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Investigations into real determinantal quartic hypersurfaces

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* (Ph.D.) at the University of Oslo. The research presented here was conducted under the supervision of professor Kristian Ranestad.

The thesis is a collection of four papers, presented in chronological order of writing. They all concern determinantal representations of quartic hypersurfaces. The papers are preceded by an introductory chapter that relates them to each other and provides context and motivation for the work. I am the sole author of three of the papers. The remaining paper is joint work with Kristian Ranestad.

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Having spent a third of my life at the Department of Mathematics, leaving is both a timely and wistful affair. During my studies, I have encountered many people making a lasting, positive impact on me. I make no attempt to list them all, in fear of the inevitable forgetful omissions. To my former colleagues and fellow students, I wish to express my deepest gratitude for all the good times.

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• **Martin Helsø**

Oslo, November 2020

List of papers

Paper I

Helsø, M. ‘Rational quartic symmetroids’. In: *Adv. Geom.* Vol. 20, no. 1 (2020), pp. 71–89. DOI: [10.1515/advgeom-2018-0037](https://doi.org/10.1515/advgeom-2018-0037).

Paper II

Helsø, M. and Ranestad, K. ‘Rational quartic spectrahedra’. To appear in: *Math. Scand.* arXiv: [1810.11235](https://arxiv.org/abs/1810.11235).

Paper III

Helsø, M. *Maximal dimension of quartic symmetroids with a double quadric of codimension 1*. 2019. Preprint. arXiv: [1905.01091](https://arxiv.org/abs/1905.01091).

Paper IV

Helsø, M. *Determinantal quartic surfaces with a definite Hermitian representation*. 2020. Preprint. arXiv: [2007.01121](https://arxiv.org/abs/2007.01121).

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Chapter 1

Introduction

1.1 Determinantal representations

Determinantal representations connect spaces of matrices with algebraic geometry. It is a classical subject with applications in areas such as linear algebra, spectral theory, operator theory, dynamical systems, convex optimisation and convex algebraic geometry.

Let F be a homogeneous polynomial of degree d in $\mathbb{C}[x_0, \dots, x_n]$. A *determinantal representation* of F or $\mathcal{V}(F) \subset \mathbb{P}^n$ is an expression

$$M := M(x) := M_0x_0 + \dots + M_nx_n, \quad (1.1.1)$$

where $M_i \in \text{Mat}_d(\mathbb{C})$ are $(d \times d)$ -matrices, such that $\det(M) = cF$ for some $c \in \mathbb{C} \setminus \{0\}$. Many common problems involving determinantal representations restrict the attention to matrices M_i with specific properties, for instance that they are all symmetric, Hermitian or have certain ranks. [Papers I to III](#) deal with symmetric representations, whereas [Paper IV](#) concerns Hermitian representations. Below is an account of some of the history of determinantal representations.

In 1844, Hesse showed that all cubic curves have a symmetric determinantal representation [[Hes44](#)]. He did this by proving that all cubics can be written as a Hessian curve in three different ways. Hesse was the first to formulate the problem of writing the equation of a hypersurface as the determinant of a symmetric matrix of linear forms [[Hes55a](#)]. He solved this problem for quartic plane curves by studying families of contact cubics of the quartics [[Hes55b](#)].

Recall that the *adjugate* or *classical adjoint* A^{adj} of a $(d \times d)$ -matrix A is the transpose of the cofactor matrix of A . That is, the (i, j) -entry of A^{adj} equals $(-1)^{i+j}A_{ji}$, where A_{ji} is the minor of A obtained by removing the j th row and the i th column of A . It satisfies

$$AA^{\text{adj}} = A^{\text{adj}}A = (\det A)I_d,$$

where I_d is the identity matrix of size d . The entries of M^{adj} are homogeneous polynomials of degree $d - 1$ in $\mathbb{C}[x_0, \dots, x_n]$. In [[Hes55a](#)], the *adjugate system* associated with M was introduced. It is defined as the $(d - 1)$ -dimensional system

$$\mathcal{C}(M) := \left\{ y^T M^{\text{adj}}(x) y \mid y \in \mathbb{C}^d \setminus \{0\} \right\}$$

of polynomials of degree $d - 1$ in $\mathbb{C}[x_0, \dots, x_n]$. Hesse showed that for a curve $\mathcal{V}(\det M) \subset \mathbb{CP}^2$, the curves defined by $\mathcal{C}(M)$ are *contact curves* for $\mathcal{V}(\det M)$; all intersection points have even multiplicity.

Dixon showed in 1902 that general projective plane curves admit a symmetric determinantal representation [[Dix02](#)]. The proof is algorithmic,

sometimes referred to as *Dixon's method*. It demonstrates that you can find a representation M for a curve C of degree d given a contact curve C' of degree $d-1$. In particular, M can be recovered from the adjugate system $\mathcal{C}(M)$. Dixon proved the existence of contact curves using ineffective theta characteristics. Barth gave the first definition of non-invertible theta characteristics on a singular curve, and extended Dixon's result to singular curves [Bar77]. This was also proven by Catanese [Cat81, Proposition 2.28]. Computing examples with the algorithm can be hard, as it can be difficult to find a contact curve C' , but there are some known applications. For instance, Edge gave a determinantal representation of the curve $\mathcal{V}(x_0^4 + x_1^4 + x_2^4)$ using Dixon's method [Edg38]. Recently, Plaumann and Vinzant proved a variation of Dixon's method that replaces the contact curve with a curve of *real contact*, and yields a Hermitian determinantal representation of the input curve C [PV13]. This is computationally simpler, since a curve of real contact can be constructed from directional derivatives of C .

Vinnikov wrote his thesis on determinantal representations of algebraic curves and published several papers on the matter during the 1980s and 1990s [Vin84; Vin86a; Vin86b; Vin88; Vin89; Vin90; Vin93]. Among other things, he gave a complete description of determinantal representations of complex smooth curves, as well as studying Hermitian and real symmetric representations of real curves in full generality. He also laid the foundations for the Helton–Vinnikov theorem, which we return to in Section 1.2.

Dickson classified in 1921 the pairs (n, d) for which a generic hypersurface of degree $d \geq 2$ in $\mathbb{P}^n$ admits a determinantal representation [Dic21]. This holds only for $(2, d)$, plane curves of any degree, and $(3, d \leq 3)$, quadratic and cubic surfaces. Beauville recovered this result by showing that the existence of a determinantal representation is equivalent to the existence of a line bundle of a certain type on the hypersurface [Bea00].

Higher-dimensional hypersurfaces with determinantal representations are always singular. The singular locus of a determinantal hypersurface in $\mathbb{P}^n$ has at least dimension $n-4$; if the representation is symmetric, then the dimension is at least $n-3$. The singularities at points where $\text{corank } M(x) \geq 2$ are called *essential*, other singularities are called *accidental*. The generic symmetric determinantal hypersurface of degree d has only essential singularities [Sal65, p. 420]. Moreover, its singular locus has degree $\binom{d+1}{3}$. All the singularities on a generic symmetric determinantal surface are nodes.

Brundu and Logar showed that all cubic surfaces have a determinantal representation, except those containing only one line [BL98]. After a choice of coordinates, the surface without a determinantal representation can be written as $\mathcal{V}(x_0x_1^2 + x_1x_2^2 + x_3^3)$. The determinantal representations of smooth cubic surfaces were classified by Buckley and Košir [BK07]. For a smooth cubic, they gave an explicit construction of a representative for each of the seventy-two equivalence classes of determinantal representations by connecting them to the seventy-two sets of six mutually skew lines on the surface. The classification is extended to all cubic surfaces in [Dol12, Section 9.3].

A quartic surface admits a determinantal representation only if it contains a projectively normal sextic curve with genus 3. This was first discovered by

Schur [Sch81], and later explained by Coble as a consequence of the fact that the Picard group of a general quartic surface is generated by a plane section [Cob82, p. 39]. The determinantal quartics thus form a subvariety of codimension 1 in the $\mathbb{P}^{34}$ of all quartics. They were studied in a series of papers by Room [Roo50].

Cayley studied quartic surfaces with a symmetric representation and introduced the term *symmetroid* for them [Cay69a]. These form a subvariety of codimension 10 in the $\mathbb{P}^{34}$ of all quartics. We use *symmetroid* as a name for any hypersurface of any degree which admits a symmetric determinantal representation. Cayley related the generic quartic symmetroid to the Jacobian surface of four quadrics. He also proved the following characterisation:

Theorem 1.1.1 ([Cay69b, p. 200]). *Let p be a node on a nodal quartic surface $S \subset \mathbb{CP}^3$. If the projection from p is ramified along the union of two cubic curves, then S is a symmetroid. Conversely, if S is a symmetroid with representation $M(x)$ and p is essential with respect to $M(x)$, then the projection from p is ramified along the union of two cubic curves.*

A quartic symmetroid S contains a sextic curve C of genus 3 such that there exists a cubic surface S' that is tangent to S along C . Indeed, given a determinantal representation (1.1.1) of S , let M' be the (3×4) -submatrix of M obtained by removing a row. The vanishing of the (3×3) -minors of M' is a sextic curve of genus 3. One of the principal (3×3) -minors of M defines a cubic symmetroid that is tangent to S along this curve. Blekherman et al. gave an algorithm for producing a symmetric representation of a 10-nodal quartic surface with such a curve C [Ble+12, Section 5].

Quartic symmetroids appear several places in mathematics. For instance, the surfaces arising as the étale double cover of generic Reye congruences are identified as generic quartic symmetroids [Cos83]. Artin and Mumford used quartic symmetroids in their counterexamples to the Lüroth problem [AM72]. For modern references on the theory of symmetroids, see the works of Beauville [Bea00] and Dolgachev [Dol12]. The content of Paper I is a classification of rational, quartic symmetroids.

In the last years, determinantal representations have gotten attention from convex optimisation, specifically from semidefinite and hyperbolic programming, which we cover in the next section.

1.2 Real algebraic geometry

Real algebraic geometry is the study of real solutions to real polynomial equations and inequalities. It is a rich subject with connections to many areas of mathematics, such as analytic geometry, algebraic topology and analysis. Real algebraic geometry also has applications in interdisciplinary fields such as robotics, computer vision and computer-aided design. A central theme in Papers II to IV is the interplay between the real and complex geometry of the objects under consideration.

1. Introduction

We use the real numbers $\mathbb{R}$ as the ground field, although much of real algebraic geometry can be carried out over any real closed field. Because the real numbers are ordered, we can study polynomial inequalities. In analogue to the variety $\mathcal{V}(F_1, \dots, F_r)$, we introduce the *basic closed semialgebraic set*

$$\mathcal{W}(F_1, \dots, F_r) := \{x \in \mathbb{R}^n \mid F_1(x) \geq 0, \dots, F_r(x) \geq 0\} \subseteq \mathbb{R}^n$$

for polynomials $F_1, \dots, F_r \in \mathbb{R}[x_1, \dots, x_n]$. A general *semialgebraic set* is a finite sequence of unions, intersections and complements of basic closed semialgebraic sets. By definition, semialgebraic sets are closed under unions, intersections and complements. Moreover, the projection of a semialgebraic set is again semialgebraic by the Tarski–Seidenberg theorem [Sei54; Tar51, Note 13].

In classical algebraic geometry, Hilbert’s Nullstellensatz is a certificate for the feasibility of a variety over an algebraically closed field: $\mathcal{V}(F_1, \dots, F_r) = \emptyset$ if and only if $-1 \in \langle F_1, \dots, F_r \rangle$. The ideal $\langle F_1, \dots, F_r \rangle$ consists of polynomials that vanish on $\mathcal{V}(F_1, \dots, F_r)$. A preorder is the corresponding notion for a basic closed semialgebraic set; it consists of polynomials that are nonnegative on $\mathcal{W}(F_1, \dots, F_r)$. For $F_1, \dots, F_r \in \mathbb{R}[x_1, \dots, x_n]$, the *preorder* $\mathcal{P}(F_1, \dots, F_r)$ is the smallest set closed under addition and multiplication, containing $F_1, \dots, F_r$ and all sums of squares in $\mathbb{R}[x_1, \dots, x_n]$. Symbolically,

$$\mathcal{P}(F_1, \dots, F_r) := \left\{ \sum_{\mathbf{e} \in \{0,1\}^r} \sigma_{\mathbf{e}} F_1^{e_1} \cdots F_r^{e_r} \mid \sigma_{\mathbf{e}} \text{ is a sum of squares in } \mathbb{R}[x_1, \dots, x_n] \right\}.$$

In general, $\mathcal{P}(F_1, \dots, F_r)$ does not contain *all* polynomials that are nonnegative on $\mathcal{W}(F_1, \dots, F_r)$, but it is sufficient for the following feasibility certificate: The Krivine–Stengle Positivstellensatz [Kri64, Théorème 7; Ste74, Theorem 3] implies that $\mathcal{W}(F_1, \dots, F_r) = \emptyset$ if and only if $-1 \in \mathcal{P}(F_1, \dots, F_r)$.

There is a plethora of refined *Stellensätze* under various assumptions. In particular, we note the following result due to Schmüdgen:

Theorem 1.2.1 ([Sch91, Theorem 1]). *Let $F, F_1, \dots, F_r \in \mathbb{R}[x_1, \dots, x_n]$ be such that $\mathcal{W}(F_1, \dots, F_r) \subset \mathbb{R}^n$ is bounded. Then $F > 0$ on $\mathcal{W}(F_1, \dots, F_r)$ implies $F \in \mathcal{P}(F_1, \dots, F_r)$.*

In 2001, Lasserre used Theorem 1.2.1 to prove that the problem of finding the unconstrained global minimum of a real-valued polynomial $F: \mathbb{R}^n \rightarrow \mathbb{R}$ reduces to solving a sequence of semidefinite programs [Las01].

In *semidefinite programming*, the objective is to minimise a linear function over a spectrahedron. A *spectrahedron* is the intersection between the cone of real positive semidefinite matrices and an affine subspace. For our purposes in Papers II to IV, it is more convenient to consider projective geometry than it is to work in an affine space. A spectrahedron can thus be expressed as the set

$$\{x \in \mathbb{RP}^n \mid M_0 x_0 + \cdots + M_n x_n \text{ is semidefinite}\}, \quad (1.2.1)$$

where $M_i \in \text{Mat}_d(\mathbb{R})$ are real, symmetric $(d \times d)$ -matrices. Note that linear optimisation problems can be formulated as a semidefinite program where the

M_i are diagonal matrices. While semidefinite programs are more general than linear programs, they are not significantly more difficult to solve. Semidefinite programs can be solved numerically with interior-point methods, and there is a duality theory for them; see for instance [WSV00].

An important property of spectrahedra is that they are convex, as the cone of positive semidefinite matrices is convex. Furthermore, every face of a spectrahedron is exposed, meaning that the face is the intersection of the spectrahedron with a supporting hyperplane [RG95, Corollary 1]. In addition, a spectrahedron is a basic closed semialgebraic set. Indeed, Theorem 1.2.2 below translates the description (1.2.1) of a spectrahedron into polynomial inequalities in $x_0, \dots, x_n$. Recall that a minor Δ_k of a $(d \times d)$ -matrix A is called *principal* if it is obtained by deleting $d - k$ rows and the $d - k$ columns with the same indices, and a minor D_k is a *leading principal minor* if it is obtained by deleting the last $d - k$ rows and columns. Different proofs of the definite statements in Theorem 1.2.2 can be found in [Gio17], and a proof of the semidefinite statements is found in [HJ13, Theorem 7.2.5].

Theorem 1.2.2 (Sylvester’s criterion). *Let A be a Hermitian matrix. Then*

- *A is positive definite $\iff D_k > 0$ for all leading principal minors;*
- *A is positive semidefinite $\iff \Delta_k \geq 0$ for all principal minors;*
- *A is negative definite $\iff (-1)^k D_k > 0$ for all leading principal minors;*
- *A is negative semidefinite $\iff (-1)^k \Delta_k \geq 0$ for all principal minors.*

Spectrahedra are not characterised by being convex, basic closed semialgebraic sets with exposed faces. The pivotal feature is hyperbolicity, which implies the other properties. A homogeneous polynomial $F \in \mathbb{R}[x_0, \dots, x_n]$ is *hyperbolic with respect to $e \in \mathbb{RP}^n$* if $F(e) \neq 0$ and every real line through e meets $\mathcal{V}_{\mathbb{C}}(F) \subset \mathbb{CP}^n$ in only real points. Hyperbolic polynomials were studied by Gårding [Går59]. The connected component of e in the complement of $\mathcal{V}_{\mathbb{R}}(F)$ in $\mathbb{RP}^n$ is called the *hyperbolicity cone* with respect to e . Suppose that (1.1.1) is a Hermitian representation, and that e is a point such that $M(e)$ is definite. Because a Hermitian matrix has only real eigenvalues, it follows that $\det(M)$ is hyperbolic with respect to e . In particular, this is true for real symmetric representations, so spectrahedra are hyperbolicity cones. The objective in *hyperbolic programming* is to minimise a linear function over an affine slice of the hyperbolicity cone, so hyperbolic programming is a generalisation of semidefinite programming.

In 1958, Lax conjectured that all hyperbolic polynomials in three variables admit a real symmetric representation [Lax58]. Helton and Vinnikov proved that all real zero polynomials F on $\mathbb{R}^2$ with $F(0) > 0$ admit a definite, real symmetric representation [HV07]. Here a *real zero polynomial* on $\mathbb{R}^n$ means a polynomial p such that for all $x \in \mathbb{R}^n$, the polynomial $p_x(\mu) := p(\mu x)$ in μ has only real zeros. This is the nonhomogeneous analogue of a hyperbolic polynomial. After seeing a preliminary version of Helton and Vinnikov’s paper, Lewis, Parrilo and Ramana showed that the Helton–Vinnikov theorem is equivalent to the

Lax conjecture [LPR05]. They also showed that the direct analogue of the Lax conjecture to higher dimensions is false. Helton and Vinnikov conjectured that all real zero polynomials F on $\mathbb{R}^n$ with $F(0) > 0$ admit a definite, real symmetric representation. Brändén found counterexamples to this conjecture [Brä11]. It follows from the Helton–Vinnikov theorem that all hyperbolicity cones in the projective plane are spectrahedra. The generalised Lax conjecture says that all hyperbolicity cones are spectrahedra [HV07, Section 6.1; Vin12, Conjecture 5]. This is a major open problem.

The projection of a spectrahedron is called a *spectrahedral shadow*. It is a convex, semialgebraic set. Spectrahedral shadows have gained a lot of interest because it is possible to reduce a linear optimisation problem over a spectrahedral shadow to a semidefinite program. Most notably, Nemirovski asked if all convex, semialgebraic sets are spectrahedral shadows [Nem07]. Helton and Nie conjectured that this is true. They gave equivalent conditions for a set to be a spectrahedral shadow, and proved that a large class of convex, semialgebraic sets are spectrahedral shadows [HN09; HN10]. Scheiderer proved that the Helton–Nie conjecture is true for subsets of the plane [Sch18a], but found counterexamples in higher dimensions [Sch18b]. The smallest possible dimension of a counterexample is still unknown.

Spectrahedra are closely related to determinantal representations, as the symmetroid hypersurface with representation (1.1.1) is the algebraic boundary of the spectrahedron in (1.2.1). In response to a question of Sturmfels, Degtyarev and Itenberg considered generic quartic symmetroids in $\mathbb{P}^3$ with a nonempty spectrahedron [DI11]. With an indirect proof that cannot be used to construct examples, they described all possible arrangements of the ten nodes:

Theorem 1.2.3 ([DI11, Theorem 1.1]). *There exists a real transversal quartic symmetroid with a nonempty spectrahedron, having ρ real nodes, of which σ nodes lie on the boundary of the spectrahedron, if and only if $0 \leq \sigma \leq \rho$, both even, and $2 \leq \rho \leq 10$.*

Ott et al. gave a new and algorithmic proof for the same result [Ott+15]. To do so, they used Dixon’s method and Theorem 1.1.1. We prove analogous results to Theorem 1.2.3 in Papers II and III, and conjecture an extension in Paper IV.

1.3 Quartic surfaces

The core objects in Papers I, II and IV are quartic surfaces. Paper III looks at higher-dimensional quartic hypersurfaces, but for the sake of this introduction, we limit the scope to surfaces in 3-space. Surfaces of degree 4 are perennial favourites among geometers. Degtyarev gave the following reason for their attraction — albeit in a more specific context — in [Deg90, p. 608]:

The interest in surfaces of degree four and curves of degree six can apparently be explained by the fact that the problem of studying them is sufficiently simple and can be expected to be solved in the foreseeable future, while the structure of such curves and surfaces is

sufficiently complicated, and the knowledge of it will probably enable us to formulate some conjectures of a general nature.

Degtyarev mentioned sextic curves because they are a common device in the study of quartic surfaces. Let $S \subset \mathbb{CP}^3$ be a quartic surface with a double point p . After a change of coordinates, we may assume that $p := [1 : 0 : 0 : 0]$. Then $S = \mathcal{V}(F_2x_0^2 + F_3x_0 + F_4)$, where $F_d \in \mathbb{C}[x_1, x_2, x_3]$ are forms of degree d . The projection $\pi_p: S \setminus \{p\} \rightarrow \mathbb{CP}^2$ is a two-to-one map ramified along the sextic curve $R_p := \mathcal{V}(F_3^2 - 4F_2F_4) \subset \mathbb{CP}^2$. Note that the conic section $C_p := \mathcal{V}(F_2) \subset \mathbb{CP}^2$ is a contact curve to R_p ; this is often useful in the analysis of these curves. Another important observation is:

Lemma 1.3.1. *The projection π_p maps singular points on $S \setminus \{p\}$ to singular points on R_p . Every singular point on R_p is the image of a singular point on $S \setminus \{p\}$, or the image of a line in S through p .*

From [Lemma 1.3.1](#), we see that the 16-nodal Kummer surfaces have the maximal number of isolated nodes, since R_p has at most fifteen isolated nodes, which occurs precisely when it is the union of six generic lines. Rohn classified quartic surfaces with at least eight nodes by examining sextic curves with at least seven nodes [[Roh87](#)]. [Theorem 1.1.1](#) says that if R_p splits into cubics, then S is a symmetroid. If R_p splits into cubics, then S has at least ten nodes. Not all 10-nodal quartic surfaces are symmetroids, as there exist irreducible 9-nodal sextic curves. However, all quartic surfaces with at least fourteen isolated nodes are symmetroids, because all sextic curves with at least thirteen nodes can be written as the union of two cubic curves.

Let S be a quartic surface with only simple singularities, for example only isolated nodes. The minimal resolution $\tilde{S} \rightarrow S$ of singularities of S is a K3-surface. A K3-surface is a surface with trivial canonical divisor and irregularity equal to 0. K3-surfaces are the simplest varieties apart from curves and abelian varieties. As such, they show up several places in mathematics and theoretical physics. For instance, K3-surfaces are one of two examples of 2-dimensional Calabi–Yau manifolds. Pjateckiĭ–Šapiro and Šafarevič showed the global Torelli theorem for K3-surfaces [[P-ŠŠ71](#)], which was later extended by Burns and Rapoport to complex analytic surfaces [[BR75](#)]. It says that a marked K3-surface is uniquely determined by its periods. This makes it possible to translate topological questions about singular or real K3-surfaces into arithmetic problems. In order to prove [Theorem 1.2.3](#), Degtyarev and Itenberg used the global Torelli theorem on the minimal resolution of a transversal quartic symmetroid. By converting the problem to the period domain for K3-surfaces, they identified strata of spectrahedra via intersection lattices.

Recall that a surface $\mathcal{V}(F)$ is *rational* over a field $\mathbb{k}$ if it is birationally equivalent to $\mathbb{P}_{\mathbb{k}}^2$. A rational complex surface $\mathcal{V}(F)$ is thus parameterised by three rational functions defined on a dense subset of $\mathbb{CP}^2$. Note that the existence of a parameterisation defined on a subset of $\mathbb{CP}^2$ of the complex surface $\mathcal{V}_{\mathbb{C}}(F)$ does not imply the existence of a parameterisation defined on a subset of $\mathbb{RP}^2$ of the real surface $\mathcal{V}_{\mathbb{R}}(F)$: [Figure II.8](#) on page 68 shows the real part of a rational

complex surface. The real surface has two connected components, hence it is not rational over $\mathbb{R}$. Castelnuovo proved that a complex surface S is rational if and only if both its arithmetic genus $p_a(S)$ and second plurigenus $P_2(S)$ are equal to 0; see [Ser58] for a modern proof due to Kodaira. Schicho developed an algorithm for complex surfaces that takes F as its input, checks if $\mathcal{V}(F)$ is rational using Castelnuovo's criterion and calculates a rational parameterisation if possible [Sch98b]. In addition, he described a similar algorithm for real tubular surfaces [Sch98a].

The quartics birational to a K3-surface are not rational. It was long known that quartic surfaces that have a triple point or are singular along either a line, conic section or cubic curve are rational. Noether proved that these, together with three families of surfaces with nonsimple double points, are the only irreducible rational quartic surfaces [Noe89]. These surfaces are the focus of [Papers I](#) and [II](#), so we give some details about each of them:

- A quartic surface with a triple point is parameterised by a linear system of plane quartic curves with twelve simple base points. The inverse map is given by projection from the triple point. The surface has at most six additional nodes.
- A quartic surface with a double line is parameterised by a linear system of plane quartic curves with eight simple base points and one double base point. The surface has at most eight isolated nodes, in which case it is called *Plücker's surface* [Jes16, Article 83].
- A quartic surface with a double conic is the projection of a quartic del Pezzo surface in $\mathbb{P}^4$ [Dol12, Theorem 8.6.4]. The surface has at most four isolated nodes.
- A quartic surface with a double cubic curve is the projection of an embedding of either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_2$ to $\mathbb{P}^5$ [Ura86a, §1]. The surface has no isolated singularities. The surface obtained by projecting the Veronese embedding of $\mathbb{P}^2$ from a line, is singular along three concurrent lines and is called the *Steiner surface*.
- The surfaces in two of the families described by Noether have an $\tilde{E}_8$ -singularity or its degeneration. A surface in one of these families is parameterised by a linear system of plane curves of degree 7, with one triple and nine double base points. A surface in the other family is parameterised by a linear system of plane curves of degree 9, with one simple, one double and eight triple base points. The surfaces in the third family have an $\tilde{E}_7$ -singularity or its degeneration. In the classical literature, an $\tilde{E}_7$ -singularity is called a *tacnode* [Jes16, Chapter VIII]. A surface in this family is parameterised by a linear system of plane sextic curves, with four simple and seven double base points.

In the 1980s, Urabe attempted to classify all combinations of singularities on quartic surfaces and plane sextic curves [Ura83; Ura84; Ura85; Ura86a;

Ura86b; Ura87; Ura88; Ura89; Ura90]. The main finding is that the Dynkin diagram obtained by performing two transformations — with some technical conditions — on one of the nine Dynkin diagrams $A_{15} + B_2$, $A_{11} + E_6$, B_{17} , $B_5 + D_{12}$, $B_3 + 2E_7$, $B_9 + E_8$, D_{16} , $2D_8$ or $2E_8$, describes the combination of singularities on a quartic surface with only simple singularities [Ura85, Theorem 2]. Urabe conjectured the converse to be true, but found counterexamples of surfaces whose combination of singularities are not covered by this list [Ura90]. He remarked that Nikulin’s lattice theory [Nik79] can be used to check if a Dynkin diagram corresponds to the combination of singularities on a quartic surface. Yang wrote a computer program to check this for all candidates, and thereby completed the list of all combinations of simple singularities on a quartic surface [Yan97].

Urabe’s work is divided into the study of normal and nonnormal surfaces. For normal surfaces with only simple singularities, he considered their minimal desingularisation, which is a K3-surface, and applied Torelli type theorems [Ura87]. Normal surfaces with a nonsimple singularity p were investigated using the projection π_p and the ramification curve R_p [Ura84]. The nonnormal surfaces have nonisolated singularities; they are either a cone over a singular plane quartic curve, a ruled surface over an elliptic curve or a rational surface with a double curve [Ura86b]. Urabe studied these via Fujita’s theory of Δ -genera [Fuj75] and intersection of divisors.

Degtyarev classified quartic surfaces with only isolated singularities and at least one nonsimple singularity, up to rigid isotopy [Deg90]. He showed that there are 4184 rigid isotopy classes of surfaces corresponding to 2523 combinations of singularities. Twelve of the classes consist of ruled surfaces, the rest consists of rational surfaces. Degtyarev’s technique involves projecting from the worst singularity and studying the sextic ramification curve. This provides a method for producing equations for the surfaces, in contrast to Urabe and Yang’s classification of surfaces with simple singularities.

Piontkowski established which combinations of isolated singularities occur on cubic and quartic symmetroids in $\mathbb{CP}^3$ [Pio06]. One of the results is that cubic and quartic symmetroids do not have D_{2k+1} -singularities, but he gave an explicit example of a quintic symmetroid with a D_5 -singularity. For the quartic symmetroids with only simple singularities, Piontkowski followed Urabe’s methods and used Yang’s software to produce a complete list of combinations of singularities. He did not provide symmetric determinantal representations realising these combinations, but speculated that it might be possible to find representations using Theorem 1.1.1. For quartic surfaces with nonsimple singularities, Piontkowski employed Degtyarev’s results. He provided concrete symmetric matrices in this case.

Piontkowski found *all* combinations of singularities on quartic symmetroids with only isolated singularities. This includes symmetroids with a triple point or a tacnode, which are rational. In Paper I, we find the *generic* combination of singularities for rational quartic symmetroids. We find the generic combination of singularities for symmetroids with a triple point, a tacnode or nonisolated singularities.

1.4 Summary of papers

Paper I classifies rational quartic symmetroids in $\mathbb{CP}^3$. We determine which members of Noether’s list of rational quartic surfaces occur as symmetroids: They are surfaces that are singular along a line or a conic section, or have a triple point or a tacnode. For the families of symmetroids having exactly one of these singular loci, we compute the number of additional, isolated nodes for the general member of each family.

We also give a symmetric determinantal representation for some examples of highly singular quartics, such as Plücker’s surface and the Steiner surface.

Paper II gives bounds on the number of real nodes lying on the boundary of a rational quartic spectrahedron. We say that a spectrahedron is *rational* if its complex symmetroid is rational. For each of the families of rational quartic symmetroids described in [Paper I](#), we know the number of isolated nodes for the general member considered as a complex variety. Restricting to spectrahedral symmetroids, let a be the number of real, isolated nodes and b the number of real nodes lying on the boundary of the spectrahedron. That is, (a, b) is just a different notation from (ρ, σ) in [Theorem 1.2.3](#). For each of the different families, we prove inequalities that the pairs (a, b) satisfy. This is an analogous result to the possible arrangements of the ten nodes of a generic quartic spectrahedron described in [Theorem 1.2.3](#). For the various solutions of the inequalities, [Paper II](#) provides examples of spectrahedra with (a, b) for all but one pair.

Paper III grew out of [Paper II](#). In order to resolve one case in that article, namely quartic symmetroids singular along a conic section with no real points, we utilised that those surfaces are cyclides with a known classification [[CDH89](#)]. A *cyclide* is a real quartic surface singular along a conic section without real points. In this paper, we show that the direct analogue of a cyclide — a real quartic hypersurface singular along a quadric of codimension 1 without real points — does not occur as a symmetroidal n -fold for $n > 2$. However, quartic symmetroids singular along a quadric of codimension 1, with no conditions on the real points, occur in higher dimensions.

Paper III finds the maximal dimension of quartic symmetroids singular along a quadric of codimension 1, under the assumption that they are not cones. We split the result into two, depending on whether the quadric is reducible or irreducible. For the cases with maximal dimension, we find the additional singularities of the symmetroid, similarly to [Paper I](#). Finally, we specialise the cases with maximal dimensions to spectrahedral symmetroids. For them, we describe the arrangements of the real singularities in relation to the spectrahedron, like in [Paper II](#) and [Theorem 1.2.3](#).

Paper IV shows a bound on the number of essential singularities on a determinantal quartic surface S_4 with only isolated, simple singularities. If

S_4 has η essential singularities, then $0 \leq \eta \leq 8$, unless S_4 is a symmetroid, in which case $\eta = 10$.

A definite Hermitian determinantal representation defines a spectrahedron. We conjecture the natural extension of [Theorem 1.2.3](#) to surfaces with a definite Hermitian determinantal representation. For a surface with such a representation and only isolated, simple singularities, let $\rho \leq \eta$ denote its number of real, essential nodes, and let σ be the number of those that lie on the boundary of the spectrahedron, as in [Theorem 1.2.3](#). For $0 \leq \eta \leq 8$, we conjecture that there are no restrictions on (η, ρ, σ) , except that ρ and η have the same parity. We prove that all triples (η, ρ, σ) occur for $0 \leq \eta \leq 3$, and provide a partial list of examples for $4 \leq \eta \leq 8$.

1.5 Editorial notes

In Sections 3.2 and 3.3 of [\[Pio06\]](#), Piontkowski studied rational quartic symmetroids with a nonsimple double point or a triple point. This research is closely related to the content of [Paper I](#), and it should have been credited there. Unfortunately, we became first aware of Piontkowski's work after [Paper I](#) was published.

[Remark I.9.2](#) states that if S is a quartic surface singular along two intersecting lines, L_1 and L_2 , then S satisfies [\(I.7.1\)](#), the equation for a tacnodal quartic surface. The assumption that $L_1 \cap L_2$ is a double point is not expressed in that remark. This requirement is not added to [Paper I](#), so as to keep the text unaltered from the published version. The clarification of the multiplicity of $L_1 \cap L_2$ is repeated in the proof of [Proposition II.2.9](#).

All figures are created using [\[surf\]](#).

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Papers

Rational quartic symmetroids

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Abstract

We classify rational, irreducible quartic symmetroids in projective 3-space. They are either singular along a line or a smooth conic section, or they have a triple point or a tacnode.

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I.1 Introduction

A *symmetroid* $S \subset \mathbb{CP}^n$ is a hypersurface $\mathcal{V}(F)$, whose defining polynomial F can be written as the determinant of a symmetric matrix of linear forms. That is, $F = \det(A)$, where

$$A := A(x) := A_0x_0 + \cdots + A_nx_n, \quad (\text{I.1.1})$$

and the A_i are symmetric $(d \times d)$ -matrices with entries in $\mathbb{C}$. The degree of S is then d . In this paper, we characterise families of rational symmetroids of degree 4 in $\mathbb{CP}^3$.

Since S is identified with the symmetric matrix A , we are able to talk about the rank of S at a point. By evaluating, every point $x \in \mathbb{P}^n$ is associated the symmetric matrix $A(x)$. The *rank* and *corank* of x are defined as $\text{rank } A(x)$ and $\text{corank } A(x)$, respectively. The symmetroid S consists of the points with corank at least 1. The *rank- k locus* of S is the set of points with rank less than or equal to k . This is precisely the zero locus of the $(k+1) \times (k+1)$ -minors of A . A rank- $(d-2)$ point is singular on S and generically it is a node. The rank- $(d-2)$ locus is not necessarily equal to the singular locus.

A generic quartic symmetroid in $\mathbb{P}^3$ has ten rank-2 points, which are nodes, and no further singularities. If a quartic surface has a finite set of nodes — or more generally, rational double points — it is birationally equivalent to a K3-surface, and is therefore irrational. Hence, a rational quartic symmetroid has either infinitely many rational double points or a more complicated singularity. In fact, the rational quartic surfaces are either double along a curve, or have a triple point or an elliptic double point; see [Jes16, Article 96; Noe89].

The generic case was first studied by Cayley in [Cay69]. He characterised the nodes on a quartic symmetroid by the fact that the projection from a node is ramified along the union of two cubic curves. An account of this work can also be found in [Jes16, Chapter IX]. For a real, generic quartic symmetroid with a nonempty spectrahedron, the possible arrangements of the ten nodes are described by Degtyarev and Itenberg in [DI11] and by Ottem et al. in [Ott+15]. However, the rational quartic symmetroids have not been studied systematically before. We prove the following:

Theorem I.1.1. *A rational, irreducible quartic symmetroid in $\mathbb{P}^3$ is either singular along a line — consisting either of rank-3 points or rank-2 points — or a smooth conic section, or it has a triple point or a tacnode.*

For each of the families of symmetroids having precisely one of these singular loci, a general member S satisfies the following:

1. *S is singular along a line of rank-3 points and has four isolated nodes,*
2. *S is singular along a line of rank-2 points and has six isolated nodes,*
3. *S is singular along a smooth conic section and has four isolated nodes,*
4. *S has a triple point and six isolated nodes,*

5. S has a tacnode and six isolated nodes.

All of the isolated nodes are rank-2 points. For the cases 1-5, the families have dimensions 21, 19, 17, 21 and 20, respectively.

Remark I.1.2. [Theorem I.1.1](#) is not an exhaustive list of all rational quartic symmetroids. Such a symmetroid can be singular along a curve with more than one component, or some of the isolated rank-2 points may coincide and form more complicated singularities, or the symmetroid may have additional isolated rank-3 nodes.

A few examples are given in [Sections I.9](#) and [I.10](#). Among them are notably the Steiner surface and Plücker's surface. ♠

I.2 Preliminaries

I.2.1 Projection from a node

The most classical tool for studying symmetroids is the projection from a node. Let $S := \mathcal{V}(F)$ be a quartic surface and p a double point on S . The projection from p defines a two-to-one map, which extends to a morphism $\pi_p: \tilde{S} \rightarrow \mathbb{P}^2$ on the blow-up $\tilde{S}$ of S with centre p . Write F as

$$F_2x_0^2 + F_3x_0 + F_4, \quad (\text{I.2.1})$$

where F_d is a form of degree d in $\mathbb{C}[x_1, x_2, x_3]$. Then π_p is ramified along the sextic curve $R_p := \mathcal{V}(F_3^2 - 4F_2F_4)$. The following is one of the earliest results about symmetroids. For a modern proof and an extension, see [\[Ott+15, Theorem 1.2\]](#).

Theorem I.2.1 ([\[Cay69\]](#)). *Let p be a node on a quartic surface $S \subset \mathbb{P}^3$. If S is a symmetroid and p is a rank-2 point, then the ramification locus $R_p = R_1 \cup R_2$ is the union of two cubic curves, R_1 and R_2 . Conversely, if R_p breaks up into cubics, then S is a symmetroid.*

The rest of the rank-2 locus is mapped into $R_1 \cap R_2$ by π_p . In the 10-nodal case, the correspondence between the rest of the rank-2 locus and $R_1 \cap R_2$ is bijective:

Corollary I.2.2 ([\[Ott+15, Corollary 3.1\]](#)). *Let p be a node on a 10-nodal quartic symmetroid $S \subset \mathbb{P}^3$. Suppose that S contains no line through p . Then the nine intersection points of R_1 and R_2 are precisely the images of the rank-2 nodes on S other than p .*

I.2.2 Linear systems of quadrics

Having identified the symmetroid S with the symmetric matrix A , we have the notions of the associated quadratic form and quadric of S at a point. If S has degree d , let $\mathbf{y} := [y_0, \dots, y_{d-1}]$. The point $x \in \mathbb{P}^n$ is associated the quadratic

I. Rational quartic symmetroids

form $q(x) := \mathbf{y}A(x)\mathbf{y}^\top$. Then $Q(x) := \mathcal{V}(q(x)) \subset \mathbb{P}^{d-1}$ is the *associated quadric* at x . The quadrics in the set

$$W(S) := \{Q(x) \mid x \in \mathbb{P}^n\}$$

form the *associated linear system of quadrics* of S . The point in $W(S)$ which corresponds to the quadric $Q \subset \mathbb{P}^{d-1}$ is denoted by $[Q]$. A symmetroid defined as the set of singular quadrics in a space W of quadrics, is called the *discriminant hypersurface* of the space. The discriminant D is given by $\det(A) = 0$, where A is the matrix parametrising W . We have that $W = W(D)$.

Remark I.2.3. In the notation $W(S)$, the choice of representation (I.1.1) is not specified. This is abuse of notation, because the associated system is in general not unique. The uniqueness holds in special cases. In particular, this is true for 10-nodal quartic symmetroids in $\mathbb{P}^3$; see [Ble+12, Proposition 11].

The greatest discrepancy among the associated systems occurs for cones. Indeed, suppose that S is defined by a matrix A involving a variable x_i that does not appear in $\det(A)$. Then S is also given by the matrix A' defined as A with $x_i = 0$. The linear system of quadrics induced by A' has lower dimension than the system induced by A .

Different matrix representations can also give rise to different systems that have the same dimension. For instance, Example I.9.6 shows a symmetroid with two representations, where the rank-2 locus of the symmetroid differs for the two representations.

Our abuse of notation is justified by the fact that we only use properties that hold for all associated linear systems of quadrics that have the same rank- k loci. ♠

Recall that the dimension of the Grassmannian $\mathbb{G}(k, n)$ of k -dimensional, linear subspaces of $\mathbb{P}^n$ is given by

$$\dim \mathbb{G}(k, n) = (k+1)(n-k). \quad (\text{I.2.2})$$

Moreover, let $Y \subseteq \mathbb{G}(k, n)$ be the subvariety consisting of the linear, k -dimensional subspaces K that intersect a fixed linear, m -dimensional subspace M such that $\dim(K \cap M) \geq l$. By [Har92, Example 11.42], the dimension of Y is given by the following formula:

$$\dim(Y) = (l+1)(m-l) + (k-l)(n-k). \quad (\text{I.2.3})$$

Consider $\mathbb{P}^9$ as the space of all quadrics in $\mathbb{P}^3$. By relating a symmetroid S with $W(S)$, quartic symmetroids in $\mathbb{P}^3$ correspond to linear subspaces $W \subset \mathbb{P}^9$ with $\dim(W) \leq 3$. Since $\dim \mathbb{G}(3, 9) = 24$ by (I.2.2), the symmetroids form a 24-dimensional variety in the $\mathbb{P}^{34}$ of all quartic surfaces.

Let $\mathcal{D}_d$ be the discriminant of the space $\mathbb{P}^N := \mathbb{P}^{\binom{d+2}{2}-1}$ of all quadrics in $\mathbb{P}^d$. The rank-2 locus X_2 of $\mathcal{D}_3$ is a sixfold of degree 10, and the rank-1 locus X_1 is a threefold of degree 8. Bézout's theorem implies that a generic linear 3-space in $\mathbb{P}^9$ intersects X_2 in ten points and avoids X_1 . A generic quartic symmetroid has

therefore ten rank-2 points, which are nodes, and no rank-1 points. Moreover, $\text{Sing}(\mathcal{D}_3) = X_2$, so it has no other singularities.

Lemma I.2.4 below is a well-known fact that is used repeatedly in our exposition. The different parts of the lemma can for instance be found in [Ili+17, Lemma 2.13] and [Wal81, Lemma 1.1]. To prove it, consider the discriminant D of a linear space $\mathbb{P}^n$ of quadrics in $\mathbb{P}^d$ as a linear section of $\mathcal{D}_d$. If $[Q]$ is a rank- d point, then it is nonsingular in $\mathcal{D}_d$. The tangent space $T_{[Q]}\mathcal{D}_d$ can be identified with the set of quadrics in $\mathbb{P}^d$ passing through the singular point of Q , see [Dol12, Equation (1.45)]. Since $T_{[Q]}D = \mathbb{P}^n \cap T_{[Q]}\mathcal{D}_d$, the lemma follows.

Lemma I.2.4. *Let $\mathbb{P}^n$ be a linear space of quadrics in a projective space $\mathbb{P}^d$ and let $B \subset \mathbb{P}^d$ be the base locus of the quadrics in $\mathbb{P}^n$. Let $D \subset \mathbb{P}^n$ be the discriminant. If $D \neq \mathbb{P}^n$, then D has degree $d + 1$. If $[Q] \in \mathbb{P}^n$ is a point such that Q is a singular quadric with a singularity at $p \in B$, then the discriminant D is singular at $[Q]$. If $[Q]$ is a rank- d point such that D is singular at $[Q]$, then $p \in B$.*

We will need some special properties of 1-dimensional linear systems of quadrics:

Lemma I.2.5. *Let P be a pencil of rank-2 quadrics in $\mathbb{P}^n$, with $n \geq 2$. The base locus of P consists of a hyperplane H and a linear subspace $L \not\subset H$ having codimension 2.*

Proof. The base locus of a general pencil generated by two rank-2 quadrics, is the union of four linear subspaces of codimension 2. Bertini's theorem implies that a general member of the pencil is smooth outside of the base locus. In the case of P , where all members have rank 2, it follows that the quadrics share a common singular subspace L' of codimension 2. Projecting from L' defines a pencil of quadrics in $\mathbb{P}^1$. If the pencil does not have a base point, then it contains at least one double point, implying that P contains a rank-1 quadric. If it does have a base point, then all quadrics in P share a hyperplane H . If $H \cup H_1$ and $H \cup H_2$ are two generators of P , then $L := H_1 \cap H_2$ is contained in the base locus of P . Since P does not contain a rank-1 quadric, $L \not\subset H$. ■

Lemma I.2.6. *Let P be a pencil of quadrics in $\mathbb{P}^3$. Assume that a general quadric in P has rank 3. Then one of the following is true:*

1. *The quadrics in P have a common singular point, and the rank-2 locus of P is a scheme of length 3.*
2. *The quadrics in P have a common tangent plane that is tangent along a line in each quadric in P , and P contains a unique rank-2 quadric.*

Proof. Assume that the quadrics share a common singular point. Projecting from the singularity gives a pencil P' of conics in $\mathbb{P}^2$. Since P is not a pencil of rank-2 quadrics, P' is not contained in the cubic discriminant $\mathcal{D}_2 \subset \mathbb{P}^5$. Hence P' intersects $\mathcal{D}_2$ in a scheme of length 3.

Assume that the quadrics do not share a singularity. It follows from Lemma I.2.4 and Bézout's theorem that the line L between the singular points of any two rank-3 quadrics is contained in the base locus of P . Furthermore, this

is the only line in the base locus, so every quadric in P is singular in a point on L . Thus the base locus is singular along L , meaning it contains the intersection of each quadric in P with their tangent plane along L . This tangent plane is therefore common to all the quadrics. Since the quadrics in P do not have a common singular point, a rank-2 quadric in P cannot be singular along L . On the other hand, the common tangent plane along L is a component in a quadric in P , which is the unique rank-2 quadric in P . ■

The following simple observation is useful for excluding possible symmetroids:

Lemma I.2.7. *Let $S \subset \mathbb{P}^n$ be a quartic symmetroid and assume that the base locus of $W(S)$ contains a curve C . Then S is reducible.*

Proof. The base locus of $W(S)$ is an intersection of quadrics in $\mathbb{P}^3$. Hence, C has either a line, a smooth conic section, a twisted cubic curve or an irreducible quartic curve as a component. Let $P \subset W(S)$ be a general pencil. Suppose that C contains a line L . Then the base locus of P is the union of L and a twisted cubic curve, hence it has two singular points. Thus P intersects the discriminant of $W(S)$ in only two points, since otherwise the base locus of P would contain more singularities. Hence S is a double quadric.

If C contains a twisted cubic curve, then $\dim W(S) \leq 2$. Moreover, the other component in the base locus of P is again a line. By the previous case, S is a double quadric.

Suppose that C contains a smooth conic section K . In the restriction of P to the plane spanned by K , one of the members is the entire plane. Hence P contains a rank-2 quadric and therefore only three singular quadrics. This can only be if S is the union of a quadric and a double hyperplane.

Finally, a quartic curve Q is the intersection of two quadrics, so $\dim W(S) \leq 1$. If C contains Q , then the equation for S is defined by only two variables. It follows that S is the union of four hyperplanes. ■

We isolate the result from the first part of the proof of [Lemma I.2.7](#) for easy reference:

Lemma I.2.8. *Let P be a linear space of quadrics and suppose that the base locus of P contains a line. Then the discriminant of P is a square.*

I.3 Quartic symmetroids with a double line

Let $p_1, \dots, p_9 \in \mathbb{P}^2$ be nine points that are not the complete intersection of two cubic curves. Consider the linear system $\mathfrak{d}$ of quartic curves passing twice through p_1 and once through each of the points $p_2, \dots, p_9$. Let $\varphi: \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ be the map induced by $\mathfrak{d}$. The image $\varphi(\mathbb{P}^2) \subset \mathbb{P}^3$ is a quartic surface with a double line. Any quartic surface $S \subset \mathbb{P}^3$ with a double line arises this way; see [[Jes16](#), Article 79]. Consequently, S is rational.

For a quartic symmetroid S with a double line L , there are two possible scenarios: The points along L are either generically rank-3 points, or they are

all rank-2 points. We show that if S is a generic symmetroid with a double line containing rank-3 points, then S has four rank-2 points outside of L and no further singularities. The family of such symmetroids is 21-dimensional. Likewise, we show that if S is a generic symmetroid with a line of rank-2 points, then it has six rank-2 points outside of L . Symmetroids of this type form a 19-dimensional family.

Lemma I.3.1. *Let $S \subset \mathbb{P}^3$ be a general, irreducible, quartic symmetroid with a line L of rank-2 points. Then $W(S)$ has four general, coplanar base points.*

Proof. The line L corresponds to a pencil $P \subset W(S)$ of rank-2 quadrics. By Lemma I.2.5, the base locus of P consists of a plane H and a line $l \not\subset H$.

Let $Q_1, Q_2 \in W(S)$ be such that Q_1, Q_2 and P generate $W(S)$. We may assume that $[Q_1]$ and $[Q_2]$ are not in the rank-2 locus of S , so the plane H is not contained in either of the Q_i . By Bézout's theorem, each Q_i intersects H in a conic C_i . Similarly, Q_i intersects the line l in two points, p_i and p'_i , each. Generically, none of the points p_1, p'_1, p_2 and p'_2 coincide. However, Bézout's theorem implies that C_1 and C_2 generically intersect in four general points. ■

The connection between symmetroids with a line of rank-2 points and webs of quadrics with four coplanar base points is also true in the other direction:

Lemma I.3.2. *Let W be a web of quadrics in $\mathbb{P}^3$ with four general, coplanar base points. Generically, the discriminant $D \subset \mathbb{P}^3$ of W , has a line of rank-2 points and six additional rank-2 points.*

Proof. Consider the $\mathbb{P}^5$ of quadrics passing through the four coplanar points $p_1, p_2, p_3, p_4 \in \mathbb{P}^3$. We will now describe the rank-2 points in $\mathbb{P}^5$ by finding the rank-2 quadrics passing through p_1, p_2, p_3 and p_4 . First, let H be the plane spanned by p_1, p_2, p_3 and p_4 . Then the union of H and any plane $H' \subset \mathbb{P}^3$ is a rank-2 quadric containing the base points. The set X of all such unions forms a $\mathbb{P}^3 \subset \mathbb{P}^5$.

Next, let H_{ij} be a plane containing the line L_{ij} spanned by the points p_i and p_j . Let H_{kl} be a plane containing the line L_{kl} spanned by the remaining two points, p_k and p_l . Then the union of H_{ij} and H_{kl} is a rank-2 quadric containing p_1, p_2, p_3 and p_4 . The set X_{ij} of all such unions forms a quadratic surface in $\mathbb{P}^5$. Since the points are in general position, there are in total three such surfaces of rank-2 points, namely X_{12}, X_{13} and X_{14} .

By Bézout's theorem, a generic, linear 3-space $W \subset \mathbb{P}^5$ intersects X in a line and the three surfaces X_{ij} in two points each. This proves the claim. ■

In Lemma I.3.2, if we omit the assumption that the coplanar base points are general, then three of them can lie on a line L . In that case, the base locus contains L and Lemma I.2.7 states that D is reducible.

The next result is immediate from Lemmas I.3.1 and I.3.2:

Proposition I.3.3. *Let $S \subset \mathbb{P}^3$ be a general quartic symmetroid with a line of rank-2 points. Then S has six additional rank-2 points.*

The construction indicated by [Lemmas I.3.1](#) and [I.3.2](#) allows us to count the number of quartic symmetroids with a line of rank-2 points:

Proposition I.3.4. *The family of quartic symmetroids with a line of rank-2 points is 19-dimensional.*

Proof. [Lemmas I.3.1](#) and [I.3.2](#) imply that a generic quartic symmetroid with a line of rank-2 points is obtained by choosing four coplanar points $p_1, p_2, p_3, p_4 \in \mathbb{P}^3$ in general position, and then choosing a general $\mathbb{P}^3$ in the $\mathbb{P}^5$ of quadrics passing through p_1, p_2, p_3 and p_4 .

In how many ways can p_1, p_2, p_3 and p_4 be chosen? The first three points can be chosen freely, and the last point p_4 must lie in the plane spanned by p_1, p_2 and p_3 . Hence four coplanar points in $\mathbb{P}^3$ correspond to a point in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$, which is an 11-dimensional space. There are points in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^2$ that do not correspond to four general, coplanar points, since the same point in $\mathbb{P}^3$ is taken more than once or since the points are not in a general position. However, excluding these exceptions does not affect the dimension.

By [\(I.2.2\)](#), the Grassmannian $\mathbb{G}(3, 5)$ of linear 3-spaces in the $\mathbb{P}^5$ of quadrics through p_1, p_2, p_3 and p_4 , is 8-dimensional. In total, the family of quartic symmetroids with a line of rank-2 points has dimension $11 + 8 = 19$. ■

The symmetroids with a line of rank-2 points only make up a small fraction of the quartic symmetroids with a double line:

Proposition I.3.5. *The family of irreducible quartic symmetroids with a double line is 21-dimensional.*

Consider the variety X of quartics with two nodes. Requiring the line spanned by the nodes to be singular imposes three more conditions. Hence the codimension of such surfaces is 3 in X . In the case of symmetroids, this gives dimension $24 - 3 = 21$. For a more rigorous treatment:

Proof. Let S be a quartic symmetroid with a double line L . The case where L consists of only rank-2 points is covered by [Proposition I.3.4](#). Assume therefore that L contains a rank-3 point $[Q]$. Let p be the singular point of Q . Then [Lemma I.2.4](#) implies that p is a base point for $W(S)$.

Consider the $\mathbb{P}^8$ of all quadrics through p , and let D be its discriminant. It imposes three conditions to require that a quadric passing through p is singular at p . Hence the set $X_p \subset \mathbb{P}^8$ of all quadrics in $\mathbb{P}^3$ that are singular at p is a linear 5-space. By [Lemma I.2.4](#), X_p is contained in $\text{Sing}(D)$. Let $W \subset \mathbb{P}^8$ be a linear 3-space that intersects X_p in a line. Then the discriminant of W is a quartic symmetroid with a double line.

By [\(I.2.3\)](#), the set of 3-spaces in $\mathbb{P}^8$ that meet X_p in a line, is 18-dimensional. In total, we obtain a 21-dimensional family of quartic symmetroids with a double line, by letting the base point p be arbitrary in $\mathbb{P}^3$. ■

The construction from the proof of [Proposition I.3.5](#) allows us to determine the number of extra singularities:

Proposition I.3.6. *Let $S \subset \mathbb{P}^3$ be a general quartic symmetroid that is singular along a line L of rank-3 points. Then S has four rank-2 nodes outside of L .*

Proof. We continue with the notation from the proof of [Proposition I.3.5](#). Choose coordinates such that $p := [1 : 0 : 0 : 0]$. Then the $\mathbb{P}^8$ of quadrics that pass through p is parametrised by the matrix

$$A := \begin{bmatrix} 0 & x_{01} & x_{02} & x_{03} \\ x_{01} & x_{11} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

Furthermore, we have that $X_p = \mathcal{V}(x_{01}, x_{02}, x_{03})$ and the rank-2 locus X_2 of D is a fivefold of degree 10.

Using this explicit description, we compute that the tangent space at a general point in X_2 is 5-dimensional, but the tangent space at a point in $X_2 \cap X_p$ is 6-dimensional. In fact, $\text{Sing}(X_2) = X_2 \cap X_p$ set-theoretically. Since $L \subset X_p$, it contains in general three rank-2 points, p_1, p_2 and p_3 . This can be seen either from the matrix A or [Lemma I.2.6](#). The web $W(S) \subset \mathbb{P}^8$ intersects the three tangent spaces $T_{p_i} X_2$ in a line each. Thus the intersection multiplicity of $W(S)$ and X_2 is at least 2 at each of the points p_1, p_2 and p_3 . Since $W(S) \cap X_2$ has length 10, it follows that $W(S)$ meets X_2 in at most four points outside of L . Moreover, the intersection multiplicity of $W(S)$ and X_2 is generically 2 at the points p_1, p_2 and p_3 . This proves the claim. $\blacksquare$

I.4 Quartic symmetroids with a double conic section

Let S be an irreducible quartic surface $S \subset \mathbb{P}^3$ with a double conic. Then S can be realised as the projection of a quartic del Pezzo surface $P \subset \mathbb{P}^4$; see [\[Dol12, Theorem 8.6.4\]](#). Consequently, S is rational.

We show that if S is an irreducible quartic symmetroid with a double conic C , then there are no rank-3 points on C . Furthermore, if S is a generic symmetroid with a double conic, then it has four rank-2 points outside of C .

Proposition I.4.1. *Let $S \subset \mathbb{P}^3$ be an irreducible quartic symmetroid that is double along a smooth conic section C . Then C is contained in the rank-2 locus of S .*

Proof. Assume for contradiction that C is not contained in the rank-2 locus of S . A generic point $[Q_1] \in C$ is then a rank-3 node. By [Lemma I.2.4](#), the singular point p of Q_1 is a base point for $W(S)$. If $[Q_2] \in C$ is another point such that Q_2 is singular at p , then all the quadrics in the pencil $\langle Q_1, Q_2 \rangle$ are singular at p . By [Lemma I.2.4](#), the line L spanned by $[Q_1]$ and $[Q_2]$ is contained in $\text{Sing}(S)$. Since C is smooth, L is not a component of C . Let H be the plane spanned by C . The intersection of H and S contains at least L and two times C , so H must be a component in S . This contradicts the irreducibility of S . In conclusion, the different rank-3 nodes on C give rise to different base points of S . Hence, the base locus of $W(S)$ contains a curve. [Lemma I.2.7](#) implies that S is reducible, which is impossible. $\blacksquare$

Remark I.4.2. Proposition I.4.1 is not true for singular conic sections, as shown by Example I.9.6. ♠

We have an analogue to Lemma I.3.1 for quartic symmetroids with a double conic:

Proposition I.4.3. *Let $S \subset \mathbb{P}^3$ be a general quartic symmetroid with a smooth conic section C of rank-2 points. Then $W(S)$ has four general base points.*

Proof. Let N be the net of quadrics corresponding to the plane spanned by C , and let $Q := H_1 \cup H_2 \in N$ be a quadric corresponding to a point on C . Restricting N to the plane H_i defines a pencil P_i of conics. Each line in N through Q is collapsed to a point in P_i . For each point in P_i , we can find a representative $Q' := H'_1 \cup H'_2 \in C$ for the corresponding line. Hence P_i is a pencil of line pairs $L_i \cup L'_i$. Since there are no smooth conic sections in P_i , it follows from Lemma I.2.5 that the base locus of P_i must contain a line, say L_i . Then L_1 and L_2 are contained in the base locus of N . The web $W(S)$ is generated by N and a quadric $K \notin N$. Generically, K intersects L_1 and L_2 in two points each, so $W(S)$ has four general base points. ■

Remark I.4.4. There is no analogue to Lemma I.3.2 for conic sections. Let W be a web of quadrics with four general base points. Generically, the discriminant of W does *not* contain a conic section of rank-2 points.

Indeed, consider the $\mathbb{P}^5$ of quadrics containing the four general points $p_1, p_2, p_3, p_4 \in \mathbb{P}^3$ and let D be its discriminant. We shall describe the rank-2 locus of D . Let $H \subset \mathbb{P}^3$ be the plane spanned by three of the points, p_i, p_j and p_k , and let $H_l \subset \mathbb{P}^3$ be a plane containing the remaining point p_l . The union of H and H_l is a rank-2 quadric passing through the four base points. The set X_l of all such unions forms a plane in $\mathbb{P}^5$. Hence there are four planes, X_1, X_2, X_3 and X_4 , in the rank-2 locus of D . In addition, the rank-2 locus of D contains the three quadratic surfaces X_{12}, X_{13} and X_{14} , as described in the proof of Lemma I.3.2. In total, the rank-2 locus of D is a surface of degree 10. By Bézout's theorem, a generic linear 3-space $W \subset \mathbb{P}^5$ contains 10 rank-2 points. Hence, W must be in a special position in order to contain a conic section of rank-2 points. ♠

We can still deduce the number of additional rank-2 points for a general quartic symmetroid with a double conic:

Proposition I.4.5. *Let $S \subset \mathbb{P}^3$ be a general quartic symmetroid with a smooth conic section of rank-2 points. Then S has four additional rank-2 nodes.*

Proof. We continue with the notation from Remark I.4.4. The union of H and a plane containing the line spanned by p_l and p_k , is a quadric that lie in $X_l \cap X_{ij}$. The intersection $X_l \cap X_{ij}$ is the line L_l of all such quadrics. Suppose that $W(S) \subset \mathbb{P}^5$ intersects X_{12} in a conic section C . Generically, $W(S)$ intersects the quadratic surfaces X_{13} and X_{14} in two points each, and the planes X_l in a point each. Since the lines L_l meet C in a point, $W(S)$ does not intersect X_l outside of C . Hence, $W(S)$ has generically four isolated rank-2 points. ■

We can count the number of symmetroids with a double conic:

Proposition I.4.6. *The family of quartic symmetroids with a double smooth conic section is 17-dimensional.*

Proof. As in the proof of Proposition I.4.5, a quartic symmetroid with a double smooth conic section corresponds to a linear 3-space $W \subset \mathbb{P}^5$ that intersects X_{12} in a conic section. This is the same as saying that W intersects the $\mathbb{P}^3$ spanned by X_{12} , in a plane. It follows from (I.2.3) that the family of 3-spaces that intersect X_{12} in a conic is 5-dimensional.

The calculation above shows the number of linear systems, with a conic section of rank-2 points, having fixed base points p_1, p_2, p_3 and p_4 . A choice of base points corresponds to a point in $\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3$, which is 12-dimensional. In total, the family of quartic symmetroids with a double conic has dimension $5 + 12 = 17$. ■

I.5 Quartic symmetroids with a double twisted cubic curve

An irreducible quartic surface with a double twisted cubic curve T is a scroll. Indeed, every nonsingular point on the surface lies on a unique secant line of T , which intersects the quartic with multiplicity greater than 4. Hence the secant line is contained in the surface. All quartic symmetroids with a double twisted cubic curve are reducible:

Proposition I.5.1. *Let $S \subset \mathbb{P}^3$ be an irreducible quartic surface with a double twisted cubic curve T . Then S is not a symmetroid.*

Proof. Assume for contradiction that S is a symmetroid. Then there are two cases: The points on T are either generically rank-3 points or they are all rank-2 points.

Suppose that there are only finitely many rank-2 points on T . Let $p_1, p_2 \in T$ be two rank-3 nodes and assume that the associated quadrics of p_1 and p_2 have a common singular point p . Then Lemma I.2.4 implies that p is a base point for $W(S)$. All the associated quadrics on the line L spanned by p_1 and p_2 , are singular at p . By Lemma I.2.4, L is contained in $\text{Sing}(S)$. But then S is singular along a quartic curve, so S is reducible, which is a contradiction. We conclude that the associated quadrics of p_1 and p_2 have different apexes. It follows from Lemma I.2.4 that T gives rise to a curve of base points for $W(S)$, but this is impossible by Lemma I.2.7.

Assume that T consists of rank-2 points. The Jacobian ideal of S defines T . Since S is a scroll, along a general line l in S , the Jacobian defines a scheme of length 2. However, l corresponds to a pencil of quadrics with rank 2 and 3. Lemma I.2.6 implies that l contains a scheme of length 3 of rank-2 points. This contradicts the fact that the rank-2 locus is contained in the singular locus. ■

I.6 Quartic symmetroids with a triple point

Let $S \subset \mathbb{P}^3$ be a quartic surface with a triple point p . Note that the projection $\pi_p: S \setminus \{p\} \dashrightarrow \mathbb{P}^2$ from p is a birational map, so S is rational.

If $p := [1 : 0 : 0 : 0]$, then the equation of S can be written as $x_0 F_3 + F_4$, where $F_3, F_4 \in \mathbb{C}[x_1, x_2, x_3]$ are polynomials of degree 3 and 4, respectively. The cubic cone $C := \mathcal{V}(F_3)$ intersects S in twelve lines, which meet at p . Let $p_1, \dots, p_{12} \in \mathbb{P}^2$ be the images of these lines under π_p . Then S can be represented as the image of the map induced by the linear system of quartic curves through the p_i . Let e_i be the linear equivalence class of the exceptional line over the point p_i .

If S is a symmetroid, then the matrix defining S can be written as

$$\begin{bmatrix} x_0 + l_{00} & l_{01} & l_{02} & l_{03} \\ l_{01} & l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & l_{33} \end{bmatrix},$$

where $l_{ij} \in \mathbb{C}[x_1, x_2, x_3]_1$ are linear forms. Moreover, F_3 equals the determinant of the submatrix

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{12} & l_{22} & l_{23} \\ l_{13} & l_{23} & l_{33} \end{bmatrix}.$$

This implies that C is tangent to S along the sextic curve given by the zero locus of the (3×3) -minors of the submatrix

$$\begin{bmatrix} l_{01} & l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & l_{33} \end{bmatrix}.$$

Hence, the twelve lines on S through p coincide in such a way that they occur with even multiplicity. The general case is that two and two lines coincide. Then six of the linear equivalence classes $e_i - e_j$ are effective. This induces six nodes on S .

Proposition I.6.1 ([Jes16, Article 93]). *Let the surface $S \subset \mathbb{P}^3$ be a general quartic symmetroid with a triple point. Then S has six additional nodes.*

Consider a quartic surface S with a triple point and six nodes that do not lie on a conic section. Let p be one of the nodes. Jessop notes that the ramification curve R_p breaks up into two cubics, each having a double point at the image of the triple point. They intersect at the images of the five remaining nodes. By [Theorem I.2.1](#), S is a symmetroid. This fact makes it straightforward to deduce the size of the family of symmetroids with a triple point:

Proposition I.6.2 ([Jes16, Article 93]). *The family of quartic symmetroids with a triple point is 21-dimensional.*

1.7 Quartic symmetroids with an elliptic double point

Let p be a normal singularity on a surface S , and let $\pi: \tilde{S} \rightarrow S$ be a minimal resolution of p . We say that p is an *elliptic* singularity if there is a curve of arithmetic genus 1 with support on the exceptional set $E := \pi^{-1}(p)$. The elliptic singularity is called *simple* if E is nonsingular, see [Sai74], and *minimal* if every connected, proper subvariety of E is the exceptional set of a rational singularity, see [Lau77]. The *degree* of p is $-E \cdot E$, where $E \cdot E$ denotes the self-intersection number. The survey article [Kas15] contains many references about elliptic singularities.

In [Noe89], Noether describes three classes, $S_4^{(1)}$, $S_4^{(2)}$ and $S_4^{(3)}$, of rational quartic surfaces having an elliptic double point. These surfaces have minimal elliptic double points of degree less than or equal to 2. Noether expresses them by linear systems of plane curves and gives explicit equations for the surfaces. He proves that together with the quartic surfaces having a double curve or a triple point, these are the only rational quartic surfaces.

Uraabe studies simple elliptic singularities in [Ura83a; Ura83b; Ura84]. In particular, in [Ura84] he considers a double cover of $\mathbb{P}^2$ branched along a sextic curve, and shows that quartic surfaces having a simple elliptic singularity $\tilde{E}_8$ of degree 1, are rational.

We show that of the surfaces in Noether's list, only $S_4^{(1)}$ can occur as a symmetroid. This is a surface with a minimal elliptic double point of degree 2. This singularity is called a *tacnode* by Jessop [Jes16, Chapter VIII]. We prove that a general tacnodal symmetroid has six additional nodes.

The rational parametrisation of the surfaces $S_4^{(1)}$, $S_4^{(2)}$ and $S_4^{(3)}$ is given by linear systems of plane curves passing through some base points p_i on a cubic curve. Let e_i be the linear equivalence class of the exceptional line over the point p_i , and let l be the class of the pullback of a line in $\mathbb{P}^2$. The linear systems can then be expressed as

$$\begin{aligned} S_4^{(1)}: \quad & 6l - 2 \sum_{i=1}^7 e_i - \sum_{i=8}^{11} e_i, \\ S_4^{(2)}: \quad & 7l - 3e_1 - 2 \sum_{i=2}^{10} e_i, \\ S_4^{(3)}: \quad & 9l - 3 \sum_{i=1}^8 e_i - 2e_9 - e_{10}. \end{aligned}$$

The first linear system defines a surface with an elliptic singularity of degree 2 and the last two define elliptic double points of degree 1, since they are blow-ups of 11 and 10 points on a cubic curve, respectively.

Choosing coordinates such that the elliptic double point is $p := [1 : 0 : 0 : 0]$, we may assume that the polynomials defining the different types of surfaces, are

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on the forms

$$S_4^{(1)}: \quad x_1^2 x_0^2 + (x_1 F_2) x_0 + F_4, \quad (\text{I.7.1})$$

$$S_4^{(2)}: \quad x_1^2 x_0^2 + (x_1 x_3 (2x_3 + B_1) + A_3) x_0 + x_3^4 + B_1 x_3^3 + B_2 x_3^2 + B_3 x_3 + B_4, \quad (\text{I.7.2})$$

$$S_4^{(3)}: \quad x_1^2 x_0^2 + 2(x_1 x_3 A_1 + A_3) x_0 - x_1 x_3^3 + B_2 x_3^2 + B_3 x_3 + B_4, \quad (\text{I.7.3})$$

where F_d is a form of degree d in $\mathbb{C}[x_1, x_2, x_3]$ and A_d, B_d are forms of degree d in $\mathbb{C}[x_1, x_2]$.

Writing the equation of a quartic surface S with a double point as in (I.2.1), the quadric $\mathcal{V}(F_2)$ is called the *tangent cone* of S at p . In the cases $S_4^{(1)}, S_4^{(2)}$ and $S_4^{(3)}$, the tangent cone is a double plane. Using this fact, we show that elliptic double points have rank 2:

Lemma I.7.1. *Let S be a general rational quartic symmetroid with an elliptic double point p . Then p is a rank-2 point.*

Proof. Note that the generality of S means, in particular, that S is not singular along a curve or has a triple point.

Assume for contradiction that p is a rank-3 singularity and choose coordinates such that $p := [1 : 0 : 0 : 0]$. Then we may assume, as in the proof of Lemma I.2.4, that the matrix defining S is

$$M = \begin{bmatrix} a_0 x_0 + l_{00} & l_{01} & l_{02} & l_{03} \\ l_{01} & a_1 x_0 + l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & a_2 x_0 + l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & 0 \end{bmatrix},$$

where $l_{ij} \in \mathbb{C}[x_1, x_2, x_3]_1$ and $a_0, a_1, a_2 \in \mathbb{C} \setminus \{0\}$. Then

$$\det(M) = -(a_1 a_2 l_{03}^2 + a_0 a_2 l_{13}^2 + a_0 a_1 l_{23}^2) x_0^2 + F_3 x_0 + F_4,$$

where F_d is a form of degree d in $\mathbb{C}[x_1, x_2, x_3]$. Since p is an elliptic double point, then $-(a_1 a_2 l_{03}^2 + a_0 a_2 l_{13}^2 + a_0 a_1 l_{23}^2)$ is a square. By scaling the l_{i3} , we have that

$$l_{03}^2 + l_{13}^2 + l_{23}^2 = l^2$$

for some linear form l . Equivalently,

$$(l_{03} - i l_{13})(l_{03} + i l_{13}) = (l - l_{23})(l + l_{23}).$$

Since $\mathbb{C}[x_1, x_2, x_3]$ is a unique factorisation domain, it follows that l_{23} is a linear combination of l_{03} and l_{13} .

Note that every term in $\det(M)$ has an $l_{i3} l_{j3}$ factor. Because l_{23} vanishes whenever both l_{03} and l_{13} vanish, S is singular along the line $\mathcal{V}(l_{03}, l_{13})$. This contradicts the generality of S . ■

1.7.1 Type $S_4^{(1)}$

The main result of this section is:

Proposition 1.7.2. *Let $S \subset \mathbb{P}^3$ be a general quartic symmetroid with a tacnode p . Then S has six additional rank-2 nodes.*

If we can show that S contains a rank-2 node p , then the ramification locus R_p splits into cubics R_1 and R_2 by [Theorem 1.2.1](#). A natural idea is to study pairs of cubics (R_1, R_2) that give rise to elliptic singularities of degree 2, and count the number of points in $R_1 \cap R_2$. However, deducing the number of rank-2 points of S from $R_1 \cap R_2$ requires an extension of [Corollary 1.2.2](#). Instead we prove [Proposition 1.7.2](#) by examining webs of quadrics.

Tacnodal surfaces are distinguished from surfaces of types $S_4^{(2)}$ and $S_4^{(3)}$ by the intersection of the reduced tangent cone at the elliptic double point, with the surface.

Lemma 1.7.3. *Let $S \subset \mathbb{P}^3$ be a general, irreducible quartic symmetroid with a tacnode p . Then the reduced tangent cone of S at p , intersects S in two double lines.*

Proof. In (1.7.1), the reduced tangent cone at p is the plane $H := \mathcal{V}(x_1)$. It is clear from (1.7.1) that H intersects S in a cone, that is, four concurrent lines, L_1, L_2, L_3 and L_4 . When S is a symmetroid, we interpret H as a net N of quadrics, where the discriminant D consists of four pencils P_i , corresponding to L_i . Let Q be the quadric satisfying $[Q] = p$.

By [Lemma 1.2.6](#), each P_i is of one of two types. We say that P_i is of type I if the quadrics in P_i have a common singularity, and that P_i is of type II if the quadrics have a common tangent plane.

First, we show that at most one of the pencils are of type I. Suppose that P_1 and P_2 are of type I. If all the rank-2 quadrics in P_1 coincide, then the base locus of P_1 contains a triple line. It follows that the line $\text{Sing}(Q)$ is contained in the base locus of P_1 . If the rank-2 quadrics in both P_1 and P_2 coincide, then $\text{Sing}(Q)$ is in the base locus of N . [Lemma 1.2.8](#) implies that D is a square. On the other hand, if P_1 contains at least one rank-2 quadric Q' different from Q , then $[Q']$ is a singular point on S . Any line L in H through $[Q']$ meets L_2, L_3 and L_4 in a point each. It follows that L is contained in S and thus that H is a component in S . This is impossible. Hence P_1 must coincide with one of the other pencils.

Next, we show that at most two of the pencils are of type II. If P_1 is of type II, then one of the planes in Q is the common tangent plane for the quadrics in P_1 . It follows that if three or more of the P_i are of type II, then there are two pencils, P_1 and P_2 , having the same common tangent plane H' . Choose coordinates such that H' is $\mathcal{V}(y_3)$ and such that H' is tangent to the quadrics in P_i along the line $\mathcal{V}(y_i, y_3)$, for $i = 1, 2$. Then the quadrics in P_1 contain no $y_0^2, y_0y_1, y_0y_2, y_1y_2$ or y_2^2 terms. Likewise, the quadrics in P_2 contain no $y_0^2, y_0y_1, y_0y_2, y_1y_2$ or y_1^2 terms. Hence the quadrics in N have no y_0^2, y_0y_1, y_0y_2 or y_1y_2

terms. Thus N is contained in the space of quadrics parametrised by the matrix

$$A := \begin{bmatrix} 0 & 0 & 0 & x_{03} \\ 0 & x_{11} & 0 & x_{13} \\ 0 & 0 & x_{22} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

The determinant is $\det(A) = -x_{03}^2 x_{11} x_{22}$. Hence the discriminant is not reduced. Moreover, note that the pencils of quadrics defined by $\mathcal{V}(x_{03})$, $\mathcal{V}(x_{11})$ and $\mathcal{V}(x_{22})$ each have a common singular point at $[1 : 0 : 0 : 0]$, $[x_{13} : -x_{03} : 0 : 0]$ and $[x_{23} : 0 : -x_{03} : 0]$, respectively. Thus they are all of type I, which is a contradiction.

The discriminant D consists of four lines, but the above paragraphs show that we can have at most three distinct pencils. In order to make sense of this, we consider pencils appearing with higher multiplicity. Consider the case where both P_1 and P_2 are of type II, whereas P_3 and P_4 coincide and are of type I. Let H_1 and H_2 be the common tangent planes of P_1 and P_2 , respectively. The base locus of P_1 consists of a double line $l_1 \subset H_1$ and a conic section $C_1 \subset H_2$. Similarly, the base locus of P_2 consists of a double line $l_2 \subset H_2$ and a conic section $C_2 \subset H_1$. Let $p' \in H_1 \cap H_2$ be the common singularity of P_3 . Since P_3 appears twice in D , [Lemma I.2.4](#) implies that p' is a base point for N . Let $p'' \in l_1 \cap C_2$ be an intersection point different from p' . Then p'' is a base point for N . The line L spanned by p' and p'' lies in H_1 , so it is tangent to all quadrics in P_1 . Since the quadrics in P_1 pass through both p' and p'' , they contain L . Moreover, L is in the base locus of P_3 . Thus L is in the base locus of N . [Lemma I.2.8](#) states that D is a square.

Finally, if a pencil of type II appears with higher multiplicity in D , then [Lemma I.2.4](#) implies that the base locus of N contains the line of singular points. By [Lemma I.2.8](#), D is a square. ■

Lemma I.7.4. *Let $S \subset \mathbb{P}^3$ be a general, irreducible quartic symmetroid with a tacnode p . Then $W(S)$ has two base points.*

Proof. By [Lemma I.7.3](#), the reduced tangent cone at p intersects S in two double lines. This corresponds to a net N of quadrics, where the discriminant D consists of two double lines, L_1 and L_2 . Because S is generic, the general quadric in each of these pencils has rank 3. [Lemma I.2.6](#) states that the quadrics along L_i have either a common singular point or the singular points form a line. By [Lemma I.2.4](#), L_i gives rise to a single base point or a line of base points for N , respectively.

Suppose that L_1 gives rise to a line of base points. Let Q be a rank-3 quadric that corresponds to a point $[Q] \in L_2$ not contained in L_1 . Let q be the singular point of Q , and let q' be one of the base points coming from L_1 . Then the line spanned by q and q' is contained in Q . It follows that the plane spanned by q and the line of base points, is contained in Q . This contradicts the assumption that Q has rank 3. We conclude that the quadrics along each line L_i have a common singularity.

Let q_i be the common singularity of the quadrics along L_i . Then the line L spanned by q_1 and q_2 , is contained in the base locus of N . Extend N with a quadric $Q' \notin N$, such that N and Q' span the web $W(S)$. Then Q' intersects L in two points. Hence $W(S)$ has two base points. $\blacksquare$

Proof of Proposition 1.7.2. Following Lemma 1.7.4, consider the $\mathbb{P}^7$ of quadrics that pass through the base points p_1 and p_2 . We shall describe the rank-2 locus of this space. It consists of two components, X_1 and X_2 . First, let H_{12} be a plane containing both p_1 and p_2 , and let H be any plane in $\mathbb{P}^3$. The union $H_{12} \cup H$ is a rank-2 quadric passing through the base points. The set $X_1 \subset \mathbb{P}^7$ of all such unions is a fourfold of degree 4. Next, let H_1 be a plane containing p_1 , and H_2 a plane containing p_2 . The union $H_1 \cup H_2$ is a rank-2 quadric passing through the base points. The set $X_2 \subset \mathbb{P}^7$ of all such unions is a fourfold of degree 6.

Let H_{12} and H'_{12} be two planes that both contain p_1 and p_2 . Then both X_1 and X_2 are singular at the point $[H_{12} \cup H'_{12}]$. From the proof of Lemma 1.7.4, it is clear that p_1 and p_2 are contained in the singular locus of the quadric associated to the tacnode p . Thus this quadric consists of two planes that both contain the base points. Hence, $W(S) \subset \mathbb{P}^7$ intersects $\text{Sing}(X_1) \cap \text{Sing}(X_2)$.

Choose coordinates such that $p_1 := [1 : 0 : 0 : 0]$ and $p_2 := [0 : 1 : 0 : 0]$. A general plane H is given by $a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and a plane H_{12} through p_1 and p_2 is given by $b_2x_2 + b_3x_3 = 0$. Consider the Segre embedding $\sigma_{3,1} : \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P}^7$ given by

$$([a_0 : a_1 : a_2 : a_3], [b_2 : b_3]) \mapsto [a_0b_2 : a_1b_2 : a_2b_2 : a_3b_2 : a_0b_3 : a_1b_3 : a_2b_3 : a_3b_3].$$

Let $\mathbb{P}^7$ have coordinates $[x_{02} : x_{12} : x_{22} : x_{32} : x_{03} : x_{13} : x_{23} : x_{33}]$. The image $\Sigma_{3,1}$ of $\sigma_{3,1}$ is then given by the (2×2) -minors of the matrix

$$M := \begin{bmatrix} x_{02} & x_{12} & x_{22} & x_{32} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

We can expand M into a (4×4) -matrix in the following manner:

$$\begin{aligned} \begin{bmatrix} x_{02} & x_{12} & x_{22} & x_{32} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix} &\rightsquigarrow \begin{bmatrix} x_{02} & x_{12} & x_{22} & x_{32} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix} \\ &\rightsquigarrow \begin{bmatrix} 0 & 0 & x_{02} & x_{03} \\ 0 & 0 & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & x_{32} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix} \\ &=: A. \end{aligned}$$

Let A' be defined as A with $x_{23} = x_{32}$. Then the (3×3) -minors of A' define X_1 . We deduce that X_1 is the Segre variety $\Sigma_{3,1}$ projected down to $\mathbb{P}^6$.

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We see from A' that the base locus of the $\mathbb{P}^6$ spanned by X_1 , contains a line. By [Lemma I.2.7](#), we may therefore assume that $W(S)$ is not contained in this $\mathbb{P}^6$. Let W' be the plane defined as the intersection of $W(S)$ and the hyperplane spanned by X_1 . The web $W(S)$ is a generic 3-space that is such that W' meets $\text{Sing}(X_1)$ in a point p . The fourfold X_1 is singular precisely at points that correspond to the union of two planes that both contain the base points p_1 and p_2 . Set-theoretically, $\text{Sing}(X_1)$ is given by

$$x_{02} = x_{03} = x_{12} = x_{13} = 0. \quad (\text{I.7.4})$$

The only minor of M that survives under the relations (I.7.4) is

$$\begin{vmatrix} x_{22} & x_{32} \\ x_{23} & x_{33} \end{vmatrix} = x_{22}x_{33} - x_{23}x_{32}.$$

Let $Q \subset \mathbb{P}^7$ be the quadric defined by this minor. The intersection of Q and the linear space V given by (I.7.4) is mapped two-to-one onto $\text{Sing}(X_1)$ under the projection $\mathbb{P}^7 \rightarrow \mathbb{P}^6$. Let $\widetilde{W}' \subset \mathbb{P}^7$ be the 3-space lying over W' . Since the degree of $\Sigma_{3,1}$ is 4, Bézout's theorem implies that $\widetilde{W}'$ meets $\Sigma_{3,1}$ in four points. The intersection $\widetilde{W}' \cap V$ is a line L lying over p . The quadric Q intersects L in two points. Therefore, $\widetilde{W}'$ meets $\Sigma_{3,1}$ in two points outside of V . It follows that W' meets X_1 in two points outside of p . Thus the same is true for $W(S)$.

An analogous argument for X_2 , considering the Segre variety $\Sigma_{2,2} \subset \mathbb{P}^8$, shows that $W(S)$ generically intersects X_2 in four points outside of p . In total, $W(S)$ contains in general six rank-2 points in addition to p . ■

Remark I.7.5. The reader may wonder why we in the proof of [Proposition I.7.2](#) argue via Segre varieties, instead of using Bézout's theorem directly on X_1 and X_2 . The reason is that X_1 and X_2 are not normal. Using the description of X_1 as the vanishing of the (3×3) -minors of A' , we calculate that $\text{Sing}(X_1)$ is given as

$$\mathcal{V}(x_{02}^2, x_{02}x_{03}, x_{02}x_{12}, x_{02}x_{13}, x_{03}^2, x_{03}x_{12}, x_{03}x_{13}, x_{12}^2, x_{12}x_{13}, x_{13}^2)$$

in the $\mathbb{P}^6$ spanned by X_1 . Thus $\text{Sing}(X_1)$ is the whole first-order infinitesimal neighbourhood of the plane $\mathcal{V}(x_{02}, x_{03}, x_{12}, x_{13})$. This has codimension 4, hence its degree is 5. The intersection $W' \cap \text{Sing}(X_1)$ contains the whole first-order infinitesimal neighbourhood of p in W' . This has codimension 2 in W' . Hence p appears in $W(S) \cap X_1$ with multiplicity 3, not 2. ♠

We can now determine the number of symmetroids with a tacnode:

Proposition I.7.6. *The family of quartic symmetroids in $\mathbb{P}^3$ with a tacnode is 20-dimensional.*

Proof. As in the proof of [Proposition I.7.2](#), given two base points p_1 and p_2 , a symmetroid with a tacnode corresponds to a $\mathbb{P}^3 \subset \mathbb{P}^7$ that intersects the plane $\text{Sing}(X_1) \cap \text{Sing}(X_2)$ in a point. It follows from (I.2.3) that there is a 14-dimensional family of such 3-spaces. A choice of base points p_1 and p_2 corresponds to a point in the 6-dimensional space $\mathbb{P}^3 \times \mathbb{P}^3$. Hence the family of quartic symmetroids with a tacnode has in total dimension $14 + 6 = 20$. ■

I.7.2 Types $S_4^{(2)}$ and $S_4^{(3)}$

Hitherto, we have primarily argued in terms of the associated linear system of quadrics, but now we turn to ramification. We specify a property of the cubics in [Theorem I.2.1](#). The idea is taken from the proof of [\[Ott+15, Theorem 1.2\]](#):

Lemma I.7.7. *Let $S := \mathcal{V}(F_2x_0^2 + F_3x_0 + F_4) \subset \mathbb{P}^3$ be an irreducible quartic symmetroid with a rank-2 point at $p := [1 : 0 : 0 : 0]$, where F_d is a form of degree d in $\mathbb{C}[x_1, x_2, x_3]$. The ramification locus R_p splits into two cubics $\mathcal{V}(r_1)$ and $\mathcal{V}(r_2)$. Then $\mathcal{V}(r_1, F_4) = 2Z$ is two times a scheme Z of length 6, and Z is not contained in a conic section.*

Proof. After conjugating with an appropriate matrix, we may assume that the matrix defining S is

$$\begin{bmatrix} l_{00} & x_0 + l_{01} & l_{02} & l_{03} \\ x_0 + l_{01} & l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & l_{33} \end{bmatrix},$$

where the l_{ij} are linear forms in $\mathbb{C}[x_1, x_2, x_3]$. Then F_4 is the determinant of

$$M := \begin{bmatrix} l_{00} & l_{01} & l_{02} & l_{03} \\ l_{01} & l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & l_{33} \end{bmatrix}.$$

Also, r_1 is the $(0,0)$ -minor and r_2 the $(1,1)$ -minor of M .

The fact that $\mathcal{V}(r_1, F_4)$ is a double scheme $2Z$ can be seen from the identity $r_1r_2 = F_3^2 - 4F_2F_4$, which shows that F_2F_4 is a square modulo r_1 . We claim that Z is equal to the scheme Z' given by the (3×3) -minors of the submatrix

$$A := \begin{bmatrix} l_{01} & l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & l_{33} \end{bmatrix}.$$

Clearly, $Z' \subseteq \mathcal{V}(r_1, F_4)$. By the symmetry of M , it follows that $2Z' \subseteq \mathcal{V}(r_1, F_4)$. Equality follows by considering degrees. Thus $Z = Z'$.

Assume for contradiction that Z is contained in a conic section. Then $Z = \mathcal{V}(q, g)$ for a quadratic form q and a cubic form g . The vector space of cubics vanishing on Z is $\langle x_1q, x_2q, x_3q, g \rangle$. Considering the syzygies between the generators, we construct the Hilbert–Burch matrix

$$A' := \begin{bmatrix} x_2 & -x_1 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 \end{bmatrix}.$$

The (3×3) -minors of A' are identically zero. The matrix A is row equivalent with A' , so $F_4 = \det(M) = 0$. This contradicts the assumption that S is irreducible. ■

We show that if surfaces of type $S_4^{(2)}$ or $S_4^{(3)}$ satisfy the conclusions of [Theorem I.2.1](#) and [Lemma I.7.7](#), then they degenerate to type $S_4^{(1)}$.

Proposition I.7.8. *Let S be a rational quartic surface of type $S_4^{(2)}$. Then S is not a symmetroid.*

Proof. We may assume that the equation defining S is as in [\(I.7.2\)](#). The branch locus R for the projection from the elliptic double point p is then given by

$$x_1(x_1(B_1^2 - 4B_2) + 4A_3)x_3^2 + x_1(2A_3B_1 - 4x_1B_3)x_3 + A_3^2 - 4x_1^2B_4. \quad (\text{I.7.5})$$

The sextic curve R has a quadruple point at $q := [0 : 0 : 1]$.

The following observation is used repeatedly throughout the proof: From [\(I.7.1\)](#) and [\(I.7.2\)](#), we see that if x_1 divides A_3 , then S has a tacnode at p . This contradicts the assumption that S is of type $S_4^{(2)}$. By [\(I.7.5\)](#), this is equivalent to $\mathcal{V}(x_1)$ being a component of R . Assume that x_1 does not divide A_3 .

Assume for contradiction that S is a symmetroid. [Lemma I.7.1](#) and [Theorem I.2.1](#) imply that R splits into two cubics, R_1 and R_2 . Since R has a quadruple point at q , there are two possibilities: Either both R_1 and R_2 have double points at q , or R_1 has a triple point and R_2 passes only once through q .

Case I – R_1 and R_2 have double points at q .

The equations for R_1 and R_2 can respectively be written as $r_1 := C_2x_3 + C_3$ and $r_2 := D_2x_3 + D_3$, where C_d and D_d are forms of degree d in $\mathbb{C}[x_1, x_2]$. Since $R_1 \cup R_2 = R$, we have that r_1r_2 equals [\(I.7.5\)](#). Equating the coefficients of the x_3^3 terms of r_1r_2 and [\(I.7.5\)](#), produces the following system of equations:

$$C_2D_2 = x_1(x_1(B_1^2 - 4B_2) + 4A_3), \quad (\text{I.7.6})$$

$$C_2D_3 + C_3D_2 = x_1(2A_3B_1 - 4x_1B_3), \quad (\text{I.7.7})$$

$$C_3D_3 = A_3^2 - 4x_1^2B_4.$$

From [\(I.7.6\)](#), we get that x_1 is a factor in either C_2 or D_2 . Suppose that x_1 divides C_2 . Inserting this into [\(I.7.7\)](#), we get that x_1 divides either C_3 or D_2 . Suppose first that x_1 is a factor in C_3 . Then $\mathcal{V}(x_1)$ is a component of R_1 , so $\mathcal{V}(x_1)$ is a component of R , which implies that S has a tacnode at p . Now suppose that x_1 is a factor in D_2 . Inserting this back into [\(I.7.6\)](#), we find that x_1 divides A_3 . Hence S has a tacnode at p .

Case II – R_1 has a triple point and R_2 has a single point at q .

The equations for R_1 and R_2 can be written as $r_1 := C_1x_3^2 + C_2x_3 + C_3$ and $r_2 := D_3$, respectively, where C_d and D_d are forms of degree d in $\mathbb{C}[x_1, x_2]$. We can assume that x_1 is not a factor in D_3 , as that would imply that $\mathcal{V}(x_1)$ is a component of R . Equating the coefficients of the x_3^3 terms of r_1r_2 and [\(I.7.5\)](#), produces the following system of equations:

$$C_1D_3 = x_1(x_1(B_1^2 - 4B_2) + 4A_3), \quad (\text{I.7.8})$$

$$C_2 D_3 = x_1(2A_3 B_1 - 4x_1 B_3), \quad (\text{I.7.9})$$

$$C_3 D_3 = A_3^2 - 4x_1^2 B_4. \quad (\text{I.7.10})$$

Since x_1 does not divide D_3 , we get from (I.7.8) that, up to scalar, $C_1 = x_1$ and $D_3 = x_1(B_1^2 - 4B_2) + 4A_3$. Similarly, (I.7.9) yields that $C_2 = x_1 C'_1$ for some linear form $C'_1 \in \mathbb{C}[x_1, x_2]$. Substituting for C_2 and D_3 into (I.7.9) and cancelling x_1 , we get

$$(x_1(B_1^2 - 4B_2) + 4A_3)C'_1 = 2A_3 B_1 - 4x_1 B_3,$$

which is equivalent to

$$x_1((B_1^2 - 4B_2)C'_1 + 4B_3) = 2A_3(B_1 - 2C'_1). \quad (\text{I.7.11})$$

Either A_3 or $B_1 - 2C'_1$ is divisible by x_1 , but we have assumed the former to be incorrect. Therefore, $C'_1 = \frac{1}{2}B_1 - ax_1$ for some $a \in \mathbb{C}$. Putting this back into (I.7.11) and cancelling x_1 , we obtain

$$(B_1^2 - 4B_2)\left(\frac{1}{2}B_1 - ax_1\right) + 4B_3 = 4aA_3,$$

so

$$B_3 = aA_3 + \frac{1}{4}(B_1^2 - 4B_2)\left(ax_1 - \frac{1}{2}B_1\right).$$

Inserting the expression we found for D_3 into (I.7.10) gives

$$(x_1(B_1^2 - 4B_2) + 4A_3)C_3 = A_3^2 - 4x_1^2 B_4,$$

which can be transformed to

$$x_1((B_1^2 - 4B_2)C_3 + 4x_1 B_4) = A_3(A_3 - 4C_3). \quad (\text{I.7.12})$$

Because x_1 is not a factor in A_3 , we conclude that $A_3 - 4C_3$ is divisible by x_1 . Thus, $C_3 = \frac{1}{4}A_3 - x_1 C'_2$ for some quadratic form $C'_2 \in \mathbb{C}[x_1, x_2]$. Putting this into (I.7.12) and cancelling x_1 , we get

$$(B_1^2 - 4B_2)\left(\frac{1}{4}A_3 - x_1 C'_2\right) + 4x_1 B_4 = 4A_3 C'_2,$$

which is equivalent to

$$x_1(4B_4 - (B_1^2 - 4B_2)C'_2) = A_3\left(4C'_2 - \frac{1}{4}(B_1^2 - 4B_2)\right). \quad (\text{I.7.13})$$

Again, A_3 is not divisible by x_1 , so x_1 is a factor in $4C'_2 - \frac{1}{4}(B_1^2 - 4B_2)$. Hence $C'_2 = \frac{1}{16}(B_1^2 - 4B_2) + x_1 C''_1$ for some linear form $C''_1 \in \mathbb{C}[x_1, x_2]$. Substituting for C'_2 in (I.7.13) and cancelling x_1 , we obtain

$$4B_4 - (B_1^2 - 4B_2)\left(\frac{1}{16}(B_1^2 - 4B_2) + x_1 C''_1\right) = 4A_3 C''_1,$$

so

$$B_4 = A_3 C''_1 + \frac{1}{64}(B_1^2 - 4B_2)^2 + \frac{1}{4}x_1(B_1^2 - 4B_2)C''_1.$$

I. Rational quartic symmetroids

We have now described the necessary conditions on B_3 and B_4 for R to split into cubics.

Using the above expressions for r_1 , r_2 , B_3 and B_4 , we can verify the relations

$$\begin{aligned} (8x_3^2 - 4(4ax_1 - B_1)x_3 - 16x_1C_1 - (B_1^2 - 4B_2))^2 &= 64f_4 - 256(ax_3 + C_1'')r_1, \\ (8x_3^2 + 4B_1x_3 - (B_1^2 - 4B_2))^2 &= 64f_4 - 16(ax_3 + C_1'')r_2, \end{aligned}$$

where $f_4 := x_3^4 + B_1x_3^3 + B_2x_3^2 + B_3x_3 + B_4$. Hence for $i = 1, 2$, we have that $\mathcal{V}(r_i, f_4) = 2Z_i$ for a scheme Z_i that is contained in a conic. This contradicts [Lemma I.7.7](#), showing that S is not a symmetroid. ■

Proposition I.7.9. *Let S be a rational quartic surface of type $S_4^{(3)}$. Then S is not a symmetroid.*

Proof. We may assume that the equation defining S is as in [\(I.7.3\)](#). The branch locus R for the projection from the elliptic double point p is then given by

$$x_1^3x_3^3 + x_1^2(A_1^2 - B_2)x_3^2 + x_1(2A_1A_3 - x_1B_3)x_3 + A_3^2 - x_1^2B_4.$$

The sextic curve R has two consecutive triple points — a triple point with an infinitely near triple point — at the point $q := [0 : 0 : 1]$, with tangent direction $x_1 = 0$. The tangent direction of R at q , corresponds to the double plane which is the tangent cone of S at p .

Assume for contradiction that S is a symmetroid. [Lemma I.7.1](#) and [Theorem I.2.1](#) imply that R splits into two cubics, R_1 and R_2 . This leaves two possibilities for consecutive triple points. One possibility is that R_2 does not pass through q , and R_1 is a triple line corresponding to the tangent direction at q . The second possibility is that R_1 breaks up into a line L and a conic C , such that L , C and R_2 all have the same tangent direction at q . In either case, the line corresponding to the tangent direction at q , is a component in R .

Writing the equation for S as $x_1^2x_0^2 + F_3x_0 + F_4$, as in [\(I.2.1\)](#), then the equation for R becomes $F_3^2 - 4x_1^2F_4$. Since the line $\mathcal{V}(x_1)$ is a component of R , then x_1 is a factor in F_3 . By [\(I.7.1\)](#), we see that S has a tacnode at p . This contradicts the assumption that S is of type $S_4^{(3)}$. ■

I.8 Proof of Theorem I.1.1

By Noether's classification in [\[Noe89\]](#), we know that the only rational quartic surfaces are those with either a double curve, a triple point or an elliptic double point of type $S_4^{(1)}$, $S_4^{(2)}$ or $S_4^{(3)}$. Since we are dealing with quartic surfaces that are irreducible, we need only consider double curves of degree up to 3. [Propositions I.4.1, I.5.1, I.7.8 and I.7.9](#) show that there are no other possible rational quartic symmetroids than the ones listed in the theorem.

The claims about the general number of additional singularities are covered by [Propositions I.3.3, I.3.6, I.4.5, I.6.1 and I.7.2](#). The dimensions of the different families are calculated in [Propositions I.3.4, I.3.5, I.4.6, I.6.2 and I.7.6](#). ■

I.9 Quartic symmetroids with two double intersecting lines

The case of a surface $S := \mathcal{V}(F)$ with two double lines, L_1 and L_2 , is an interesting example, so we treat it with special care. Because of the following proposition, we are only interested in $L_1 \cap L_2 \neq \emptyset$:

Proposition I.9.1. *Let $S \subset \mathbb{P}^3$ be a quartic surface with two double, skew lines L_1 and L_2 . Then S is not rational.*

Proof. Let $p \in \mathbb{P}^3$ be a point outside of L_1 and L_2 . Let H be a plane that does not contain p . Let L'_i be the projection of L_i from p to H . The lines L'_1 and L'_2 meet in a point, which corresponds to a line L through p that intersects both L_1 and L_2 . If $p \in S$, then Bézout's theorem implies that $L \subset S$, since L_1 and L_2 are skew and double. It follows that S is a scroll.

Let H be a plane that does not contain any of the lines in S . Then H intersects each line in S in a point. The hyperplane section $C := H \cap S$ is a plane quartic curve with two double points. The curve C has genus 1, so S has genus 1. This proves the claim. $\blacksquare$

Remark I.9.2. For our purposes, we may therefore assume that $L_1 := \mathcal{V}(x_1, x_2)$ and $L_2 := \mathcal{V}(x_1, x_3)$. Since S is singular along these, the terms in F have either x_1^2 , $x_1x_2x_3$ or $x_2^2x_3^2$ as a factor. It follows that F satisfies the equation (I.7.1) of a tacnodal surface. However, a tacnode is defined as an isolated singularity. We may regard S as a degeneration of tacnodal surfaces. In addition, S is a degeneration of surfaces with one double line and a degeneration of surfaces with a double smooth conic section. $\spadesuit$

Next, we prove similar results to [Theorem I.1.1](#) for the different possible ranks:

Proposition I.9.3. *Let $S \subset \mathbb{P}^3$ be a generic, irreducible quartic symmetroid that is singular along two intersecting lines, L_1 and L_2 , of rank-3 points. Then S has two isolated rank-2 nodes.*

Proof. By [Lemmas I.2.4](#) and [I.2.7](#), the quadrics along each line L_i have a common singular point. [Lemma I.2.6](#) states that the rank-2 locus along each line has length 3.

As noted in [Remark I.9.2](#), the equation for S satisfies (I.7.1). It follows that all the arguments in the proof of [Proposition I.7.2](#) hold here as well, showing that the rank-2 locus is of length 6 outside of $p := L_1 \cap L_2$. Moreover, p is counted with multiplicity 2 in each of the rank-2 loci of the pencils defined by the L_i . Hence each L_i contains only one additional rank-2 point p_i .

The proof of [Proposition I.3.6](#) shows that p_i is counted with multiplicity 2 in the rank-2 locus of S . Thus the rank-2 locus has length 2 outside of L_1 and L_2 , which proves the claim. $\blacksquare$

Proposition I.9.4. *Let $S \subset \mathbb{P}^3$ be a generic, irreducible quartic symmetroid with two intersecting lines, L_1 and L_2 , of rank-2 points. Then S has four additional, isolated rank-2 points.*

Proof. Let P_i be the pencil of quadrics that corresponds to the line L_i . By [Lemma I.2.5](#), the base locus of P_i consists of a plane H_i and a line $l_i \not\subset H_i$. The associated quadric at the point $L_1 \cap L_2$ is $H_1 \cup H_2$. It follows that $l_1 \subset H_2$ and $l_2 \subset H_1$. Thus the base locus of the net N spanned by P_1 and P_2 , consists of the three lines l_1 , l_2 and $H_1 \cap H_2$. The web $W(S)$ is the net N extended with another quadric $Q \notin N$. Then Q intersects the lines l_1 , l_2 and $H_1 \cap H_2$ in two points each. Hence the base locus of $W(S)$ consists of six points. The symmetroid S is therefore uniquely determined by the base locus of $W(S)$.

Choose coordinates such that $H_1 := \mathcal{V}(x_2)$, $H_2 := \mathcal{V}(x_3)$, $l_1 := \mathcal{V}(x_0, x_3)$ and $l_2 := \mathcal{V}(x_1, x_2)$. We can assume that the six base points are

$$\begin{aligned} [0 : 1 : 1 : 0], [0 : 1 : -1 : 0] &\in l_1, \\ [1 : 0 : 0 : 1], [1 : 0 : 0 : -1] &\in l_2, \\ [1 : 1 : 0 : 0], [1 : -1 : 0 : 0] &\in H_1 \cap H_2. \end{aligned}$$

The quadrics passing through these base points are parametrised by the matrix

$$M := \begin{bmatrix} x_{00} & 0 & x_{02} & 0 \\ 0 & -x_{00} & 0 & x_{13} \\ x_{02} & 0 & x_{00} & x_{23} \\ 0 & x_{13} & x_{23} & -x_{00} \end{bmatrix}.$$

The symmetroid defined by M has four isolated rank-2 points in addition to the lines L_1 and L_2 . ■

Proposition I.9.5. *Let $S \subset \mathbb{P}^3$ be a generic, irreducible quartic symmetroid that is singular along two intersecting lines, L_1 and L_2 . Suppose that the points along L_1 are generically rank-3 points, and that L_2 consists of rank-2 points. Then S has two isolated rank-2 nodes.*

Proof. By [Proposition I.3.3](#), the rank-2 locus of S has length 6 outside of L_2 . As in the proof of [Proposition I.3.6](#), the two rank-2 points in $L_1 \setminus L_2$ are counted with multiplicity 2 each in the rank-2 locus. Hence there are two rank-2 points outside of L_1 and L_2 . ■

In the case of [Proposition I.9.5](#), the rank along the lines depends on the matrix representation ([I.1.1](#)):

Example I.9.6. The symmetroid S defined by the matrix

$$A_1 := \begin{bmatrix} 0 & x_0 & 4x_1 & 2x_2 \\ x_0 & 4x_3 & 2x_1 - 2x_3 & 0 \\ 4x_1 & 2x_1 - 2x_3 & -4x_1 & -x_2 \\ 2x_2 & 0 & -x_2 & x_3 \end{bmatrix}$$

is singular along two lines, L_1 and L_2 , and it has four isolated nodes, p_1, p_2, p_3 and p_4 . Only L_1, p_1, p_2 and two points on $L_2 \setminus L_1$ are contained in the rank-2 locus of S .

The matrix

$$A_2 := \begin{bmatrix} 0 & x_0 - 8x_3 & 4x_3 & 2x_2 \\ x_0 - 8x_3 & 4x_1 + 8x_3 & -2x_1 - 6x_3 & -2x_2 \\ 4x_3 & -2x_1 - 6x_3 & 4x_3 & x_2 \\ 2x_2 & -2x_2 & x_2 & -x_1 \end{bmatrix}$$

has the same determinant as A_1 . The matrices are not conjugates of each other, which can be verified by evaluating A_1 and A_2 at a point and checking that they have different eigenvalues. With the representation given by A_2 , the rank-2 locus of S equals L_2, p_1, p_3 and two points on $L_1 \setminus L_2$. $\blacklozenge$

I.10 Examples

We end with a few examples, including demonstrations showing that Plücker's surface and the Steiner surface are symmetroids.

Example I.10.1. Let $S \subset \mathbb{P}^3$ be a quartic symmetroid with a tacnode at the point $p := [1 : 0 : 0 : 0]$. By [Lemma I.7.1](#), we may assume that the matrix defining S is on the form

$$A := \begin{bmatrix} a_0x_0 + l_{00} & l_{01} & l_{02} & l_{03} \\ l_{01} & a_1x_0 + l_{11} & l_{12} & l_{13} \\ l_{02} & l_{12} & l_{22} & l_{23} \\ l_{03} & l_{13} & l_{23} & l_{33} \end{bmatrix},$$

where $a_0, a_1 \in \mathbb{C} \setminus \{0\}$ and the l_{ij} are linear forms in $\mathbb{C}[x_1, x_2, x_3]$. The tangent cone of S at p is given by

$$\begin{vmatrix} l_{22} & l_{23} \\ l_{23} & l_{33} \end{vmatrix} = l_{22}l_{33} - l_{23}^2. \quad (\text{I.10.1})$$

Since p is an elliptic double point, (I.10.1) is a square. If l_{22}, l_{23} and l_{33} are scalar multiples of each other, not all zero, then $\det(A)$ is on the form (I.7.1). In general, A does not define a tacnode for other ways of realising (I.10.1) as a square.

Concretely, the matrix

$$\begin{bmatrix} x_0 + x_1 & x_2 & x_3 & x_1 \\ x_2 & -x_0 + x_1 & x_2 & x_2 \\ x_3 & x_2 & x_1 & 0 \\ x_1 & x_2 & 0 & x_1 \end{bmatrix}$$

defines a symmetroid with a tacnode and six additional nodes. $\blacklozenge$

Example I.10.2 (Plücker's surface). The maximal number of isolated nodes on a quartic surface with a double line is eight. The surface S satisfying that

description is known as *Plücker's surface* in the classical literature; see [Jes16, Article 83]. It is represented as a symmetroid by the matrix

$$\begin{bmatrix} 0 & x_0 - x_1 + x_2 & x_0 - x_1 + x_3 & x_0 \\ x_0 - x_1 + x_2 & 0 & x_3 & x_1 \\ x_0 - x_1 + x_3 & x_3 & 0 & x_2 \\ x_0 & x_1 & x_2 & 0 \end{bmatrix}.$$

The double line and six of the nodes are contained in the rank-2 locus.

In [Ott+15, Remark 5.4], the Kummer surfaces are given the following interpretation: They are the nodal symmetroids where the associated web of quadrics contains a net that defines a twisted cubic curve. Hence the Kummer surfaces have six general base points. The base locus of $W(S)$ consists of six points, four of which are coplanar. Plücker's surface is therefore a degeneration of Kummer surfaces. $\blacklozenge$

Example I.10.3 (Steiner surface). The Veronese surface $V \subset \mathbb{P}^5$ can be identified with the rank-1 locus of the symmetroid S defined by the matrix

$$\begin{bmatrix} x_{00} & x_{01} & x_{02} \\ x_{01} & x_{11} & x_{12} \\ x_{02} & x_{12} & x_{22} \end{bmatrix}.$$

Moreover, S is the secant variety of V ; see [Dol12, Section 2.1.1].

The general projection R of V to $\mathbb{P}^3$ is known as the *Roman surface* or the *Steiner surface*. It is an irreducible surface with three concurrent double lines, L_1 , L_2 and L_3 . They meet in a triple point p . After a suitable choice of coordinates, we may assume that R is given by [Dol12, Equation (2.1)], which is

$$x_0x_1x_2x_3 + x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2 = 0.$$

The determinant of the matrix

$$A := \begin{bmatrix} 0 & x_0 & x_1 & x_0 \\ x_0 & 0 & -2x_2 & 4x_0 + 2x_2 \\ x_1 & -2x_2 & 0 & 2x_1 \\ x_0 & 4x_0 + 2x_2 & 2x_1 & 6x_0 + 2x_1 + 2x_2 - x_3 \end{bmatrix}$$

is

$$\det(A) = 4(x_0x_1x_2x_3 + x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2),$$

so the Steiner surface is a symmetroid. The triple point p is a rank-1 point. There is also a single rank-2 point, with multiplicity 2, on each line L_i .

The base locus of $W(R)$ is a scheme of length 6. It consists of three points and a direction through each. As with Plücker's surface in Example I.10.2, the Steiner surface is a degeneration of Kummer surfaces. $\blacklozenge$

Example I.10.4. The symmetroid S defined by

$$\begin{bmatrix} 0 & x_0 & x_1 & x_0 \\ x_0 & 0 & 2x_2 - x_3 & x_2 \\ x_1 & 2x_2 - x_3 & 0 & x_0 \\ x_0 & x_2 & x_0 & x_3 \end{bmatrix}$$

is double along a smooth conic section C and a line L . The rank-2 locus consists of C and a point $p \in L \setminus C$. The base locus of $W(S)$ is a scheme of length 4 with support in three points. $\blacklozenge$

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Appendix I.A Pictures of rational quartic symmetroids

These pictures are not included in the published paper.

Figure I.1: General quartic symmetroids with a double line. The surface in (a) has four real nodes, and the surface in (b) has six real nodes.

Figure I.2: A general quartic symmetroid with a double smooth conic section. The surface has four real nodes.

Figure I.3: A general quartic symmetroid with a triple point, which is the central singularity. The surface has six real nodes.

Figure I.4: A general quartic symmetroid with a tacnode. The figure to the right shows three plane sections meeting at the tacnode. The surface has six real nodes.

Figure I.5: The Steiner surface.

Paper II

Rational quartic spectrahedra

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Abstract

Rational quartic spectrahedra in 3-space are semialgebraic convex subsets in $\mathbb{R}^3$ of semidefinite, real symmetric (4×4) -matrices, whose boundary admits a rational parameterization. The Zariski closure in $\mathbb{CP}^3$ of the boundary of a rational spectrahedron is a rational complex symmetroid. We give necessary conditions on the configurations of singularities of the corresponding real symmetroids in $\mathbb{RP}^3$ of rational quartic spectrahedra. We provide an almost exhaustive list of examples realizing the configurations, and conjecture that the missing example does not occur.

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II.1 Introduction

Spectrahedra are important basic objects in polynomial optimization and in convex algebraic geometry [BPT13]. They are intersections of the cone of positive-semidefinite matrices in the space of real symmetric $(n \times n)$ -matrices by an affine

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subspace. Quartic spectrahedra are the case of (4×4) -matrices intersected with a 3-dimensional affine space that contains a positive definite matrix. We identify the affine space with $\mathbb{R}^3$. The boundary of a quartic spectrahedron has a Zariski closure $V(f_A) \subset \mathbb{RP}^3$ defined by the determinant $f_A(x) := f_A(x_0, x_1, x_2, x_3)$ of a symmetric matrix $A(x)$, where explicitly

$$A(x) := A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3, \quad (\text{II.1.1})$$

and each A_i is a real symmetric (4×4) -matrix. Since the matrix $A(x)$ is symmetric, the surface $V(f_A)$ is called a *(real) symmetroid*. Similarly, the complex algebraic boundary $V_{\mathbb{C}}(f_A) \subset \mathbb{CP}^3$ defined by f_A is called a *complex symmetroid* to distinguish it from its real points $V(f_A)$. We say that a real symmetroid $V(f_A)$ is *spectrahedral* if it is the Zariski closure of the boundary of a nonempty spectrahedron, i.e., if $A(x_p)$ is definite for some $x_p \in \mathbb{RP}^3$. If the complex algebraic boundary of a quartic spectrahedron is a rational complex symmetroid, we say that the spectrahedron is *rational*. The general quartic symmetroid is not rational.

For a general matrix $A(x)$, the singular points of $V(f_A)$ is a finite set of double points, quadratic singularities called *nodes*. The possible arrangements of the nodes of general quartic spectrahedral symmetroids were identified by Degtyarev and Itenberg [DI11] and further investigated by Ottem et al. in [Ott+15]. In [Ott+15], the authors describe their paper as a “first step towards the classification of all spectrahedra of a given degree and dimension”. A quartic surface with a finite set of nodes, or more generally rational double points, is irrational, in fact birational to a K3-surface. This paper sets out to fill in part of the classification of quartic spectrahedra in 3-space by studying the rational members.

The significant feature of a rational spectrahedron is that its boundary allows a parameterization of rational functions from a subset $X \subset \mathbb{CP}^2$. This is immediate from the fact that the Zariski closure in $\mathbb{CP}^3$ of the boundary allows a parameterization of rational functions from $\mathbb{CP}^2$. We do not provide any explicit boundary parameterizations. *A priori*, X does not have to be a subset of $\mathbb{RP}^2$ even though the boundary of the spectrahedron is real.

A quartic surface is rational only if it has a triple point, an elliptic double point or is singular along a curve cf. [Jes16; Noe89]. The first author identified families of rational quartic symmetroids in [Hel20]. We state the results of that paper, after making a note about ranks and quadrics.

At every point $x_p \in V(f_A) \subset \mathbb{CP}^3$, the matrix $A(x_p)$ has rank at most 3. We say that $x_p \in V(f_A)$ is a *rank- k point*, if $A(x_p)$ has rank k . The symmetroid $V(f_A)$ has a double point at each rank-2 point, and a triple point at each rank-1 point. It may, however, also be singular at rank-3 points. This phenomenon is characterized in Lemma II.1.1 by properties of the web of quadrics associated to the symmetroid:

If $y := (y_0, y_1, y_2, y_3)$, then $q_A(x_p) := y \cdot A(x_p) \cdot y^T$ is a quadratic form, and its vanishing $Q_A(x_p) := V(q_A(x_p)) \subset \mathbb{CP}_y^3$ is a quadric surface. The set $Q_A(x) := \{Q_A(x_p) \mid x_p \in \mathbb{CP}^3\}$ is called a *web of quadrics*.

Lemma II.1.1 ([Ili+17, Lemma 2.13], [Wal81, Lemma 1.1]). *The symmetroid $V_{\mathbb{C}}(f_A)$ has a singularity at a rank-3 point x_p if and only if the web of quadrics $Q_A(x)$ has a basepoint at the singular point of $Q_A(x_p)$.*

The main results of [Hel20] can be summarized as follows:

Theorem II.1.2 ([Hel20]). *The rational complex quartic symmetroids in $\mathbb{CP}^3$ form irreducible families whose general members are of the following types:*

- A. $V_{\mathbb{C}}(f_A)$ has a triple point and six additional nodes;
- B. $V_{\mathbb{C}}(f_A)$ has a tacnode and six additional nodes, and the web $Q_A(x)$ has two basepoints;
- C. $V_{\mathbb{C}}(f_A)$ is singular along a conic and has four additional nodes, and the web $Q_A(x)$ has four linearly independent basepoints;
- D. $V_{\mathbb{C}}(f_A)$ has rank 2 along a line and has six additional nodes, and the web $Q_A(x)$ has four coplanar basepoints;
- E. $V_{\mathbb{C}}(f_A)$ is singular of rank 3 along a line and has four additional nodes, and the web $Q_A(x)$ has one basepoint.

Remark II.1.3. For the general members in the families mentioned in Theorem II.1.2, i.e., symmetroids of types A, B, C, D, E, all the additional nodes are rank-2 points and they are isolated. Nongeneral members can have more nodes that are rank-3 points, or some of the singularities may coincide. In this paper an isolated node will always be a rank-2 point. ♠

Remark II.1.4. No symmetroids of type E of Theorem II.1.2 are spectrahedral. A consequence of Lemma II.1.1 is that singular rank-3 points appear in complex conjugate pairs on spectrahedral symmetroids. A spectrahedral symmetroid which is singular of rank 3 along a line, is therefore singular along two lines. ♠

Proposition II.1.5. *The complex quartic symmetroids in $\mathbb{CP}^3$ singular of rank 3 along two intersecting lines, L_1, L_2 , form an irreducible family whose general members $V_{\mathbb{C}}(f_A)$ have a rank-2 point at $L_1 \cap L_2$, two isolated nodes and the base locus of the web $Q_A(x)$ is a scheme of length 4 with support in two points.*

Proof. The statement about the singularities is [Hel20, Proposition 9.3] and the claim about the base locus follows from Lemma II.1.1 and [Hel19, Lemma 2.6]. ■

Definition II.1.6. A complex quartic symmetroid in $\mathbb{CP}^3$ singular of rank 3 along two lines intersecting in a rank-2 point, with two additional, isolated nodes is said to be of *type F*.

In this paper, we show with reference to Theorem II.1.2 and Definition II.1.6:

Theorem II.1.7. *Let $S := V(f_A) \subset \mathbb{RP}^3$ be a spectrahedral symmetroid, and suppose that the complex symmetroid $V_{\mathbb{C}}(f_A) \subset \mathbb{CP}^3$ is rational. Let $a \geq 0$ denote the number of real, isolated nodes on S and let $0 \leq b \leq a$ denote the number of nodes on the boundary of the spectrahedron.*

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A. If $V_{\mathbb{C}}(f_A)$ is of type **A**, then S has a triple point on the boundary of the spectrahedron and $0 \leq b \leq a \leq 6$, with a even.

B. If $V_{\mathbb{C}}(f_A)$ is of type **B**, then either

- 1) S has a tacnode on the boundary of the spectrahedron;
- 2) S has a tacnode disjoint from the spectrahedron.

In either case, $0 \leq b \leq a \leq b + 2 \leq 6$, both even.

C. If $V_{\mathbb{C}}(f_A)$ is of type **C**, then either

- 1) S is singular along a smooth conic section with a real point that lies on the boundary of the spectrahedron and $0 \leq b \leq a \leq b + 2 \leq 4$, both even and $a \geq 2$;
- 2) S is singular along a smooth conic section with a real point that is disjoint from the spectrahedron and $a = b = 2$ or $a = b = 4$;
- 3) S is singular along a smooth conic section with no real points and $a = b = 2$.

D. If $V_{\mathbb{C}}(f_A)$ is of type **D**, then S has rank 2 along a line disjoint from the boundary of the spectrahedron and $0 \leq b \leq a \leq b + 2 \leq 6$, both even.

F. If $V_{\mathbb{C}}(f_A)$ is of type **F**, then either

- 1) S is singular of rank 3 along two conjugated intersecting lines whose intersection point lies on the boundary of the spectrahedron;
- 2) S is singular of rank 3 along two conjugated intersecting lines whose intersection point is disjoint from the spectrahedron.

In either case, $0 \leq a = b \leq 2$ even.

Theorem II.1.7 provides necessary conditions for the pair (a, b) to be realized by real, rational spectrahedral symmetroids. This is proven case-by-case in [Sections II.2](#) and [II.2.1](#). After that, in [Section II.2.2](#), we discuss deformation relations between the symmetroids appearing in [Theorem II.1.7](#). We show that symmetroids of type **B** degenerate into a symmetroid of type **F**, but that no other types degenerate into each other.

Sufficient conditions for the pair (a, b) are given by explicit examples of rational spectrahedral symmetroids. [Section II.3](#) contains examples realizing all solutions of the bounds given by [Theorem II.1.7](#), except spectrahedral symmetroids of type **D** with $(a, b) = (0, 0)$. We conjecture that they do not exist:

Conjecture II.1.8. *A quartic symmetroid in $\mathbb{CP}^3$ with a nonempty spectrahedron and a line of rank-2 points has—or is a degeneration of one with—at least two real, isolated nodes.*

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II.2 Real singularities of rational spectrahedral symmetroids

Before we can prove [Theorem II.1.7](#), we need some preliminary results. We restrict the attention to real rational quartic symmetroids $V(f_A)$ with a nonempty spectrahedron, i.e., with $A(x_p)$ definite for some $x_p \in \mathbb{R}^3$. First note that if $A(x_p)$ is definite, then $Q_A(x_p)$ has no real points. So if the quartic spectrahedron of $A(x)$ is nonempty, then the web of quadratic surfaces $Q_A(x)$ has no common real points, i.e., no real basepoints, so they have an even number of complex conjugate basepoints. Therefore, when we consider real singularities of the symmetroid $V(f_A)$, they represent real rank-2 quadrics in a web of quadrics $Q_A(x)$ with complex basepoints. We say that a real quadric is *semidefinite* resp. *indefinite*, when the associated symmetric matrix is. We begin by specializing the base loci mentioned in [Theorem II.1.2](#) to pairs of complex conjugated points.

Lemma II.2.1. *Let $p_1, \bar{p}_1, p_2, \bar{p}_2$, be two pairs of complex conjugate points in $\mathbb{CP}^3$ that do not all lie in a line. Then a real rank-2 quadric that contains both pairs of points is indefinite if and only if it contains the real lines $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$.*

Proof. Assume that $Q := M \cup N$ is a real rank-2 quadric, the union of two planes M and N . If M and N are both real, then Q is indefinite, while if M and N are complex conjugates, then Q is semidefinite.

If M and N each contains only two of the four points $p_1, \bar{p}_1, p_2, \bar{p}_2$, the lemma follows. If M contains exactly three of the points, say $p_1, \bar{p}_1, p_2$ and is not real, then N must contain $p_1, \bar{p}_1, \bar{p}_2$, so Q is semidefinite. If M contains all four basepoints, M is real, so Q is indefinite. ■

Lemma II.2.2. *Let $p_1, \bar{p}_1, p_2, \bar{p}_2$, be two pairs of complex conjugate points in $\mathbb{CP}^3$ that do not all lie in a line, and let $Q(x)$ be the 5-dimensional linear system of all quadratic surfaces with basepoints at these four points.*

If the basepoints are not coplanar, then the rank-2 quadrics in $Q(x)$ form three quadratic surfaces, Q_i, Q_{s1}, Q_{s2} , and four planes, $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$, where the real quadrics in Q_i are indefinite and the real quadrics in $Q_{s1}, Q_{s2}, H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$ are semidefinite.

If the basepoints are coplanar, then the rank-2 quadrics in $Q(x)$ form three quadratic surfaces, as in the nonplanar case, and in addition a web W , whose real quadrics are indefinite. In this case, the double plane containing the basepoints is a rank-1 quadric that lies in the closure of each component of rank-2 quadrics in $Q(x)$.

Proof. First, note that the lines $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$ are real and distinct, so if they intersect, they do so in a real point.

The quadrics in Q_i contain the two lines $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$. Likewise, the quadrics in Q_{s1} contain the lines $\langle p_1, p_2 \rangle, \langle \bar{p}_1, \bar{p}_2 \rangle$, and the quadrics in Q_{s2} contain the lines $\langle p_1, \bar{p}_2 \rangle, \langle \bar{p}_1, p_2 \rangle$. The quadrics in $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$ contain

the plane $\langle \bar{p}_1, p_2, \bar{p}_2 \rangle$, $\langle p_1, p_2, \bar{p}_2 \rangle$, $\langle p_1, \bar{p}_1, \bar{p}_2 \rangle$, $\langle p_1, \bar{p}_1, p_2 \rangle$, respectively. The first part of the claim follows from [Lemma II.2.1](#).

Assume now that the basepoints span a plane M , which is real. Then W consists of all quadrics $Q = M \cup N$, where N is any plane. If Q is real, then N is also real, so Q is indefinite when N is distinct from M . On the other hand, the semidefinite double plane $2M$ is contained in W , Q_i , Q_{s1} and Q_{s2} . ■

Remark II.2.3. If $V_{\mathbb{C}}(f_A)$ is a symmetroid of type [C](#), then [Theorem II.1.2](#) implies that $Q_A(x) \subset Q(x)$. A generic 3-space in $Q(x)$ intersects H_{p_1} , $H_{\bar{p}_1}$, H_{p_2} , $H_{\bar{p}_2}$ in a point each outside of Q_i , Q_{s1} and Q_{s2} . Since $Q_A(x)$ is a special 3-space that intersects one of the quadratic surfaces Q_i , Q_{s1} or Q_{s2} in a conic section C , it meets H_{p_1} , $H_{\bar{p}_1}$, H_{p_2} , $H_{\bar{p}_2}$ in points on C [[Hel20](#), Proof of Proposition 4.5]. Hence H_{p_1} , $H_{\bar{p}_1}$, H_{p_2} , $H_{\bar{p}_2}$ are not relevant in the analysis of the isolated nodes of $V_{\mathbb{C}}(f_A)$. ♠

We now give a preliminary analysis of real singularities for spectrahedral symmetroids with nonisolated singularities.

Lemma II.2.4. *Let $S = V(f_A)$ be a rational quartic spectrahedral symmetroid with nonisolated singularities. Then $V_{\mathbb{C}}(f_A)$ has rank 2 along a real line or a real conic, or it is singular and has rank 3 along two intersecting complex conjugate lines. Furthermore:*

1. *A line of rank-2 points on S is disjoint from the spectrahedron.*
2. *A real conic of rank-2 points on S may have no real points, or have a real point and be disjoint from the spectrahedron, or lie on the boundary of the spectrahedron.*

Proof. If the complex symmetroid $V_{\mathbb{C}}(f_A)$ is singular along a curve, then, by [Theorem II.1.2](#), this curve contains a line or a smooth conic section. Furthermore, when $V_{\mathbb{C}}(f_A)$ is singular along a line, the matrix $A(x)$ may have rank 2 or 3 along the line.

In the first case, when $A(x)$ has rank 2 along the line, the quadrics $Q_A(x)$ have four coplanar basepoints and the line is real. By [Lemma II.2.2](#), the matrix $A(x)$ is indefinite along the line, so on the real spectrahedral symmetroid $V(f_A)$, the singular line must be disjoint from the spectrahedron.

In the second case, when $A(x)$ has rank 3 along the singular line, the web of quadrics $Q_A(x)$ contains a pencil $L \subset Q_A(x)$ of quadrics that are all singular at a basepoint cf. [[Hel20](#), Proof of Proposition 3.5]. Since this basepoint cannot be real, the complex conjugate is also a basepoint. But then, the complex conjugate pencil $\bar{L} \subset Q_A(x)$ must be distinct from L , and $V_{\mathbb{C}}(f_A)$ must be singular of rank 3 along two complex conjugate lines. If these lines do not intersect, the symmetroid $V_{\mathbb{C}}(f_A)$ is a scroll of lines. The lines of this scroll form a curve of bidegree $(2, 2)$ on a quadratic surface in the Grassmannian of lines in $\mathbb{CP}_y^3$, so the scroll is birational to an elliptic scroll, i.e., irrational. Therefore, the symmetroid $V_{\mathbb{C}}(f_A)$ that is singular, but of rank 3 along two lines, is rational only if the two

lines intersect. When the lines are complex conjugates, they of course intersect in a real point. Thus the real symmetroid $V(f_A)$ is singular at this point.

If $V(f_A)$ is a rational spectrahedral symmetroid singular along a smooth conic section, then, by [Theorem II.1.2](#), $A(x)$ must have rank 2 along this curve and the web of quadrics $Q_A(x)$ has two pairs of complex conjugate basepoints that are linearly independent. Clearly the conic section is real and the *a priori* listed possibilities follow. ■

We separate the proof of nonexistence of some cases from the proof of [Theorem II.1.7](#). First we note that symmetroids of type [C.3](#)) belong to a well-known class of real surfaces. A real quartic surface singular along a conic section with no real points in the plane at infinity, is known as a *cyclide* [[Jes16](#), Chapter V].

Not all cyclides are symmetroids. By [Theorem II.1.2.C](#), a symmetroid singular along a conic has four additional nodes. The cyclide $V((x_0^2 + x_1^2 + x_2^2)^2 - x_3^4)$ has no singularities outside of the conic $V(x_3, x_0^2 + x_1^2 + x_2^2)$, hence it is not a symmetroid. For cyclides with four additional nodes, we have the following result:

Proposition II.2.5 ([[Jes16](#), Article 68]). *If a cyclide has four additional nodes, then at most two of the isolated nodes are real.*

Corollary II.2.6. *A general, real, quartic symmetroid singular along a smooth conic section with no real points has either two or no real nodes.*

In a paper by Chandru, Dutta and Hoffmann, the authors summarize classical works by Cayley [[Cay73](#)] and Maxwell [[Max68](#)]. This is used to produce a classification of the various forms of the cyclides [[CDH89](#), Section 6]. Apart from the degenerated cases of a cone or a cylinder, the cyclides are divided into three forms, *horned cyclides*, *ring cyclides* and *spindle cyclides*. Of these, only the ring cyclides have no real nodes. The ring cyclides resemble squashed tori and do not bound a convex region. Hence they do not occur as spectrahedral symmetroids. For horned cyclides and spindle cyclides with precisely two real nodes, the nodes connect two components of the real cyclide. If these occur as spectrahedral symmetroids, one of the components is the boundary of the spectrahedron. We conclude:

Proposition II.2.7. *Let $S := V(f_A)$ be a real quartic symmetroid with a nonempty spectrahedron that is singular along a real conic section with no real points. Then S has two real nodes, both on the boundary of the spectrahedron.*

We now disprove the existence of all spectrahedral symmetroids of type [C](#) with $(a, b) = (0, 0)$.

Proposition II.2.8 ([[Ste20](#)]). *Let $S := V(f_A) \subset \mathbb{RP}^3$ be a spectrahedral symmetroid, and suppose that the complex symmetroid $V_{\mathbb{C}}(f_A) \subset \mathbb{CP}^3$ is of type [C](#). Then S has a real, isolated node.*

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Proof. The symmetroids with a given pair (a, b) form a full-dimensional, Zariski open set in the families of symmetroids of types C.1), C.2) or C.3). Hence it suffices to show the nonexistence of symmetroids with $(a, b) = (0, 0)$ on a Zariski open set in each of the families.

By Theorem II.1.2.C, $Q_A(x)$ has four linearly independent basepoints. Since S is spectrahedral, the basepoints appear in complex conjugate pairs, $p_1, \bar{p}_1$ and $p_2, \bar{p}_2$. After a change of coordinates, we may assume that $p_1 := [1 : i : 0 : 0]$ and $p_2 := [0 : 0 : 1 : i]$. Consider the space $Q(x)$ of all quadrics with $p_1, \bar{p}_1, p_2, \bar{p}_2$ as basepoints. The quadrics in $Q(x)$ have matrices on the form

$$M(x) := \begin{bmatrix} x_{00} & 0 & x_{02} & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & 0 \\ x_{03} & x_{13} & 0 & x_{22} \end{bmatrix}.$$

The quadratic surfaces described in Lemma II.2.2 are

$$\begin{aligned} Q_i &= V(x_{00}, x_{22}, x_{02}x_{13} - x_{03}x_{12}), \\ Q_{s1} &= V(x_{02} - x_{13}, x_{03} + x_{12}, x_{00}x_{22} - x_{12}^2 - x_{13}^2), \\ Q_{s2} &= V(x_{02} + x_{13}, x_{03} - x_{12}, x_{00}x_{22} - x_{12}^2 - x_{13}^2). \end{aligned}$$

By Remark II.2.3, we do not have to consider $H_{p_1}, H_{\bar{p}_1}, H_{p_2}, H_{\bar{p}_2}$.

For symmetroids of type C.1), $Q_A(x)$ intersects either Q_{s1} or Q_{s2} in a conic section; say Q_{s1} . Then the hyperplane spanned by Q_{s1} and $Q_A(x)$ equals $V(\lambda(x_{02} - x_{13}) + \mu(x_{03} + x_{12}))$ for some constants λ, μ . After conjugating $M(x)$ with the matrix

$$\begin{bmatrix} \lambda & -\mu & 0 & 0 \\ \mu & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we may assume that $\mu = 0$. Thus $x_{02} = x_{13}$ in the hyperplane $\langle Q_{s1}, Q_A(x) \rangle$. If $Q_A(x)$ is sufficiently general, we may after a projective linear transformation assume that $A(x)$ is on the form

$$\begin{bmatrix} x_0 & 0 & x_1 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 \\ 0 & x_0 & x_2 & x_1 \\ x_1 & x_2 & x_3 & 0 \\ a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 & x_1 & 0 & x_3 \end{bmatrix}, \quad (\text{II.2.1})$$

for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The rank-2 locus of $V_{\mathbb{C}}(f_A)$ is then

$$\begin{aligned} Q_A(x) \cap Q_{s1} &= V(a_0x_0 + a_1x_1 + (a_2 + 1)x_2 + a_3x_3, x_1^2 + x_2^2 - x_0x_3), \\ Q_A(x) \cap Q_{s2} &= V(x_1, a_0x_0 + (a_2 - 1)x_2 + a_3x_3, x_2^2 - x_0x_3), \\ Q_A(x) \cap Q_i &= V(x_0, x_3, x_1^2 - a_1x_1x_2 - a_2x_2^2). \end{aligned}$$

Assume for contradiction that S has no real, isolated nodes. Then $Q_A(x) \cap Q_{s2}$ and $Q_A(x) \cap Q_i$ are not real. For $a_0 = 1$, this means that the discriminants

$$D_{s2} := (a_2 - 1)^2 - 4a_3 \quad (\text{II.2.2})$$

$$D_i := a_1^2 + 4a_2 \quad (\text{II.2.3})$$

are negative. Moreover, for $a_0 = 1$, Sylvester's criterion implies that the conic section $Q_A(x) \cap Q_{s1}$ is positive definite if and only if $D_{s2} + D_i < 0$. In other words, if S has no real, isolated nodes, then $Q_A(x) \cap Q_{s1}$ has no real points, so S is not of type C.1).

For symmetroids of type C.2), $Q_A(x)$ intersects Q_i in a conic section. The hyperplane spanned by Q_i and $Q_A(x)$ equals $V(\lambda x_{00} + \mu x_{22})$ for some constants λ, μ . After rescaling x_{22} , we may assume that $x_{22} = \pm x_{00}$ in this hyperplane. If $x_{22} = -x_{00}$, then Sylvester's criterion implies that the hyperplane contains no definite matrix. Assume therefore that $x_{22} = x_{00}$. If $Q_A(x)$ is sufficiently general, we may after a projective linear transformation assume that $A(x)$ is on the form

$$\begin{bmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_3 & a_0x_0 + a_1x_2 + a_2x_2 + a_3x_3 \\ x_1 & x_3 & x_0 & 0 \\ x_2 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 & 0 & x_0 \end{bmatrix} \quad (\text{II.2.4})$$

for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The isolated nodes of $V_{\mathbb{C}}(f_A)$ are then

$$Q_A(x) \cap Q_{s1} = V(x_2 + x_3, a_0x_0 + (a_1 - 1)x_1 - (a_2 - a_3)x_3, x_0^2 - x_1^2 - x_3^2),$$

$$Q_A(x) \cap Q_{s2} = V(x_2 - x_3, a_0x_0 + (a_1 + 1)x_1 + (a_2 + a_3)x_3, x_0^2 - x_1^2 - x_3^2).$$

For $a_0 = 1$, these are not real if the discriminants

$$D_{s1} := a_1^2 + a_2^2 + a_3^2 - 2a_1 - 2a_2a_3 \quad (\text{II.2.5})$$

$$D_{s2} := a_1^2 + a_2^2 + a_3^2 + 2a_1 + 2a_2a_3 \quad (\text{II.2.6})$$

are negative. But if $D_{s1} < 0$ and $D_{s2} < 0$, then $D_{s1} + D_{s2} = 2(a_1^2 + a_2^2 + a_3^2) < 0$, which is impossible. Hence at least one of $Q_A(x) \cap Q_{s1}$ and $Q_A(x) \cap Q_{s2}$ consists of real points.

The case for symmetroids of type C.3) is covered by Proposition II.2.7. ■

II.2.1 Proof of Theorem II.1.7

II.2.1.1 Type A

Triple points are rank-1 points, hence the real ones are semidefinite and therefore on the boundary of the spectrahedron.

By Theorem II.1.2.A, a general complex symmetroid with a triple point has six nodes. Since a spectrahedral symmetroid is a real surface, the number a of real nodes is even. There are no further restraints on b , the number of real semidefinite nodes, as the examples in Table II.1 show.

II.2.1.2 Type B

By [Theorem II.1.2.B](#), the web $Q_A(x)$ has two basepoints. Since S is spectrahedral, the basepoints are complex conjugates, p and $\bar{p}$. Let $Q(x)$ be the 7-dimensional linear system of all quadratic surfaces with p and $\bar{p}$ as basepoints. The rank-2 locus of $Q(x)$ consists of two fourfolds, X_i and X_s . The quadrics in X_i are pairs of planes, where one of the planes contains the line $\langle p, \bar{p} \rangle$. In X_s , the quadrics consist of two planes, where the planes contain one basepoint each. The set $\text{Sing}(X_i) = \text{Sing}(X_s)$ consists of pairs of planes that both contain $\langle p, \bar{p} \rangle$. The real quadrics in $X_i \setminus \text{Sing}(X_i)$ are indefinite and the real quadrics in $X_s \setminus \text{Sing}(X_s)$ are semidefinite. The real quadrics in $\text{Sing}(X_i) = \text{Sing}(X_s)$ are either semidefinite or indefinite.

In the proof of [\[Hel20, Proposition 7.4\]](#), it is shown that the tacnode corresponds to a point in $\text{Sing}(X_i) = \text{Sing}(X_s)$, and that $Q_A(x)$ intersects $X_i \setminus \text{Sing}(X_i)$ in two points and $X_s \setminus \text{Sing}(X_s)$ in four points. The claim follows.

II.2.1.3 Type C

Let C denote the conic section in the singular locus of S . By [Theorem II.1.2.C](#), S is the discriminant of a web $Q_A(x)$ of quadrics with four linearly independent basepoints. In the notation of [Lemma II.2.2](#), $Q_A(x)$ intersects one of the surfaces Q_i , Q_{s1} or Q_{s2} in C and the remaining two surfaces in two points each. By [Remark II.2.3](#), S does not have any other rank-2 points. Hence $a \leq 4$.

Assume first that C contains a real point. If C is disjoint from the spectrahedron, then $C \subset Q_i$. The real quadrics in Q_{s1} and Q_{s2} are semidefinite, so $a = b$. If C lies on the boundary of the spectrahedron, then C is contained in either Q_{s1} or Q_{s2} . It follows that $a \leq b + 2$, since the real quadrics in Q_i are indefinite. We get $a \geq 2$ from [Proposition II.2.8](#).

The case where C has no real points is dealt with in [Proposition II.2.7](#).

II.2.1.4 Type D

[Lemma II.2.4](#) states that a line of rank-2 points is disjoint from the spectrahedron.

By [Theorem II.1.2.D](#), S is the discriminant of a web $Q_A(x)$ of quadrics with four coplanar basepoints. Since S has a nonempty spectrahedron, the basepoints occur in two complex conjugate pairs. In the notation of [Lemma II.2.2](#), $Q_A(x)$ intersects W in a line and the surfaces Q_i , Q_{s1} , Q_{s2} in two points each. These are the only singularities on S , so we get that

$$0 \leq b \leq a \leq b + 2 \leq 6,$$

and a, b are even from [Lemma II.2.2](#).

II.2.1.5 Type F

It follows from [\[Hel19, Remark 4.5\]](#) that S is the intersection of a 3-space with $S' \subset \mathbb{CP}^5$, where S' is a quartic spectrahedral symmetroid that is singular of rank 3 along two complex conjugate 3-spaces intersecting in a plane. The claim

is immediate from [Hel19, Proposition 4.6], which says that S' is singular along an additional quadratic surface whose real points lie on the spectrahedron. ■

II.2.2 Deformation relations between rational symmetroids

It is natural to ask whether the different types of rational symmetroids listed in Theorem II.1.7 are deformations of each other. That is, does there exist a flat family of symmetroids where one of the symmetroids is of one type, while all the others are of a different type? The following result addresses this:

Proposition II.2.9. *Let $V(f_A) \subset \mathbb{RP}^3$ be a quartic spectrahedral symmetroid and suppose that the complex symmetroid $V_{\mathbb{C}}(f_A) \subset \mathbb{CP}^3$ is rational. If $V_{\mathbb{C}}(f_A)$ is of type B, then it can degenerate into a symmetroid of type F. This is the only instance of a symmetroid of type A, B, C, D or F that degenerates into one of the other types. Moreover, a real symmetroid $V(f_A)$ does not degenerate from type B.1) to B.2); from C.1) to C.2) or C.3); or from F.1) to F.2), or vice versa. In addition, a real symmetroid $V(f_A)$ with a real nodes, b of which lie on the spectrahedron, does not degenerate into a symmetroid $V(f_{A'})$ with a' real nodes, b' of which lie on the spectrahedron, if $(a, b) \neq (a', b')$.*

Proof. We start with the complex symmetroids. In [Hel20, Remark 9.2], it is argued that if S is a quartic surface singular along two intersecting lines, L_1 and L_2 , with the implicit assumption that $L_1 \cap L_2$ is not a triple point, then S satisfies the equation of a tacnodal surface. Hence a symmetroid of type F can arise as a degeneration of surfaces of type B of Theorem II.1.2. Consider the explicit deformation

$$M(t) := \begin{bmatrix} l_{00} & 0 & l_{02} & l_{03} \\ 0 & l_{00} & -l_{03} + tl_{12} & l_{02} + tl_{13} \\ l_{02} & -l_{03} + tl_{12} & a_{22}x_0 + l_{22} & a_{23}x_0 + l_{23} \\ l_{03} & l_{02} + tl_{13} & a_{23}x_0 + l_{23} & a_{33}x_0 + l_{33} \end{bmatrix},$$

where l_{ij} are linear forms in x_1, x_2, x_3 , and $a_{22}, a_{23}, a_{33}, t$ are scalars. For all t , the matrix $M(t)$ is on the form (II.3.2) of a tacnodal symmetroid. For $t = 0$, the matrix $M(0)$ takes the form (II.3.4) of a symmetroid singular along two lines.

No other type of complex symmetroids is a degeneration of one of the other types. Most of the possible degenerations can be excluded by simple reasons, for instance a surface does not degenerate into one whose singular locus has lower degree or dimension, or into one with only singularities of lower multiplicity. We check the remaining cases:

- A symmetroid of type A is not a degeneration of symmetroids of type B. This is because a web of quadrics $Q_A(x)$ with basepoints does not degenerate into a web with no basepoints.
- Symmetroids of types C or D are not degenerations of symmetroids of type B. Indeed, as noted in the proof of Theorem II.1.7, the associated quadric at a tacnode is singular along the line spanned by two basepoints of

$Q_A(x)$. In a degeneration of tacnodal symmetroids, one quadric in $Q_A(x)$ is singular along the line spanned by two basepoints. For symmetroids of types **C** or **D**, $Q_A(x)$ has four basepoints. Assume that $Q_A(x)$ has four basepoints and recall the notation from [Lemma II.2.2](#). If the basepoints are independent and a quadric is singular along the line spanned by two of them, then it lies in one of the intersections $Q_i \cap Q_{s1}$, $Q_i \cap Q_{s2}$ or $Q_{s1} \cap Q_{s2}$. If the basepoints are coplanar and a quadric is singular along the line spanned by two of them, then it lies in one of the intersections $W \cap Q_i$, $W \cap Q_{s1}$ or $W \cap Q_{s2}$. In either case, this implies that the surface has one fewer isolated rank-2 point, than one of type **C** or **D**.

- A symmetroid of type **C** is not a degeneration of symmetroids of type **D**. This is because a web of quadrics $Q_A(x)$ with linearly dependent basepoints does not degenerate into one with independent basepoints.
- A symmetroid of type **F** is not a degeneration of symmetroids of types **C** or **D**. This is because a symmetroid with a curve in its rank-2 locus does not degenerate into one without a curve in its rank-2 locus.

For the real symmetroids, we note that a positive semidefinite rank-2 matrix has two positive eigenvalues, while an indefinite rank-2 matrix has eigenvalues with different signs. In a degeneration from one into the other, one of the eigenvalues changes to 0, causing the rank to drop. This implies that a symmetroid with singularities on the spectrahedron does not degenerate into one with singularities outside of the spectrahedron, and vice versa.

Finally, in a degeneration of a surface with two complex conjugate nodes into one with two real nodes, the imaginary part of the coordinates of the nodes changes to 0, while the coordinates have the same real part for both nodes. Hence the two nodes coincide. Thus a symmetroid with a real nodes does not degenerate into one with $a' \neq a$ real nodes. ■

II.3 Examples of existence

This section lists matrices that define spectrahedral symmetroids with the different values of (a, b) described by [Theorem II.1.7](#). There is one missing instance, namely **D** with $(a, b) = (0, 0)$. For brevity in the tables, we write $(a_{00}, a_{01}, a_{02}, a_{03}, a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33})$ for the symmetric matrix

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

[Tables II.1](#), [II.2](#) and [II.4](#) provide one such 10-tuple for each of the matrices A_2 and A_3 in [\(II.1.1\)](#). The matrices A_0 and A_1 are fixed, and therefore not listed.

II.3.1 Spectrahedral symmetroids with a triple point

Triple points correspond precisely to rank-1 points. Thus (II.1.1) defines a spectrahedral symmetroid with a triple point if A_0 is a positive definite matrix, A_1 is a rank-1 matrix, and A_2, A_3 are any symmetric matrices. To find examples, we let A_0 be the identity matrix I_4 ,

$$A_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and used a pseudorandom generator to draw symmetric matrices A_2 and A_3 with integer entries between -9 and 9 . Doing this, we found all values of (a, b) specified by Theorem II.1.7.A. The results are listed in Table II.1, a sample image is given in Figure II.1.

Table II.1: Examples of matrices defining spectrahedral symmetroids of type A.

(a, b)	A_2	A_3
(6, 6)	$(-9, -1, 2, -2, -3, 7, 7, 5, 5, -4)$	$(3, -4, -7, 3, -1, -2, -9, 4, 8, 6)$
(6, 5)	$(3, -7, 5, 4, 3, 2, 9, 5, 7, -8)$	$(-3, 5, 5, -5, -3, -6, 3, -2, 0, -7)$
(6, 4)	$(-7, 7, -6, 6, 5, 5, 5, -8, -2, 1)$	$(-3, 8, -2, 1, -4, 1, 5, -8, 8, -7)$
(6, 3)	$(0, 0, 3, -3, 0, -2, 8, 6, 7, 1)$	$(-2, 1, 5, 4, 5, 6, -8, 5, 1, 6)$
(6, 2)	$(9, 7, 2, 3, -5, -9, -2, -2, 3, -5)$	$(-9, 6, -3, 3, -7, 2, -1, 2, -7, 5)$
(6, 1)	$(8, -5, 2, -9, 1, -1, 2, -5, 9, -8)$	$(-9, 6, -3, 3, -7, 2, -1, 2, -7, 5)$
(6, 0)	$(-3, 6, -4, 1, 2, 6, 9, 0, -7, 8)$	$(8, 6, 3, -4, 5, 9, 7, 3, 7, -9)$
(4, 4)	$(1, -6, -6, 4, 6, 2, 5, -5, -1, -8)$	$(-5, 1, -7, 6, 9, 9, 7, -9, -8, -2)$
(4, 3)	$(6, 3, 9, 9, -8, 9, 0, -7, 6, -7)$	$(-1, 3, 3, 3, -9, 5, -6, 5, 4, -9)$
(4, 2)	$(6, 8, -3, 9, 2, -2, -9, 4, 6, 7)$	$(-2, 9, -4, -2, 8, -1, 9, 1, -4, 4)$
(4, 1)	$(2, 9, -1, -8, 1, 0, -1, -8, 6, -2)$	$(2, -6, 0, -6, -5, 2, -1, 6, -1, 9)$
(4, 0)	$(-8, 0, -9, 6, 3, -6, 3, -7, 6, 6)$	$(1, -5, -1, 8, -9, 0, 4, -2, 8, -3)$
(2, 2)	$(4, -4, 1, 6, 3, 2, 0, -5, 1, -3)$	$(4, 3, 9, 5, -6, -3, -5, 2, 0, 4)$
(2, 1)	$(-5, 8, -1, -6, 8, 7, 9, 5, 9, 6)$	$(3, 1, 4, 5, -4, -6, -8, -2, 8, 5)$
(2, 0)	$(-9, -6, 5, -1, 4, 0, 1, 8, 1, 6)$	$(-7, 5, -7, 4, 5, 0, 3, 6, 4, 7)$
(0, 0)	$(-5, 9, 5, 1, 1, -2, 9, -2, 0, 2)$	$(7, -6, 5, 1, -4, 1, 3, -9, 8, -5)$

II.3.2 Spectrahedral symmetroids with a tacnode

Consider the space $Q(x)$ of all quadrics with basepoints $p, \bar{p}$. After a change of coordinates, we may assume that $p := [1 : \mathbf{i} : 0 : 0]$. Then quadrics in $Q(x)$ have matrices on the form

$$\begin{bmatrix} x_{00} & 0 & x_{02} & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}. \quad (\text{II.3.1})$$

Using the notation from [Section II.2.1.2](#), a 3-space $Q_A(x) \subset Q(x)$, corresponding to a tacnodal symmetroid, intersects $\text{Sing}(X_i) = V(x_{00}, x_{02}, x_{03}, x_{12}, x_{13})$ in a point. Hence $A(x)$ has the form

$$A(x) := \begin{bmatrix} l_{00} & 0 & l_{02} & l_{03} \\ 0 & l_{00} & l_{12} & l_{13} \\ l_{02} & l_{12} & a_{22}x_0 + l_{22} & a_{23}x_0 + l_{23} \\ l_{03} & l_{13} & a_{23}x_0 + l_{23} & a_{33}x_0 + l_{33} \end{bmatrix}, \quad (\text{II.3.2})$$

where each l_{ij} is a linear form in x_1, x_2, x_3 , and $a_{ij} \in \mathbb{R}$. Moreover, if $V(f_A)$ is spectrahedral, we can take $A_1 = A([0 : 1 : 0 : 0])$ to be positive definite.

To find examples, we let A_1 in [\(II.1.1\)](#) be the identity matrix I_4 , and

$$A_0 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}.$$

For symmetroids with a tacnode on the boundary of the spectrahedron, we chose $\alpha = 1$ and $\beta = 2$. Likewise, we chose $\alpha = 1$ and $\beta = -2$ for symmetroids with a tacnode disjoint from the spectrahedron. We used a pseudorandom generator to draw symmetric matrices A_2 and A_3 on the form [\(II.3.1\)](#) with integer entries between -9 and 9 . Doing this, we found all values of (a, b) specified by [Theorem II.1.7.B.1\)](#) and [Theorem II.1.7.B.2\)](#). The results are listed in [Table II.2](#) and a sample image in [Figure II.2](#).

Table II.2: Examples of matrices defining spectrahedral symmetroids of types [B.1\)](#) and [B.2\)](#).

(a, b)	B.1)	B.2)
$(6, 4)$	$(6, 0, -3, 6, 6, -6, -4, 0, 3, 6)$	$(8, 0, -6, -8, 8, 7, 1, -7, 4, 7)$
	$(3, 0, 5, -8, 3, -3, -4, 2, -5, 8)$	$(6, 0, 2, -2, 6, -7, 6, -1, 9, 7)$
$(4, 4)$	$(0, 0, 4, -4, 0, 3, 3, 1, -8, 4)$	$(4, 0, -8, -2, 4, 5, -3, 5, -8, 9)$
	$(5, 0, -2, 4, 5, -5, 2, 1, -5, -8)$	$(3, 0, 2, -4, 3, 0, 4, 0, 2, 6)$
$(4, 2)$	$(8, 0, -5, 8, 8, 8, -3, 5, -3, 2)$	$(1, 0, 5, -7, 1, 6, 4, -6, -3, 6)$
	$(-8, 0, 1, 9, -8, -9, 7, -8, 8, 9)$	$(6, 0, 2, -8, 6, 1, -3, 5, 4, -8)$

Continued on the next page.

(a, b)	B.1)	B.2)
$(2, 2)$	$(6, 0, -8, -4, 6, 1, 9, -8, 0, 3)$ $(3, 0, -2, 7, 3, 7, 6, -6, 4, 1)$	$(7, 0, 6, 5, 7, 2, -4, 0, -7, 2)$ $(0, 0, -8, 2, 0, 3, 7, 8, -3, 7)$
$(2, 0)$	$(6, 0, 6, -5, 6, 6, -1, 1, -1, 7)$ $(8, 0, 5, -8, 8, -6, 1, 2, -2, -9)$	$(5, 0, 6, -6, 5, -6, 8, -8, 7, 6)$ $(4, 0, 2, -1, 4, 6, -6, -2, 3, 5)$
$(0, 0)$	$(-1, 0, 2, 4, -1, 0, 1, 7, 5, 5)$ $(7, 0, 4, -6, 7, 5, 8, -2, 0, -4)$	$(9, 0, 3, -9, 9, 6, -7, -4, 1, 1)$ $(4, 0, 5, -9, 4, -2, 5, 3, 1, -4)$

II.3.3 Spectrahedral symmetroids with a double conic

To find examples of symmetroids $V(f_A)$ of type C.1), we can take $A(x)$ to be on the form (II.2.1). For $a_0 = 1$, we get $(a, b) = (4, 2)$ if both discriminants (II.2.2) and (II.2.3) are positive, $(a, b) = (2, 2)$ if (II.2.2) is positive and (II.2.3) is negative, and $(a, b) = (2, 0)$ if (II.2.2) is negative and (II.2.3) is positive. It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. In particular, the following examples are spectrahedral:

- $a_0 := 1, a_1 := 0, a_2 := 1, a_3 := -1$ gives a symmetroid with $(a, b) = (4, 2)$;
- $a_0 := 1, a_1 := 0, a_2 := -2, a_3 := -4$ gives a symmetroid with $(a, b) = (2, 2)$;
- $a_0 := 1, a_1 := -3, a_2 := 0, a_3 := 1$ gives a symmetroid with $(a, b) = (2, 0)$.

To find examples of symmetroids $V(f_A)$ of type C.2), we can take $A(x)$ to be on the form (II.2.4). For $a_0 = 1$, we get $(a, b) = (4, 4)$ if both discriminants (II.2.5) and (II.2.6) are positive, and $(a, b) = (2, 2)$ if only one of (II.2.5) and (II.2.6) is positive. It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. In particular, the following examples are spectrahedral:

- $a_0 := 1, a_1 := 3, a_2 := 0, a_3 := 0$ gives a symmetroid with $(a, b) = (4, 4)$;
- $a_0 := 1, a_1 := \frac{1}{2}, a_2 := 0, a_3 := 0$ gives a symmetroid with $(a, b) = (2, 2)$.

To find examples of symmetroids $V(f_A)$ of type C.3), we can again take $A(x)$ to be on the form (II.2.1). For $a_0 = 1$, we get the correct type if the discriminant D_{s2} from (II.2.2) is positive, D_i from (II.2.3) is negative and $D_{s2} < -D_i$. It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. In particular, (II.2.1) gives a spectrahedral symmetroid of type C.3) for $a_0 := 1, a_1 := 2, a_2 := -5$ and $a_3 := 6$. See Figure II.5.

II.3.4 Spectrahedral symmetroids with rank 2 along a double line

Consider the space $Q(x)$ of quadrics with coplanar basepoints $p_1, \bar{p}_1, p_2, \bar{p}_2$. After a change of coordinates, we may assume that $p_1 := [1 : i : 0 : 0]$ and

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$p_2 := [1 : 0 : i : 0]$. Then the quadrics in $Q(x)$ have matrices on the form

$$\begin{bmatrix} x_{00} & 0 & 0 & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ 0 & x_{12} & x_{00} & x_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

If $Q_A(x) \subset Q(x)$ is a generic 3-space, we may after a projective linear transformation assume that $A(x)$ is on the form

$$\begin{bmatrix} x_0 & 0 & 0 & x_1 \\ 0 & x_0 & x_2 & x_3 \\ 0 & x_2 & x_0 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 \\ x_1 & x_3 & a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 & b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 \end{bmatrix}, \quad (\text{II.3.3})$$

for $a_i, b_i \in \mathbb{R}$. Using the notation from [Lemma II.2.2](#), the rank-2 locus of $Q_A(x)$ is

$$\begin{aligned} Q_A(x) \cap W &= V(x_0, x_2), \\ Q_A(x) \cap Q_i &= V(x_0, x_1, b_2x_2^2 + (-2a_2 + b_3)x_2x_3 - 2a_3x_3^2), \\ Q_A(x) \cap Q_{s1} &= V(x_0 - x_2, a_1x_1 + (a_0 + a_2)x_2 + (a_3 - 1)x_3, \\ &\quad x_1^2 - b_1x_1x_2 - (b_0 + b_2)x_2^2 - b_3x_2x_3 + x_3^2), \\ Q_A(x) \cap Q_{s2} &= V(x_0 + x_2, a_1x_1 - (a_0 - a_2)x_2 + (a_3 + 1)x_3, \\ &\quad x_1^2 + b_1x_1x_2 - (b_0 - b_2)x_2^2 + b_3x_2x_3 + x_3^2). \end{aligned}$$

Hence the matrix $A(x)$ defines a symmetroid $V(f_A)$ of type [D](#).

For $a_1 = 1$, the reality of the isolated nodes is determined by the discriminants

$$\begin{aligned} D_i &:= (2a_2 + b_3)^2 + 8b_2a_3, \\ D_{s1} &:= (a_3(2a_0 + 2a_2 + b_1) - 2a_0 - 2a_2 - b_1 - b_3)^2 \\ &\quad - 4(a_3^2 - 2a_3 + 2)((a_0 + a_2)(a_0 + a_2 + b_1) - b_0 - b_2), \\ D_{s2} &:= (a_3(-2a_0 + 2a_2 - b_1) - 2a_0 + 2a_2 - b_1 + b_3)^2 \\ &\quad - 4(a_3^2 + 2a_3 + 2)((a_0 - a_2)(a_0 - a_2 + b_1) - b_0 + b_2). \end{aligned}$$

More precisely,

- $(a, b) = (6, 4)$ if D_i , D_{s1} and D_{s2} are positive;
- $(a, b) = (4, 4)$ if D_{s1} and D_{s2} are positive and D_i is negative;
- $(a, b) = (4, 2)$ if D_i is positive and either D_{s1} or D_{s2} is positive;
- $(a, b) = (2, 2)$ if either D_{s1} or D_{s2} is positive and D_i is negative;
- $(a, b) = (2, 0)$ if D_i is positive and D_{s1} and D_{s2} are negative;
- $(a, b) = (0, 0)$ if D_i , D_{s1} , D_{s2} are negative.

It remains to check that $A(x)$ contains a definite matrix to conclude that $V(f_A)$ is in fact spectrahedral. Spectrahedral examples are given in Table II.3, except $(a, b) = (0, 0)$. Some images are given in Figure II.6.

Remark II.3.1. If we let $a_0 := 3$, $a_1 := 1$, $a_2 := 0$, $a_3 := 1$, $b_0 := 0$, $b_1 := 0$, $b_2 := -1$, $b_3 := 0$ then (II.3.3) defines a symmetroid with rank 2 along a line and no real, isolated nodes. By Sylvester's criterion, it is not spectrahedral. We have not been able to find a spectrahedral symmetroid of type D with $(a, b) = (0, 0)$, nor prove its nonexistence. ♠

Table II.3: Examples of parameters for (II.3.3) that define spectrahedral symmetroids of type D.

(a, b)	a_0	a_1	a_2	a_3	b_0	b_1	b_2	b_3
(6, 4)	0	1	0	1	0	0	0	1
(4, 4)	0	1	0	-1	1	0	1	1
(4, 2)	0	1	-1	2	0	0	0	1
(2, 2)	0	1	0	1	0	0	-1	1
(2, 0)	0	1	-1	1	0	0	0	1

II.3.5 Spectrahedral symmetroids with rank 3 along two double lines

After a change of coordinates, Proposition II.1.5 implies that if $V_C(f_A)$ is of type F, then $Q_A(x)$ is contained in the 7-space $Q(x)$ defined by (II.3.1). The discriminant of $Q(x)$ is singular of rank 3 along the complex conjugate 4-spaces $H_4 := V(x_{00}, x_{02} - ix_{12}, x_{03} - ix_{13})$ and $\overline{H}_4 := V(x_{00}, x_{02} + ix_{12}, x_{03} + ix_{13})$. In order to find examples of 3-spaces $Q_A(x)$ corresponding to symmetroids of type F, we consider the 5-space $H := V(x_{02} - x_{03}, x_{03} + x_{13})$. Because H intersects H_4 and $\overline{H}_4$ in a 3-space each, a generic 3-space $Q_A(x) \subset H$ corresponds to a symmetroid of type F. We therefore consider matrices on the form

$$A(x) := \begin{bmatrix} l_{00} & 0 & l_{02} & l_{03} \\ 0 & l_{00} & -l_{03} & l_{02} \\ l_{02} & -l_{03} & l_{22} & l_{23} \\ l_{03} & l_{02} & l_{23} & l_{33} \end{bmatrix},$$

where l_{ij} are linear forms in x_0, x_1, x_2, x_3 . In order to make $A(x)$ resemble (II.3.2), we let $A_0 = A([1 : 0 : 0 : 0])$ correspond to the intersection between the lines in the singular locus of $V(f_A)$. Hence we reduce to

$$A(x) := \begin{bmatrix} l_{00} & 0 & l_{02} & l_{03} \\ 0 & l_{00} & -l_{03} & l_{02} \\ l_{02} & -l_{03} & a_{22}x_0 + l_{22} & a_{23}x_0 + l_{23} \\ l_{03} & l_{02} & a_{23}x_0 + l_{23} & a_{33}x_0 + l_{33} \end{bmatrix}, \quad (\text{II.3.4})$$

where l_{ij} are linear forms in x_1, x_2, x_3 . Furthermore, if $V(f_A)$ is spectrahedral, we can take $A_1 = A([0 : 1 : 0 : 0])$ to be a definite matrix.

We chose A_1 to be the positive definite matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

For type [F.1](#)), we let $a_{22} := 1$, $a_{23} := 1$, $a_{33} := 3$ in A_0 . For type [F.2](#)), we let $a_{22} := 1$, $a_{23} := 1$ and $a_{33} := 0$. [Table II.4](#) shows A_2 and A_3 realizing all values of (a, b) specified by [Theorem II.1.7.F.1](#)) and [Theorem II.1.7.F.2](#)). Some corresponding images are given in [Figures II.7](#) and [II.8](#).

Table II.4: Examples of matrices defining spectrahedral symmetroids of types [F.1](#)) and [F.2](#)).

(a, b)	F.1	F.2
(2, 2)	$(0, 0, 1, 0, 0, 0, 1, 0, 1, 0)$	$(0, 0, 1, 0, 0, 0, 1, 0, 1, 0)$
	$(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$	$(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$
(0, 0)	$(0, 0, 1, 0, 0, 0, 1, 1, 1, 0)$	$(0, 0, 1, 0, 0, 0, 1, 0, 1, 0)$
	$(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$	$(0, 0, 0, 1, 0, -1, 0, 0, 0, 0)$

II.3.6 Pictures

Figure II.1: A symmetroid of type [A](#) with $(a, b) = (6, 4)$.

Figure II.2: A symmetroid of type [B.1](#)) with $(a, b) = (6, 4)$.

Figure II.3: A symmetroid of type C.1) with $(a, b) = (2, 2)$.

Figure II.4: A symmetroid of type C.2) with $(a, b) = (4, 4)$.

Figure II.5: A symmetroid of type C.3) with $(a, b) = (2, 2)$. The surface is known as a “spindle cyclide”.

Figure II.6: Two symmetroids of type D. The surface to the left has $(a, b) = (2, 2)$ and the surface to the right has $(a, b) = (2, 0)$.

Figure II.7: A symmetroid of type F.1) with $(a, b) = (0, 0)$.

Figure II.8: A symmetroid of type F.2) with $(a, b) = (0, 0)$.

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Maximal dimension of quartic symmetroids with a double quadric of codimension 1

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Abstract

We prove that the dimension of a quartic symmetroid singular along a quadric of codimension 1 is at most 4, if it is not a cone. In the case with maximal dimension, the quadric is reducible and consists of rank-3-points. If the quadric is irreducible, it consists of rank-2 points and the symmetroid is at most 3-dimensional.

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III.1 Introduction

In semidefinite programming, the objective is to optimise a linear function over the intersection of the cone of positive semidefinite matrices in the space of real, symmetric $(d \times d)$ -matrices with an affine subspace. The feasible region of a semidefinite program is called a *spectrahedron*. Spectrahedra are important, elementary objects in convex algebraic geometry [BPT13].

Quartic spectrahedra are the case of (4×4) -matrices intersected with an affine space that contains a positive definite matrix. The affine space is identified

with $\mathbb{R}^n$. The algebraic boundary of a spectrahedron in $\mathbb{R}^n$ is a hypersurface $\mathcal{V}(f)$ in $\mathbb{CP}^n$. The polynomial f is the determinant of a matrix

$$A(x) := A_0x_0 + A_1x_1 + \cdots + A_nx_n, \quad (\text{III.1.1})$$

where A_i is a real, symmetric (4×4) -matrix. Classically, the hypersurface $\mathcal{V}(f)$ is called a *symmetroid* [Cay69; Dol12; Jes16]. Given a matrix $A(x)$ as in (III.1.1), with $f := \det(A(x))$, the *associated spectrahedron* is the set

$$S(f) := \{x \in \mathbb{RP}^n \mid A(x) \text{ is semidefinite}\}.$$

We say the symmetroid $\mathcal{V}(f)$ is *spectrahedral* or has a *nonempty spectrahedron* if $S(f)$ contains a positive definite matrix.

For a given dimension n , the generic quartic symmetroid $\mathcal{V}(f) \subset \mathbb{CP}^n$ has a fixed number of singularities. Some of the singularities may be real, and some of the real singularities may lie on the topological boundary of the spectrahedron. For generic quartic symmetroids in $\mathbb{CP}^3$ with a nonempty spectrahedron, the possible arrangements of singularities are characterised in [DI11]. This result is recovered in [Ott+15], using an algorithmic proof. The singular loci of rational quartic symmetroids in $\mathbb{CP}^3$ are described in [Hel20]. The possible locations of the singularities of rational quartic symmetroids with a nonempty spectrahedron are further specified by [HR18].

Rational quartic symmetroids in $\mathbb{CP}^3$ include quartic surfaces with a double conic section. In this paper, we consider the analogous situation in higher dimensions. Our main result is:

Theorem III.1.1. *Let $S \subset \mathbb{P}^n$ be an irreducible, quartic, symmetroidal hypersurface which is not a cone. Assume that Q is an $(n-2)$ -dimensional, quadratic component in $\text{Sing}(S)$. Then $n \leq 4$ if Q is irreducible and $n \leq 5$ otherwise.*

This is proved by combining Corollary III.3.3 and Proposition III.4.1. The bounds are only valid for symmetroids that are not cones, as (III.1.1) defines a symmetroid in $\mathbb{CP}^m$ for any $m \geq n$.

The rest of the paper is organised as follows: Section III.2 introduces notation and basic facts about symmetroids and linear systems of quadrics. Section III.3 demonstrates that if S is a symmetroid with an irreducible, double quadric Q of codimension 1, then points in Q have at most rank 2 and $\dim(Q) \leq 2$. Section III.3.1 deals with the case where S is a threefold. First we find the additional singularities outside of Q . Then we describe the different configurations of singularities when S is spectrahedral. In Section III.3.2, we argue that Q must contain real points if S is a threefold. Section III.4 shows that if we allow Q to be reducible with rank-3 points, then $\dim(Q) \leq 3$. Section III.4.1 determines the number of additional singularities outside of Q when S is a fourfold, and examines them further in the spectrahedral case.

III.2 Preliminaries

Here we present our main tool, the linear system of quadrics associated to a symmetroid, and sundry useful results.

First, note that throughout the paper, we assume that the symmetroids under consideration are not cones.

Our notation is as follows: Let $S \subset \mathbb{CP}^n$ be a quartic symmetroid with representation (III.1.1). For $x \in \mathbb{CP}^n$ and $\mathbf{y} := [y_0, y_1, y_2, y_3]$, let $q(x) := \mathbf{y}A(x)\mathbf{y}^\top$. Then $Q(x) := \mathcal{V}(q(x)) \subset \mathbb{CP}^3$ is a quadratic surface. The set

$$W(S) := \{Q(x) \mid x \in \mathbb{P}^n\}$$

is the *associated linear system of quadrics* of S . For a subset $U \subset S$, we let

$$W(U) := \{Q(x) \mid x \in \langle U \rangle\} \subset W(S).$$

For a quadric $Q \subset \mathbb{CP}^3$, let $[Q]$ denote the corresponding point in $W(S) = \mathbb{CP}^n$. For a space W of quadrics, the base locus is written as $\text{Bl}(W)$. For a subset $\text{Bl}(W) \subset \mathbb{P}^3$, we let $X_{\text{Bl}(W)} \subset \mathbb{P}^9$ denote the space of all quadrics containing $\text{Bl}(W)$.

The *rank* and *corank* of a point $x \in \mathbb{CP}^n$ are defined as $\text{rank } A(x)$ and $\text{corank } A(x)$, respectively. The *rank- k locus* of S — or of $W(S)$ — is the set of points with rank less than or equal to k . If S is a symmetroid of degree d , then the rank- $(d-2)$ locus is contained in $\text{Sing}(S)$, but equality does not always hold. The following well-known connection between the base locus of $W(S)$ and the singular rank- $(d-1)$ points can be found in [Wal81, Lemma 1.1] and [Ili+17, Lemma 2.13]:

Lemma III.2.1. *Let $S \subset \mathbb{P}^n$ be a symmetroid. If $[Q] \in \mathbb{P}^n$ is a point such that Q is a singular quadric with a singularity at a point $p \in \text{Bl}(W(S))$, then $[Q] \in \text{Sing}(S)$. If $[Q] \in \text{Sing}(S)$ is a corank-1 point and p is the singular point of Q , then $p \in \text{Bl}(W(S))$.*

The following lemma is useful for eliminating possible symmetroids:

Lemma III.2.2 ([Hel20, Lemma 2.7]). *Let $S \subset \mathbb{P}^n$ be a quartic symmetroid and assume that $\text{Bl}(W(S))$ contains a curve. Then S is reducible.*

The next result elaborates on the relationship between singular corank-1 points and $\text{Bl}(W(S))$ for quartics:

Lemma III.2.3. *Let $S \subset \mathbb{P}^n$ be an irreducible quartic symmetroid and suppose that $H \subseteq \text{Sing}(S)$ is a linear space of rank-3 points. Then the quadrics in $W(H)$ share a common singularity, which is a base point for $W(S)$.*

Proof. By Lemma III.2.1, the singularities of the quadrics in $W(H)$ are base points for $W(S)$. If the quadrics in $W(H)$ have different singularities, then $\text{Bl}(W(S))$ contains a curve. This is impossible by Lemma III.2.2. ■

We also need some facts about special linear systems of quadrics, beginning with pencils:

Lemma III.2.4. *Let P be a pencil of quadrics in $\mathbb{P}^n$ and assume that the generic member is a rank-2 quadric. Then one of the following is true:*

1. *There are no rank-1 quadrics in P , and $\text{Bl}(P)$ consists of a hyperplane H and a linear subspace $L \not\subset H$ of codimension 2;*
2. *There is one rank-1 quadric in P , and $\text{Bl}(P)$ consists of a hyperplane H and a double linear subspace $L \subset H$ of codimension 2;*
3. *There are two rank-1 quadrics in P , and $\text{Bl}(P)$ contains a double linear subspace L of codimension 2.*

Proof. The first two statements can be read off of the proof of [Hel20, Lemma 2.5]. The last statement follows by considering the two rank-1 quadrics as generators for P . ■

Using Lemma III.2.4 as the base case for induction, we generalise the statement to linear systems of any dimension:

Lemma III.2.5. *Let W be a linear system of quadrics in $\mathbb{P}^n$ and assume that the generic member is a rank-2 quadric. Then one of the following is true:*

1. *There are no rank-1 quadrics in W , and $\text{Bl}(W)$ contains a hyperplane;*
2. *There is one rank-1 quadric in W , and $\text{Bl}(W)$ contains a hyperplane;*
3. *There is a quadratic hypersurface of rank-1 points in W , and $\text{Bl}(W)$ contains a double linear subspace L of codimension 2.*

Proof. We prove this by induction on $\dim(W)$. The base case $\dim(W) = 1$ is true by Lemma III.2.4. Let $k := \dim(W)$ and assume that the assertions hold for linear systems of dimension less than or equal to $k - 1$.

1. If there are no rank-1 quadrics in W , then $\text{Bl}(W')$ contains a hyperplane for all proper, nonempty linear subsets $W' \subset W$. Let $W_1, W_2, W_3 \subset W$ be three linear subsets of codimension 1 containing the quadric $H_1 \cup H_2$. Then one of the hyperplanes H_1, H_2 is contained in at least two of the loci $\text{Bl}(W_1), \text{Bl}(W_2)$ and $\text{Bl}(W_3)$. That hyperplane is contained in $\text{Bl}(W)$.
2. If $2H$ is the only rank-1 quadric in W , then $H \subset \text{Bl}(W')$ for all proper, linear subsets $W' \subset W$ containing $2H$. Thus $H \subset \text{Bl}(W)$.
3. Assume that there is more than one rank-1 quadric in W and let $2H$ be one of them. Since there is no hyperplane contained in $\text{Bl}(W)$, by the previous case, there is at most one hyperplane $W' \subset W$ containing $2H$ and no other rank-1 quadrics. Hence W contains a hypersurface S of rank-1 quadrics. A generic hyperplane $W' \subset W$ intersects S in a quadric, so S has degree 2.

Restricting W to H defines a linear system W'' of quadrics in $\mathbb{P}^{n-1}$. Each line in W through $[2H]$ is collapsed to a point in W'' . For each point

in W'' , we can find a representative in S . It follows that W'' is a linear system of double hyperplanes in $\mathbb{P}^{n-1}$. Thus W'' consists of a single double hyperplane $L \subset \mathbb{P}^{n-1}$, which is contained in $\text{Bl}(W)$. ■

Lemma III.2.6. *Let W be a linear system of quadrics in $\mathbb{P}^3$ with base point p . Let $V \subset W$ be the subspace of quadrics singular at a point p . Suppose that $\dim(V) = \dim(W) - 2$. Then $\text{Bl}(W)$ contains a scheme of length 2 with support in p .*

Proof. Let Q_1, Q_2 be two quadrics that generate W together with V . Let P_i be a pencil generated by Q_i and a quadric Q'_i in V . Since p is a base point for P_i and Q'_i is singular at p , the base locus $\text{Bl}(P_i)$ is a quartic curve C_i with a singularity at p . Let T_i be the tangent plane of C_i at p . It follows that T_i is a common tangent plane to all the quadrics in P_i . Hence all quadrics in P_i contain the first order infinitesimal neighbourhood of p in T_i . Since all quadrics in V are singular at p , we get that $\text{Bl}(W)$ contains the first order infinitesimal neighbourhood of p in the line $T_1 \cap T_2$. ■

Remark III.2.7. Repeatedly in this paper, we encounter a symmetroid S where the base locus $\text{Bl}(W(S))$ contains four points. Therefore, we make a general remark here about the space X_{Bl} of all quadrics containing four base points $\text{Bl} := \{p_1, p_2, p_3, p_4\}$. We describe the rank-2 locus of X_{Bl} in the two cases where Bl consists of four general or four coplanar points.

In both cases, there are three quadratic surfaces, X_{12}, X_{13}, X_{14} in the rank-2 locus of X_{Bl} . Indeed, let H_{ij} and H_{kl} be two planes containing the lines $\langle p_i, p_j \rangle$ and $\langle p_k, p_l \rangle$, respectively. The set of pairs $H_{ij} \cup H_{kl}$ forms a smooth quadratic surface $X_{ij} \subset X_{\text{Bl}}$ of rank-2 points.

If Bl spans $\mathbb{P}^3$, let H_i be any plane passing through p_i . The set X_i of all unions $H_i \cup \langle p_j, p_k, p_l \rangle$ forms a plane in X_{Bl} . On the other hand, if the base points are coplanar, the rank-2 locus contains a linear 3-space X instead of the planes X_1, X_2, X_3, X_4 . The quadrics in X are the unions of any plane in $\mathbb{P}^3$ with the plane $\langle p_1, p_2, p_3, p_4 \rangle$. ♠

III.3 Maximal dimension of irreducible quadrics

In this section, we prove that the dimension of an irreducible quadric that can be a component in the singular locus of a quartic symmetroid is bounded by 2. Hence a quartic symmetroid with a double irreducible quadric of codimension 1 is at most 3-dimensional.

We give the first characterisation of singular quadrics of codimension 1 by modifying the proof of [Hel20, Proposition 4.1] slightly to be valid in any dimension:

Lemma III.3.1. *Let $S \subset \mathbb{P}^n$ be an irreducible, quartic symmetroid that is double along an irreducible, $(n-2)$ -dimensional quadric Q . Then Q is contained in the rank-2 locus of S .*

Proof. Assume for contradiction that Q is not contained in the rank-2 locus of S . A generic point in Q is then a rank-3 node. Let $[Q_0], \dots, [Q_{n-2}] \in Q$ be rank-3 points that span a linear subspace $H \subset \mathbb{P}^n$ of codimension 2. Suppose that $Q_0, \dots, Q_{n-2}$ share a common singular point p . Then all the quadrics in H are singular at p . By [Lemma III.2.1](#), this implies that H is contained in $\text{Sing}(S)$. Since Q is irreducible, H is not a component of Q . The space spanned by Q intersects S in Q and H , and it follows that it is contained in S . This is impossible since S is irreducible. We conclude that $Q_0, \dots, Q_{n-2}$ do not share a common singularity. Then there exists a curve of singularities of rank-3 nodes in Q . [Lemmas III.2.1](#) and [III.2.2](#) imply that S is reducible, which again is impossible. Hence Q is contained in the rank-2 locus of S . ■

[Lemma III.3.1](#) leads us to find the maximal dimension of an irreducible quadratic variety consisting of rank-2 points. We know *a priori* that the rank-2 quadrics form a 6-dimensional variety in the $\mathbb{P}^9$ of all quadrics in $\mathbb{P}^3$, but requiring that the quadratic variety should be a component in the rank-2 locus lowers the possible dimension:

Proposition III.3.2. *Let $S \subset \mathbb{P}^n$ be a quartic symmetroid and assume that Q is an $(n-2)$ -dimensional, singular quadratic component in the rank-2 locus of S . Assume that Q contains at most finitely many rank-1 points. Then $n \leq 3$.*

Proof. It suffices to show that $n \neq 4$. Assume for contradiction that $n = 4$. We proceed by eliminating case by case:

1. Assume that Q is reducible. Then Q contains a plane H . Because Q contains finitely many rank-1 points, it follows from [Lemma III.2.5](#) that $\text{Bl}(W(H))$ is a plane. Since $W(S)$ equals the span of $W(H)$ and two quadrics, we have that $\text{Bl}(W(S))$ contains four coplanar points. In the notation of [Remark III.2.7](#), $W(S)$ intersects X in Q . Bézout's theorem implies that the 3-space X is contained in the rank-2 locus of S , so Q is not a component in the rank-2 locus.
2. Assume that Q is an irreducible cone. Let $p := \text{Sing}(Q)$ be the apex of Q . Let $H_1 \cup H_2 \subset \mathbb{P}^3$ be the quadric corresponding to p . Since Q contains at most finitely many rank-1 points, there are at least three lines through p that correspond to pencils of type 1 or 2 in the numbering of [Lemma III.2.4](#). The base loci of each of these pencils contain either H_1 or H_2 . Hence there are two pencils having a plane in common, say H_1 . Then $H_1 \subseteq \text{Bl}(N)$, where N is the net spanned by these two pencils. The elements of N are rank-2 quadrics, so N corresponds to a plane contained in Q , contradicting the irreducibility. ■

Corollary III.3.3. *Let $S \subset \mathbb{P}^n$ be a quartic symmetroid and assume that Q is an $(n-2)$ -dimensional, smooth quadratic component in the rank-2 locus of S . Then $n \leq 4$.*

Proof. Assume for contradiction that $n = 5$, so $\dim(Q) = 3$. Let $p \in Q$ be a point, and L a linear subspace that strictly contains the tangent space $T_p Q$, but not Q . Let Q' and S' be the intersections $Q \cap T_p Q$ and $S \cap L$, respectively. Then Q' is a quadratic surface which is singular at p .

Assume for contradiction that Q contains a surface T of rank-1 points. Then the secant variety $\text{Sec}(T)$ is contained in the rank-2 locus of S , and is thus a component of either S or Q . This is impossible, since both are irreducible. Since Q does not contain a surface of rank-1 points, Q' contains at most finitely many rank-1 points for a general choice of p . Moreover, Q' is contained in the rank-2 locus of the quartic symmetroid S' . Thus Q' contradicts [Proposition III.3.2](#). ■

The assumption about finitely many rank-1 points in [Proposition III.3.2](#) is necessary, as shown below:

Example III.3.4. The matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 & 0 \\ x_1 & x_3 & 0 & x_4 \\ x_2 & 0 & 0 & x_2 + x_4 \\ 0 & x_4 & x_2 + x_4 & 0 \end{bmatrix}$$

defines a quartic symmetroid in $\mathbb{CP}^4$ which has the double plane $\mathcal{V}(x_2 + x_4, x_4^2)$ in its rank-2 locus and contains the smooth conic section $\mathcal{V}(x_2, x_4, x_0 x_3 - x_1^2)$ of rank-1 points.

A quartic symmetroid cannot contain a 3-space H of rank-2 points with a quadratic surface $Q \subset H$ of rank-1 points. Indeed, if that were the case, $\text{Bl}(W(H)) \subset \mathbb{P}^3$ would contain a line by [Lemma III.2.5](#). This is impossible, since a line in $\mathbb{P}^3$ is only common to a conic of rank-1 quadrics. This example has thus maximal dimension. ♦

III.3.1 Threefolds

Let S be a quartic symmetroid of maximal dimension that is singular along an irreducible quadric Q of codimension 1. Then S is a threefold and Q is a smooth surface of rank-2 points by [Corollary III.3.3](#). In this section, we describe the singular locus of S , first in general and then specialise to spectrahedral symmetroids. The existence of S is shown in [Example III.3.9](#).

Note first that the set of quadrics associated to Q has a nice description:

Remark III.3.5. Let $L_1, L_2 \subset \mathbb{P}^3$ be two lines, and let H_i be a plane containing L_i . The set of pairs $H_1 \cup H_2$ forms a smooth quadratic surface in the $\mathbb{P}^9$ of all quadrics in $\mathbb{P}^3$. Any smooth quadratic surface $Q \subset \mathbb{P}^9$ of rank-2 quadrics arises this way. Indeed, it is straightforward to see that Q contains at most one rank-1 quadric. Thus by [Lemma III.2.4](#), if P_1 and P_2 are pencils from each of the two rulings on Q , then $\text{Bl}(P_i)$ consists of a plane H_i and a line L_i . It follows that $L_1 \cup L_2 \subseteq \text{Bl}(Q)$. ♠

Since Q has codimension 1 in S , [Remark III.3.5](#) imposes significant conditions on $W(S)$, and thus on S :

Lemma III.3.6. *Let $S \subset \mathbb{P}^4$ be a general quartic symmetroid singular along a smooth quadratic surface Q . Then $\text{Bl}(W(S))$ consists of four general points.*

Proof. The linear system $W(S)$ is spanned by $W(Q)$ and a quadric $K \notin W(Q)$. By Remark III.3.5, $\text{Bl}(W(Q))$ is the union of two lines, L_1 and L_2 . The assertion follows since K intersects L_1 and L_2 in two points each. ■

Proposition III.3.7. *Let $S \subset \mathbb{P}^4$ be a general quartic symmetroid singular along a smooth quadratic surface Q . Then S is singular along two additional conic sections.*

Proof. Since S is general, the only other singularities are rank-2 points outside of Q . By Lemma III.3.6, $X_{\text{Bl}(W(S))} = X_{\text{Bl}}$ in the notation of Remark III.2.7. We can identify the rank-2 points of S with the intersection of $W(S)$ with the rank-2 locus of X_{Bl} .

Continuing with the notation of Remark III.2.7, consider the intersection $X_{ij} \cap X_k$. Since $X_{ij} = X_{kl}$, we may assume that the indices i, j and k are distinct. The intersection consists of all quadrics $H_{kl} \cup H$, where H_{kl} is any plane containing $\langle p_k, p_l \rangle$, and $H := \langle p_i, p_j, p_l \rangle$. Thus $X_{ij} \cap X_k$ is a line.

The linear system $W(S)$ is a hyperplane in $X_{\text{Bl}} = \mathbb{P}^5$. By Bézout's theorem, a generic hyperplane intersects X_{ij} in a conic section and X_i in a line. Suppose that the quadratic surface Q contained in $W(S)$ is X_{12} . Then $W(S)$ intersects X_{13} and X_{14} in a conic section each. Since X_{12} intersects each plane X_i in a line, $W(S)$ does not meet X_i outside of $X_{12} = Q$. ■

III.3.1.1 Spectrahedra

Proposition III.3.7 specifies the singularities of S considered as a complex variety. If S is a real, spectrahedral symmetroid, then some of those singularities are real and some of the real singularities lie on the boundary of the spectrahedron. Here we examine the possible configurations of real singularities.

If S has a nonempty spectrahedron, then $W(S)$ contains a positive definite quadric. It follows that $\text{Bl}(W(S))$ consists of complex conjugate pairs, $p_1, \bar{p}_1$ and $p_2, \bar{p}_2$. As pointed out in [HR18, Lemma 2.1], the real quadrics containing $\langle p_1, \bar{p}_1 \rangle$ and $\langle p_2, \bar{p}_2 \rangle$ are indefinite, whereas the real quadrics containing $\langle p_1, p_2 \rangle$ and $\langle \bar{p}_1, \bar{p}_2 \rangle$ or $\langle p_1, \bar{p}_2 \rangle$ and $\langle \bar{p}_1, p_2 \rangle$ are semidefinite. We rename X_{12}, X_{13} and X_{14} from Remark III.2.7 as X_i, X_{s1} and X_{s2} , indicating whether the real points are indefinite or semidefinite.

Proposition III.3.8. *Let $S \subset \mathbb{P}^4$ be a quartic spectrahedral symmetroid singular along a quadratic surface Q . Then S is also singular along two conic sections with real points, and one of the following is true:*

1. Q is disjoint from the spectrahedron, and both conics lie on the boundary of the spectrahedron.
2. Q and one of the conics lie on the boundary of the spectrahedron, while the other conic is disjoint from the spectrahedron.

Proof. Choose coordinates such that $p_1 := [1 : i : 0 : 0]$ and $p_2 := [0 : 0 : 1 : i]$. Then $X_{\text{Bl}(W(S))}$ is parametrised by the matrix

$$\begin{bmatrix} x_{00} & 0 & x_{02} & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ x_{02} & x_{12} & x_{22} & 0 \\ x_{03} & x_{13} & 0 & x_{22} \end{bmatrix},$$

and X_i is given by

$$\mathcal{V}(x_{00}, x_{22}, x_{02}x_{13} - x_{03}x_{12}). \quad (\text{III.3.1})$$

For the first case, assume that the quadratic surface Q contained in $W(S)$ is X_i . Then $W(S)$ is in the pencil of hyperplanes $\mathcal{V}(\delta x_{00} - \lambda x_{22})$. If either $\delta = 0$ or $\lambda = 0$, then the rank-2 locus of S contains two additional planes, so we may assume that $W(S)$ equals $\mathcal{V}(x_{00} - \lambda x_{22})$ for $\lambda \neq 0$. Hence $W(S)$ is parametrised by

$$\begin{bmatrix} x_{00} & 0 & x_{02} & x_{03} \\ 0 & x_{00} & x_{12} & x_{13} \\ x_{02} & x_{12} & \lambda x_{00} & 0 \\ x_{03} & x_{13} & 0 & \lambda x_{00} \end{bmatrix}.$$

Thus $W(S)$ intersects X_{s_1} and X_{s_2} in the conic sections

$$\begin{aligned} &\mathcal{V}(x_{02} + x_{13}, x_{03} - x_{12}, \lambda x_{00}^2 - x_{02}^2 - x_{12}^2), \\ &\mathcal{V}(x_{02} - x_{13}, x_{03} + x_{12}, \lambda x_{00}^2 - x_{02}^2 - x_{12}^2), \end{aligned}$$

both of which have real points if and only if $\lambda > 0$. Clearly, $W(S)$ contains a positive definite matrix for $\lambda > 0$, so S is spectrahedral. By Sylvester's criterion, $W(S)$ does not contain a positive definite matrix for $\lambda < 0$, in which case S is not spectrahedral.

For the second case, assume that the quadratic surface Q contained in $W(S)$ is X_{s_1} . The surfaces X_{s_1} and X_{s_2} intersect in the two points

$$[\langle p_1, \bar{p}_1, p_2 \rangle \cup \langle \bar{p}_1, p_1, \bar{p}_2 \rangle] \quad \text{and} \quad [\langle p_1, p_2, \bar{p}_2 \rangle \cup \langle \bar{p}_1, \bar{p}_2, p_2 \rangle],$$

which are both real. So any hyperplane containing X_{s_1} intersects X_{s_2} in a conic section with real points. Finally, by (III.3.1), all conic sections on X_i have real points. ■

Example III.3.9. The matrix

$$\begin{bmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_3 & x_4 \\ x_1 & x_3 & x_0 & 0 \\ x_2 & x_4 & 0 & x_0 \end{bmatrix}$$

defines a quartic spectrahedral symmetroid in $\mathbb{CP}^4$ which is singular along the smooth quadratic surface $\mathcal{V}(x_0, x_1x_4 - x_2x_3)$. The quadric is disjoint from the spectrahedron.

The matrix

$$\begin{bmatrix} x_0 & 0 & x_1 & x_2 \\ 0 & x_0 & x_2 & x_3 \\ x_1 & x_2 & x_4 & 0 \\ x_2 & x_3 & 0 & x_4 \end{bmatrix}$$

defines a quartic spectrahedral symmetroid in $\mathbb{CP}^4$ which is singular along the smooth quadratic surface $\mathcal{V}(x_1 + x_3, x_0x_4 - x_2^2 - x_3^2)$. The quadric lies on the boundary of the spectrahedron. $\blacklozenge$

III.3.2 Maximal dimension of cyclides

In the classical literature, a real quartic surface singular along a conic section without real points is known as a *cyclide* [Jes16, Chapter V]. Typically, the plane at infinity is chosen to be the plane spanned by the conic. Cyclides were originally studied by Dupin [Dup22], and subsequently by Darboux in a more general setting [Dar73]. In [HR18, Proposition 2.7], it is shown that a so-called *spindle cyclide* with two real nodes is the only type of cyclide that can occur as a spectrahedral symmetroid. We extend the definition of a cyclide to higher dimensions:

Definition III.3.10. A *cyclide* is a real, quartic hypersurface in $\mathbb{CP}^n$ which is singular along a smooth, quadratic $(n - 2)$ -fold with no real points.

We know from Corollary III.3.3 that a symmetroidal cyclide is at most 3-dimensional. In this section, we show that such a threefold does not exist.

Suppose for contradiction that there exists a symmetroid $S \subset \mathbb{CP}^4$ that is singular along a quadratic surface Q with no real points. Then Q is necessarily smooth and $\text{Bl}(W(Q))$ contains two lines, L_1 and L_2 , by Remark III.3.5. Since Q contains no real points, the associated quadric $H_1 \cup H_2$ at any point on Q is not the union of two real or two complex conjugate planes. This imposes conditions on L_1 and L_2 :

Lemma III.3.11. *A line $L \subset \mathbb{CP}^n$ is contained in a real plane if and only if L meets its complex conjugate line $\bar{L}$.*

Proof. A real line is trivially contained in a real plane. If L is not real and meets $\bar{L}$ in a point, then L and $\bar{L}$ span a plane which is real. If L is contained in a real plane H , then $\bar{L}$ is also contained in H and therefore meets L . $\blacksquare$

Lemma III.3.12. *Let $L_1, L_2 \subset \mathbb{CP}^n$ be two lines. Then there exists a plane H such that $L_1 \subset H$ and $L_2 \subset \bar{H}$ if and only if L_1 meets $\bar{L}_2$.*

Proof. If $L_1 = \bar{L}_2$, then H trivially exists. If $L_1 \neq \bar{L}_2$ meet, then they span a plane H . Hence, L_2 is contained in the complex conjugate plane $\bar{H}$. Conversely, if $L_1 \subset H$ and $L_2 \subset \bar{H}$, then $\bar{L}_2 \subset H$, so L_1 and $\bar{L}_2$ meet. $\blacksquare$

Remark III.3.13. Consider $\mathbb{P}^9$ as the set of quadrics in $\mathbb{P}^3$. Let $Q \subset \mathbb{P}^9$ be a smooth quadratic surface of rank-2 quadrics. By Remark III.3.5, $\text{Bl}(Q)$ consists

of two lines, L_1 and L_2 . Suppose that Q contains no real points. A rank-2 quadric is real if and only if it is the union of two real or two complex conjugate planes. By [Lemmas III.3.11](#) and [III.3.12](#), Q contains no real points if and only if $L_1 \cap \bar{L}_2 = \emptyset$ and at least one of the intersections $L_1 \cap \bar{L}_1$ or $L_2 \cap \bar{L}_2$ is empty. In particular, L_1 and L_2 are not complex conjugates and at most one of them is real. $\spadesuit$

All symmetroids that are cyclides are covered by the original 2-dimensional definition of a cyclide:

Proposition III.3.14. *Let $S \subset \mathbb{CP}^n$ be a symmetroidal cyclide. Then S is a surface.*

Proof. Assume for contradiction that S is a threefold. Let $Q \subseteq \text{Sing}(S)$ be the quadratic surface with no real points. Since S is real, so is Q . Hence $W(Q)$ spans a real web $W \subset \mathbb{P}^9$ of quadrics. Because W is real and linear, it is spanned by real quadrics. Thus $\text{Bl}(W)$ is real. Since W is spanned by $W(Q)$, we have $\text{Bl}(W) = \text{Bl}(W(Q))$. Hence the two lines in $\text{Bl}(W(Q))$ are either both real or complex conjugates. This contradicts [Remark III.3.13](#). $\blacksquare$

III.4 Maximal dimension of reducible quadrics

[Section III.3](#) finds the maximal dimension of irreducible quadratic components in the singular locus of a quartic symmetroid. It is natural to ask whether this is the maximal dimension among all double quadrics. We show here that there exists a quartic symmetroid $S \subset \mathbb{P}^5$ that is singular along a reducible, quadratic threefold Q and that S has maximal dimension.

We argue that the points in Q are generically rank-3 points: Let H be a 3-space in Q . Assume for contradiction that H is contained in the rank-2 locus of S . By [Example III.3.4](#), H cannot contain infinitely many rank-1 points. Because H contains finitely many rank-1 points, [Lemma III.2.5](#) implies that $\text{Bl}(W(H))$ contains a plane. Since $W(S)$ is generated by $W(H)$ and two quadrics, it follows that $\text{Bl}(W(S))$ consists of four coplanar points, so $X_{\text{Bl}(W(S))}$ is 5-dimensional. Hence $W(S) = X_{\text{Bl}(W(S))}$ and S is the discriminant of $X_{\text{Bl}(W(S))}$. In the notation of [Remark III.2.7](#), the only threefold that the discriminant of $X_{\text{Bl}(W(S))}$ is singular along is X , which is not a quadric.

Proposition III.4.1. *Let $S \subset \mathbb{P}^n$ be an irreducible, quartic symmetroid. Assume that Q is an $(n-2)$ -dimensional quadric in the singular locus of S . Then $n \leq 5$.*

Proof. Consider $\mathbb{P}^9$ as the set of all quadrics in $\mathbb{P}^3$. Our aim is to find a linear subspace $W \subset \mathbb{P}^9$ of maximal dimension such that the discriminant of W is singular along a quadric of dimension $\dim(W) - 2$.

Assume first that $Q := H_1 \cup H_2$ is the union of two linear subspaces of rank-3 points. [Lemma III.2.3](#) implies that the quadrics associated to H_i share a common singular point p_i , which is a base point for $W(S) \subset \mathbb{P}^9$. Hence $W(S) \subset X_{\{p_1, p_2\}} = \mathbb{P}^7$. Let $Y_i \subset X_{\{p_1, p_2\}}$ be the 4-space of quadrics that are

singular at p_i . We identify H_i with $W(S) \cap Y_i$. The set of quadrics that are singular at both p_1 and p_2 forms a double plane. Hence Y_1 and Y_2 intersect in a plane, so the span $\langle Y_1 \cup Y_2 \rangle$ of $Y_1 \cup Y_2$ is a 6-space. All quadrics in $\langle Y_1 \cup Y_2 \rangle$ contain the line spanned by p_1 and p_2 , so the discriminant of $\langle Y_1 \cup Y_2 \rangle$ is reducible by [Lemma III.2.2](#). Thus $W \neq \langle Y_1 \cup Y_2 \rangle$ and we conclude that $n \leq 5$.

By [Lemma III.2.6](#), if Q is a double linear subspace, the situation is analogous. Indeed, $\text{Bl}(W(S))$ contains a scheme $2p$ of length 2 with support in one point p . Let $Y \subset X_{2p} = \mathbb{P}^7$ be the 4-space of quadrics that are singular at p and contain the line spanned by $2p$. Let D be the discriminant of X_{2p} . The singular locus of D contains a double subspace $2Y$ whose underlying, reduced space is Y . The fourfold $2Y$ spans a 6-space. By letting $2Y$ play the role of $Y_1 \cup Y_2$, we get the same conclusion as in the previous paragraph. ■

III.4.1 Fourfolds

Let S be a quartic symmetroid of maximal dimension that is singular along a reducible quadric of codimension 1 in S . Then S is at most 4-dimensional by [Proposition III.4.1](#). In this section, we describe the rest of the singular locus of such a fourfold. The existence is demonstrated in [Examples III.4.4](#) and [III.4.7](#).

Proposition III.4.2. *Let $S \subset \mathbb{P}^5$ be a general quartic symmetroid singular along two 3-spaces, H_1 and H_2 . Then S is singular along an additional smooth quadratic surface.*

Proof. Since S is general, the only other singularities are rank-2 points outside of H_1 and H_2 . We can identify the rank-2 points of S with the intersection of $W(S)$ with the rank-2 locus of $X_{\text{Bl}(W(S))}$.

[Lemma III.2.3](#) tells us that the quadrics associated to H_i , for $i = 1, 2$, have a common singularity p_i . By applying [Lemma III.2.6](#) to H_1 and H_2 , we get that $\text{Bl}(W(S))$ consists of two schemes $2p_i$ of length 2 with support in p_i , for $i = 1, 2$. Since $\text{Bl}(W(S))$ is of length 4, $X_{\text{Bl}(W(S))}$ is a degeneration of X_{Bl} in [Remark III.2.7](#). Moreover, $W(S) = X_{\text{Bl}(W(S))}$.

For a rank-2 quadric $H \cup H' \subset \mathbb{P}^3$ to contain the scheme $2p_1$, there are two possibilities: Either the line spanned by $2p_1$ is contained in the plane H , or both H and H' contain p_1 . In the latter case, $H \cup H'$ is singular at p_1 , so $[H \cup H']$ is contained in H_1 , the set of all quadrics in $W(S)$ singular at p_1 . Hence only one of the quadratic surfaces described in [Remark III.2.7](#) is not contained in H_1 or H_2 : The set of all unions $H \cup H'$, where H contains the line spanned by $2p_1$ and H' contains the line spanned by $2p_2$. ■

Proposition III.4.3. *Let $S \subset \mathbb{P}^5$ be a general quartic symmetroid with a double 3-space $2H$ in $\text{Sing}(S)$. Then S is singular along an additional plane.*

Proof. Since S is general, the only other singularities are rank-2 points outside of $2H$. We can identify the rank-2 points of S with the intersection of $W(S)$ with the rank-2 locus of $X_{\text{Bl}(W(S))}$.

Recall from the discussion at the start of [Section III.4](#) that the points in H are generically rank-3 points. Thus [Lemma III.2.3](#) implies that the quadrics associated to H are singular at a point p . By [Lemma III.2.6](#), $\text{Bl}(W(S))$ contains a scheme $2p$ of length 2 with support in p , so $X_{\text{Bl}(W(S))} \subseteq X_{2p}$. For a rank-2 quadric $H_1 \cup H_2 \subset \mathbb{P}^3$ to contain $2p$, there are two possibilities: Either the line L spanned by $2p$ is contained in the plane H_1 , or both H_1 and H_2 contain p . In the latter case, $H_1 \cup H_2$ is singular at p , so $[H_1 \cup H_2]$ is contained in H . Hence if $H_1 \cup H_2$ is the associated quadric to a rank-2 point outside of H , then H_1 contains L .

The set of pairs $H_1 \cup H_2$, where H_1 contains L , forms a fourfold Z_L of degree 4 in $X_{2p} = \mathbb{P}^7$. The singular locus of Z_L is the plane of pairs $H_1 \cup H_2$, where both H_1 and H_2 contain L . Let $Y \subset X_{2p}$ be the 4-space of quadrics that are singular at p and contains L , as in the proof of [Proposition III.4.1](#). We identify H as $Y \cap W(S)$. The intersection $Z_L \cap Y$ is the set of all unions $H_1 \cup H_2$, where H_1 contains L and H_2 contains p . This is a divisor on Z_L that contains $\text{Sing}(Z_L)$; it is a threefold of degree 3. The linear system $W(S) = \mathbb{P}^5$ intersects Z_L in a quartic surface. Since $W(S)$ intersects Y in the 3-space H , it meets $Z_L \cap Y$ in a cubic surface. Hence $W(S)$ intersects Z_L in a plane outside of H . This proves the claim. $\blacksquare$

Example III.4.4. The matrix

$$\begin{bmatrix} x_0 + x_1 & x_0 + x_2 & x_3 & x_2 \\ x_0 + x_2 & x_0 - x_1 & x_3 & x_2 \\ x_3 & x_3 & x_4 & x_5 \\ x_2 & x_2 & x_5 & 0 \end{bmatrix}$$

defines a quartic symmetroid $S \subset \mathbb{CP}^5$ whose Jacobian ideal is contained in the ideal $\langle x_2^2, x_5 \rangle$. In other words, the singular locus of S contains a double $\mathbb{P}^3$. The symmetroid is also singular along the plane $H_1 := \mathcal{V}(x_0, x_1, x_2)$.

The rank-2 locus of S consists of H_1 , three times the plane $H_2 := \mathcal{V}(x_2, x_4, x_5)$ and six times the plane $H_3 := \mathcal{V}(x_1, x_2, x_5)$. Note that H_3 equals $\text{Sing}(Z_L)$ in the proof of [Proposition III.4.3](#). It contains the conic $\mathcal{V}(x_1, x_2, x_5, x_0x_4 - x_3^2)$ of rank-1 points. $\blacklozenge$

Remark III.4.5. If S is a quartic symmetroid singular along the union of two distinct linear subspaces with codimension 1 in S , then [Lemma III.2.6](#) implies that $\text{Bl}(W(S))$ is a scheme of length 4. Hence $X_{\text{Bl}(W(S))} = \mathbb{P}^5$. If S is a fourfold, it follows that $W(S) = X_{\text{Bl}(W(S))}$. Thus S is the discriminant of $X_{\text{Bl}(W(S))}$, so it is uniquely determined by $\text{Bl}(W(S))$.

However, if $S \subset \mathbb{P}^5$ is a quartic symmetroid with a double 3-space in its singular locus, then $\text{Bl}(W(S))$ is too small to be unique for S . In particular, we compute that $\text{Bl}(W(S))$ is a scheme of length 3 for S in [Example III.4.4](#). $\spadesuit$

III.4.1.1 Spectrahedra

Here we examine the real singularities in the case of spectrahedral symmetroids. If S has a nonempty spectrahedron, then the base points of $W(S)$ appear in

complex conjugate pairs. It follows that the 3-spaces in the singular locus of S are also complex conjugates, since they correspond to quadrics in $W(S)$ that are singular at one of the base points. This means that $\text{Sing}(S)$ cannot contain a double 3-space, only two distinct 3-spaces.

Proposition III.4.6. *Let $S \subset \mathbb{P}^5$ be a quartic spectrahedral symmetroid singular along two complex conjugate 3-spaces, H and $\overline{H}$. The plane $H \cap \overline{H}$ of real points intersects the spectrahedron in a region bounded by a conic section of rank-1 points.*

The symmetroid S is also singular along a smooth quadratic surface Q whose real points lie on the spectrahedron.

Proof. The plane $H \cap \overline{H}$ corresponds to quadrics that are singular along the real line L spanned by the two reduced base points of $W(S)$, p and $\bar{p}$. These include all real double planes containing L , the set of which corresponds to a conic section C of real rank-1 points in $H \cap \overline{H}$. Let l_1, l_2 be some linear forms such that $[\mathcal{V}(l_1^2)]$ and $[\mathcal{V}(l_2^2)]$ are two points in C . Consider the line spanned by the points: The quadric $\mathcal{V}(al_1^2 + bl_2^2)$ is semidefinite if and only if the constants a and b have the same sign. Hence the intersection between $H \cap \overline{H}$ and the spectrahedron is bounded by C .

From the proof of [Proposition III.4.2](#), we know that the points in Q correspond to pairs of planes $H_1 \cup H_2$, where H_1 contains the line spanned by $2p$ and H_2 contains the line spanned by $2\bar{p}$. If $H_1 \cup H_2$ is real, then $H_1 = \overline{H_2}$, hence $H_1 \cup H_2$ is semidefinite. ■

Example III.4.7. The matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_4 & -x_3 & x_2 \\ x_2 & -x_3 & x_5 & 0 \\ x_3 & x_2 & 0 & x_5 \end{bmatrix}$$

defines a quartic spectrahedral symmetroid in $\mathbb{CP}^5$ which is singular along the two complex conjugate 3-spaces $\mathcal{V}(x_2 \pm ix_3, x_5)$ and the quadratic surface $\mathcal{V}(x_0 - x_4, x_1, x_2^2 + x_3^2 - x_4x_5)$. ◆

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Paper IV

Determinantal quartic surfaces
with a definite Hermitian
representation

Martin Helsø

arXiv: [2007.01121](#).

Abstract

We give a bound on the number of isolated, essential singularities of determinantal quartic surfaces in 3-space. We also provide examples of different configurations of real singularities on quartic surfaces with a definite Hermitian determinantal representation, and conjecture an extension of a theorem by Degtyarev and Itenberg.

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IV.1 Introduction

Representing a polynomial as the determinant of a linear matrix is a problem dating back at least to Hesse [Hes44]. Determinantal representations have applications in areas such as linear algebra, operator theory, convex optimisation and algebraic geometry.

A homogeneous polynomial $f \in \mathbb{C}[x_0, \dots, x_n]_d$ of degree d has a *determinantal representation* if $f(\mathbf{x}) = \det(M_d(\mathbf{x}))$, where

$$M_d(\mathbf{x}) := M_d(x_0, \dots, x_n) := M_{d,0}x_0 + \dots + M_{d,n}x_n \quad (\text{IV.1.1})$$

for some $(d \times d)$ -matrices $M_{d,0}, \dots, M_{d,n}$. We say that the representation is *Hermitian* if $M_{d,0}, \dots, M_{d,n}$ are Hermitian matrices, and it is *symmetric* if $M_{d,0}, \dots, M_{d,n}$ are symmetric matrices. The hypersurface $\mathcal{V}(f) \subset \mathbb{CP}^n$ is called *determinantal* if f possesses a determinantal representation; $V(f)$ is called a *symmetroid* if the representation is symmetric.

Let $f \in \mathbb{R}[x_0, \dots, x_n]_d$ be a real polynomial with a Hermitian determinantal representation (IV.1.1). The representation is *definite* if the matrix $M_d(\mathbf{e})$ is positive definite for some point $\mathbf{e} \in \mathbb{RP}^n$. The eigenvalues of a Hermitian matrix are real. It follows that every real line through $\mathbf{e}$ only meets the hypersurface $\mathcal{V}(f) \subset \mathbb{CP}^n$ in real points. A polynomial with this property is called *hyperbolic with respect to $\mathbf{e}$* . The connected component of $\mathbf{e}$ in $\mathbb{RP}^n \setminus V_{\mathbb{R}}(f)$ is called the *hyperbolicity cone* of f with respect to $\mathbf{e}$.

If $M_{d,0}, \dots, M_{d,n}$ in (IV.1.1) are real, symmetric matrices, then the set

$$\{\mathbf{x} \in \mathbb{RP}^n \mid M_d(\mathbf{x}) \text{ is semidefinite}\}$$

is called a *spectrahedron*. It is easy to see that all spectrahedra are hyperbolicity cones. The converse statement, that all hyperbolicity cones are spectrahedra, is called the generalised Lax conjecture and is an object of much interest. A partial result is the Helton–Vinnikov theorem, which implies that all hyperbolicity cones in $\mathbb{RP}^2$ are spectrahedra [HV07; LPR05]. Let $f \in \mathbb{R}[x_0, \dots, x_n]_d$ be a polynomial with a definite Hermitian representation (IV.1.1), and suppose that $\mathbf{e} \in \mathbb{RP}^n$ is such that $M_d(\mathbf{e})$ is positive definite. We note that the hyperbolicity cone of f with respect to $\mathbf{e}$ is a spectrahedron, by following [PV13, Corollary 5.1]: We can write $M_d(\mathbf{x}) = A_d(\mathbf{x}) + \mathbf{i}B_d(\mathbf{x})$, where $A_d(\mathbf{x})$ is real symmetric and $B_d(\mathbf{x})$ is real antisymmetric. We define the real symmetric $(2d \times 2d)$ -matrix

$$A_{2d}(\mathbf{x}) := \begin{bmatrix} A_d(\mathbf{x}) & B_d^T(\mathbf{x}) \\ B_d(\mathbf{x}) & A_d(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} A_d(\mathbf{x}) & -B_d(\mathbf{x}) \\ B_d(\mathbf{x}) & A_d(\mathbf{x}) \end{bmatrix}. \quad (\text{IV.1.2})$$

Let

$$U := \frac{\sqrt{2}}{2} \begin{bmatrix} I_d & \mathbf{i}I_d \\ \mathbf{i}I_d & I_d \end{bmatrix},$$

where I_d is the identity matrix of size d . After the change of coordinates

$$\bar{U}A_{2d}U^T = \begin{bmatrix} A + \mathbf{i}B & 0 \\ 0 & A - \mathbf{i}B \end{bmatrix} = \begin{bmatrix} M_d & 0 \\ 0 & \bar{M}_d \end{bmatrix},$$

we see that

$$\det(A_{2d}(\mathbf{x})) = \det(M_d(\mathbf{x})) \det(\overline{M}_d(\mathbf{x})) = f^2.$$

The polynomials f and f^2 have the same hyperbolicity cone with respect to $\mathbf{e}$, which is the spectrahedron defined by $A_{2d}(\mathbf{x})$.

The *rank* and *corank* of a point $\mathbf{x} \in \mathbb{CP}^n$ are defined as $\text{rank } M_d(\mathbf{x})$ and $\text{corank } M_d(\mathbf{x})$, respectively. The rank- k locus of $M_d(\mathbf{x})$ is the set of points in $\mathbb{CP}^n$ with rank less than or equal to k . The hypersurface $\mathcal{V}(f)$ is equal to the rank- $(d-1)$ locus. The points $\mathbf{x} \in \mathcal{V}(f)$ with $\text{corank}(\mathbf{x}) \geq 2$ with respect to $M_d(\mathbf{x})$ are always singular on $\mathcal{V}(f)$, and they are called *essential singularities*. A point $\mathbf{x}$ with $\text{corank}(\mathbf{x}) = 1$ is generally not singular, but if $\mathbf{x} \in \text{Sing } \mathcal{V}(f)$, then $\mathbf{x}$ is called an *accidental singularity*. The multiplicity of a point $\mathbf{x} \in \mathcal{V}(f)$ is greater than or equal to its corank. Since the rank- $(d-2)$ locus is given by the vanishing of the $(d-1) \times (d-1)$ -minors of $M_d(\mathbf{x})$, the singular locus of $\mathcal{V}(f)$ is at least $(n-4)$ -dimensional, and it is at least $(n-3)$ -dimensional if $\mathcal{V}(f)$ is a symmetroid. Moreover, if $\mathcal{V}(f)$ is a generic symmetroid of degree d , then $\text{Sing } \mathcal{V}(f)$ has degree $\binom{d+1}{3}$ and contains no accidental singularities [Pio06, Proposition 1.5; Sal65, p. 420].

We restrict the attention to quartic determinantal surfaces in $\mathbb{P}^3$. A generic determinantal surface is smooth, while a generic quartic symmetroid has ten rank-2 points that are nodes, that is, isolated quadratic singularities. A nodal quartic symmetroid is called *transversal* if it has ten rank-2 nodes and no further singularities. The study of quartic symmetroids originated with Cayley [Cay69a]. Recently, real quartic symmetroids with a nonempty spectrahedron have gained attention. Using the global Torelli theorem for K3-surfaces, Degtyarev and Itenberg proved the following:

Theorem IV.1.1 ([DI11, Theorem 1.1]). *There exists a real transversal quartic symmetroid with a nonempty spectrahedron, having ρ real nodes, of which σ nodes lie on the boundary of the spectrahedron, if and only if $0 \leq \sigma \leq \rho$, both even, and $2 \leq \rho \leq 10$.*

Ott et al. presented an algorithmic proof of Theorem IV.1.1, and for each pair (ρ, σ) satisfying the inequalities in the theorem, they gave an example of a symmetroid with the corresponding configuration of nodes [Ott+15]. An analogue to Theorem IV.1.1 for rational quartic symmetroids is proven in [HR18, Theorem 1.7].

We prove the following bound on the number of isolated rank-2 points of a determinantal quartic surface in Section IV.2:

Theorem IV.1.2. *Let $S_4 \subset \mathbb{P}^3$ be a determinantal quartic surface with only isolated, simple singularities. Let η be the number of essential singularities of S_4 . Then $\eta \leq 8$, unless S_4 is a symmetroid, in which case $\eta = 10$.*

We expect that Theorem IV.1.2 is well-known, but we have not been able to find a reference in the literature. As remarked above, a definite Hermitian determinantal representation gives rise to a spectrahedron. The real singularities may lie on

IV. Determinantal quartic surfaces with a definite Hermitian representation

or off the spectrahedron. We surmise a generalisation of [Theorem IV.1.1](#) to Hermitian representations:

Conjecture IV.1.3. *The following statements are equivalent:*

- *There exists a real quartic surface $\mathcal{V}(f) \subset \mathbb{RP}^3$, where*
 - *f admits a definite Hermitian determinantal representation $M_4(\mathbf{x})$;*
 - *the complex surface $V_{\mathbb{C}}(f) \subset \mathbb{CP}^3$ has η isolated nodes, all of which are essential with respect to $M_4(\mathbf{x})$;*
 - *$\mathcal{V}(f)$ has ρ real nodes, of which σ real nodes lie on the spectrahedron defined by $M_4(\mathbf{x})$.*
- *$0 \leq \sigma \leq \rho \leq \eta \leq 8$ and $\rho \equiv \eta \pmod{2}$, or $0 \leq \sigma \leq \rho \leq \eta = 10$, $\rho \geq 2$ and $\sigma \equiv \rho \equiv \eta \pmod{2}$.*

In [Section IV.3](#), we consider the real singularities of quartic surfaces with a Hermitian determinantal representation. In particular, [Section IV.3.1](#) explains why we expect fewer restrictions on ρ and σ for $0 \leq \eta \leq 8$ than for $\eta = 10$ in [Conjecture IV.1.3](#). After that, we describe our strategy for finding examples of surfaces with a given triple (η, ρ, σ) . [Table IV.1](#) shows the progress towards proving [Conjecture IV.1.3](#). The existence of all cases for $\eta = 10$ is given by [Theorem IV.1.1](#), and explicit examples are given in [\[Ott+15\]](#). The known examples for $0 \leq \eta \leq 8$ are listed in [Section IV.4](#).

Table IV.1: Progress on [Conjecture IV.1.3](#).

η	(ρ, σ)
0	Known examples: (0, 0)
	Missing examples: None
1	Known examples: (1, 1), (1, 0)
	Missing examples: None
2	Known examples: (2, 2), (2, 1), (2, 0), (0, 0)
	Missing examples: None
3	Known examples: (3, 3), (3, 2), (3, 1), (3, 0), (1, 1), (1, 0)
	Missing examples: None
4	Known examples: (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (2, 2), (2, 1), (0, 0)
	Missing examples: (2, 0)
5	Known examples: (5, 3), (5, 2), (5, 1)
	Missing examples: (5, 5), (5, 4), (5, 0), (3, 3), (3, 2), (3, 1), (3, 0), (1, 1), (1, 0)

Continued on the next page.

η	(ρ, σ)
6	Known examples: (6, 4), (6, 3)
	Missing examples: (6, 6), (6, 5), (6, 2), (6, 1), (6, 0), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (2, 2), (2, 1), (2, 0), (0, 0)
7	Known examples: (7, 5), (7, 4)
	Missing examples: (7, 7), (7, 6), (7, 3), (7, 2), (7, 1), (7, 0), (5, 5), (5, 4), (5, 3), (5, 2), (5, 1), (5, 0), (3, 3), (3, 2), (3, 1), (3, 0), (1, 1), (1, 0)
8	Known examples: (8, 5), (8, 4), (6, 4)
	Missing examples: (8, 8), (8, 7), (8, 6), (8, 3), (8, 2), (8, 1), (8, 0), (6, 6), (6, 5), (6, 3), (6, 2), (6, 1), (6, 0), (4, 4), (4, 3), (4, 2), (4, 1), (4, 0), (2, 2), (2, 1), (2, 0), (0, 0)
10	Known examples: (10, 10), (10, 8), (10, 6), (10, 4), (10, 2), (10, 0), (8, 8), (8, 6), (8, 4), (8, 2), (8, 0), (6, 6), (6, 4), (6, 2), (6, 0), (4, 4), (4, 2), (4, 0), (2, 2), (2, 0)
	Missing examples: None

IV.2 Essential singularities on determinantal quartic surfaces

A quartic surface $\mathcal{V}(f) \subset \mathbb{P}^3$ with only isolated singularities has between zero and sixteen nodes. If f has a symmetric determinantal representation, exactly ten of the nodes — counted with multiplicity — are essential singularities. For each of the sixteen nodes on a Kummer surface $\mathcal{V}(f)$, there exists a symmetric determinantal representation of f such that the node is essential in that representation [Ott+15, p. 597]. On the other hand, [Jes16, Article 9] describes a quartic symmetroid with eleven nodes, where one of the nodes is an accidental singularity in every symmetric determinantal representation. It is natural to ask how many nodes can be essential singularities when we consider *nonsymmetric* determinantal representations. We show that the maximum number of isolated, essential singularities is obtained precisely with a symmetric determinantal representation.

Determinantal quartic surfaces are characterised by containing a projectively normal sextic curve with genus 3 [Sch81]. Coble noted that this follows because the Picard group of a general quartic surface is generated by a plane section [Cob82, p. 39]. Given a determinantal representation

$$M_4(\mathbf{x}) := \begin{bmatrix} m_{00} & m_{10} & m_{20} & m_{30} \\ m_{01} & m_{11} & m_{21} & m_{31} \\ m_{02} & m_{12} & m_{22} & m_{32} \\ m_{03} & m_{13} & m_{23} & m_{33} \end{bmatrix}$$

of a quartic surface $S_4 := \mathcal{V}(\det(M_4(\mathbf{x})))$, two families of genus 3 sextics on S_4 can be described as follows: The (3×3) -minors of a (4×3) -submatrix of

IV. Determinantal quartic surfaces with a definite Hermitian representation

$M_4(\mathbf{x})$ define a sextic curve of genus 3; the four curves obtained by the different (4×3) -submatrices span the first family $\mathcal{C}_1$. The second family $\mathcal{C}_2$ is defined similarly by replacing the (4×3) -submatrices with (3×4) -submatrices. If $M_4(\mathbf{x})$ is symmetric, $\mathcal{C}_1$ and $\mathcal{C}_2$ coincide; if the representation is Hermitian, the two families are complex conjugates. Because these curves are given by the vanishing of some (3×3) -minors, they contain the rank-2 locus of S_4 . That is, they contain the set of essential singularities on S_4 . Moreover, for each curve $C_1 \in \mathcal{C}_1$ there is a curve $C_2 \in \mathcal{C}_2$ such that $C_1 \cup C_2$ is the complete intersection of S_4 and a cubic surface S_3 . In particular, the union of the curves defined by the (3×3) -minors of

$$A_1 := \begin{bmatrix} m_{00} & m_{10} & m_{20} \\ m_{01} & m_{11} & m_{21} \\ m_{02} & m_{12} & m_{22} \\ m_{03} & m_{13} & m_{23} \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} m_{00} & m_{10} & m_{20} & m_{30} \\ m_{01} & m_{11} & m_{21} & m_{31} \\ m_{02} & m_{12} & m_{22} & m_{32} \end{bmatrix}$$

is the intersection of S_4 and the surface S_3 defined by their common (3×3) -minor

$$\begin{vmatrix} m_{00} & m_{10} & m_{20} \\ m_{01} & m_{11} & m_{21} \\ m_{02} & m_{12} & m_{22} \end{vmatrix} = 0.$$

The curves in $\mathcal{C}_1$ and $\mathcal{C}_2$ are nonhyperelliptic. Indeed, a sextic curve of genus 3 in $\mathbb{P}^3$ is nonhyperelliptic if and only if it is projectively normal [Dol12, Exercise 4.10]. We give an elementary argument showing that the curve C defined by A_1 , is nonhyperelliptic: Suppose that the entries m_{ij} in A_1 are linear forms in the variables x_0, x_1, x_2 and x_3 . Note that C is the solution set to the equation

$$A_1 \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{IV.2.1})$$

We can rewrite (IV.2.1) as

$$A'_1 \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (\text{IV.2.2})$$

where A'_1 is a (4×4) -matrix with linear entries in y_0, y_1 and y_3 . Both (IV.2.1) and (IV.2.2) define the same curve K in $\mathbb{P}^2 \times \mathbb{P}^3$. Then C is the projection of K to $\mathbb{P}^3$, and the curve C' given by $\det(A'_1) = 0$ is the projection to $\mathbb{P}^2$. The curve C' is a smooth planar quartic curve, hence nonhyperelliptic [Har77, Example IV.5.2.1]. It follows that C is nonhyperelliptic as well.

We are now ready to prove the bound on the possible number of essential singularities on a quartic surface.

Proof of Theorem IV.1.2. Without loss of generality, we may assume that S_4 has only essential singularities, $P_1, \dots, P_\eta$. Since S_4 is determinantal, there

are smooth, sextic curves $C_1, C_2 \subset S_4$ of genus 3 passing through $P_1, \dots, P_\eta$. Moreover, we may assume that $C_1 \cup C_2 = S_4 \cap S_3$ for some cubic surface S_3 .

Let $\pi: \tilde{S}_4 \rightarrow S_4$ be the blow-up of S_4 at $P_1, \dots, P_\eta$. Then $\tilde{S}_4$ is a smooth K3-surface. The exceptional divisor E_i over P_i is a (-2) -curve satisfying $h \cdot E_i = 0$, where h is the class of the preimage of a plane section of S_4 . Because C_1 passes through P_i , the strict transform $\tilde{C}_1$ satisfies $\tilde{C}_1 \cdot E_i = 1$. Furthermore, the adjunction formula gives

$$\tilde{C}_1 \cdot \tilde{C}_1 = 2g_{\tilde{C}_1} - 2 = 2 \cdot 3 - 2 = 4,$$

since the canonical divisor on $\tilde{S}_4$ is trivial. It follows from $C_1 \cup C_2 = S_4 \cap S_3$ that the total transform $\pi^{-1}(C_1 \cup C_2) = 3h$. The curve $C_1 \cup C_2$ is double at $P_1, \dots, P_\eta$, so $\pi^{-1}(C_1 \cup C_2) = \tilde{C}_1 + \tilde{C}_2 - \sum_{i=1}^\eta E_i$. Thus

$$\tilde{C}_1 + \tilde{C}_2 = \pi^{-1}(C_1 \cup C_2) - \sum_{i=1}^\eta E_i = 3h - \sum_{i=1}^\eta E_i. \quad (\text{IV.2.3})$$

We intersect both sides of (IV.2.3) with $\tilde{C}_1$:

$$\tilde{C}_1 \cdot (\tilde{C}_1 + \tilde{C}_2) = \tilde{C}_1 \cdot \left(3h - \sum_{i=1}^\eta E_i \right). \quad (\text{IV.2.4})$$

By using the facts above and that $\tilde{C}_1 \cdot h = \deg(\tilde{C}_1) = 6$, Equation (IV.2.4) yields $\tilde{C}_1 \cdot \tilde{C}_2 = 14 - \eta$.

The linear system $|\tilde{C}_1|$ gives rise to a morphism $\varphi: \tilde{S}_4 \rightarrow \mathbb{P}^3$. Because $\tilde{C}_1$ is a nonhyperelliptic curve of genus 3 with $\tilde{C}_1 \cdot \tilde{C}_1 = 4$, it is mapped to a plane curve $C'_1 := \varphi(\tilde{C}_1)$ of degree 4. Thus the image $S'_4 := \varphi(\tilde{S}_4)$ is a quartic surface. Assume first that S_4 is not a symmetroid, so $C_1 \neq C_2$. Then $C'_2 := \varphi(\tilde{C}_2)$ is not a plane section. Because C'_2 is nonhyperelliptic and spans $\mathbb{P}^3$, we have $\deg(C'_2) \geq 6$. Since C'_1 is a plane section, we have $\deg(C'_2) = \tilde{C}_1 \cdot \tilde{C}_2$. Therefore

$$14 - \eta = \tilde{C}_1 \cdot \tilde{C}_2 \geq 6, \quad (\text{IV.2.5})$$

so we get $\eta \leq 8$.

Assume now that S_4 is a symmetroid, so $C_1 = C_2$. Then $\tilde{C}_1 \cdot \tilde{C}_2 = \tilde{C}_1 \cdot \tilde{C}_1 = 4$. Adjusting for this in the right-hand side of (IV.2.5) gives $\eta = 10$. Hence we have recovered the well-known fact that a quartic symmetroid with only isolated singularities has ten rank-2 points. $\blacksquare$

IV.3 Real singularities on quartics with a Hermitian determinantal representation

Consider $\mathbb{CP}^{15}$ as the projectivisation of the vector space over $\mathbb{C}$ spanned by Hermitian (4×4) -matrices. Then the rank-2 locus X_2 of $\mathbb{CP}^{15}$ is given by

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the vanishing of the (3×3) -minors of a general Hermitian (4×4) -matrix; it is an elevenfold of degree 20. A quartic surface $S_4 \subset \mathbb{P}^3$ with a Hermitian determinantal representation corresponds to a linear 3-space $H \subset \mathbb{CP}^{15}$, and essential singularities on S_4 corresponds to the intersection of H with X_2 . This is helpful for finding examples of Hermitian determinantal representations with a specific singular locus.

In [Hel19; Hel20; HR18], symmetroids are studied via the associated quadratic form. We wish to use this technique to determine the real part of X_2 . Let $\mathbf{y} := [y_0, y_1, y_2, y_3]$. The form $h_{M_4}(\mathbf{y}) := \bar{\mathbf{y}} M_4 \mathbf{y}^T$ associated to a Hermitian (4×4) -matrix M_4 is not polynomial in y_0, y_1, y_2 and y_3 . However, we can associate a quadratic form to the symmetric matrix A_8 from (IV.1.2). Incidentally, $h_{M_4} = \mathbf{y}' A_8 \mathbf{y}'^T$, where

$$\mathbf{y}' := [\operatorname{Re}(y_0), \operatorname{Re}(y_1), \operatorname{Re}(y_2), \operatorname{Re}(y_3), \operatorname{Im}(y_0), \operatorname{Im}(y_1), \operatorname{Im}(y_2), \operatorname{Im}(y_3)].$$

Since M_4 is Hermitian, it is unitarily diagonalisable. If M_4 is diagonal, then so is A_8 . From this we see that M_4 and A_8 have the same eigenvalues λ_i , but the algebraic multiplicity $\mu_{A_8}(\lambda_i)$ is $2\mu_{M_4}(\lambda_i)$. Hence $\operatorname{rank} A_8 = 2\operatorname{rank}(M_4)$. In addition, if $M_4(\mathbf{x})$ is a Hermitian determinantal representation of a real polynomial f , then we can define its spectrahedron in terms of $M_4(\mathbf{x})$ only, because

$$\{\mathbf{x} \in \mathbb{RP}^3 \mid M_4(\mathbf{x}) \text{ is semidefinite}\} = \{\mathbf{x} \in \mathbb{RP}^3 \mid A_8(\mathbf{x}) \text{ is semidefinite}\}.$$

The construction of A_8 from M_4 allows us to view the point

$$\mathbf{x} := [x_{00} : x_{01} : x_{02} : x_{03} : x_{11} : x_{12} : x_{13} : x_{22} : x_{33} : y_{01} : y_{02} : y_{03} : y_{12} : y_{13} : y_{23}]$$

in $\mathbb{CP}^{15}$ as the symmetric (8×8) -matrix

$$A_8(\mathbf{x}) := \begin{bmatrix} x_{00} & x_{01} & x_{02} & x_{03} & 0 & -y_{01} & -y_{02} & -y_{03} \\ x_{01} & x_{11} & x_{12} & x_{13} & y_{01} & 0 & -y_{12} & -y_{13} \\ x_{02} & x_{12} & x_{22} & x_{23} & y_{02} & y_{12} & 0 & -y_{23} \\ x_{03} & x_{13} & x_{23} & x_{33} & y_{03} & y_{13} & y_{23} & 0 \\ 0 & y_{01} & y_{02} & y_{03} & x_{00} & x_{01} & x_{02} & x_{03} \\ -y_{01} & 0 & y_{12} & y_{13} & x_{01} & x_{11} & x_{12} & x_{13} \\ -y_{02} & -y_{12} & 0 & y_{23} & x_{02} & x_{12} & x_{22} & x_{23} \\ -y_{03} & -y_{13} & -y_{23} & 0 & x_{03} & x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

Let $\mathbf{y} := [y_0, \dots, y_7]$. Denote by $Q_{A_8}(\mathbf{x})$ the quadric $\mathcal{V}(\mathbf{y} A_8(\mathbf{x}) \mathbf{y}^T) \subset \mathbb{CP}^7$ associated to $A_8(\mathbf{x})$, and let

$$W := \{Q_{A_8(\mathbf{x})} \mid \mathbf{x} \in \mathbb{CP}^{15}\}.$$

After a [Macaulay2] calculation, we find that the baselocus $\operatorname{Bl}(W) \subset \mathbb{CP}^7$ of W consists of the two disjoint, complex conjugate 3-spaces $H_{\operatorname{Bl}} := \mathcal{V}(l_0, l_1, l_2, l_3)$ and $\overline{H}_{\operatorname{Bl}} := \mathcal{V}(\bar{l}_0, \bar{l}_1, \bar{l}_2, \bar{l}_3)$, where l_j is the linear form $y_j + \mathbf{i}y_{j+4}$ for $j = 0, 1, 2, 3$. We summarise the correspondence between M_4 and A_8 :

Lemma IV.3.1. *There is an isomorphism between the projectivisation of the vector space over $\mathbb{C}$ spanned by Hermitian (4×4) -matrices of rank k , and the space of rank- $2k$ quadrics in $\mathbb{CP}^7$ passing through two given disjoint 3-spaces.*

A point in X_2 corresponds to a rank-4 quadric in W . We will now count the rank-4 quadrics in W . Let Q be a rank-4 quadric in W . By the rank-nullity theorem, $\text{Sing}(Q)$ is a 3-space. Since Q contains $\text{Bl}(W)$ and is irreducible, it follows that $\text{Sing}(Q)$ intersects H_{Bl} and $\overline{H}_{\text{Bl}}$ in a line each. A 3-space H which intersects H_{Bl} and $\overline{H}_{\text{Bl}}$ in a line each, is spanned by those lines, since H_{Bl} and $\overline{H}_{\text{Bl}}$ are disjoint. The Grassmannian $\mathbb{G}(1, 3)$ of lines in a 3-space is 4-dimensional. Hence there is an 8-dimensional space of 3-spaces that are spanned by a line in H_{Bl} and a line in $\overline{H}_{\text{Bl}}$. The 3-space H is the common singular locus of a web $W_H \subset W$ of quadrics. This can be seen by projecting $\mathbb{P}^7 \rightarrow \mathbb{P}^3$ with H as projection centre. The 3-spaces H_{Bl} and $\overline{H}_{\text{Bl}}$ are projected onto two skew lines $L_{\text{Bl}}, \overline{L}_{\text{Bl}} \subset \mathbb{P}^3$. A rank-4 quadric $Q \subset \mathbb{P}^7$ singular at H and containing H_{Bl} and $\overline{H}_{\text{Bl}}$ is projected onto a quadric in $\mathbb{P}^3$ containing L_{Bl} and $\overline{L}_{\text{Bl}}$. There is a web of quadrics in $\mathbb{P}^3$ containing L_{Bl} and $\overline{L}_{\text{Bl}}$. In total, we get that the space of rank-4 quadrics in W has dimension $8 + 3 = 11$. Since there is a bijection between this space and X_2 , this is as expected.

A real point in X_2 corresponds to a real rank-4 quadric Q in W . Then $H := \text{Sing}(Q)$ is real. Thus H intersects H_{Bl} and $\overline{H}_{\text{Bl}}$ in two complex conjugate lines, $L, \overline{L}$. Hence $H_L := \langle L, \overline{L} \rangle$ is the unique real 3-space which contains L and is the singular locus of a rank-4 quadric in W . As noted above, there is a web of quadrics singular at H_L . Since the Grassmannian $\mathbb{G}(1, 3)$ of lines in H_{Bl} is 4-dimensional, we get that the locus of real rank-4 quadrics in W has dimension $3 + 4 = 7$. Hence the real part of the rank-2 locus X_2 is 7-dimensional.

Proposition IV.3.2. *The rank-2 locus X_2 of $\mathbb{CP}^{15}$, the projectivisation of the vector space over $\mathbb{C}$ spanned by Hermitian (4×4) -matrices, is 11-dimensional. The real part of X_2 is 7-dimensional.*

We derive a parameterisation of the real rank-4 quadrics in W , which in turn corresponds to a parameterisation of the real part of X_2 . A line L in H_{Bl} is given by

$$\mathcal{V}\left(y_0 + iy_4, y_1 + iy_5, y_2 + iy_6, y_3 + iy_7, \sum_{j=0}^7 (a_j + ib_j)y_j, \sum_{j=0}^7 (c_j + id_j)y_j\right)$$

for $a_j, b_j, c_j, d_j \in \mathbb{R}$. Since H_L is the only real 3-space containing L , we deduce that $H_L = \mathcal{V}(\ell_0, \ell_1, \ell_2, \ell_3)$, where

$$\begin{aligned} \ell_0 &:= \sum_{j=0}^3 \left((a_j - b_{j+4})y_j + (a_{j+4} + b_j)y_{j+4} \right), \\ \ell_1 &:= \sum_{j=0}^3 \left((a_j - b_{j+4})y_{j+4} - (a_{j+4} + b_j)y_j \right), \end{aligned}$$

$$\begin{aligned}\ell_2 &:= \sum_{j=0}^3 \left((c_j - d_{j+4})y_j + (c_{j+4} + d_j)y_{j+4} \right), \\ \ell_3 &:= \sum_{j=0}^3 \left((c_j - d_{j+4})y_{j+4} - (c_{j+4} + d_j)y_j \right).\end{aligned}$$

The quadrics that are singular at H_L are given by quadratic polynomials in $\ell_0, \ell_1, \ell_2, \ell_3$. Moreover, to be contained in W , the quadrics must contain H_{Bl} and $\overline{H}_{\text{Bl}}$. We compute that the quadrics in W that are singular at H_L , generate the ideal

$$I := \langle \ell_0^2 + \ell_1^2, \ell_0\ell_2 + \ell_1\ell_3, \ell_0\ell_3 - \ell_1\ell_2, \ell_2^2 + \ell_3^2 \rangle.$$

Hence a real rank-4 quadric in W is on the form

$$\mathcal{V}\left(a(\ell_0^2 + \ell_1^2) + b(\ell_0\ell_2 + \ell_1\ell_3) + c(\ell_0\ell_3 - \ell_1\ell_2) + d(\ell_2^2 + \ell_3^2)\right) \quad (\text{IV.3.1})$$

for $a, b, c, d \in \mathbb{R}$. It would be interesting to have equations for the real part of X_2 , but deriving them from (IV.3.1) has been too computationally demanding.

IV.3.1 Definite representations

Let $\mathcal{V}(f) \subset \mathbb{RP}^3$ be a quartic surface with a definite Hermitian determinantal representation $M_4(\mathbf{x})$. Then $M_4(\mathbf{x})$ gives rise to the spectrahedron

$$\{\mathbf{x} \in \mathbb{RP}^3 \mid M_4(\mathbf{x}) \text{ is semidefinite}\}.$$

If $M_4(\mathbf{x})$ is semidefinite but not definite at a point $\mathbf{x}$, then $\mathbf{x} \in \mathcal{V}(f)$, and in particular, $\mathbf{x}$ lies on the boundary of the spectrahedron. Since $\mathcal{V}(f)$ is real, singularities are either real or occur in complex conjugate pairs. Hence we have $\rho \equiv \eta \pmod{2}$ in [Conjecture IV.1.3](#). For $\eta < 10$, we do not expect any further restrictions on ρ and σ , as we explain below.

Hermitian determinantal representations are subtly different from symmetric representations, which is why we do not have that σ is even and $\rho \geq 2$ for all Hermitian determinantal representations with $\eta < 10$. For instance, the defining characteristic of a transversal symmetroid $\mathcal{V}(f)$ is that the projection from one of the nodes P is ramified along the union of two cubic curves $R_1, R_2 \subset \mathbb{P}^2$ [[Cay69b](#), p. 200]. It is showed in [[Ott+15](#), Proof of Theorem 1.1, p. 600] that if $\mathcal{V}(f)$ has a nonempty spectrahedron S and $P \notin S$, then the nodes on S are projected onto the intersection points of the ovals of R_1 and R_2 . These intersect in an even number of points by the Jordan curve theorem, hence σ is even. *A priori*, the ramification curve for the projection from P does not impose restrictions on σ if $\mathcal{V}(f)$ is not a symmetroid.

In the special case of a transversal quartic symmetroid which contains a line, a simple reason for $\rho \neq 0$, is that the line passes through an odd number of rank-2 points, at least one of which must be real. A general proof of $\rho \neq 0$ for a transversal quartic symmetroid with a nonempty spectrahedron can be found in [[Ott+15](#), Lemma 4.2]. Not all lines on a quartic surface with a Hermitian determinantal representation contain an odd number of rank-2 points:

Example IV.3.3. The pencil

$$\begin{bmatrix} 0 & x_0 - ix_0 & x_1 & 0 \\ x_0 + ix_0 & x_0 & x_1 & x_0 + ix_0 \\ x_1 & x_1 & x_0 & x_0 \\ 0 & x_0 - ix_0 & x_0 & x_0 \end{bmatrix}$$

has rank 3 at all points, except at $[1 : 0]$ and $[0 : 1]$, where the rank is 2. $\blacklozenge$

IV.3.1.1 Finding examples

In [Section IV.4](#), we list examples of definite Hermitian determinantal representations with different values of (η, ρ, σ) . The list is missing the sixty-four triples in [Table IV.1](#) predicted by [Conjecture IV.1.3](#). Both [\[HR18; Ott+15\]](#) present examples of symmetroids with a nonempty spectrahedron, that are found using random searches. Our examples are not found in this way. The problem with drawing random Hermitian (4×4) -matrices $M_{4,i}$ in search of a definite determinantal representation ([IV.1.1](#)) with a given triple (η, ρ, σ) , is that we do not know of a matrix form for $M_{4,i}$ that will guarantee at least $\eta \geq 4$ essential nodes. For $\eta = 0, 1, 2, 3$ it is straightforward to achieve at least η nodes: Let $M_{4,0}$ be a definite matrix, and let η of $M_{4,1}, M_{4,2}, M_{4,3}$ be rank-2 matrices. Moreover, we can control σ by choosing the definiteness of the rank-2 matrices. This will in general yield a representation with (η, η, σ) . Note that $M_{4,i}$ corresponds to a real point by construction, hence $\rho = \eta$. To get $\rho < \eta$, we let $M_{4,0}$ and $M_{4,1}$ be two rank-2 matrices that span a pencil that contains two complex conjugate rank-2 points.

For $4 \leq \eta \leq 8$, our strategy has been the following: A transversal quartic symmetroid with a nonempty spectrahedron corresponds to a 3-space $H \subset \mathbb{CP}^{15}$ that intersects the rank-2 locus X_2 in ten points. We try to deform H whilst keeping a definite point and some nodes. More precisely, we choose four points P_1, P_2, P_3, P_4 in $H \cap X_2$ that span H . Let $Q \subset \mathbb{P}^7$ be the associated quadric at P_1 . There is a web of rank-4 quadrics singular at $\text{Sing}(Q)$, which corresponds to a 3-space $Y_2 \subset X_2$. We replace P_1 with a point P'_1 in Y_2 . The 3-space H' spanned by P'_1, P_2, P_3, P_4 intersects X_2 in $\eta \geq 4$ points. Then H' corresponds to a Hermitian representation with η essential singularities. There is a complete list of examples of all possible values of ρ and σ for $\eta = 10$ in [\[Ott+15\]](#). We used these as a starting point to get different values of ρ and σ for $4 \leq \eta \leq 8$.

IV.4 Examples of quartics with a definite Hermitian determinantal representation

Below is a list of definite Hermitian determinantal representations that define quartic surfaces with only essential singularities. For each matrix $M_4(\mathbf{x})$, we specify the configuration (η, ρ, σ) of essential singularities and a point $\mathbf{e} \in \mathbb{RP}^3$ such that $M_4(\mathbf{e})$ is definite. We omit examples with $\eta = 10$. A list of examples for $\eta = 10$ with all possible values for ρ and σ is found in [\[Ott+15\]](#).

IV.4.1 Zero rank-2 points

$(0, 0, 0)$:

$$\begin{bmatrix} x_3 & x_0 & x_2 + ix_1 & 0 \\ x_0 & x_3 & x_1 & x_1 - ix_2 \\ x_2 - ix_1 & x_1 & 2x_0 + x_3 & x_1 + x_2 \\ 0 & x_1 + ix_2 & x_1 + x_2 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

Figure IV.1: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (0, 0, 0)$.

IV.4.2 One rank-2 point

$(1, 1, 1)$:

$$\begin{bmatrix} x_0 + x_3 & 0 & x_2 + ix_1 & 0 \\ 0 & 2x_0 + x_3 & x_1 & x_1 - ix_2 \\ x_2 - ix_1 & x_1 & x_2 + x_3 & x_1 + x_2 \\ 0 & x_1 + ix_2 & x_1 + x_2 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

$(1, 1, 0)$:

$$\begin{bmatrix} x_3 & x_0 & x_2 + ix_1 & 0 \\ x_0 & x_3 & x_1 & x_1 - ix_2 \\ x_2 - ix_1 & x_1 & x_2 + x_3 & x_1 + x_2 \\ 0 & x_1 + ix_2 & x_1 + x_2 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

Figure IV.2: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (1, 1, 0)$.

IV.4.3 Two rank-2 points

$(2, 2, 2)$:

$$\begin{bmatrix} x_0 + x_1 + x_3 & 0 & x_2 + ix_1 & 0 \\ 0 & 2x_0 + x_1 + x_3 & x_1 & x_1 - ix_2 \\ x_2 - ix_1 & x_1 & 2x_1 + x_3 & x_1 + x_2 \\ 0 & x_1 + ix_2 & x_1 + x_2 & x_1 + x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(2, 2, 1)$:

$$\begin{bmatrix} x_1 + x_3 & x_0 & x_2 + ix_1 & 0 \\ x_0 & x_1 + x_3 & x_1 & x_1 - ix_2 \\ x_2 - ix_1 & x_1 & 2x_1 + x_3 & x_1 + x_2 \\ 0 & x_1 + ix_2 & x_1 + x_2 & x_1 + x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(2, 2, 0)$:

$$\begin{bmatrix} x_3 & x_0 & x_2 & x_1 \\ x_0 & x_3 & 0 & -ix_2 \\ x_2 & 0 & x_3 & x_2 + ix_1 \\ x_1 & ix_2 & x_2 - ix_1 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(2, 0, 0)$:

$$\begin{bmatrix} 2x_3 & x_2 & x_2 + x_3 & x_0 + ix_1 \\ x_2 & x_3 & x_0 + ix_1 & 0 \\ x_2 + x_3 & x_0 - ix_1 & x_2 + x_3 & x_2 \\ x_0 - ix_1 & 0 & x_2 & x_2 + x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

Figure IV.3: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (2, 2, 1)$.

IV.4.4 Three rank-2 points

$(3, 3, 3)$:

$$\begin{bmatrix} x_0 + x_1 + x_2 + x_3 & x_1 + x_2 & -x_1 + x_2 + ix_0 & ix_2 \\ x_1 + x_2 & x_0 + 2x_1 + 2x_2 + x_3 & x_0 - x_1 + x_2 & x_0 + x_1 + x_2 \\ -x_1 + x_2 - ix_0 & x_0 - x_1 + x_2 & 2x_0 + x_1 + x_2 + x_3 & x_0 + ix_2 \\ -ix_2 & x_0 + x_1 + x_2 & x_0 - ix_2 & x_0 + x_1 + 3x_2 + x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(3, 3, 2)$:

$$\begin{bmatrix} x_0 + x_2 + x_3 & x_0 & x_0 & 0 \\ x_0 & 2x_0 + 2x_2 + x_3 & x_0 & x_0 - ix_1 \\ x_0 & x_0 & x_0 + x_3 & x_1 \\ 0 & x_0 + ix_1 & x_1 & x_0 + x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(3, 3, 1)$:

$$\begin{bmatrix} x_2 + x_3 & x_0 & ix_2 & 0 \\ x_0 & x_2 + x_3 & x_2 & x_2 - ix_1 \\ -ix_2 & x_2 & 2x_2 + x_3 & x_1 + x_2 \\ 0 & x_2 + ix_1 & x_1 + x_2 & x_2 + x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(3, 3, 0)$:

$$\begin{bmatrix} x_3 & x_0 & x_2 & 0 \\ x_0 & x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(3, 1, 1)$:

$$\begin{bmatrix} x_2 + 2x_3 & 0 & x_3 + ix_2 & x_0 + ix_1 \\ 0 & x_2 + x_3 & x_0 + x_2 + ix_1 & x_2 \\ x_3 - ix_2 & x_0 + x_2 - ix_1 & 2x_2 + x_3 & x_2 \\ x_0 - ix_1 & x_2 & x_2 & x_2 + x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

$(3, 1, 0)$:

$$\begin{bmatrix} 2x_3 & 0 & x_3 & x_0 + x_2 + ix_1 \\ 0 & x_3 & x_0 + ix_1 & x_2 \\ x_3 & x_0 - ix_1 & x_3 & x_2 \\ x_0 + x_2 - ix_1 & x_2 & x_2 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

Figure IV.4: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (3, 3, 1)$.

IV.4.5 Four rank-2 points

$(4, 4, 4)$: $M_4(\mathbf{x}) := M_{4,0}x_0 + M_{4,1}x_1 + M_{4,2}x_2 + M_{4,3}x_3$, where

$$M_{4,0} := \begin{bmatrix} 1 & 9 & 24 & 18 \\ 9 & -23 & 0 & -30 \\ 24 & 0 & -36 & 36 \\ 18 & -30 & 36 & 0 \end{bmatrix} \quad M_{4,1} := \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix}$$

$$M_{4,2} := \begin{bmatrix} 31 & -9 & -12 + 36i & -18 \\ -9 & 43 & 36 & 42 \\ -12 - 36i & 36 & 72 & 0 \\ -18 & 42 & 0 & 0 \end{bmatrix} \quad M_{4,3} := \begin{bmatrix} 2 & 3 & 6 & 3i \\ 3 & 2 & 3 & -3 \\ 6 & 3 & -6 & 3i \\ -3i & -3 & -3i & 0 \end{bmatrix}$$

$\mathbf{e} := [0 : 12 : 0 : 1]$

IV. Determinantal quartic surfaces with a definite Hermitian representation

(4, 4, 3): $M_4(\mathbf{x}) := M_{4,0}x_0 + M_{4,1}x_1 + M_{4,2}x_2 + M_{4,3}x_3$, where

$$M_{4,0} := \begin{bmatrix} 26 & 0 & 12 & 18 \\ 0 & -4 & 0 & -6 \\ 12 & 0 & -9 & 9 \\ 18 & -6 & 9 & 0 \end{bmatrix} \quad M_{4,1} := \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 4 & 0 & 6 \\ -3 & 0 & 9 & 0 \\ 0 & 6 & 0 & 9 \end{bmatrix}$$

$$M_{4,2} := \begin{bmatrix} -2 & 0 & -1+i & -2 \\ 0 & 1 & 1 & 1 \\ -1-i & 1 & 2 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix} \quad M_{4,3} := \begin{bmatrix} 2 & 3 & 6 & 3i \\ 3 & 2 & 3 & -3 \\ 6 & 3 & -6 & 3i \\ -3i & -3 & -3i & 0 \end{bmatrix}$$

$\mathbf{e} := [0 : 4 : 03]$

(4, 4, 2):

$$\begin{bmatrix} x_0 + x_3 & x_0 & x_2 & 0 \\ x_0 & 2x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

(4, 4, 1):

$$\begin{bmatrix} x_0 + 2x_2 - 5x_3 & -x_0 + x_2 - 2x_3 & x_0 + x_3 & 2x_0 + ix_3 \\ -x_0 + x_2 - 2x_3 & 2x_1 - 2x_2 + 8x_3 & 2x_1 - 2x_2 + 7x_3 & x_1 - x_2 + 4x_3 \\ x_0 + x_3 & 2x_1 - 2x_2 + 7x_3 & 2x_1 - 2x_2 + 7x_3 & x_1 + ix_3 \\ 2x_0 - ix_3 & x_1 - x_2 + 4x_3 & x_1 - ix_3 & x_2 \end{bmatrix}$$

$\mathbf{e} := [0 : 5 : 32 : 10]$

(4, 4, 0):

$$\begin{bmatrix} x_3 & x_0 & x_1 + ix_2 & 0 \\ x_0 & x_3 & 0 & -ix_1 \\ x_1 - ix_2 & 0 & x_3 & x_1 + ix_2 \\ 0 & ix_1 & x_1 - ix_2 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

(4, 2, 2):

$$\begin{bmatrix} -77x_1 + x_2 & 27x_1 & -12x_1 - 3x_2 & x_0 + 81x_1 + ix_3 \\ 27x_1 & -74x_1 + 4x_2 & x_0 + ix_3 & 78x_1 + 6x_2 \\ -12x_1 - 3x_2 & x_0 - ix_3 & -45x_1 + 9x_2 & 54x_1 \\ x_0 + 81x_1 - ix_3 & 78x_1 + 6x_2 & 54x_1 & 9x_2 \end{bmatrix}$$

$\mathbf{e} := [0 : -1 : 18 : 1]$

(4, 2, 1):

$$\begin{bmatrix} 26x_1 + x_2 & 0 & 12x_1 - 3x_2 & x_0 + 18x_1 + ix_3 \\ 0 & -4x_1 + 4x_2 & x_0 + ix_3 & -6x_1 + 6x_2 \\ 12x_1 - 3x_2 & x_0 - ix_3 & -9x_1 + 9x_2 & 9x_1 \\ x_0 + 18x_1 - ix_3 & -6x_1 + 6x_2 & 9x_1 & 9x_2 \end{bmatrix}$$

$\mathbf{e} := [-45 : 1 : 201 : 9]$

$(4, 0, 0)$:

$$\begin{bmatrix} 2x_3 & x_2 & x_2 + x_3 & x_0 + ix_1 \\ x_2 & x_3 & x_0 + ix_1 & 0 \\ x_2 + x_3 & x_0 - ix_1 & x_2 + x_3 & 0 \\ x_0 - ix_1 & 0 & 0 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

Figure IV.5: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (4, 4, 2)$.

IV.4.6 Five rank-2 points

$(5, 5, 3)$:

$$\begin{bmatrix} x_0 + x_3 & ix_0 & x_2 & 0 \\ -ix_0 & x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

$(5, 5, 2)$:

$$\begin{bmatrix} x_0 + x_2 + x_3 & x_0 + x_2 & x_0 - x_2 & 0 \\ x_0 + x_2 & 2x_0 + 2x_2 + x_3 & x_0 - x_2 & x_0 + x_2 - ix_1 \\ x_0 - x_2 & x_0 - x_2 & x_0 + x_2 + x_3 & x_1 \\ 0 & x_0 + x_2 + ix_1 & x_1 & x_0 + x_2 + x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

$(5, 5, 1)$:

$$\begin{bmatrix} x_0 + 9x_1 - x_2 - x_3 & -x_0 + 3x_1 + x_2 - x_3 & x_0 + ix_3 & 2x_0 \\ -x_0 + 3x_1 + x_2 - x_3 & x_3 & x_3 & x_3 \\ x_0 - ix_3 & x_3 & x_1 - x_2 + 2x_3 & 4x_1 \\ 2x_0 & x_3 & 4x_1 & 4x_2 \end{bmatrix}$$

$\mathbf{e} := [0 : 21 : 22 : 80]$

Figure IV.6: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (5, 5, 2)$.

IV.4.7 Six rank-2 points

$(6, 6, 4)$:

$$\begin{bmatrix} x_3 & x_0 & x_1 & 0 \\ x_0 & x_3 & 0 & -ix_1 \\ x_1 & 0 & x_2 + x_3 & x_1 + ix_2 \\ 0 & ix_1 & x_1 - ix_2 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(6, 6, 3)$:

$$\begin{bmatrix} -x_2 + x_3 & x_0 & x_1 & ix_2 \\ x_0 & x_0 - x_2 + x_3 & x_1 & x_0 \\ x_1 & x_1 & -x_2 + x_3 & ix_2 \\ -ix_2 & x_0 & -ix_2 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 1 : 3]$$

Figure IV.7: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (6, 6, 3)$. Not all nodes are visible from this angle.

IV.4.8 Seven rank-2 points

$(7, 7, 5)$:

$$\begin{bmatrix} x_0 + x_3 & 0 & x_2 & 0 \\ 0 & x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

$(7, 7, 4)$:

$$\begin{bmatrix} -x_2 + x_3 & x_0 - i(x_0 + x_2 - x_3) & x_1 & -x_0 + x_3 \\ x_0 + i(x_0 + x_2 - x_3) & -x_0 - x_2 + 2x_3 & x_1 & x_0 + i(x_0 + x_2 - x_3) \\ x_1 & x_1 & x_0 & x_2 \\ -x_0 + x_3 & x_0 - i(x_0 + x_2 - x_3) & x_2 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [1 : 0 : 0 : 2]$$

Figure IV.8: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (7, 7, 4)$. Not all nodes are visible from this angle.

IV.4.9 Eight rank-2 points

$(8, 8, 5)$:

$$\begin{bmatrix} -x_2 + x_3 & x_0 - i(x_0 + x_2 - x_3) & x_1 & -x_0 + x_3 \\ x_0 + i(x_0 + x_2 - x_3) & 2x_0 & x_1 & x_0 + i(x_0 + x_2 - x_3) \\ x_1 & x_1 & x_0 & x_2 \\ -x_0 + x_3 & x_0 - i(x_0 + x_2 - x_3) & x_2 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [1 : 0 : 0 : 2]$$

$(8, 8, 4)$:

$$\begin{bmatrix} 2x_0 + x_3 & ix_0 & x_2 & 0 \\ -ix_0 & x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}$$

$$\mathbf{e} := [0 : 0 : 0 : 1]$$

(8, 6, 4):

$$\begin{bmatrix} x_0 + x_3 & 0 & x_2 & 0 \\ 0 & 2x_0 + x_3 & ix_2 & -ix_1 \\ x_2 & -ix_2 & x_2 + x_3 & x_1 \\ 0 & ix_1 & x_1 & x_3 \end{bmatrix}$$

$\mathbf{e} := [0 : 0 : 0 : 1]$

Figure IV.9: A real, quartic surface with a definite Hermitian determinantal representation and $(\eta, \rho, \sigma) = (8, 8, 5)$. Not all nodes are visible from this angle.

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