \vee \psi_m^I$. Accordingly $\lambda \vdash \varphi^I \supset (\psi_1^I \vee \dots \vee \psi_k^I)$. $\square$

Theorem 5.4. Assume $\lambda \vdash \varphi^I \equiv (\psi_1^I \vee \dots \vee \psi_k^I)$. $M \models \varphi^I$ iff $M \models \lambda \wedge \psi_i^I$.

Proof. Follows immediately from Lemmata 5.2 and 5.3. $\square$

Corollary 5.5. Assume $\lambda \vdash \varphi^I \equiv (\psi_1^I \vee \dots \vee \psi_k^I)$. φ^I is satisfiable iff $k > 0$ and valid iff $k = m$.

5.4 Complexity

In $\mathcal{A}_{\top}$ with $|I| = 1$, if φ does not contain C or O , existence of λ -expansions is equivalent to deciding if φ has a stable expansion [9], which is known to be Σ_2^p -complete. Niemelä proved the upper bounds (membership) in [13], and Gottlob the lower (hardness) in [6]. $\mathcal{A}_{\top}$ -satisfiability of $O\varphi$ is equivalent to $\mathcal{ONL}$ -satisfiability of $O\varphi$. $\mathcal{ONL}$ -satisfiability was proven Σ_2^p -complete by Rosati in [16]. Not surprisingly, the existence of λ -expansions in $\mathcal{A}_{\rho}$ also proves to be Σ_2^p -complete, assuming the characteristic formula of λ is specified instead of λ itself. To find the upper bounds, we need an algorithm deciding whether an O^I -atom is satisfiable.

Algorithm 5.6. *Input:* An O_I -atom φ^I . *Output:* Is φ^I $\mathcal{A}_{\rho}$ -satisfiable?

Nondeterministically generate a modal valuation V of φ^I . Then (1) determine whether $\varphi^I[V]$ is consistent, and (2) for each β in the domain of V , determine whether $\varphi^I[V] \wedge (\beta \equiv V(\beta))$ is consistent. If both (1) and (2) are true, $\varphi^I[V]$ is an expansion, else it is not.

Proof of correctness. Lemma 4.12 tells us that φ^I is equivalent to a disjunction where $\varphi^I[V]$ and for every modal atom β occurring in $\varphi^I[V]$, $\beta \equiv V(\beta)$ are conjuncts in a disjunct. $\square$

Theorem 5.7. $\mathcal{A}_{\rho}$ -satisfiability of $O\varphi$ is in Σ_2^p .

Proof. We need to show that Algorithm 5.6 nondeterministically in polynomial time with access to an NP-oracle can decide whether an O^I -atom is $\mathcal{A}_{\rho}$ -satisfiable.

(1) and (2) of Algorithm 5.6 can be done with $(n - 1)$ and $|M|$ propositional validity tests respectively. $\varphi^I[V]$ is inconsistent iff it violates persistence. As $\varphi^I[V]$ is on the form $O_1\psi_1 \wedge \dots \wedge O_n\psi_n$ for propositional $\psi_1, \dots, \psi_n$, persistence is violated iff $O_i\psi_i$ and $O_k\psi_k$ are conjuncts, $i \prec k$ and $\Vdash_{\rho} \psi_k \supset \psi_i$. $\square$

Lemma 5.8. Let $n \leq |\Phi|$, $A = \{\pm p_1 \wedge \dots \wedge \pm p_n \mid p_i \in \Phi\}$, $\Gamma^n = \bigwedge_{i=1}^n (p_i \equiv Bp_i)$ and $\rho = \top$. Then

$$\vdash_{\rho} O\Gamma^n \equiv \bigvee_{\alpha \in A} O\alpha \quad (5.1)$$

Proof. Substituting Bp_i with $\top$ and $\perp$ in $p_i \equiv Bp_i$ evaluates to p_i and $\neg p_i$ respectively. This means that every expansion is on the form $(Bp_i \wedge O(\psi_1 \wedge p_i \wedge \psi_2)) \vee (\neg Bp_i \wedge O(\psi_1 \wedge \neg p_i \wedge \psi_2))$, and since p_i is independent of the propositional variables occurring in ψ_1 and ψ_2 , no collapse will result in $\perp$. $\square$

We will use ρ instead of λ as parameter to the collapse rule to signal that the full logical space need not be specified.

Example 5.9. Let $\varphi = O((p \equiv Bp) \wedge (q \equiv Bq))$ and $\rho = \top$. See Fig. 5.1 for the expanded tree.

Figure 5.1: $O((p \equiv Bp) \wedge (q \equiv Bq))$ fully expanded.

No collapse results in $\perp$.

$$\begin{aligned} O(p \wedge q) \wedge Bq &\xrightarrow[\mathcal{C}]{\rho} O(p \wedge q) & O(\neg p \wedge q) \wedge Bq &\xrightarrow[\mathcal{C}]{\rho} O(\neg p \wedge q) \\ O(p \wedge \neg q) \wedge \neg Bq &\xrightarrow[\mathcal{C}]{\rho} O(p \wedge \neg q) & O(\neg p \wedge \neg q) \wedge \neg Bq &\xrightarrow[\mathcal{C}]{\rho} O(\neg p \wedge \neg q) \\ O(p \wedge q) \wedge Bp &\xrightarrow[\mathcal{C}]{\rho} O(p \wedge q) & O(\neg p \wedge q) \wedge Bp &\xrightarrow[\mathcal{C}]{\rho} O(\neg p \wedge q) \\ O(p \wedge \neg q) \wedge \neg Bp &\xrightarrow[\mathcal{C}]{\rho} O(p \wedge \neg q) & O(\neg p \wedge \neg q) \wedge \neg Bp &\xrightarrow[\mathcal{C}]{\rho} O(\neg p \wedge \neg q) \end{aligned}$$

Every potential model is a $\lambda_{\top}$ -expansion.

$$\vdash_{\rho} O\varphi \equiv [O(\neg p \wedge \neg q) \vee O(\neg p \wedge q) \vee O(p \wedge \neg q) \vee O(p \wedge q)]$$

Theorem 5.10. $\mathbb{A}_{\rho}$ -satisfiability of $O\varphi$ is Σ_2^P -hard.

Proof. We use the same reduction to $2QBF_{\exists}$ as in [6].

Let $\Phi = \{p_1, \dots, p_n, q_1, \dots, q_m\}$, let $A = \{\pm p_1 \wedge \dots \wedge \pm p_n\}$ and $\Gamma^n = \bigwedge_{i=1}^n (p_i \equiv Bp_i)$, let $\rho = \top$ and let $\psi \in \mathcal{L}_{PL}$. A and Γ^n are defined as

in Lemma 5.8. In order to show whether $\exists\{p_1, \dots, p_n\} \forall\{q_1, \dots, q_m\} \psi$ is true, we need to show whether there exists an $\alpha \in A$ such that $\vdash \alpha \supset \psi$, equivalently $\vdash_\rho O\alpha \supset B\psi$ (Theorem 2.21). This we show by computing the $\lambda_\top$ -expansions of $O\varphi$ for $\varphi = B\psi \wedge \Gamma^n$.

$$\begin{aligned} O\varphi &\xrightarrow[E]{B\psi} [O(\top \wedge \Gamma^n) \wedge B\psi] \vee [O(\perp \wedge \Gamma^n) \wedge \neg B\psi] \\ &\xrightarrow[S]{} [O\Gamma^n \wedge B\psi] \vee [O\perp \wedge \neg B\psi] \end{aligned}$$

Thus $O\perp \wedge \neg B\psi \xrightarrow[C]{\rho} \perp$ and $[O\Gamma^n \wedge B\psi] \vee \perp \xrightarrow[\perp]{} O\Gamma^n \wedge B\psi$.

We know from Lemma 5.8 that $O\Gamma^n$ is equivalent to the disjunction of all $O(\pm p_1 \wedge \dots \wedge \pm p_n)$: $O\alpha_1 \vee \dots \vee O\alpha_{2^n}$.

$$(O\alpha_1 \vee \dots \vee O\alpha_{2^n}) \wedge B\psi \xrightarrow[D]{} (O\alpha_1 \wedge B\psi) \vee \dots \vee (O\alpha_{2^n} \wedge B\psi)$$

This provides us with 2^n collapses, each on one of the following forms.

$$O\alpha_i \wedge B\psi \xrightarrow[C]{\rho} O\alpha \text{ or } O\alpha_i \wedge B\psi \xrightarrow[C]{\rho} \perp \quad (5.2)$$

If not every collapse yields $\perp$, there exists such an α_i , and our $2\text{QBF}_\exists$ formula is true. $\square$

As $\mathbb{A}_\rho$ -satisfiability of an O_I -atom is in Σ_2^p , and $\mathbb{A}_\rho$ -satisfiability of $O\varphi$ is Σ_2^p -hard (thus $\mathbb{A}_\rho$ -satisfiability of an O_I -atom is Σ_2^p -hard), we get the following Corollary.

Corollary 5.11. $\mathbb{A}_\rho$ -satisfiability is Σ_2^p -complete.

5.5 Algorithms

Algorithm 5.6 was designed specifically for proving the upper bounds of $\mathbb{A}_\rho$ -satisfiability. Also, it does not make explicit use of the general collapse rule. Thus we can give a more efficient algorithm for modal reduction.

The first algorithm computes the disjunction of λ -expansions of some $O\varphi$, with only one confidence level, using the simple rule set.

The algorithm takes a logical space λ , a formula $O\varphi$ and modal atom β as a parameter. The base case is when $O\varphi$ is prime, and thus a model candidate; $O\varphi$ and β are collapsed (3). When first calling the function, we let β be a modal atom such that the result of the collapse is $O\varphi$. In the recursive case, a modal atom β' with extensional subformula occurrence in φ is chosen (6), and $O\varphi$ expanded (7). A recursive call is made for both generated O -formulae (8/9). Upon the return of these calls, the two resulting disjunctions

of model candidates are joined using the disjunct simplification rule (10). For each model candidate $O\psi_i$, $O\psi_i$ and β are collapsed (13), and the resulting disjunction is again cleansed for $\perp$'s, again using the disjunct simplification rule (14).

Algorithm 5.12. *Input:* A logical space λ , some $O\varphi$ and a dummy modal atom $\beta = O\varphi$. *Output:* A disjunction of λ -expansions of $O\varphi$.

```

1  function reduce( $\lambda, O\varphi, \beta$ )
2    if  $\varphi \in \mathcal{L}_{\text{PL}}$  then
3       $O\varphi \wedge \beta \xrightarrow[C]{\lambda} \psi$ 
4      return  $\psi$ 
5    else
6      let  $\beta' \in \text{ext}(\varphi)$ 
7       $O\varphi \xrightarrow[E]{\beta'} (O\varphi_{\top} \wedge \beta') \vee (O\varphi_{\perp} \wedge \neg\beta')$ 
8      let  $\varphi_L = \text{reduce}(\lambda, O\varphi_{\top}, \beta')$ 
9      let  $\varphi_R = \text{reduce}(\lambda, O\varphi_{\perp}, \neg\beta')$ 
10      $\varphi_L \vee \varphi_R \xrightarrow{\perp} O\psi_1 \vee \dots \vee O\psi_m$  (where  $m \geq 0$ )
11      $(O\psi_1 \vee \dots \vee O\psi_m) \wedge \beta \xrightarrow{D} (O\psi_1 \wedge \beta) \vee \dots \vee (O\psi_m \wedge \beta)$ 
12     for  $1 \leq i \leq m$ 
13        $O\psi_i \wedge \beta \xrightarrow[C]{\lambda} \chi_i$ 
14      $\chi_1 \vee \dots \vee \chi_m \xrightarrow{\perp} \psi$ 
15     return  $\psi$ 

```

We can base the algorithm for computing general λ -expansions on this algorithm, with some minor modifications. The collapse rule will always apply in lines 3 and 13, but this is not the case in the general case. Thus, in line 10 we will not get disjunctions of model candidates, but disjunctions of conjunctions of model candidates of and modal atoms.

Algorithm 5.13. *Input:* A logical space λ , some $O_i\varphi$ and a dummy modal atom $\beta = O_i\varphi$. *Output:* A disjunction of conjunctions of model candidates and modal atoms equivalent to $O\varphi$.

```

1  function reduce( $\lambda, O_i\varphi, \beta$ )
2    if  $\varphi \in \mathcal{L}_{\text{PL}}$  then
3       $O_i\varphi \wedge \beta \xrightarrow[C]{\lambda} \psi$  if applicable, else  $\psi = O_i\varphi \wedge \beta$ 
4    return  $\psi$ 
5  else
6    let  $\beta' \in \text{mod}(\varphi)$  such that  $d(\beta') = 1$ 
7       $O_i\varphi \xrightarrow[E]{\beta'} (O_i\varphi_{\top} \wedge \beta') \vee (O_i\varphi_{\perp} \wedge \neg\beta')$ 
8    let  $\varphi_L = \text{reduce}(\lambda, O_i\varphi_{\top}, \beta')$ 
9    let  $\varphi_R = \text{reduce}(\lambda, O_i\varphi_{\perp}, \neg\beta')$ 
10    $\varphi_L \vee \varphi_R \xrightarrow[\perp]{} (O_i\psi_1 \wedge M_1) \vee \dots \vee (O_i\psi_m \wedge M_m)$ 
11    $((O_i\psi_1 \wedge M_1) \vee \dots \vee (O_i\psi_m \wedge M_m)) \wedge \beta \xrightarrow[D]{} \psi$ 
12    $(O_i\psi_1 \wedge \beta \wedge M_1) \vee \dots \vee (O_i\psi_m \wedge \beta \wedge M_m)$ 
13   for  $1 \leq i \leq m$ 
14      $O_i\psi_i \wedge \beta \xrightarrow[C]{\lambda} \chi_i$  if applicable, else  $\chi_i = O_i\psi_i \wedge \beta$ 
15    $(\chi_1 \wedge M_1) \vee \dots \vee (\chi_m \wedge M_m) \xrightarrow[\perp]{} \psi$ 
16   return  $\psi$ 

```

reduce can be used to reduce a formula on the form $O_i\varphi$, but since this formula may not be fully reducible, we need a function that reduces each conjunct in an O_I -atom φ^I (using **reduce**), puts the resulting formula on DNF, and does the final reduction, thus computing the λ -expansions of φ^I .

Function **reduce-con** takes as a parameter a logical space and an O_I -atom $O_1\varphi_1 \wedge \dots \wedge O_n\varphi_n$. Each $O_i\varphi_i$ is reduced (3) and the conjunction of the reduced formulae is put on DNF (4). The disjuncts of this formula is a conjunction of model candidates and modal atoms. For each modal atom (6) β in each conjunction (5) that is not part of the model candidate ψ_i^I , β and the O_i -formula with the same modality as β are collapsed (8). The disjuncts that proves inconsistent are then removed using the disjunct simplification rule (10).

Algorithm 5.14. *Input:* A logical space λ and an O_I -atom $O_1\varphi_1 \wedge \dots \wedge O_n\varphi_n$. *Output:* A disjunction of λ -expansions of $O_1\varphi_1 \wedge \dots \wedge O_n\varphi_n$.

```

1  function reduce-con( $\lambda, O_1\varphi_1 \wedge \dots \wedge O_n\varphi_n$ )
2    for  $1 \leq i \leq n$ 
3       $\chi_i = \text{reduce}(\lambda, O_i\varphi_i)$ 
4     $\chi_1 \wedge \dots \wedge \chi_n \xrightarrow{\text{DNF}} (\psi_1^I \wedge M_1) \vee \dots \vee (\psi_m^I \wedge M_m)$ 
5    for  $1 \leq i \leq m$ 
6      for each  $\beta \in M_i$  ( $\beta$  is on the form  $B_k\chi$ ,  $C_k\chi$  or  $O_k\chi$ )
7        let  $O_k\psi_k$  denote the  $k$ -modalized conjunct of  $\psi_i^I$ 
8         $O_k\psi_k \wedge \beta \xrightarrow[C]{\lambda} \chi$ 
9        let  $M'_i$  denote  $M_i$  collapsed
10      $(\psi_1^I \wedge M'_1) \vee \dots \vee (\psi_m^I \wedge M'_m) \xrightarrow{\perp} \psi$ 
11    return  $\psi$ 

```

5.6 LISP implementation

A Common LISP implementation of these algorithms can be found at

<http://folk.uio.no/elian/thesis/modred.tar.gz>

Chapter 6

Final words

6.1 Conclusion

We have seen that by adding an axiom and an inference rule to $\mathcal{AEL}$ we get a system $(\mathcal{AEL}_\rho)$ more similar to $\mathcal{O}\mathcal{N}\mathcal{L}$, but with a closer correspondence to $\mathcal{AEL}$.

We have also seen that generating the λ -expansions of an O_I -atom can be done entirely within $\mathcal{AEL}$, using equivalence-preserving rewriting rules. The problem of deciding whether such a λ -expansion exists (in $\mathcal{AEL}_\rho$), has proven to be Σ_2^P -complete, just as the problem of deciding whether a stable expansion of an autoepistemic theory exists.

6.2 Further work

Default logic [15] is a nonmonotonic formalism, where instead of modal formulae, there are default rules, which can be roughly translated into AEL as $(B\alpha \wedge \neg B\neg\beta) \supset \gamma$. The extension of a default theory roughly parallels the stable expansion of an autoepistemic theory.

A general scheme for representing prioritized default theories [4] in $\mathcal{AEL}$ is developed in [5]. Finding the extensions of such a theory is done by representing it as an O_I -atom and generating the (unique) expansion. $(I, \prec)$ is a tree, and the extensions are those $O_i\psi$ such that i is a leaf node, and $O_i\psi$ does not imply some specific predicates. This approach proves conceptually promising, as the sequence defaults are applied in, and why and where the default is blocked, can be seen just by looking at the $\lambda_\top$ -expansion of the O_I -atom.

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Appendix A

Mathematics

The *power set* of a set S is the set of all subsets of S and is denoted 2^S .

A *partially ordered set* or *poset* is a pair $(I, \preceq)$ such that $\preceq$ is a partial (reflexive, antisymmetric, and transitive) order on I . The irreflexive counterpart of $\preceq$ is denoted $\prec$, and is a *strict* partial order.

Let $\mathbf{n}$ denote the set $\{0, \dots, n-1\}$, specifically $\mathbf{2} = \{0, 1\}$. The set of bit vectors of length n is denoted $\mathbf{2}^n$. If $\mathbf{a} = [a_0 \ a_1 \ \dots \ a_{n-1}]$ is a vector of length n , $\mathbf{a}_i = a_i$, so bit 0 is the leftmost and bit $n-1$ is the rightmost. Let $m_{[n]}$ denote the vector of length n where every element is m , and let $\otimes$ denote the concatenation of two vectors. Let $\mathbf{a}, \mathbf{b} \in \mathbf{2}^n$. Define *inclusive disjunction*, *conjunction*, *negation*, *equivalence* and *exclusive disjunction* (XOR) on vectors as

$$(\mathbf{a} \vee \mathbf{b})_i = 0 \text{ iff } \mathbf{a}_i = \mathbf{b}_i = 0 \quad (\text{A.1})$$

$$(\mathbf{a} \wedge \mathbf{b})_i = 1 \text{ iff } \mathbf{a}_i = \mathbf{b}_i = 1 \quad (\text{A.2})$$

$$(\neg \mathbf{a})_i = 1 \text{ iff } \mathbf{a}_i = 0 \quad (\text{A.3})$$

$$(\mathbf{a} \equiv \mathbf{b})_i = 1 \text{ iff } \mathbf{a}_i = \mathbf{b}_i \quad (\text{A.4})$$

$$(\mathbf{a} \oplus \mathbf{b})_i = 1 \text{ iff } \mathbf{a}_i \neq \mathbf{b}_i \quad (\text{A.5})$$

In addition, let $j \in \mathbb{N}$ and define the *logical left shift* of a vector as

$$(\mathbf{a} \ll j)_i = \begin{cases} \mathbf{a}_{i+j} & \text{if } i+j \leq |\mathbf{a}| \\ 0 & \text{else} \end{cases} \quad (\text{A.6})$$

A bit vector corresponds to a positive integer; just view it as a binary number. Let $t : \mathbb{N} \rightarrow \mathbb{N}^n$ be the function such that $t(k)_i = 2^{k-i}$ for $1 \leq i \leq k$. Define $\text{int} : \mathbf{2}^n \rightarrow \mathbb{N}$ and $\text{vec} : \mathbb{N} \rightarrow \mathbf{2}^n$ as

$$\text{int } \mathbf{a} = t(|\mathbf{a}|) \cdot \mathbf{a}^T \quad (\text{A.7})$$

$$\text{vec } i = \text{int}^{-1}(i) \quad (\text{A.8})$$

Example A.1. Let $S = \{1, 2, 3\}$. Then

$$2^S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Example A.2. $1_{[1]} \otimes 0_{[2]} \otimes 1_{[1]} \otimes 0_{[3]} = [1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0].$

Example A.3. $[1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1] \ll 3 = [1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0].$

Example A.4.

$$\begin{aligned} \text{int } [0 \ 1 \ 0 \ 1] &= [2^3 \ 2^2 \ 2^1 \ 2^0] \cdot [0 \ 1 \ 0 \ 1]^T \\ &= 0 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1 = 5 = 0101_2 \\ \text{vec } 5 &= \text{vec } 0101_2 = [0 \ 1 \ 0 \ 1] \end{aligned}$$

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