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Computations in $\mathbb{A}^1$ -homotopy theory

Contractibility and enumerative geometry

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Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo, under the supervision of professor Paul Arne Østvær and professor Kirsten Wickelgren.

The thesis is a collection of three papers, presented in chronological order of writing. The papers are preceded by an introductory chapter that relates them together and provides background information and motivation for the work. The first paper is joint work with Adrien Dubouloz and Paul Arne Østvær. I am the sole author of the remaining paper.

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List of Papers

Paper I

Dubouloz, A. and Pauli, S., Østvær, P. A. “ $\mathbb{A}^1$ -contractibility of affine modifications”. In: *International Journal of Mathematics*. Vol. 30, no. 14 (2019), DOI: 10.1142/S0129167X19500691.

Paper II

Pauli, S. “Quadratic types and the dynamic Euler number of lines on a quintic threefold”. Preprint.

Paper III

Pauli, S. “Computing $\mathbb{A}^1$ -Euler numbers with Macaulay2”. 2020. arXiv:2003.01775. Submitted for publication.

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Chapter 1

Introduction

In this thesis, we apply tools from $\mathbb{A}^1$ -homotopy theory to other areas of mathematics and compute explicit examples of $\mathbb{A}^1$ -contractible spaces and $\mathbb{A}^1$ -enumerative counts. The idea of $\mathbb{A}^1$ -homotopy theory is to have the notion of a homotopy on smooth algebraic varieties Sm_k over a field k . This yields new invariants, for example $\mathbb{A}^1$ -homotopy groups and cohomology theories. This thesis consists of the three papers titled $\mathbb{A}^1$ -contractibility of affine modifications Paper I, *Quadratic types and dynamic Euler numbers of lines on quintic threefolds* Paper II and *Computing $\mathbb{A}^1$ -Euler numbers with Macaulay2* Paper III in which we apply some of these new tools to problems in affine algebraic geometry and enumerative geometry.

In Paper I we show how one can construct new examples of $\mathbb{A}^1$ -contractible varieties using affine modifications. These serve as potential counterexamples to the Zariski cancellation problem in affine algebraic geometry.

The $\mathbb{A}^1$ -analog of the Brouwer degree allows us to define an Euler number valued in $\mathrm{GW}(k)$. In Paper III we compute several $\mathbb{A}^1$ -degrees and $\mathbb{A}^1$ -Euler numbers with Macaulay2. In Paper II we define the *dynamic $\mathbb{A}^1$ -Euler number* and provide a quadratic count of the lines on the Fermat quintic threefold which deform with a general deformation. Furthermore, we give a geometric interpretation of the contribution of the lines to the count.

In this introduction, we first recall the construction of the $\mathbb{A}^1$ -homotopy category together with some important results. Then we explain why $\mathbb{A}^1$ -contractible varieties play an important role in affine algebraic geometry and recall the proof of the $\mathbb{A}^1$ -contractibility of the Koras-Russell threefold. Then we give an introduction to $\mathbb{A}^1$ -enumerative geometry via the $\mathbb{A}^1$ -degree map and to the $\mathbb{A}^1$ -Euler number as the sum of local $\mathbb{A}^1$ -degrees of the zeros of a general section. We also introduce our dynamic Euler number from Paper II and, as an example, we compute the dynamic Euler number of the count of lines on the union of three hyperplanes in $\mathbb{P}^3$ that deform with a general deformation. We finish this introduction with summaries of the three papers Paper I, Paper II and Paper III.

1.1 $\mathbb{A}^1$ -homotopy theory

Let Sm_k be the category of smooth, separated schemes of finite type over k . We call the objects of Sm_k *smooth varieties* over k . The idea of $\mathbb{A}^1$ -homotopy theory is to have the notion of a ‘homotopy’ of smooth varieties over k such that the affine line $\mathbb{A}^1$ plays the role of the unit interval in topology. Additionally, we want to construct the $\mathbb{A}^1$ -homotopy category by adding inverses to a subcategory

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of *weak equivalences*. In particular, the projection $X \times \mathbb{A}^1 \rightarrow X$ should be a weak equivalence for any smooth variety X .

The $\mathbb{A}^1$ -homotopy category has first been constructed in [MV99]. We give a short introduction to how it arises and refer the interested reader to [Dun+07], [Lev08] and [WW19] for more detailed introductions to $\mathbb{A}^1$ -homotopy theory.

1.1.0.1 Model categories

In order to be able to define the ‘homotopy category’, one has to make sure that the morphisms remain a set after adding inverses to the weak equivalences. One way to achieve this is to work in a model category. A *model category* is a category C with small limits and colimits together with three subcategories, called *weak equivalences*, *cofibrations* and *fibrations* satisfying several axioms (see [Hov99]). In a model category, one can define the notion of a *homotopy* and the upshot is that a morphism between *fibrant-cofibrant* objects C_{cf} in C is a weak equivalence if and only if it is a homotopy equivalence. Thus one can define the homotopy category of C_{cf} to be the category $C_{cf}/\sim$ where $\sim$ is the homotopy equivalence relation. Since every object in C is weakly equivalent to a fibrant-cofibrant object, the homotopy category $ho(C)$ exists and is equivalent to $C_{cf}/\sim$.

Two standard examples of model categories are the category Top of (nice) topological spaces and the category $sSet$ of simplicial sets. Their standard model structures are indeed *Quillen equivalent* which implies that the associated homotopy categories are equivalent and the notion of a homotopy and a weak equivalence in Top agrees with the classical ones.

1.1.1 Model structures on simplicial presheaves on smooth varieties over k

It is crucial for constructions in homotopy theory to have small limits and colimits as we will see in the following sections. The standard way to add limits and colimits to the category of smooth varieties over k is to embed Sm_k into the category of *simplicial presheaves* $sPre(Sm_k)$ on Sm_k , that is, the category of contravariant functors from Sm_k to $sSet$, with the Yoneda embedding

$$Sm_k^{op} \xrightarrow{\text{Yoneda}} sPre(Sm_k), X \mapsto (U \mapsto \text{Hom}_{Sm_k}(U, X)).$$

By abuse of notation, we denote the image of a smooth variety X under the Yoneda embedding by X . Note that any simplicial set K defines a *constant* simplicial presheaf $U \mapsto K$ which we denote by K as well.

One main idea of $\mathbb{A}^1$ -homotopy theory is that $\mathbb{A}^1$ plays the role of the unit interval $[0, 1]$. Whence, we want $X \times \mathbb{A}^1 \rightarrow X$ to be a weak equivalence. Furthermore, we want to give the objects of Sm_k a Grothendieck topology in the homotopy category. For this purpose we use the *Nisnevich topology*, that is, the Grothendieck topology on Sm_k generated by the following pullback squares

in Sm_k

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow p \in Sm_k \\ U & \xrightarrow{i} & X \end{array} \quad (1.1)$$

where $p : Y \rightarrow X$ is étale, $i : U \rightarrow X$ is an open immersion and $p^{-1}(X \setminus U) \rightarrow X \setminus U$ is an isomorphism in the reduced scheme structure. We call such squares *elementary distinguished squares*.

The category $sPre(Sm_k)$ has several (Quillen equivalent) model structures induced by the model structure on $sSet$. We use Bousfield localization [Hir03] to impose additional weak equivalences: Let L_{Nis} be the Bousfield localization that enforces the Nisnevich topology up to weak equivalence. In particular, (1.1) is a pushout square in the homotopy category of $L_{Nis}(sPre(Sm_k))$. We call objects in the category $L_{Nis}(sPre(Sm_k))$ *sheaves* over k and denote the category by $Shv(Sm_k)$. Let $L_{\mathbb{A}^1}$ be the Bousfield localization that adds the weak equivalences $X \times \mathbb{A}^1 \rightarrow X$ for X a smooth variety over k . So the category $Spc_k = L_{\mathbb{A}^1}(L_{Nis}(sPre(Sm_k)))$ of *motivic spaces* over k has the same objects as $sPre(Sm_k)$ but additional weak equivalences.

$$Sm_k \hookrightarrow sPre(Sm_k) \xrightarrow{L_{Nis}} Shv(Sm_k) \xrightarrow{L_{\mathbb{A}^1}} Spc_k.$$

The category Spc_k is *pointed* with zero object $* := \text{Spec } k$, and the model structure on Spc_k naturally lifts to a model structure on the category of *pointed motivic spaces* $(Spc_k)_*$ [Hov99, Proposition 1.1.8]. For X an object in Spc_k , denote by X_+ the object in $(Spc_k)_*$ consisting of X and a disjoint base point and denote by $- \wedge -$ the *smash product*.

Definition 1.1.1. The $\mathbb{A}^1$ -homotopy category over k is the homotopy category $\mathcal{H}(k) := ho(Spc_k)$ of motivic spaces Spc_k over k . We denote the *pointed* $\mathbb{A}^1$ -homotopy category by $\mathcal{H}_\bullet(k) := ho((Spc_k)_*)$ and its morphisms by

$$[-, -]_{\mathbb{A}^1} := \text{Hom}_{\mathcal{H}_\bullet(k)}(-, -).$$

We call an isomorphism in the (pointed) $\mathbb{A}^1$ -homotopy category a *(pointed) $\mathbb{A}^1$ -weak equivalence*.

1.1.1.1 $\mathbb{A}^1$ -homotopy sheaves

Definition 1.1.2. Let $\mathcal{X}$ be a motivic space over k . The $\mathbb{A}^1$ -connected components of $\mathcal{X}$ is the Nisnevich sheaf associated to the presheaf

$$U \mapsto \text{Hom}_{\mathcal{H}(k)}(U, \mathcal{X}).$$

Let $(\mathcal{X}, x)$ be a pointed motivic space over k and $i \geq 1$. The *i th $\mathbb{A}^1$ -homotopy sheaf* of $(\mathcal{X}, x)$ is the Nisnevich sheaf associated to the presheaf

$$U \mapsto \text{Hom}_{\mathcal{H}_\bullet(k)}(S^i \wedge U_+, (\mathcal{X}, x)) = [S^i \wedge U_+, (\mathcal{X}, x)]_{\mathbb{A}^1}$$

where S^i denotes the i -th simplicial sphere.

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Remark 1.1.3. The i -th $\mathbb{A}^1$ -homotopy sheaves are sheaves of groups for $i \geq 1$ and sheaves of abelian groups for $i \geq 2$.

Proposition 1.1.4 (§3 Proposition 2.14 in [MV99]). *A morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of $\mathbb{A}^1$ -connected motivic spaces (that means $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \cong \text{Spec } k \cong \pi_0^{\mathbb{A}^1}(\mathcal{Y})$) is an $\mathbb{A}^1$ -weak equivalence if and only if for any base point x of $\mathcal{X}$ the induced morphism of Nisnevich sheaves*

$$\pi_i^{\mathbb{A}^1}(\mathcal{X}, x) \rightarrow \pi_i^{\mathbb{A}^1}(\mathcal{Y}, f(x))$$

is an isomorphism for all $i \geq 1$.

1.1.2 Spheres and homotopy purity

For an inclusion $U \hookrightarrow X$ of smooth varieties, define

$$X/U := \text{colim} \left(\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \\ * & & \end{array} \right)$$

and note that the *quotient* X/U is naturally pointed. Let $\pi : E \rightarrow X$ be a vector bundle of smooth varieties over k . The *Thom space* $Th(E)$ of E is the quotient $E/(E \setminus s_0(X))$ where s_0 denotes the zero section. We recall the following proposition from [MV99, Proposition III.2.17].

Proposition 1.1.5.

1.1.5.1. *For two vector bundles $\pi_1 : E_1 \rightarrow X_1$ and $\pi_2 : E_2 \rightarrow X_2$, there is a canonical isomorphism $Th(E_1 \times E_2) \simeq Th(E_1) \wedge Th(E_2)$ in $\mathcal{H}_\bullet(k)$.*

1.1.5.2. *Let $\mathcal{O}_X^n$ be the trivial rank n vector bundle on X , then $Th(\mathcal{O}_X^n) \simeq X_+ \wedge (\mathbb{P}^1)^{\wedge n}$ in $\mathcal{H}_\bullet(k)$ (where $\mathbb{P}^1$ is pointed at ∞).*

1.1.5.3. *Let $\pi : E \rightarrow X$ be a vector bundle and $\mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus \mathcal{O}_X)$ be the closed embedding at infinity. Then there is a canonical isomorphism $\mathbb{P}(E \oplus \mathcal{O}_X)/\mathbb{P}(E) \simeq Th(E)$ in $\mathcal{H}_\bullet(k)$.*

We want to define a *sphere object* in $\mathbb{A}^1$ -homotopy theory. Proposition 1.1.5.2 indicates that $\mathbb{P}^1$ behaves like the 1-sphere S^1 in topology and the smash product $(\mathbb{P}^1)^{\wedge n} = \mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1$ like the n -sphere S^n .

Definition 1.1.6. We call $(\mathbb{P}^1)^{\wedge n}$ the *motivic n -sphere*.

Proposition 1.1.5 yields several equivalent models for the motivic n -sphere

$$\mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1 \simeq Th(\mathbb{A}^n) = \mathbb{A}^n/(\mathbb{A}^n - 0) \simeq \mathbb{P}(\mathbb{A}^n \oplus \mathbb{A}^1)/\mathbb{P}(\mathbb{A}^n) = \mathbb{P}^n/\mathbb{P}^{n-1}$$

in $\mathcal{H}_\bullet(k)$, namely $\mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1$, $\mathbb{P}^n/\mathbb{P}^{n-1}$ and $\mathbb{A}^n/(\mathbb{A}^n - 0)$.

Remark 1.1.7. Note that taking $\mathbb{R}$ -points of the different models for the motivic n -spheres, $\mathbb{P}^1(\mathbb{R}) \wedge \cdots \wedge \mathbb{P}^1(\mathbb{R})$, $\mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R})$ and $\mathbb{A}^n(\mathbb{R})/(\mathbb{A}^n(\mathbb{R}) \setminus \{0\})$, we get the n -sphere S^n in classical topology.

Finally, we recall the homotopy purity theorem in $\mathbb{A}^1$ -homotopy theory from [MV99, Theorem 2.23].

Theorem 1.1.8 (Homotopy Purity). *Let $i : Z \hookrightarrow X$ be a closed embedding of smooth varieties over k . Denote by $N_{X,Z} \rightarrow Z$ the normal bundle to i . Then there is a canonical isomorphism in $\mathcal{H}_\bullet(k)$*

$$X/(X - i(Z)) \simeq Th(N_{X,Z})$$

called the purity isomorphism.

1.1.3 The Grothendieck-Witt ring of k

The main input of our applications of $\mathbb{A}^1$ -homotopy theory comes from the $\mathbb{A}^1$ -degree which is the analog of the Brouwer degree in classical topology. It assigns to a pointed $\mathbb{A}^1$ -homotopy class of maps between motivic n -spheres an element of the Grothendieck-Witt ring $\mathrm{GW}(k)$ of k . We recall the definition and some examples of $\mathrm{GW}(k)$ from [Lam05].

Definition 1.1.9. The Grothendieck-Witt ring $\mathrm{GW}(k)$ of a field k is the group completion of the semiring of isometry classes of finite rank non-degenerate symmetric bilinear forms where the addition is the direct sum $\oplus$ and the product is the tensor product $\otimes$.

In other words, $\mathrm{GW}(k)$ is obtained by formally adding additive inverses to the semi-ring of isometry classes of finite rank non-degenerate symmetric bilinear forms. Since any finite rank non-degenerate symmetric bilinear form can be diagonalized, i.e., is isometric to

$$((x_1, \dots, x_n), (y_1, \dots, y_n)) \mapsto a_1 x_1 y_1 + a_2 x_2 y_2 + \dots + a_n x_n y_n \quad (1.2)$$

for $a_1, \dots, a_n \in k^\times$, the Grothendieck-Witt ring $\mathrm{GW}(k)$ is generated by the classes $\langle a \rangle$ of rank 1 non-degenerate symmetric bilinear forms $(x, y) \mapsto axy$ for $a \in k^\times$ subject to the following relations [MH73].

1. $\langle a \rangle = \langle ab^2 \rangle$ for $a, b \in k^\times$
2. $\langle a \rangle \langle b \rangle = \langle ab \rangle$ for $a, b \in k^\times$
3. $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$ for $a, b \in k^\times$ and $a + b \neq 0$
4. $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$ for $a \in k^\times$

The last relation follows from the first three relations, but simplifies computations. We call the important element $\langle 1 \rangle + \langle -1 \rangle$ of $\mathrm{GW}(k)$ the *hyperbolic form* and denote it by $\mathbb{H}$.

Example 1.1.10. For an algebraically closed field like the complex numbers $\mathbb{C}$, any a_i in (1.2) has a square root. By the first relation an element in $\mathrm{GW}(\mathbb{C})$ is completely determined by its rank n and hence $\mathrm{GW}(\mathbb{C}) \cong \mathbb{Z}$.

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Example 1.1.11. An element in $\mathrm{GW}(\mathbb{R})$ is determined by its rank r and its signature s . By the signature, we mean the number of positive a_i in (1.2) minus the number of negative a_i in (1.2). More precisely,

$$\mathrm{GW}(\mathbb{R}) \cong \{(r, s) \in \mathbb{Z} \times \mathbb{Z} : r + s \equiv 0 \pmod{2}\}$$

which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Example 1.1.12. The Grothendieck-Witt ring of a finite field $\mathbb{F}_q$ is isomorphic to $\mathbb{Z} \times \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2 \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ where the isomorphism is given by the rank and the discriminant, that is, the determinant of the Gram matrix, of the form.

Example 1.1.13 (Springer's theorem). The map

$$\frac{\mathrm{GW}(k) \oplus \mathrm{GW}(k)}{\mathbb{Z}(\mathbb{H}, -\mathbb{H})} \xrightarrow{(i,j)} \mathrm{GW}(k((t)))$$

where $i(\langle a \rangle) = \langle a \rangle$ and $j(\langle b \rangle) = \langle tb \rangle$ is an isomorphism.

1.1.3.1 The $\mathbb{A}^1$ -degree

Morel shows that for $n \geq 2$, there is a canonical isomorphism of sheaves

$$\pi_n^{\mathbb{A}^1}((\mathbb{P}^1)^{\wedge n}) \cong \pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \cong \mathbf{K}_n^{\mathrm{MW}} \quad (1.3)$$

where $\mathbf{K}_n^{\mathrm{MW}}$ is the n -th Milnor-Witt K-theory sheaf [Mor12, Theorem 6.40]. For the definition of $\mathbf{K}_n^{\mathrm{MW}}$ we refer to [Mor12], but we want to remark that the 0-th Milnor-Witt K-theory of a field k is isomorphic to the Grothendieck-Witt ring of k

$$K_0^{\mathrm{MW}}(k) \cong \mathrm{GW}(k).$$

In [Mor12, Corollary 6.43] Morel concludes from (1.3) that there is a canonical isomorphism

$$\deg^{\mathbb{A}^1} : [(\mathbb{P}^1)^{\wedge n}, (\mathbb{P}^1)^{\wedge n}]_{\mathbb{A}^1} \xrightarrow{\cong} K_0^{\mathrm{MW}}(k) \cong \mathrm{GW}(k) \quad (1.4)$$

for $n \geq 2$ which we call the $\mathbb{A}^1$ -degree. For $n = 1$, $\deg^{\mathbb{A}^1} : [\mathbb{P}^1, \mathbb{P}^1]_{\mathbb{A}^1} \rightarrow \mathrm{GW}(k)$ is an epimorphism. This $\mathbb{A}^1$ -degree is the central idea behind $\mathbb{A}^1$ -enumerative geometry.

Similarly, there is an isomorphism

$$\deg^{\mathbb{A}^1} : [\mathbb{A}^n \setminus \{0\}, \mathbb{A}^n \setminus \{0\}]_{\mathbb{A}^1} \xrightarrow{\cong} \mathrm{GW}(k) \quad (1.5)$$

when $n \geq 2$ which we will use in 1.2.4.4.

1.1.4 The stable $\mathbb{A}^1$ -homotopy category

We know from classical topology that calculations simplify in the *stable homotopy category*, that means after inverting the *suspension functor*. We invert the $\mathbb{P}^1$ -suspension $-\wedge \mathbb{P}^1$ by considering the category of *motivic spectra* Sp_k , that is

the category of sequences E_n of pointed motivic spaces over k together with *structure maps* $\mathbb{P}^1 \wedge E_n \rightarrow E_{n+1}$ for $n \geq 0$. A morphism of motivic spectra $f : (E_n)_n \rightarrow (F_n)_n$ consists of morphisms $f_n : E_n \rightarrow F_n$ of pointed motivic spaces that commute with the structure maps. A smooth variety X over k gives rise to the motivic spectrum $\sum_{\mathbb{P}^1}^\infty X_+$ where $(\sum_{\mathbb{P}^1}^\infty X_+)_n = X_+ \wedge (\mathbb{P}^1)^{\wedge n}$. There is a *stable model structure* on Sp_k and we denote the resulting homotopy category by $\mathcal{SH}(k)$.

1.2 $\mathbb{A}^1$ -contractible varieties

Definition 1.2.1. A smooth variety X over k is $\mathbb{A}^1$ -*contractible* if the structure map $X \rightarrow \mathrm{Spec} k$ is an $\mathbb{A}^1$ -weak equivalence.

We give a short introduction to the motivation and techniques to study $\mathbb{A}^1$ -contractible varieties. For a more detailed introduction to this topic, we refer to the survey [AØ19].

1.2.1 Motivation from Topology

Whitehead discovered an example of an open contractible 3-manifold W not homeomorphic to $\mathbb{R}^3$, the *Whitehead manifold* [Whi35]. Unlike $\mathbb{R}^3$, W is not *simply connected at infinity*, meaning that for a sufficiently large compact subset $K \subset W$, the fundamental group $\pi_1(W \setminus K)$ is not trivial. However, the product $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. In fact, for any open contractible n -manifold M , the cylinder $M \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ is the only open contractible n -manifold that is simply connected at infinity. For $n \leq 2$, $\mathbb{R}^n$ is the unique open contractible n -manifold [AØ19, §1].

In affine algebraic geometry, the Zariski Cancellation problem asks the following question.

Question 1.2.2 (Zariski Cancellation). *Assume X is an affine variety of dimension n over a field k such that $X \times \mathbb{A}^1$ is isomorphic to $\mathbb{A}^{n+1}$. Does this imply that X is isomorphic to $\mathbb{A}^n$?*

The answer is yes, when $n \leq 2$ and no when $n \geq 3$ and $\mathrm{char} k > 0$. When $n \geq 3$ and $\mathrm{char} k = 0$, the Zariski Cancellation problem is still unsolved (see [Gup15] for a survey on Zariski cancellation). One hopes that there is an $\mathbb{A}^1$ -*fundamental group at infinity* and a similar classification of smooth affine $\mathbb{A}^1$ -contractible varieties as for real open contractible n -manifolds.

One way to rule out a potential counterexample X to Zariski Cancellation, is to check whether X is $\mathbb{A}^1$ -contractible. If there is an isomorphism $X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ then $X \times \mathbb{A}^1$ is $\mathbb{A}^1$ -contractible (since $\mathbb{A}^n$ is) and therefore also X . It follows that a smooth variety X that is not $\mathbb{A}^1$ -contractible, cannot be a counterexample to Zariski cancellation.

1.2.2 Examples of $\mathbb{A}^1$ -weak equivalences

In order to find examples of smooth $\mathbb{A}^1$ -contractible varieties, we need to understand $\mathbb{A}^1$ -weak equivalences. The simplest example of an $\mathbb{A}^1$ -weak equivalence is $X \times \mathbb{A}^1 \rightarrow X$. Consequently, $\mathbb{A}^1 \rightarrow \operatorname{Spec} k$ is an $\mathbb{A}^1$ -weak equivalence and $\mathbb{A}^1$ is $\mathbb{A}^1$ -contractible. Since compositions of $\mathbb{A}^1$ -weak equivalences are $\mathbb{A}^1$ -weak equivalences, affine space $\mathbb{A}^n$ is $\mathbb{A}^1$ -contractible for $n \geq 0$.

More generally, a morphism $f : X \rightarrow Y$ of smooth k -varieties that is *Nisnevich locally trivial with affine space fibers*, is an $\mathbb{A}^1$ -weak equivalence [AØ19, Lemma 3.1.3]. By Nisnevich locally trivial with affine space fibers we mean that there is a Nisnevich covering $u : U \rightarrow Y$ such that $X \times_Y U \cong U \times_k \mathbb{A}^n$ for some n . For example, vector bundle maps are Nisnevich locally trivial with affine space fibers and thus $\mathbb{A}^1$ -weak equivalences.

Example 1.2.3. In [AD07] Asok and Doran find examples of quasi-affine smooth varieties over k of dimension $n \geq 4$ not isomorphic to $\mathbb{A}^n$ which are the quotient of $\mathbb{A}^{n+1}$ by a scheme-theoretically free action of the additive group scheme $\mathbb{G}_a$. The quotient map is Nisnevich locally trivial with $\mathbb{A}^1$ -fibers and thus the quotient is $\mathbb{A}^1$ -contractible.

If the field k embeds into $\mathbb{C}$, the complex points $X(\mathbb{C})$ of a smooth k -variety X have the structure of a smooth complex manifold and an $\mathbb{A}^1$ -weak equivalence $f : X \rightarrow Y$ of smooth k -varieties yields a weak equivalence of topological spaces. In particular, any $\mathbb{A}^1$ -contractible variety X over k is contractible as a smooth complex manifold. Thus contractibility in the classical sense is a necessary condition for $\mathbb{A}^1$ -contractibility.

1.2.3 Stably $\mathbb{A}^1$ -contractible varieties

Another way to test whether a (pointed) smooth variety (X, x) over k is potentially $\mathbb{A}^1$ -contractible, is to ask whether it is *stably $\mathbb{A}^1$ -contractible*, that is, the suspension spectrum $\sum_{\mathbb{P}^1}^\infty (X, x)$ is isomorphic to $\sum_{\mathbb{P}^1}^\infty \operatorname{Spec} k_+$ in the stable $\mathbb{A}^1$ -homotopy category $\mathcal{SH}(k)$. Clearly, $\mathbb{A}^1$ -contractibility implies stable $\mathbb{A}^1$ -contractibility.

Hoyois, Krishna and Østvær show that a pointed smooth variety (X, x) over k is stably $\mathbb{A}^1$ -contractible if and only if there is an $n \geq 0$ such that $(X, x) \wedge (\mathbb{P}^1)^{\wedge n}$ is $\mathbb{A}^1$ -contractible [HKØ16, Lemma 4.3], see also [AD07, Lemma 2.5.1]. Furthermore, they provide the following criterion for stable $\mathbb{A}^1$ -contractibility [HKØ16, §4].

Theorem 1.2.4. *Let k be algebraically closed of characteristic 0 and (X, x) a pointed smooth variety over k . If the projection $X \times Y \rightarrow Y$ induces an isomorphism in motivic cohomology for every smooth affine variety Y then (X, x) is stably $\mathbb{A}^1$ -contractible.*

Remark 1.2.5. By [Voe02] motivic cohomology groups are isomorphic to higher Chow groups defined in [Blo86]. So to prove that (X, x) is stably $\mathbb{A}^1$ -contractible,

we need to show that $X \times Y \rightarrow Y$ induces an isomorphism of higher Chow groups for every smooth affine variety Y over k .

1.2.4 The Koras-Russell cubic

The *Koras-Russell cubic* KR is the hypersurface in $\mathbb{A}^4$ defined by the equation $x^2y + z^2 + t^3 + x = 0$. It was first discovered by Koras and Russell as an example of a smooth affine 3-fold over $\mathbb{C}$ with a non-linearizable $\mathbb{G}_m$ -action. So if KR were isomorphic to $\mathbb{A}^3$, it would be a counterexample to the linearization conjecture in dimension 3 [Kal+97]. All the usual invariants failed to distinguish KR from $\mathbb{A}^3$. The cubic KR is for example diffeomorphic to $\mathbb{R}^6$ as a smooth manifold. Makar-Limanov introduced a new invariant of affine varieties, the Makar-Limanov invariant

$$\text{ML}(X) := \bigcap_{\mathbb{G}_a\text{-action on } X} \mathbb{G}_a\text{-invariant functions on } X$$

and showed that $KR \rightarrow \mathbb{C}, (x, y, z, t) \mapsto x$ is invariant under any $\mathbb{G}_a$ -action. The only functions that are invariant under any $\mathbb{G}_a$ -action on $\mathbb{A}^3$, are constant. Hence, $\mathbb{C}[x] \subset \text{ML}(KR)$ (this is indeed an equality) and $\text{ML}(\mathbb{A}^3) = \mathbb{C}$ and consequently $KR \not\cong \mathbb{A}^3$ [KM97]. Because of its similarity with $\mathbb{A}^3$, KR is a promising candidate for a counterexample to Zariski cancellation in characteristic 0. The Makar-Limanov invariant of $KR \times \mathbb{A}^1$ is $\mathbb{C}$ and cannot distinguish between $KR \times \mathbb{A}^1$ and $\mathbb{A}^4$ [Dub09]. An attempt to rule out KR as a counterexample, was to try to show that KR was not $\mathbb{A}^1$ -contractible. However, Dubouloz and Fasel show in [DF18] that KR is $\mathbb{A}^1$ -contractible which makes KR the first example of an affine (not quasi-affine as the examples in 1.2.3) smooth $\mathbb{A}^1$ -contractible variety not isomorphic to affine n -space. In the rest of this section, we sketch their proof since we will use many of their ideas in Paper I.

1.2.4.1 Stable $\mathbb{A}^1$ -contractibility

Before Dubouloz and Fasel proved $\mathbb{A}^1$ -contractibility of KR , Hoyois, Krishna and Østvær used Theorem 1.2.4 to show that $(KR, 0)$ is stably $\mathbb{A}^1$ -contractible [HKØ16]. Their result follows from the following Proposition, Theorem 1.2.4 and Remark 1.2.5.

Proposition 1.2.6 (Proposition 3.3 in [HKØ16]). *For any smooth complex affine variety Y , the pullback map*

$$\text{CH}^*(Y, i) \rightarrow \text{CH}^*(X \times Y, i)$$

is an isomorphism for every $i \geq 0$ where $\text{CH}^(-, i)$ denotes Bloch's higher Chow groups.*

1.2.4.2 An $\mathbb{A}^1$ -weak equivalence $KR \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$

Let $L = \{x = z = t = 0\} \subset KR$. Dubouloz and Fasel find the following $\mathbb{A}^1$ -weak equivalence $KR \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$ in [DF18, §3.2]. There is a $\mathbb{G}_a$ -

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action on KR that acts freely on $KR \setminus L$ and such that $KR \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$, $(x, y, z, t) \mapsto (x, t)$ factors through the geometric quotient $\mathfrak{S} := (KR \setminus L)/\mathbb{G}_a$. Let $V := \{x^2y + z^2 + t + x = 0\}$. There is a similar $\mathbb{G}_a$ -action on V and $V \setminus L' \rightarrow \mathbb{A}^2 \setminus \{0\}$, $(x, y, z, t) \mapsto (x, t)$ with $L' = \{x = t = z = 0\} \subset V$ factors through $\mathfrak{S}$, too. The quotient $\mathfrak{S}$ is not a scheme, only an ‘algebraic space’. However, the fiber product $W := (KR \setminus L) \times_{\mathfrak{S}} (V \setminus L')$ is a smooth variety and both $W \rightarrow (KR \setminus L)$ and $W \rightarrow (V \setminus L')$ are Nisnevich locally trivial with $\mathbb{A}^1$ -fibers and thus $\mathbb{A}^1$ -weak equivalences.

$$\begin{array}{ccccc}
 & & W & & \\
 & \swarrow \simeq & & \searrow \simeq & \\
 KR \setminus L & & & & V \setminus L' \\
 & \searrow & & \swarrow & \\
 & & \mathfrak{S} & & \\
 & \searrow \downarrow & & \swarrow \downarrow & \\
 & & \mathbb{A}^2 \setminus \{0\} & &
 \end{array} \tag{1.6}$$

Note that $V \cong \mathbb{A}^3$ and $V \setminus L' \cong \mathbb{A}^3 \setminus \{x = z = 0\} \cong (\mathbb{A}^2 \setminus \{0\}) \times \mathbb{A}^1$ is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{A}^2 \setminus \{0\}$. Consequently, there is an $\mathbb{A}^1$ -weak equivalence $KR \setminus L \rightarrow \mathbb{A}^2 \setminus \{0\}$.

1.2.4.3 The weak 5-lemma argument

Note that $\{y = 0\} \subset KR$ is isomorphic to $\mathbb{A}^2$ and let $i : \mathbb{A}^2 \setminus \{0\} \hookrightarrow KR \setminus L$ be the inclusion of $\{y = 0\}$. Consider the following diagram.

$$\begin{array}{ccccccc}
 \mathbb{A}^2 \setminus \{0\} & \longrightarrow & \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2/(\mathbb{A}^2 \setminus \{0\}) \\
 \downarrow i & & \downarrow & & \downarrow \\
 KR \setminus L & \longrightarrow & KR & \longrightarrow & KR/(KR \setminus L)
 \end{array} \tag{1.7}$$

By Proposition 1.1.5, $\mathbb{A}^2/(\mathbb{A}^2 \setminus \{0\})$ is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{P}^1 \wedge \mathbb{P}^1$. Denote by $N_L KR$ the normal bundle of L in KR . Homotopy Purity 1.1.8 and Proposition 1.1.5 imply that

$$KR/(KR \setminus L) \stackrel{1.1.8}{\simeq} N_L KR/(N_L KR \setminus L) \cong L \times \mathbb{A}^2/(L \times \mathbb{A}^2 \setminus L \times \{0\}) \stackrel{1.1.5}{\simeq} L_+ \wedge \mathbb{P}^1 \wedge \mathbb{P}^1$$

in $\mathcal{H}_\bullet(k)$. Since $L \cong \mathbb{A}^1$, the right most vertical map in (1.7) is an $\mathbb{A}^1$ -weak equivalence. The following lemma [DF18, Lemma 2.1] follows from a weaker version of the 5-lemma.

Lemma 1.2.7 (Weak 5-lemma). *Let*

$$\begin{array}{ccccc}
 A & \longrightarrow & B & \longrightarrow & C \\
 \downarrow f & & \downarrow g & & \downarrow h \\
 A' & \longrightarrow & B' & \longrightarrow & C'
 \end{array}$$

be a commutative diagram of pointed motivic spaces (or objects in any pointed model category) such that the rows are cofiber sequences. If f and h are pointed $\mathbb{A}^1$ -weak equivalences and B is $\mathbb{A}^1$ -contractible, then B' is $\mathbb{A}^1$ -contractible.

It follows that if i was an $\mathbb{A}^1$ -weak equivalence, then by the weak 5-Lemma, KR would be $\mathbb{A}^1$ -contractible. We have already seen that $KR \setminus L$ is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{A}^2 \setminus \{0\}$. We use the $\mathbb{A}^1$ -degree to show that $i : \mathbb{A}^2 \setminus \{0\} \rightarrow KR \setminus L$ is an $\mathbb{A}^1$ -weak equivalence.

1.2.4.4 The $\mathbb{A}^1$ -degree argument

Recall from 1.1.3.1 that there is an isomorphism

$$\deg^{\mathbb{A}^1} : [\mathbb{A}^n \setminus \{0\}, \mathbb{A}^n \setminus \{0\}]_{\mathbb{A}^1} \xrightarrow{\cong} \mathrm{GW}(k)$$

when $n \geq 2$. By [DF18, Lemma 2.2]

$$H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{K}_n^{\mathrm{MW}}) \cong \mathrm{GW}(k) \cdot \xi$$

for a generator ξ of $H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{K}^{\mathrm{MW}})$ and $\deg^{\mathbb{A}^1}(f : \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{A}^n \setminus \{0\})$ is the ‘degree’ of

$$f^* : H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{K}_n^{\mathrm{MW}}) \rightarrow H^{n-1}(\mathbb{A}^n \setminus \{0\}, \mathbf{K}_n^{\mathrm{MW}}),$$

that is, $\deg^{\mathbb{A}^1}(f)$ is the element of $\mathrm{GW}(k)$ such that $f^*(\xi) = \deg^{\mathbb{A}^1}(f) \cdot \xi$. Consequently, f is an $\mathbb{A}^1$ -weak equivalence if and only if $\deg^{\mathbb{A}^1}(f)$ is invertible if and only if f^* is an isomorphism.

It follows that in order to show that $i : \mathbb{A}^2 \setminus \{0\} \rightarrow KR \setminus L$ is an $\mathbb{A}^1$ -weak equivalence, it suffices to show that

$$i^* : H^{2-1}(\mathbb{A}^2 \setminus \{0\}, \mathbf{K}_2^{\mathrm{MW}}) \cong H^{2-1}(KR \setminus L, \mathbf{K}_2^{\mathrm{MW}}) \rightarrow H^{2-1}(\mathbb{A}^2 \setminus \{0\}, \mathbf{K}_2^{\mathrm{MW}})$$

is an isomorphism. The diagram (1.7) of cofiber sequences induces a diagram of long exact sequences

$$\begin{array}{ccccccc} \dots & \rightarrow & H^1(KR \setminus L, \mathbf{K}_2^{\mathrm{MW}}) & \xrightarrow{\partial} & H^2(KR/(KR \setminus L), \mathbf{K}_2^{\mathrm{MW}}) & \rightarrow & H^2(KR, \mathbf{K}_2^{\mathrm{MW}}) \rightarrow \dots \\ & & \downarrow i^* & & \downarrow \cong & & \downarrow \\ \dots & \rightarrow & H^1(\mathbb{A}^2 \setminus \{0\}, \mathbf{K}_2^{\mathrm{MW}}) & \xrightarrow[\cong]{\partial'} & H^2(\mathbb{P}^1 \wedge \mathbb{P}^1, \mathbf{K}_2^{\mathrm{MW}}) & \longrightarrow & H^2(\mathbb{A}^2, \mathbf{K}_2^{\mathrm{MW}}) \rightarrow \dots \end{array} \quad (1.8)$$

We already know that the middle vertical map in (1.8) is an isomorphism. Since $H^j(\mathbb{A}^2, \mathbf{K}_2^{\mathrm{MW}}) = 0$ for $j \geq 1$ (because $\mathbb{A}^2$ is $\mathbb{A}^1$ -contractible), ∂' is an isomorphism. So i^* is an isomorphism if and only if ∂ is an isomorphism. We show that ∂ is an isomorphism by showing that $H^1(KR, \mathbf{K}_2^{\mathrm{MW}}) = H^2(KR, \mathbf{K}_2^{\mathrm{MW}}) = 0$.

Recall that $(KR, 0)$ is stably $\mathbb{A}^1$ -contractible and thus there is an $n \geq 0$ such that $(KR, 0) \wedge (\mathbb{P}^1)^{\wedge n}$ is $\mathbb{A}^1$ -contractible. By [Fas13] $H^i(KR, \mathbf{K}_2^{\mathrm{MW}}) = H^{i+n}(KR_+ \wedge (\mathbb{P}^1)^{\wedge n}, \mathbf{K}_{2+n}^{\mathrm{MW}})$ and the cofiber sequence

$$(\mathbb{P}^1)^{\wedge n} \rightarrow KR_+ \wedge (\mathbb{P}^1)^{\wedge n} \rightarrow KR \wedge (\mathbb{P}^1)^{\wedge n} \simeq *$$

[ADF17, Proposition 2.2.4] implies that

$$H^i(KR, \mathbf{K}_2^{\text{MW}}) = H^{i+n}((\mathbb{P}^1)^{\wedge n}, \mathbf{K}_{2+n}^{\text{MW}}) = 0$$

for $i \geq 1$ by [Fas13]. So ∂' and i^* are isomorphisms and i is an $\mathbb{A}^1$ -weak equivalence. By the weak 5-lemma the Koras-Russell cubic KR is $\mathbb{A}^1$ -contractible.

1.3 The $\mathbb{A}^1$ -degree

Recall from 1.1.3.1 that the $\mathbb{A}^1$ -degree is a canonical morphism from the $\mathbb{A}^1$ -homotopy classes of maps between motivic n -spheres to $\text{GW}(k)$. The Brouwer degree assigns to a (classical) homotopy class of maps between (classical) n -spheres, an integer. As a motivation we recall how to express the Brouwer degree as a sum of *local degrees* and how to compute these local degrees in the classical setting. Then we introduce the $\mathbb{A}^1$ -analog of the local degree called the *local $\mathbb{A}^1$ -degree*. In 1.4 we use the local $\mathbb{A}^1$ -degree to define the $\mathbb{A}^1$ -Euler number. For a more detailed introduction to the (local) $\mathbb{A}^1$ -degree and the $\mathbb{A}^1$ -Euler number, we refer to [KW19] and [WW19].

1.3.1 The Brouwer degree

We recall the definition and some properties of the Brouwer degree in topology from [Hat02]. Let

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \dots + x_n^2 = 1\}$$

be the n -sphere for $n \geq 1$. Since S^n is orientable, its top homology group $H_n(S^n)$ is isomorphic to $\mathbb{Z}$ and choosing an orientation of S^n is equivalent to choosing a generator α of $H_n(S^n)$. Let $f : S^n \rightarrow S^n$ be a map of n -spheres and let $f_* : H_n(S^n) \rightarrow H_n(S^n)$ be the induced map in homology. Then $f_*(\alpha) = d \cdot \alpha$ with $d \in \mathbb{Z}$.

Definition 1.3.1. The *Brouwer degree* of f is the integer $\deg f := d$.

Let $\pi_n(S^n) = [S^n, S^n]$ be the n -th homotopy group of S^n which is equal to the homotopy classes of maps from S^n to S^n . Two homotopy equivalent maps $f, g : S^n \rightarrow S^n$ induce the same map in homology and thus $\deg f = \deg g$. Indeed the Brouwer degree establishes a well-defined homomorphism

$$\deg : [S^n, S^n] \rightarrow \mathbb{Z}$$

which turns out to be an isomorphism.

1.3.1.1 The Brouwer degree as a sum of local degrees

We are interested in computing the Brouwer degree, but this is in general a difficult task. One way to simplify computations is to express the Brouwer degree

as a sum of local degrees

$$\deg f = \sum_{q \in f^{-1}(p)} \deg_q f$$

for $p \in S^n$ a regular value of f . Recall from [Hat02, Chapter 2.2] that the *local degree* $\deg_q f$ at q is the Brouwer degree of

$$S^n \simeq U/\partial U \xrightarrow{\bar{f}} V/\partial V \simeq S^n$$

induced by f where U is a small ball around q and V is a small ball around p . Since p is a regular value, $\bar{f}$ is a homeomorphism and $\deg \bar{f} \in \{\pm 1\}$ depending on whether $\bar{f}$ is orientation preserving (+1) or not (-1).

1.3.1.2 An algebraic formula for the local degree

We can choose coordinates around q and p compatible with the orientation of S^n . Then locally f is given by $(f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $Jf(q)$ be the determinant of the jacobian $\frac{\partial f_i}{\partial x_j}$ of $(f_1, \dots, f_n)$ evaluated at q . Since p is a regular value, $Jf(q) \neq 0$, and we get an algebraic formula for $\deg_q f$

$$\deg_q f = \begin{cases} +1 & \text{if } Jf(q) > 0 \\ -1 & \text{if } Jf(q) < 0. \end{cases} \quad (1.9)$$

1.3.2 Morel's $\mathbb{A}^1$ -Brouwer degree

Recall from 1.1.2 and 1.1.3.1 that in $\mathbb{A}^1$ -homotopy theory the n -sphere is given by

$$\mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{A}^n/(\mathbb{A}^n - \{0\})$$

and that Morel has found a canonical morphism

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n/\mathbb{P}^{n-1}, \mathbb{P}^n/\mathbb{P}^{n-1}]_{\mathbb{A}^1} \rightarrow \text{GW}(k)$$

which is an isomorphism when $n \geq 2$ and an epimorphism for $n = 1$ [Mor12], called the $\mathbb{A}^1$ -degree.

1.3.2.1 The local $\mathbb{A}^1$ -degree

Again it is hard to compute the $\mathbb{A}^1$ -degree of a map $f : \mathbb{P}^n/\mathbb{P}^{n-1} \rightarrow \mathbb{P}^n/\mathbb{P}^{n-1}$ and to increase our understanding and simplify computations we wish to express it as a sum of local degrees. This has been done by Kass and Wickelgren in [KW19] and we recall some of their results. Let $p \in \mathbb{P}^n$ be a k -rational point, $q \in f^{-1}(p)$ and let U and V be Zariski open neighborhoods of p and q . Then

$$\begin{array}{ccc} U - \{q\} & \longrightarrow & \mathbb{P}^n - \{q\} \\ \downarrow & & \downarrow \\ U & \longrightarrow & \mathbb{P}^n \end{array} \quad (1.10)$$

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is an elementary distinguished square (see (1.1)) and thus a pushout square in the homotopy category $\mathcal{H}(k)$. Consequently, the quotient of the horizontal maps in (1.10)

$$U/(U - \{q\}) \rightarrow \mathbb{P}^n/(\mathbb{P}^n - \{q\})$$

is an $\mathbb{A}^1$ -weak equivalence. Similarly, we get a canonical $\mathbb{A}^1$ -weak equivalence $V/(V - \{p\}) \rightarrow \mathbb{P}^n/(\mathbb{P}^n - \{p\})$. We recall the definition of the local $\mathbb{A}^1$ -degree from (see [KW19, §2]).

Definition 1.3.2. The *local $\mathbb{A}^1$ -degree* $\deg_q^{\mathbb{A}^1} f$ of f at q is the $\mathbb{A}^1$ -degree of

$$\bar{f} : \mathbb{P}^n/\mathbb{P}^{n-1} \simeq \mathbb{P}^n/(\mathbb{P}^n - \{q\}) \simeq U/(U \setminus \{q\}) \rightarrow V/V \setminus \{p\} \simeq \mathbb{P}^n/(\mathbb{P}^n - \{q\}) \simeq \mathbb{P}^n/\mathbb{P}^{n-1}.$$

1.3.2.2 An algebraic formula for the local $\mathbb{A}^1$ -degree

Again we can choose local coordinates around q and p such that f is locally given by $(f_1, \dots, f_n) : \mathbb{A}^n \rightarrow \mathbb{A}^n$. As before define $Jf(q) := \det \frac{\partial f_i}{\partial x_j}(q)$. If q is a k -rational point and $Jf(q) \neq 0$ then the local $\mathbb{A}^1$ -degree at q is

$$\deg_q^{\mathbb{A}^1} f = \langle Jf(q) \rangle \in \mathrm{GW}(k) \quad (1.11)$$

by [KW19].

Remark 1.3.3. Note that this agrees with (1.9) since there are exactly 2 isometry classes of rank 1 non-degenerate symmetric bilinear forms over $\mathbb{R}$, namely $\langle 1 \rangle$ and $\langle -1 \rangle$ since $\mathbb{R}^\times/(\mathbb{R}^\times)^2 = \{\pm 1\}$.

For a finite separable field extension L/k there is a morphism

$$\mathrm{Tr}_{L/k} : \mathrm{GW}(L) \rightarrow \mathrm{GW}(k)$$

which sends a finite rank non-degenerate symmetric bilinear form $\beta : V \times V \rightarrow L$ over L to its *trace form*

$$V \times V \xrightarrow{\beta} L \xrightarrow{\mathrm{Tr}_{L/k}} k$$

where $\mathrm{Tr}_{L/k} : L \rightarrow k$ denotes the field trace. If the residue field $k(q)$ of q is a finite, separable field extension of k and $Jf(q) \neq 0$ then

$$\deg_q^{\mathbb{A}^1} f = \mathrm{Tr}_{k(q)/k}(\langle Jf(q) \rangle) \in \mathrm{GW}(k) \quad (1.12)$$

by [KW19, Proposition 15].

Remark 1.3.4. One can define the trace $\mathrm{Tr}_{L/k}$ and thus the trace form in a more general setting, namely when L is a finite étale algebra over k . Assume $q_1, \dots, q_r$ are finitely many closed points in the preimage of p under $(f_1, \dots, f_n)$, such that the coordinate ring E of the closed subscheme $Z := \{q_1, \dots, q_r\}$ is a finite étale algebra over k . Then the sum of local $\mathbb{A}^1$ -degrees at the points in Z is equal to $\mathrm{Tr}_{E/k}(\langle (Jf)_E \rangle)$ where $(Jf)_E$ is the image of $Jf = \frac{\partial f_i}{\partial x_j}$ in E .

In Paper III we provide Macaulay2 code that computes the trace form.

1.3.3 The EKL-class

We want to provide an algebraic formula for the local $\mathbb{A}^1$ -degree in the case that $Jf(q) = 0$. Again the motivation for this formula comes from classical topology. Both Eisenbud and Levine [EL77] and Khimshiashvili [Khi77] find a way to compute the local (classical Brouwer) degree as the signature of a certain non-degenerate, symmetric bilinear form whose definition is purely algebraic. Hence, it can be defined over any field k . We call the class of this form in $\text{GW}(k)$ the *EKL-class*, short for Eisenbud-Khimshiashvili-Levine.

We recall the definition of the EKL-class from [KW19]. Let $f = (f_1, \dots, f_n) : \mathbb{A}^n \rightarrow \mathbb{A}^n$ have an isolated zero at the origin. Find $a_{ij} \in k[x_1, \dots, x_n]$ such that $f_i = \sum_{j=1}^n a_{ij}x_j$ and define $E := \det a_{ij}$. For any choice of a_{ij} , E generates the *socle*, that is, the sum of the minimal nonzero ideals, of $\mathcal{Q} := \frac{k[x_1, \dots, x_n]_0}{(f_1, \dots, f_n)}$ [KW19, Lemma 4] and we call E the *distinguished socle element*. Let $\phi : \mathcal{Q} \rightarrow k$ be any k -linear map that sends E to 1.

Definition 1.3.5. The *EKL-class* of f at zero is the class $\omega \in \text{GW}(k)$ of the symmetric bilinear form $\beta_\phi : \mathcal{Q} \times \mathcal{Q} \rightarrow k$ defined by $(x, y) \mapsto \phi(xy)$.

By [KW19, Lemma 6], β_ϕ is non-degenerate and ω does not depend on the choice of ϕ , so ω is a well-defined element of $\text{GW}(k)$.

Remark 1.3.6. Let $r := \text{rank}_k \mathcal{Q}$. Then $r \cdot E = Jf$ where $Jf := \det \frac{\partial f_i}{\partial x_j}$ [SS75, (4.7) Korollar]. So when $\text{char } k$ does not divide r , we can compute the EKL-class of f at zero as follows: Let $1 = b_1, b_2, \dots, b_{r-1}, Jf$ be a k -basis of $\mathcal{Q}$. Then the EKL-class of f at zero is the class of the non-degenerate symmetric bilinear form $(x, y) \mapsto \beta_\phi(xy)$ where $\beta_\phi : \mathcal{Q} \rightarrow k$ is the k -linear map that sends $1, b_2, \dots, b_{r-1}$ to 0 and Jf to r .

Example 1.3.7. Consider $f = x^r : \mathbb{A}^1 \rightarrow \mathbb{A}^1$. Then $\mathcal{Q} = \frac{k[x]_0}{(x^r)}$ and $1, x, x^2, \dots, x^{r-2}, rx^{r-1} = Jf$ is a basis of $\mathcal{Q}$. The Gram matrix of the symmetric bilinear form $\beta_\phi : (x, y) \mapsto \phi(xy)$ where $\phi(x^i) = 0$ for $i = 0, \dots, r-2$ and $\phi(rx^{r-1}) = r$, in the chosen basis is

$$\begin{pmatrix} 0 & 0 & \dots & 0 & r \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ r & 0 & \dots & 0 & 0 \end{pmatrix}$$

and hence, the EKL-class of f at zero is equal to

$$\begin{cases} \frac{r}{2}\mathbb{H} & \text{if } r \text{ even} \\ \frac{r-1}{2}\mathbb{H} + \langle 1 \rangle & \text{if } r \text{ odd.} \end{cases}$$

Again remembering the whole form, not only the signature gives us the local $\mathbb{A}^1$ -degree: When q is k -rational, choose local coordinates around q

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and p such that $q = 0 \in \mathbb{A}^n$ and $p = 0 \in \mathbb{A}^n$ and f is locally defined by $(f_1, \dots, f_n) : \mathbb{A}_k^n \rightarrow \mathbb{A}_k^n$. Then

$$\deg_q^{\mathbb{A}^1} f = \omega \in \text{GW}(k),$$

where ω is the EKL-class of $(f_1, \dots, f_n) : \mathbb{A}^n \rightarrow \mathbb{A}^n$ at zero [KW19, Main Theorem].

In [Bra+19] this result is generalized to the case where $k(q)$ is a finite separable field extension of k , that is,

$$\deg_q^{\mathbb{A}^1} f = \text{Tr}_{k(q)/k}(\omega) \in \text{GW}(k)$$

where ω is the EKL-class of the (base changed) map $(f_1, \dots, f_n) : \mathbb{A}_{k(q)}^n \rightarrow \mathbb{A}_{k(q)}^n$ where we again have chosen local coordinates around $q = 0 \in \mathbb{A}_{k(q)}^n$ and $p = 0 \in \mathbb{A}_{k(q)}^n$.

In Paper III we provide Macaulay2 code that computes the form in Remark 1.3.6 representing the EKL-class and thus the local $\mathbb{A}^1$ -degree at a k -rational point when $\text{char } k$ does not divide the rank of the local algebra $\mathcal{Q}$.

1.4 The $\mathbb{A}^1$ -Euler number

In this section we introduce the $\mathbb{A}^1$ -Euler number valued in $\text{GW}(k)$. As an application, we explain how one can use the $\mathbb{A}^1$ -Euler number to count lines on a smooth cubic surface, or more generally, on a degree $(2n - 1)$ -hypersurface in $\mathbb{P}^{n+1}$ enriched in $\text{GW}(k)$. Furthermore, we introduce our *dynamic Euler number* from Paper II and redo the count of lines on a general cubic surface using enriched dynamic intersection theory. Before we define the $\mathbb{A}^1$ -Euler number, we recall the definition of the Euler number in topology.

1.4.1 The Euler number of an oriented real vector bundle

We recall the definitions of an orientation of a vector bundle and the Euler class from [MS74]. Recall that an *orientation* of a vector space is an equivalence class of ordered bases where two bases are equivalent if the basis change matrix has positive determinant. A real vector bundle $\pi : E \rightarrow X$ of rank r is *orientable* if there is a continuous choice of orientation of the fibers F , or equivalently, a continuous choice of generators $u_F \in H^r(F, F \setminus \{0\}; \mathbb{Z})$. A specific choice of generators determines an element $u \in H^r(E, E \setminus s_0(X); \mathbb{Z})$ where $s_0 : X \rightarrow E$ denotes the zero section. The *Euler class* $e(E)$ is the image of the *Thom class* u under the pullback of $(X, \emptyset) \xrightarrow{s_0} (E, \emptyset) \hookrightarrow (E, E \setminus s_0(X))$, that is,

$$e(E) = s_0^*(u|_E) \in H^r(X; \mathbb{Z})$$

where $u|_E$ is the restriction of $u \in H^r(E, E \setminus s_0(X); \mathbb{Z})$ to $H^r(E; \mathbb{Z})$.

Assume that X is an oriented smooth connected closed manifold of dimension equal to the rank r of $\pi : E \rightarrow X$ and $\sigma : X \rightarrow E$ is a smooth general section,

that is, σ intersects the zero section s_0 transversally. Then the homology class $[\sigma^{-1}(0)] \in H_0(X) \cong \mathbb{Z}$ of zeros of σ is the Poincaré dual of $e(E)$ and is called the *Euler number* of E . By abuse of notation, we denote the Euler number of E by $e(E)$, too.

1.4.1.1 The Euler number as the sum of local indices

Just like the Brouwer degree, the Euler number $e(E)$ can be expressed as the sum of the local contributions of the zeros of σ [Mil97]

$$e(E) = \sum_{x \in \sigma^{-1}(0)} \text{ind}_x \sigma \quad (1.13)$$

where the *local index* $\text{ind}_x \sigma$ is defined as follows: Choose a small ball U around x and a trivialization $\phi : E|_U \xrightarrow{\cong} U \times \mathbb{R}^r$ such that preimage of the standard basis of $\mathbb{R}^r$ via ϕ is an oriented basis of every fiber over U . Then $\text{ind}_x \sigma$ is the local degree at x of the composition

$$U \xrightarrow{\sigma|_U} E|_U \xrightarrow{\phi} U \times \mathbb{R}^r \xrightarrow{\pi_2} \mathbb{R}^r.$$

Inspired by (1.13), Kass and Wickelgren define the $\mathbb{A}^1$ -Euler number of a rank r sufficiently oriented vector bundle over a smooth proper r -dimensional variety X over k as the sum of local indices valued in $\text{GW}(k)$ [KW17].

1.4.2 The $\mathbb{A}^1$ -Euler number

1.4.2.1 Relative orientation and Nisnevich coordinates

In order to define the local index $\text{ind}_x \sigma \in \text{GW}(k)$, we need orientation data and coordinates. We recall the definitions of a *relative orientation* and *Nisnevich coordinates* from [KW17].

A real vector bundle $\pi : E \rightarrow X$ is orientable if and only if its determinant bundle $\det E$ is isomorphic to the trivial rank 1 bundle and an orientation is an isomorphism of line bundles $\phi : \det E \xrightarrow{\cong} X \times \mathbb{R}$. For vector bundles of varieties over an arbitrary field k , a weaker notion of orientation is sufficient.

Definition 1.4.1. An *orientation* of a vector bundle $\pi : E \rightarrow X$ of varieties over k is an isomorphism $\phi : \det E \xrightarrow{\cong} L^{\otimes 2}$ for a line bundle $L \rightarrow X$.

Since any real line bundle $L \rightarrow X$ is isomorphic to its dual $L^* \rightarrow X$, the square $L^{\otimes 2}$ of L is isomorphic to the trivial bundle and the definition of an orientation of a real vector bundle agrees with Definition 1.4.1.

To define the Euler number in the real case, we needed both E and X to be oriented. However, it suffices that E is relatively oriented with respect to X .

Definition 1.4.2. A *relative orientation* of a vector bundle $\pi : E \rightarrow X$ over a smooth variety X over k is an orientation of $\text{Hom}(TX, E)$ where $TX \rightarrow X$ denotes the tangent bundle of X . In other words, a relative orientation of

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$\pi : E \rightarrow X$ is an isomorphism $\phi : \text{Hom}(\det TX, \det E) \xrightarrow{\cong} L^{\otimes 2}$ for a line bundle $L \rightarrow X$.

Assume now that X is smooth, proper and of dimension $r = \text{rank } E$ over k . To define the local index $\text{ind}_x \sigma$ of a general section $\sigma : X \rightarrow E$ at x , we need *coordinates* around x .

Definition 1.4.3. An étale map $\psi : U \rightarrow \mathbb{A}^r$ from a Zariski neighborhood U of x in X which induces an isomorphism on the residue field of x is called *Nisnevich coordinates* around x .

Nisnevich coordinates exist when the residue field $k(x)$ is separable over k and $r \geq 1$ by [KW17, Lemma 18]. Since ψ is étale, the standard basis for the tangent space of $\mathbb{A}^r$ defines a trivialization of $TX|_U$.

Definition 1.4.4. A trivialization of $E|_U$ is *compatible* with the relative orientation ϕ and the Nisnevich coordinates ψ if the element of $\text{Hom}(\det TX|_U, \det E|_U)$ that takes the distinguished basis of $\det TX|_U$ to the distinguished basis of $\det E|_U$ is a square, i.e., is sent to a square by ϕ .

1.4.2.2 The $\mathbb{A}^1$ -Euler number as the sum of local indices

Let $\sigma : X \rightarrow E$ be a general section of a rank r vector bundle $\pi : E \rightarrow X$ over a smooth, proper, r -dimensional variety X over k equipped with a relative orientation ϕ . Choose Nisnevich coordinates $\psi : U \rightarrow \mathbb{A}^r$ and a trivialization of $E|_U$ compatible with ϕ and ψ .

Definition 1.4.5. The *local index* $\text{ind}_x \sigma$ of σ at x is the local $\mathbb{A}^1$ -degree of σ at x in the chosen Nisnevich coordinates and trivialization of $E|_U$.

Explicitly, in the local (Nisnevich) coordinates, the section σ is given by r polynomial functions $(f_1, \dots, f_r) : \mathbb{A}^r \rightarrow \mathbb{A}^r$ and the local index $\text{ind}_x \sigma$ is the local $\mathbb{A}^1$ -degree of $(f_1, \dots, f_r)$ at x .

Definition 1.4.6. The $\mathbb{A}^1$ -Euler number of $\pi : E \rightarrow X$ is the sum of the local indices at the zeros of a general section $\sigma : X \rightarrow E$

$$e^{\mathbb{A}^1}(E) := \sum_{x \in \sigma^{-1}(0)} \text{ind}_x \sigma \in \text{GW}(k).$$

The $\mathbb{A}^1$ -Euler number $e^{\mathbb{A}^1}(E)$ can be defined whenever the chosen section σ has only isolated zeros. By [BW20, Theorem 1.1] it does not depend on the chosen section.

Remark 1.4.7. As in the real case, there are several equivalent definitions of the $\mathbb{A}^1$ -Euler number and the $\mathbb{A}^1$ -Euler class (see for example [AF16] and [BW20]) but in this thesis we only use Definition 1.4.6.

1.4.2.3 Lines on a smooth cubic surface

Kass and Wickelgren use the $\mathbb{A}^1$ -Euler number to count lines on a smooth cubic surface in $\text{GW}(k)$. Let $\text{Gr}(2, 4)$ be the Grassmannian of lines in $\mathbb{P}^3$ and $\mathcal{S} \rightarrow \text{Gr}(2, 4)$ be its tautological bundle. Define $\mathcal{E} := \text{Sym}^3 \mathcal{S}^* \rightarrow \text{Gr}(2, 4)$. The Grassmannian $\text{Gr}(2, 4)$ is smooth, proper and 4-dimensional. The rank of $\mathcal{E}$ is 4 and by [KW17, Corollary 45] $\mathcal{E}$ is relatively orientable. So the $\mathbb{A}^1$ -Euler number of $\mathcal{E}$ is defined and Kass and Wickelgren find that $e^{\mathbb{A}^1}(\mathcal{E}) = 15\langle 1 \rangle + 12\langle -1 \rangle \in \text{GW}(k)$ in [KW17].

A smooth cubic surface $X = \{f = 0\} \subset \mathbb{P}^3$, where f is a homogeneous degree 3 polynomial in 4 variables, defines a section σ_f of $\mathcal{E}$ by restricting f to the lines in $\mathbb{P}^3$. Since a line l lies on X if and only if $f|_l = 0$, the zeros of σ_f are the lines on X .

Segre finds that there are two *types* of real lines on a cubic surface which he calls *hyperbolic* and *elliptic* [Seg42]. Both, Okonek and Teleman [OT14] and Finashin and Kharlamov [FK15], observe from Segre's computations that the number of real hyperbolic lines minus the number of real elliptic lines on a smooth cubic surface is always equal to 3. Kass and Wickelgren generalize Segre's definition of the type of a real line to the type of a line defined over a field of characteristic not equal to 2: Let $l \subset X \subset \mathbb{P}^3$ be a line on a smooth cubic surface with field of definition $k(l)$. The *Gauß map*

$$C : l \cong \mathbb{P}_{k(l)}^1 \rightarrow \mathbb{P}_{k(l)}^1 \cong \text{2-planes in } \mathbb{P}_{k(l)}^3 \text{ containing } l$$

that maps $p \in l$ to its tangent space $T_p X$ identified with a 2-plane in $\mathbb{P}_{k(l)}^3$, is a degree 2 map and thus to each $p \in l$ there is $q \in l$ such that $T_p X = T_q X$. This defines an involution $i : l \rightarrow l$ of l that sends p to q . We call i the *Segre involution* and its fixed points the *Segre fixed points* of l . The Segre fixed points of l are defined over a quadratic field extension $k(l)(\sqrt{\alpha})$ of $k(l)$ with $\alpha \in k(l)^\times / (k(l)^\times)^2$.

Definition 1.4.8. The *type* of l in X is $\langle \alpha \rangle \in \text{GW}(k(l))$.

Note that $\mathbb{R}^\times / (\mathbb{R}^\times)^2 \cong \{\pm 1\}$. A real line is hyperbolic if its type is $\langle 1 \rangle \in \text{GW}(\mathbb{R})$ and it is elliptic if its type is $\langle -1 \rangle \in \text{GW}(\mathbb{R})$ in the sense of Segre. Kass and Wickelgren show that the type of l agrees with the local index $\text{ind}_l \sigma_f$ of the section σ_f of $\text{Sym}^3 \mathcal{S}^* \rightarrow \text{Gr}(2, 4)$ defined by $X = \{f = 0\} \subset \mathbb{P}^3$ at l and conclude that

$$\begin{aligned} e^{\mathbb{A}^1}(\text{Sym}^3 \mathcal{S}^*) &= \sum_{l \subset X} \text{ind}_l \sigma_f = \sum_{l \subset X} \text{Tr}_{k(l)/k}(\text{ind}_{l_{k(l)}} \sigma_{f_{k(l)}}) \\ &= \sum_{l \subset X} \text{Tr}_{k(l)/k}(\text{Type}(l)) = 15\langle 1 \rangle + 12\langle -1 \rangle \in \text{GW}(k) \end{aligned}$$

where $f_{k(l)}$ and $l_{k(l)}$ denote the base change of f and l to the field $k(l)$, respectively.

Smoothness of X implies that σ_f has only isolated zeros. Hence, for any smooth cubic surface X we have the following.

Theorem 1.4.9 (Kass-Wickelgren). *The enriched count of lines on a smooth cubic surface X is*

$$\sum_{l \subset X} \mathrm{Tr}_{k(l)/k}(\mathrm{Type}(l)) = 15\langle 1 \rangle + 12\langle -1 \rangle \in \mathrm{GW}(k). \quad (1.14)$$

Taking the rank of (1.14) yields 27 which gives us the classical complex count [Cay09] and taking the signature of (1.14) gives the real count = 3 [Seg42], see also [FK15] and [OT14].

Kass and Wickelgren get the count (1.14) by summing up the local indices at the lines on 2 particularly nice smooth cubic surfaces. In Paper III we redo their count by providing Macaulay2 code that computes the $\mathbb{A}^1$ -Euler number $e^{\mathbb{A}^1}(\mathcal{E})$ as the sum of local indices at the lines on a cubic surface defined by a random homogeneous degree 3 polynomial in 4 variables.

1.4.2.4 Lines on a degree $(2n - 1)$ -hypersurface in $\mathbb{P}^{n+1}$

More generally, the Grassmannian $\mathrm{Gr}(2, n + 2)$ of lines in $\mathbb{P}^{n+1}$ has dimension $2n$. Let $\mathcal{S} \rightarrow \mathrm{Gr}(2, n + 2)$ be the tautological bundle. The vector bundle $\mathrm{Sym}^{2n-1} \mathcal{S}^* \rightarrow \mathrm{Gr}(2, n + 2)$ has rank $2n$ and is relatively orientable. So its $\mathbb{A}^1$ -Euler number is well-defined and by the same reasoning as above, $e^{\mathbb{A}^1}(\mathrm{Sym}^{2n-1} \mathcal{S}^*)$ is equal to the sum of local indices at the lines on a general degree $(2n - 1)$ -hypersurface in $\mathbb{P}^{n+1}$.

In Paper II we find a geometric interpretation for the local index at a line on a general quintic threefold which we call the *type*. We want to get a count of lines on a general quintic threefold weighted by the types similar to (1.14). However, it turns out to be incredibly hard to find all the lines on a general quintic threefold, even with the Macaulay2 code from Paper III. That is why, we introduce the *dynamic Euler number*.

1.4.3 The dynamic Euler number

In practice, it often happens that our preferred section σ of the vector bundle $\pi : E \rightarrow X$ we want to compute the $\mathbb{A}^1$ -Euler number of, has infinitely many non-isolated zeros. Deforming the section σ to $\sigma_t = \sigma + t\sigma'$ for a general section σ' , yields finitely many points in $\sigma^{-1}(0)$ that deform. In Paper II we define the *dynamic Euler number* to be the sum of the local indices of the deformed section at the deformed zeros valued in $\mathrm{GW}(k((t)))$, that is, the $\mathbb{A}^1$ -Euler number of the base changed bundle $E_{k((t))}$.

$$e^{\mathrm{dynamic}}(E) := e^{\mathbb{A}^1}(E_{k((t))}) = \sum_{x_t \in \sigma_t^{-1}(0)} \mathrm{ind}_{x_t} \sigma_t \in \mathrm{GW}(k((t)))$$

The dynamic Euler number determines the $\mathbb{A}^1$ -Euler number completely: Euler numbers commute with base change. So the $\mathbb{A}^1$ -Euler number is the unique element in $\mathrm{GW}(k)$ that is sent to the dynamic Euler number by the injective map $i : \mathrm{GW}(k) \hookrightarrow \mathrm{GW}(k((t)))$ from Example 1.1.13, that is, $i(e^{\mathbb{A}^1}(E)) = e^{\mathrm{dynamic}}(E) \in \mathrm{GW}(k((t)))$.

1.4.3.1 The dynamic Euler number of $\mathrm{Sym}^3 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 4)$

Before we finish this introduction with the summaries of the three papers, we want to give one example of a computation of the $\mathbb{A}^1$ -Euler number in which we use both, the dynamic Euler number from Paper II and the Macaulay2 code from Paper III.

We redo the count of lines on a general cubic surface: The (non-smooth) cubic surface which is the union of 3 hyperplanes $X = \{xyz = 0\} \subset \mathbb{P}^3 = \mathbb{P}(k[w, x, y, z])$ contains infinitely many lines, namely the lines lying in the 3 hyperplanes $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$. For a general degree 3 homogeneous polynomial f in the 4 variables w, x, y, z , let X_t be the deformation $X_t = \{f_t = xyz + tf = 0\} \subset \mathbb{P}_{k((t))}^3$ of X .

Definition 1.4.10. We say that a line l on X *deforms* with X_t if there is a family of lines l_t on the family of cubic surfaces $\{xyz + tf = 0\} \subset \mathbb{P}_{k[[t]]}^3$ over $\mathrm{Spec} k[[t]]$, such that $l_t|_{t=0} = l$ and we call l_t a *deformation* of l .

For such an l_t , the base change $(l_t)_{k((t))}$ is a line on X_t and the dynamic Euler number of $\mathrm{Sym}^3 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 4)$ is the sum of local indices at the base changed deformations valued in $\mathrm{GW}(k((t)))$.

Since f is general, $f_x(x) := f(1, x, 0, 0)$, $f_y(y) := f(1, 0, y, 0)$ and $f_z(z) := f(1, 0, 0, z)$ each have 3 distinct zeros defined over $\bar{k}$ which we call $\{p_1, p_2, p_3\}$, $\{q_1, q_2, q_3\}$ and $\{r_1, r_2, r_3\}$, respectively. Segre finds that the complex lines on the hyperplane $\{x = 0\} \subset \mathbb{P}^3$ that deform with X_t are the 9 lines through q_i and r_j for $i, j = 1, 2, 3$ (see Figure 1.1) [Seg42]. Similarly, there are 9 lines on $\{y = 0\}$ and 9 lines on $\{z = 0\}$ that deform. So in total there are 27 lines defined over $\mathbb{C}$ that deform and 27 equals the classical count of lines on a general cubic surface over the complex numbers. We will see that this works over any field k , that is, the same lines deform when the base field k is arbitrary (Proposition 1.4.11) and we get the same result as Kass and Wickelgren in Theorem 1.4.9 when we sum up the local indices at the (base-changed) deformed lines.

Let l be the line through $q := q_1$ and $r := r_1$ and L its field of definition, that is, the residue field of the corresponding point in the Grassmannian. We show in Proposition 1.4.11 that l deforms, that is, there is a line l_t defined over $L[[t]]$ with $l_t|_{t=0} = l$.

In order to compute the local index at $(l_t)_{L((t))}$, we choose local coordinates $U \cong \mathbb{A}_{L((t))}^4$ around $l_{L((t))}$ and a trivialization of $\mathcal{E}_{L((t))}|_U$ compatible with the distinguished relative orientation of $\mathcal{E}_{L((t))}$ defined in [KW17, Definition 46]. Let $U = \mathbb{A}_{L((t))}^4 = \mathrm{Spec}(L((t))[a, a', b, b']) \subset \mathrm{Gr}(2, 4)_{L((t))}$ be the lines spanned by $ae_1 + be_2 + e_3$ and $a'e_1 + b'e_2 + e_4$ for the standard basis e_1, e_2, e_3, e_4 of $(L((t)))^4$. The line $l_{L((t))} = (u + v : 0 : qu : rv)$ through q and r is the point $(\frac{1}{q}, \frac{1}{r}, 0, 0) \in \mathbb{A}_{L((t))}^4$ in these coordinates and for a deformation l_t of l , the base change $(l_t)_{L((t))}$ is of the form $(a_t, a'_t, b_t, b'_t) \in \mathbb{A}_{L((t))}^4$ with

$$a_t = \frac{1}{q} + ta_1 + t^2 a_2 + \dots$$

Figure 1.1: Lines that deform in $x = 0$

$$\begin{aligned} a'_t &= \frac{1}{r} + a'_1 t + a'_2 t^2 + \dots \\ b_t &= 0 + b_1 t + b_2 t^2 + \dots \\ b'_t &= 0 + b'_1 t + b'_2 t^2 + \dots \end{aligned}$$

in the chosen coordinates.

Let $\tilde{\phi}_3$ and $\tilde{\phi}_4$ be the duals of $ae_1 + be_2 + e_3$ and $a'e_1 + b'e_2 + e_4$ in $(L((t))[a, a', b, b'])^4$, respectively. Then $\tilde{\phi}_3^3, \tilde{\phi}_3^2 \tilde{\phi}_4, \tilde{\phi}_3 \tilde{\phi}_4^2, \tilde{\phi}_4^3$ is a basis of $\mathcal{E}_{L((t))}|_U$ which defines a trivialization of $\mathcal{E}_{L((t))}|_U$ compatible with the distinguished relative orientation of $\mathcal{E}_{L((t))}$ and our chosen coordinates by [KW17, Lemma 42].

The section σ_{f_t} of $\mathcal{E}_{L((t))}$ in the chosen coordinates and trivialization is equal to the coefficients f_1, f_2, f_3, f_4 of u^3, u^2v, uv^2, v^3 of

$$\begin{aligned} f_t(au + a'v, bu + b'v, u, v) &= (xyz + tf)(au + a'v, bu + b'v, u, v) \\ &= (bu + b'v)uv + tf(au + a'v, bu + b'v, u, v). \end{aligned}$$

A computation shows that f_1, f_2, f_3, f_4 are equal to

$$\begin{aligned} f_1 &= tf(a, b, 1, 0) = tg_1(a, b) \\ f_2 &= b + tg_2(a, a', b, b') \\ f_3 &= b' + tg_3(a, a', b, b') \\ f_4 &= tf(a', b', 0, 1) = tg_4(a', b') \end{aligned}$$

for suitable $g_1, g_2, g_3, g_4 \in L((t))[a, a', b, b']$.

Proposition 1.4.11. *The line $l_{L((t))}$ through r and q deforms, that is, there are unique $a_1, a_2, \dots, a'_1, a'_2, \dots, b_1, b_2, \dots$ and $b'_1, b'_2, \dots$ in L such that $(f_1, f_2, f_3, f_4)(a_t, a'_t, b_t, b'_t) = 0$.*

Proof. The t -term in $f_1(a_t, a'_t, b_t, b'_t)$ has coefficient $g_1(\frac{1}{q}, 0) = f(\frac{1}{q}, 0, 1, 0) = \frac{1}{q^3} f(1, 0, q, 0) = 0$ since q is a zero of $f_y(y) = f(1, 0, y, 0)$. Similarly the t -term in $f_4(a_t, a'_t, b_t, b'_t)$ is equal to zero. The constant terms in $f_2(a_t, a'_t, b_t, b'_t)$ and $f_3(a_t, a'_t, b_t, b'_t)$ are $b_0 = 0$ and $b'_0 = 0$. Setting the t -term in $f_2(a_t, a'_t, b_t, b'_t)$ equal to zero, we get a unique solution for b_1 and the t -term in $f_3(a_t, a'_t, b_t, b'_t)$ yields a unique b'_1 .

We show inductively that one can find unique $a_{n-1}, a'_{n-1}, b_n, b'_n$ such that the coefficient of the t^n -term in $(f_1, f_2, f_3, f_4)(a_t, a'_t, b_t, b'_t)$ equals 0 for $n \geq 2$. Assume we have solved for $a_1, \dots, a_{n-2}, a'_1, \dots, a'_{n-2}, b_1, \dots, b_{n-1}, b'_1, \dots, b'_{n-1}$. The t^n -coefficient in $f_1(a_t, a'_t, b_t, b'_t)$ is a linear equation in a_{n-1} with coefficients determined by $a_1, \dots, a_{n-2}, b_1, \dots, b_{n-1}$. So there is a unique a_{n-1} such that the t^n -coefficient in $f_1(a_t, a'_t, b_t, b'_t)$ is 0. Similarly, there is a unique a'_{n-1} such that the t^n -coefficient in $f_4(a_t, a'_t, b_t, b'_t)$ is 0, and the t^n -terms in $f_2(a_t, a'_t, b_t, b'_t)$ and $f_3(a_t, a'_t, b_t, b'_t)$ determine b_n and b'_n uniquely. ■

The local index at $(l_t)_{L((t))}$ is equal to

$$\langle \det(\text{jacobian}(f_1, f_2, f_3, f_4))(a_t, a'_t, b_t, b'_t) \rangle \in \text{GW}(L((t)))$$

by (1.11). The class of a rank 1 non-degenerate form $\langle \sum_{i=m}^{\infty} a_i t^i \rangle \in \text{GW}(L((t)))$ is determined by its lowest non-zero term, that is, $\langle \sum_{i=m}^{\infty} a_i t^i \rangle = \langle a_m t^m \rangle \in \text{GW}(L((t)))$ (see Claim 5.3 in Paper II). So the local index at $(l_t)_{L((t))}$ is determined by the lowest non-zero term of the power series $\det(\text{jacobian}(f_1, f_2, f_3, f_4))(a_t, a'_t, b_t, b'_t)$. We highlight the entries of the jacobian of (f_1, f_2, f_3, f_4) evaluated at (a_t, a'_t, b_t, b'_t) that contribute to the lowest term of its determinant.

$$\begin{pmatrix} \frac{\partial g_1}{\partial a}(\frac{1}{q}, 0)t + t^2(\dots) & 0 & t(\dots) & 0 \\ t(\dots) & t(\dots) & 1 + t(\dots) & t(\dots) \\ t(\dots) & t(\dots) & t(\dots) & 1 + t(\dots) \\ 0 & \frac{\partial g_4}{\partial a'}(\frac{1}{r}, 0)t + t^2(\dots) & 0 & t(\dots) \end{pmatrix} \quad (1.15)$$

It follows that the lowest term is equal to

$$1 \cdot 1 \cdot \frac{\partial g_1}{\partial a}(\frac{1}{q}, 0) \cdot \frac{\partial g_4}{\partial a'}(\frac{1}{r}, 0) \cdot t^2.$$

A calculation shows that

$$\frac{\partial g_1}{\partial a}(\frac{1}{q}, 0) = \frac{1}{q^2} \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, q, 0)$$

and

$$\frac{\partial g_4}{\partial a'}(\frac{1}{r}, 0) = \frac{1}{r^2} \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, 0, r).$$

The calculation in Proposition 1.4.11 works for any of Segre's 9 distinguished lines on $\{x = 0\}$ and thus the lines that deform on $\{x = 0\}$ over $\bar{k}$ are exactly those 9 lines (since there cannot be more than 27 in total). Over k the coordinate ring of

1. Introduction

the lines on $\{x = 0\}$ that deform is the 9-dimensional k -algebra $E := \frac{k[y,z]}{(f_y(y), f_z(z))}$ which is étale for a general f . Thus $E \cong L_1 \times \cdots \times L_s$ for finite separable field extensions L_i/k and $\sum_{i=1}^s [L_i : k] = 9$. The L_i are the fields of definition of the s lines $l_1, \dots, l_s$ through $q_{l_i} \in \{q_1, q_2, q_3\}$ and $r_{l_i} \in \{r_1, r_2, r_3\}$ on $\{x = 0\}$ that deform and the contribution of the deformations of those s lines to $e^{\text{dynamic}}(\mathcal{E})$ is

$$\begin{aligned} \sum_{i=1}^s \text{Tr}_{L_i((t))/k((t))} \left(\left\langle \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, q_{l_i}, 0) \cdot \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, 0, r_{l_i}) \right\rangle \right) = \\ \text{Tr}_{E((t))/k((t))} \left(\left\langle \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, y, 0) \cdot \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, 0, z) \right\rangle \right) \in \text{GW}(k((t))) \end{aligned} \quad (1.16)$$

by Remark 1.3.4. Note that (1.16) does not depend on t . So we use our Macaulay2 code from Paper III to compute the trace form $\text{Tr}_{E/k} \left(\left\langle \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, y, 0) \cdot \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, 0, z) \right\rangle \right)$ which agrees with (1.16) viewed as a form over $k((t))$.

The following Macaulay2 code generates a random degree 3 polynomial f in 4 variables over the rational numbers $\mathbb{Q}$.

```
R=QQ[w,x,y,z];
f=random(3,R);
```

We define J to be $\frac{\partial f(w, x, y, z)}{\partial w}(1, 0, y, 0) \cdot \frac{\partial f(w, x, y, z)}{\partial w}(1, 0, 0, z)$ and I to be the ideal in $C = \mathbb{Q}[y, z]$ generated by $f(1, 0, y, 0)$ and $f(1, 0, 0, z)$. So C/I is equal to E in (1.16).

```
C=QQ[y,z];
m1=map(C,R,{1,0,y,0});
m2=map(C,R,{1,0,0,z});
J=(m1 diff(w,f))*(m2 diff(w,f));
I=ideal(m1 f, m2 f);
```

The trace form Macaulay2 code in Paper III computes a form representing $\text{Tr}_{(C/I)/\mathbb{Q}}(\langle J \rangle)$ which has rank 9 and signature 1 (viewed as a form over $\mathbb{R}$).

```
traceForm(C,I,J,QQ)
```

Doing the same over a finite field $\mathbb{F}_p$ (e.g. $p = 32003$) gives us a rank 9 form of discriminant $1 \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$.

Using a similar code for the 9 distinguished lines on $\{y = 0\}$ and $\{z = 0\}$, we again get rank 9 forms of signature 1 (when $k = \mathbb{R}$) and discriminant 1 (when k is a field of characteristic $p = 32003$). Summing up those 3 forms we get that the dynamic Euler number $e^{\text{dynamic}}(\mathcal{E}) \in \text{GW}(k((t)))$ has rank 27, signature 3 and discriminant 1. The $\mathbb{A}^1$ -Euler number $e^{\mathbb{A}^1}(\mathcal{E})$ of $\mathcal{E}$ is the unique element of $\text{GW}(k)$ that is mapped to the dynamic Euler number by $i : \text{GW}(k) \rightarrow \text{GW}(k((t)))$ defined by $\langle a \rangle \mapsto \langle a \rangle$. It follows that $e^{\mathbb{A}^1}(\mathcal{E}) \in \text{GW}(k)$ has rank 27, signature 3 and discriminant 1. By [BW20, Theorem 5.8] the $\mathbb{A}^1$ -Euler number is either of the form

$$\frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2} \langle 1 \rangle + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2} \langle -1 \rangle \in \text{GW}(k) \quad (1.17)$$

or

$$\frac{n_{\mathbb{C}} + n_{\mathbb{R}}}{2} \langle 1 \rangle + \frac{n_{\mathbb{C}} - n_{\mathbb{R}}}{2} \langle -1 \rangle + \langle 2 \rangle - \langle 1 \rangle \in \mathrm{GW}(k). \quad (1.18)$$

Here $n_{\mathbb{C}}$ is the complex count, that is, the rank 27, and $n_{\mathbb{R}}$ the real count, that is, the signature 3. Note that 2 is not a square in $\mathbb{F}_p$ with $p = 32003$ and we can conclude that $e^{\mathbb{A}^1}(\mathcal{E})$ is of the form (1.17), that is,

$$e^{\mathbb{A}^1}(\mathcal{E}) = 15\langle 1 \rangle + 12\langle -1 \rangle \in \mathrm{GW}(k).$$

1.5 Summary of the papers

1.5.1 $\mathbb{A}^1$ -contractibility of affine modifications

In Paper I we provide new examples of $\mathbb{A}^1$ -contractible motivic spaces with similar properties as the Koras-Russell threefold KR . We work over an algebraically closed field k of characteristic 0.

The proof that shows that KR is contractible as a smooth manifold in the classical sense, uses that it is an *affine modification* of a contractible space.

1.5.1.1 Affine modifications

We recall the definition of an affine modification from [KZ99, Definition 1.1].

Definition 1.5.1. Let $X = \mathrm{Spec} A$ be an affine variety, $I \subset A$ an ideal and $f \in I$ not a zero divisor. The *affine modification* of X along f with center I is

$$\tilde{X} := \mathrm{Spec}(A[It]/(1 - ft))$$

where $A[It] = A \oplus \bigoplus_{n=1}^{\infty} (It)^n$.

Remark 1.5.2. In Paper I we provide a more general definition of an affine modification of an arbitrary, not necessarily affine, scheme over k .

Example 1.5.3. The Koras-Russell cubic $KR = \{x^2y + z^2 + t^3 + x = 0\} \subset \mathbb{A}^4$ is the affine modification of $\mathbb{A}^3$ along x^2 with center $(-x^2, z^2 + t^3 + x)$.

Let $Bl_I X = \mathbb{P}(A[It])$ be the blow up of X along I . The affine modification $\tilde{X}$ is an open subset of the blow up $Bl_I X$, namely, $\tilde{X}$ is equal to $Bl_I X \setminus D^{pr}$ where $D^{pr} := \{ft = 0\} \subset Bl_I X$ is the *proper transform* of $D := \{f = 0\}$. The blow up morphism $\sigma : Bl_I X \rightarrow X$ restricts to $\pi := \sigma|_{\tilde{X}} : \tilde{X} \rightarrow X$. Let $Z = \mathrm{Spec}(A/I)$ and $E' = \pi^{-1}(Z) = \sigma^{-1}(Z) \cap \tilde{X}$. It follows that the affine modification morphism π restricts to an isomorphism $\pi|_{\tilde{X} \setminus E'} : \tilde{X} \setminus E' \xrightarrow{\cong} X \setminus D$ since the blow up morphism σ is an isomorphism away from Z . In [KZ99, §3] Kaliman and Zaidenberg use a 5-lemma argument to show that X is contractible (in the classical sense) if and only if $\tilde{X}$ is contractible, given that $\pi|_{E'} : E' \rightarrow D$ is a homotopy equivalence.

1.5.1.2 $\mathbb{A}^1$ -contractibility results of affine modifications

In Paper I we find similar $\mathbb{A}^1$ -contractibility results for affine modifications. Assume that X , D and Z are all smooth and consider the diagram of cofiber sequences.

$$\begin{array}{ccccc} \tilde{X} \setminus E' & \longrightarrow & \tilde{X} & \longrightarrow & \tilde{X}/(\tilde{X} \setminus E') \\ \downarrow \cong & & \downarrow \pi & & \downarrow \\ X \setminus D & \longrightarrow & X & \longrightarrow & X/(X \setminus D) \end{array} \quad (1.19)$$

By the Purity Theorem 1.1.8 the right vertical map is $\mathbb{A}^1$ -weakly equivalent to $E_+ \wedge \mathbb{P}^1 \rightarrow D_+ \wedge \mathbb{P}^1$ and thus an $\mathbb{A}^1$ -weak equivalence if $E' \rightarrow D$ is an $\mathbb{A}^1$ -weak equivalence. Smoothness of Z and D implies that I is generated by a regular sequence $(f, a_1, \dots, a_r)$ and it follows that $E' \cong Z \times \mathbb{A}^r$, in particular, E' is $\mathbb{A}^1$ -weakly equivalent to Z . We use the weak 5-lemma 1.2.7 to conclude the following.

Theorem 1.5.4 (Special case of Theorem 2.17 in Paper I). *With the notation and assumptions above, assume further that D and E' are irreducible, $Z \hookrightarrow D$ is an $\mathbb{A}^1$ -weak equivalence and that $\tilde{X}$ is $\mathbb{A}^1$ -contractible. Then X is $\mathbb{A}^1$ -contractible.*

Higher Chow groups can be defined for non-smooth varieties. Applying Theorem 1.2.4 and Remark 1.2.5, we get much more general criteria for stable $\mathbb{A}^1$ -contractibility of affine modifications since we do not need to assume that Z or D are smooth.

Theorem 1.5.5 (Theorem 3.2 in Paper I). *With the same notation as above, assume I is generated by a regular sequence $(f, a_1, \dots, a_r)$ and that X and $\tilde{X}$ are both smooth and affine varieties. Now D and Z do not to be smooth. Assume further that both projections $Z \times Y \rightarrow Y$ and $D \times Y \rightarrow Y$ induce isomorphisms in higher Chow groups for any smooth affine variety Y . Then X is stably $\mathbb{A}^1$ -contractible if and only if $\tilde{X}$ is stably $\mathbb{A}^1$ -contractible.*

1.5.1.3 New families of $\mathbb{A}^1$ -contractible motivic spaces

Aside from proving the unstable and stable $\mathbb{A}^1$ -contractibility theorems of affine modifications, we find several families of (stably) $\mathbb{A}^1$ -contractible varieties which arise as affine modifications of $\mathbb{A}^1$ -contractible varieties like affine space or KR . To prove $\mathbb{A}^1$ -contractibility of those families we also use the $\mathbb{A}^1$ -degree argument 1.2.4.4 and the weak 5-lemma 1.2.7 in addition to the $\mathbb{A}^1$ -contractibility theorems 1.5.4 and 1.5.5 of affine modifications.

1.5.2 Quadratic types and dynamic Euler numbers of lines on quintic threefolds

We have seen in 1.4.2.4 that the count of lines on a general quintic threefold enriched in $\mathrm{GW}(k)$ is equal to the $\mathbb{A}^1$ -Euler number of $\mathrm{Sym}^5 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 5)$ where

$\mathrm{Gr}(2, 5)$ is the Grassmannian of lines in $\mathbb{P}^4$ and $\mathcal{S} \rightarrow \mathrm{Gr}(2, 5)$ its tautological bundle.

Paper II consists of two parts. First we give a geometric interpretation of the local index at a line on a general quintic threefold X , called the *type*, which is similar to the Segre type of a line on a cubic surface. Then we introduce the *dynamic Euler number* (see 1.4.3) to compute $e^{\mathbb{A}^1}(\mathrm{Sym}^5 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 5))$ as the sum of local indices at the 2875 distinguished lines on the Fermat quintic threefold

$$\{X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0\} \subset \mathbb{P}^4 \quad (1.20)$$

that deform with a general deformation.

1.5.2.1 The type of a line on a quintic threefold

Let $l \subset X \subset \mathbb{P}^4$ be a line on a general quintic threefold $X = \{f = 0\} \subset \mathbb{P}^4$. In Paper II we find a geometric interpretation of $\mathrm{ind}_l \sigma_f$ called the *type* of l valued in $\mathrm{GW}(k(l))$ where σ_f is the section of $\mathrm{Sym}^5 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 5)$ defined by X and $k(l)$ is the field of definition of l . We assume that $\mathrm{char} k(l) \neq 2$. As for lines on smooth cubic surfaces, this is a generalization of the (Segre) type of a real line on a quintic threefold (or more generally the (Segre) type of a line on a degree $(2n - 1)$ -hypersurface in $\mathbb{P}^{n+1}$) defined by Finashin and Kharlamov in [FK19].

The Gauß map

$$C : l \cong \mathbb{P}_{k(l)}^1 \rightarrow \mathbb{P}_{k(l)}^2 \cong \text{3-planes in } \mathbb{P}_{k(l)}^4 \text{ containing } l$$

that maps $p \in l$ to its tangent space $T_p X$ in $\mathbb{P}_{k(l)}^4$ is a degree 4 map. So its image is a degree 4 rational curve which has 4 nodal points by the Castenuovo count (see for example [Arb+85]). Let H_t be the pencil of lines through one of the nodal points M . Then an element of the pencil H_{t_0} intersects C in M and two additional points $C(p)$ and $C(q)$ and we get an involution i_M of l that swaps p and q . Assume for simplicity that all 3 nodal points M_1, M_2 and M_3 of C are defined over $k(l)$. Then the fixed points of the involutions i_{M_j} are defined over $k(l)(\sqrt{\alpha_j})$ for $\alpha_j \in k(l)^\times / (k(l)^\times)^2$ and $j = 1, 2, 3$.

Definition 1.5.6. The *type* of l on X is $\langle \alpha_1 \alpha_2 \alpha_3 \rangle \in \mathrm{GW}(k(l))$.

Theorem 1.5.7 (Theorem 4.1 in Paper II). *The type of l is equal to $\mathrm{ind}_l \sigma_f$ in $\mathrm{GW}(k(l))$.*

1.5.2.2 Dynamic Euler numbers and the lines on the Fermat Quintic Threefold

Now that we know what geometric information $\mathrm{ind}_l \sigma_f$ gives us, we want to count lines on $X = \{f = 0\} \subset \mathbb{P}^4$ in $\mathrm{GW}(k)$ as

$$\begin{aligned} e^{\mathbb{A}^1}(\mathrm{Sym}^5 \mathcal{S}^*) &= \sum_{l \subset X} \mathrm{ind}_l \sigma_f = \sum_{l \subset X} \mathrm{Tr}_{k(l)/k}(\mathrm{ind}_{l_{k(l)}} \sigma_{f_{k(l)}}) \\ &= \sum_{l \subset X} \mathrm{Tr}_{k(l)/k}(\mathrm{Type}(l)) \in \mathrm{GW}(k). \end{aligned} \quad (1.21)$$

The problem is to find a nice enough quintic threefold X on which we know all 2875 (=classical count over $\mathbb{C}$) lines defined over $\bar{k}$.

In [AK91] Albano and Katz study the lines on the Fermat quintic threefold (1.20) and find 2875 *distinguished lines* that are the *limits* of lines on a general deformation

$$X_t = \{F_t := F + tG + t^2H + \cdots = 0\} \subset \mathbb{P}^4$$

where F is the equation defining the Fermat (1.20). Since there are infinitely many lines on (1.20), the section σ_F of $\mathrm{Sym}^5 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 5)$ defined by F has non-isolated zeros and we cannot compute $e^{\mathbb{A}^1}(\mathrm{Sym}^5 \mathcal{S}^* \rightarrow \mathrm{Gr}(2, 5))$ as the sum of local indices at the zeros of σ_F . For this reason, we introduce the *dynamic Euler number* in Paper II which we have already used in this introduction in 1.4.3 to redo the count of lines on a cubic surface. We find that

$$e^{\mathrm{dynamic}}(\mathrm{Sym}^5 \mathcal{S}^*) = \sum_{l_t \subset X_t} \mathrm{ind}_{l_t} \sigma_{F_t} = 1445\langle 1 \rangle + 1430\langle -1 \rangle \in \mathrm{GW}(k((t)))$$

when $\mathrm{char} k \neq 2, 5$ and can conclude that (1.21) is equal to

$$e^{\mathbb{A}^1}(\mathrm{Sym}^5 \mathcal{S}^*) = 1445\langle 1 \rangle + 1430\langle -1 \rangle \in \mathrm{GW}(k).$$

1.5.3 Computing $\mathbb{A}^1$ -Euler numbers with Macaulay2

In Paper III we provide Macaulay2 code that can compute the trace form $\mathrm{Tr}_{E/k}(\langle a \rangle) \in \mathrm{GW}(k)$ (see Remark 1.3.4) of $\langle a \rangle \in \mathrm{GW}(E)$ where E is a finite dimensional étale algebra over a field k . We use this code to compute several $\mathbb{A}^1$ -Euler numbers, some of which are already known, like the count of lines on a smooth cubic surface or the count of lines meeting three general lines in $\mathbb{P}^3$ [SW18], and some new examples like the count of lines on a quadratic surface meeting one general line and the count of singular elements on a pencil of degree d -surfaces.

We also provide Macaulay2 code that computes the EKL-class (see 1.3.3) and compute several $\mathbb{A}^1$ -Milnor numbers (see [KW19]) as an application.

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Papers

Paper I

$\mathbb{A}^1$ -contractibility of affine modifications

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Abstract

We introduce Koras-Russell fiber bundles over algebraically closed fields of characteristic zero. After a single suspension, this exhibits an infinite family of smooth affine $\mathbb{A}^1$ -contractible 3-folds. Moreover, we give examples of stably $\mathbb{A}^1$ -contractible smooth affine 4-folds containing a Brieskorn-Pham surface, and a family of smooth affine 3-folds with a higher dimensional $\mathbb{A}^1$ -contractible total space.

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I.1 Introduction

Suppose that $\mathcal{X}$ is some d -dimensional smooth affine $\mathbb{A}^1$ -contractible scheme of finite type over an algebraically closed field F of characteristic zero. When $d = 1$, then $\mathcal{X}$ is isomorphic to $\mathbb{A}^1$ [AD07, Claim 5.7]. Any $\mathbb{A}^1$ -chain connected smooth affine surface in the sense of Asok-Morel [AM11] is $\mathbb{A}^1$ -ruled, i.e., it contains a cylinderlike open subscheme of the form $\mathcal{X}' \times \mathbb{A}^1$, and hence is of logarithmic Kodaira dimension $-\infty$ [DK15]. Thus, when $d = 2$, deep classification results in birational geometry due to Miyanishi-Sugie and Fujita [MS80] imply that $\mathcal{X}$ is isomorphic to $\mathbb{A}^2$ (see subsection I.2.7). Thus, up to isomorphism, $\mathbb{A}^d$ is the unique smooth affine $\mathbb{A}^1$ -contractible and $\mathbb{A}^1$ -chain connected scheme of dimension $d \leq 2$. Such a characterization fails for $\mathbb{A}^3$, as recently witnessed by Koras-Russell 3-folds of the first kind [DF18], [HKØ16]. Initially used in the monumental work of showing every $\mathbb{G}_m$ -action on the affine 3-space is linearizable [Kal+97], these 3-folds are of interest in their own right as potential counterexamples to the long-standing Zariski cancellation problem [Kra96], [Zai99]. Moreover, they admit hyperbolic $\mathbb{G}_m$ -actions with a unique fixed point, and can be distinguished from $\mathbb{A}^3$ by the Derksen and Makar-Limanov invariants recording locally nilpotent derivations on coordinate rings [Fre06].

For integers $n, \alpha_i \geq 2$, where α_1 and α_2 are coprime, $a \in F^\times$, recall that Koras-Russell 3-folds of the first kind are defined in $\mathbb{A}^4$ by the equation:

$$\mathcal{X}(n, \alpha_i, a) := \{x^n z = y^{\alpha_1} + t^{\alpha_2} + ax\}. \quad (\text{I.1})$$

In [HKØ16] it is shown that Koras-Russell 3-folds of the first kind are stably $\mathbb{A}^1$ -contractible, and consequently $\mathbb{A}^1$ -contractible after a finite suspension by the projective line $\mathbb{P}^1$ pointed at infinity:

$$\mathcal{X}(n, \alpha_i, a) \wedge \mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} *. \quad (\text{I.2})$$

Similarly by [HKØ16], for $m \geq 1$, $n, \alpha_i \geq 2$, where α_1 and $n\alpha_2$ are coprime, $a \in F^\times$, the same $\mathbb{A}^1$ -contractibility result holds for Koras-Russell 3-folds of the second kind, defined in $\mathbb{A}^4$ by the equation:

$$\mathcal{X}(m, n, \alpha_i, a) := \{(x^n + y^{\alpha_1})^m z = t^{\alpha_2} + ax\}. \quad (\text{I.3})$$

As shown in [DF18], $\mathcal{X}(n, \alpha_i, a)$ is in fact $\mathbb{A}^1$ -contractible. It is an open question whether Koras-Russell 3-folds of the second kind (I.3) are $\mathbb{A}^1$ -contractible. This makes (I.1) the first family of examples of smooth affine $\mathbb{A}^1$ -contractible schemes that are not isomorphic to affine spaces. One cannot make such a distinction topologically because $\mathcal{X}(n, \alpha_i, a)$ is diffeomorphic to $\mathbb{R}^6$ when equipped with the Euclidean topology; this was shown independently by Dimca and Ramanujam [Zai99, §3]. From the viewpoint of motivic homotopy theory, the remarkable properties of a Koras-Russell 3-fold of the first kind makes it a potential algebro-geometric analogue of the Whitehead manifold.

The latter is an open contractible 3-manifold non-homeomorphic to $\mathbb{R}^3$, and the first counterexample to the assertion that every contractible manifold is homeomorphic to a ball, as stated in a purported proof of the Poincaré conjecture [Whi34], [Whi35].

In this paper we extend the results in [DF18] and [HKØ16] to classes of examples such as deformed Koras-Russell 3-folds of the first kind

$$\mathcal{X}(n, \alpha_i, p) := \{x^n z = y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t)\}, \quad (\text{I.4})$$

where $n, \alpha_i \geq 2$ are integers, α_1 and α_2 are coprime, and $p(x, y, t) \in F[x, y, t]$ satisfies $p(0, 0, 0) \in F^*$. The special case $p(x, y, t) = a \in F^*$ recovers (I.1). By [DF18] it is known that $\mathcal{X}(n, \alpha_i, p)$ is $\mathbb{A}^1$ -contractible when $p(x, y, t) = q(x) \in F[x]$ and $q(0) \in F^*$. Note that $\mathcal{X}(n, \alpha_i, p)$ is smooth according to the Jacobian criterion since we assume $p(0, 0, 0) \in F^*$. Moreover, the unique singular fiber of the projection map

$$\text{pr}_x : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}_x^1 \quad (\text{I.5})$$

is a cylinder on the cuspidal curve $\Gamma_{\alpha_1, \alpha_2} := \{y^{\alpha_1} + t^{\alpha_2} = 0\} \subset \mathbb{A}^2$, which is $\mathbb{A}^1$ -contractible:

$$\text{pr}_x^{-1}(0) = \Gamma_{\alpha_1, \alpha_2} \times \mathbb{A}_z^1 \sim_{\mathbb{A}^1} *. \quad (\text{I.6})$$

Here (I.5) is a flat $\mathbb{A}^2$ -fibration, i.e., a flat surjection with general fibers isomorphic to $\mathbb{A}^2$, restricting to a trivial $\mathbb{A}^2$ -bundle over $\mathbb{A}_x^1 \setminus \{0\}$ and $\mathcal{X}(n, \alpha_i, p)$ is factorial (see e.g., [Gup14, Lemma 3.1]). The $\mathbb{A}^1$ -homotopy theory of deformed Koras-Russell 3-folds of the first kind (I.4) is essentially governed by (I.5) and (I.6). More generally, we make the following definition.

Definition I.1.1. Suppose $s(x) \in F[x]$ has positive degree and let $R(x, y, t) \in F[x, y, t]$. Define the closed subscheme $\mathcal{X}(s, R)$ of $\mathbb{A}_x^1 \times \mathbb{A}^3 = \text{Spec}(F[x][y, z, t])$ by the equation:

$$\{s(x)z = R(x, y, t)\}.$$

We say that the projection map

$$\rho := \text{pr}_x : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1 \quad (\text{I.7})$$

defines a *Koras-Russell fiber bundle* if

- (a) $\mathcal{X}(s, R)$ is a smooth scheme, and
- (b) For every zero x_0 of $s(x)$, the zero locus in $\mathbb{A}^2 = \text{Spec}(F[y, t])$ of the polynomial $R(x_0, y, t)$ is an integral rational plane curve with a unique place at infinity and at most unibranch singularities.

Remark I.1.2. Theorem I.2.4 and Theorem I.2.6 show that a Koras-Russell fiber bundle is isomorphic to $\mathbb{A}^3$ if and only if for every zero x_0 of $s(x)$ the curve $\{R(x_0, y, t) = 0\}$ is isomorphic to $\mathbb{A}^1$. For the theory of unibranch singularities, we refer to [Zar86].

As examples of Koras-Russell fiber bundles we consider smooth affine 3-folds with degenerate fibers. For $1 \leq i \leq m$, choose distinct linear forms $l_i(x) = (x - x_i) \in F[x]$, $n_i, \alpha_i, \beta_i \geq 2$, where α_i and β_i are coprime, and $a \in F^\times$. We define $\mathcal{X}_m(n_i, \alpha_i, \beta_i, a)$ or simply $\mathcal{X}_m$ by the equation in $\mathbb{A}^4$:

$$\begin{aligned} \mathcal{X}_m &= \mathcal{X}_m(n_i, \alpha_j, \beta_j, a) := \left\{ \left(\prod_{i=1}^m l_i(x)^{n_i} \right) z \right. \\ &= \sum_{i=1}^m \left(\left(\prod_{j \neq i} l_j(x) \right) (y^{\alpha_i} + t^{\beta_i}) \right) + a \prod_{i=1}^m l_i(x) \Big\}. \end{aligned} \quad (\text{I.8})$$

In the case of two degenerate fibers, (I.8) takes the form:

$$\begin{aligned} \mathcal{X}_2 &= \{(x - x_1)^{n_1} (x - x_2)^{n_2} z = (x - x_1)(y^{\alpha_2} + t^{\beta_2}) \\ &+ (x - x_2)(y^{\alpha_1} + t^{\beta_1}) + a(x - x_1)(x - x_2)\}. \end{aligned} \quad (\text{I.9})$$

For all m the Makar-Limanov invariant of $\mathcal{X}_m(n_i, \alpha_j, \beta_j, a)$ equals $F[x]$ (see Example I.2.11); hence we can distinguish it from $\mathbb{A}^3$. Moreover, the projection map

$$\text{pr}_x : \mathcal{X}_m(n_i, \alpha_j, \beta_j, a) \rightarrow \mathbb{A}_x^1 \quad (\text{I.10})$$

defines a trivial $\mathbb{A}^2$ -bundle over the punctured affine line $\mathbb{A}_x^1 \setminus \{x_1, \dots, x_m\}$. Its fiber over the closed point $x_i \in \mathbb{A}_x^1$ is isomorphic to the cylinder on the cuspidal curve $\Gamma_{\alpha_i, \beta_i} : \{y^{\alpha_i} + t^{\beta_i} = 0\}$. By counting closed fibers non-isomorphic to $\mathbb{A}^2$ we show $\mathcal{X}_m$ and $\mathcal{X}_{m'}$ are non-isomorphic when $m \neq m'$ (see Lemma I.2.12).

Theorem I.1.3. *Suppose $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ is a Koras-Russell fiber bundle with basepoint the origin. The S^1 -suspension and hence the $\mathbb{P}^1$ -suspension of $\mathcal{X}(s, R)$ are $\mathbb{A}^1$ -contractible:*

$$\mathcal{X}(s, R) \wedge S^1 \sim_{\mathbb{A}^1} \mathcal{X}(s, R) \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} *.$$

Remark I.1.4. In general, $\mathbb{A}^1$ -weak equivalences do not desuspend, e.g., by homotopy purity we have $(\mathbb{G}_m \vee \mathbb{G}_m) \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} (\mathbb{P}^1 \setminus \{0, 1, \infty\} = \text{Spec}(F[t^{\pm 1}, (1-t)^{-1}])) \wedge \mathbb{P}^1$, but there is no $\mathbb{A}^1$ -weak equivalence between $\mathbb{G}_m \vee \mathbb{G}_m$ and $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ by [Wic16]. Theorem I.1.3 does not allow us to conclude that $\mathcal{X}(s, R)$ is $\mathbb{A}^1$ -contractible.

A systematic study of $\mathbb{A}^1$ -contractible varieties presents itself by noting that deformed Koras-Russell 3-folds are examples of affine modifications [KZ99]. It turns out these "affine blow-ups" are intertwined with questions about $\mathbb{A}^1$ -contractibility. This provides a useful technique for constructing families smooth affine varieties which become $\mathbb{A}^1$ -contractible after a single simplicial suspension. By way of example, we consider affine modifications of deformed Koras-Russell 3-folds of the first kind (I.4). This yields iterated Koras-Russell 3-folds of the first kind defined in $\mathbb{A}^4$ by

$$\mathcal{Y}(m, n_i, \alpha_i, p) := \{x^{n_1} z = (x^{n_2} y + z^m)^{\alpha_1} + t^{\alpha_2} + xp(x, x^{n_2} y + z^m, t)\}, \quad (\text{I.11})$$

where $m, n_i, \alpha_i \geq 2$, $m\alpha_1$ and α_2 are coprime, and $p(x, w, t) \in F[x, w, t]$ satisfies $p(0, 0, 0) \in F^*$. The projection map $\text{pr}_x : \mathcal{Y}(m, n_i, \alpha_i, p) \rightarrow \mathbb{A}_x^1$ has a unique singular fiber $\mathbb{A}^1 \times \Gamma_{m\alpha_1, \alpha_2}$ at the origin and restricts to a trivial $\mathbb{A}^2$ -bundle over $\mathbb{A}_x^1 \setminus \{0\}$. By Theorem I.2.4 it follows that $\mathcal{Y}(m, n_i, \alpha_i, p)$ in (I.11) is not isomorphic to $\mathbb{A}^3$.

Theorem I.1.5. *For every iterated Koras-Russell 3-fold of the first kind (I.11) we have:*

$$\mathcal{Y}(m, n_i, \alpha_i, p) \wedge S^1 \sim_{\mathbb{A}^1} \mathcal{Y}(m, n_i, \alpha_i, p) \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} *.$$

Inspired by these 3-folds and the classical Brieskorn-Pham surfaces [Pha65]

$$\mathcal{S}_{\alpha_i} := \{y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3} = 0\}, \quad (\text{I.12})$$

for pairwise coprime integers $\alpha_i \geq 2$, we form the smooth affine modification

$$\mathcal{Y}(n, \alpha_i, p) := \{x^n z = y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3} + xp(x, y_1, y_2, y_3)\}, \quad (\text{I.13})$$

where $n \geq 2$, and $p(x, y_1, y_2, y_3) \in F[x, y_1, y_2, y_3]$ satisfies $p(0, 0, 0, 0) \in F^*$. The 4-fold (I.13) is not an affine space since it has a nontrivial Makar-Limanov invariant (see e.g., [KM07, Proposition 11.1]). In a special case we show (I.13) is stably $\mathbb{A}^1$ -contractible, and conclude:

Theorem I.1.6. *If $\alpha_3 = m\alpha_1\alpha_2 + 1$, $m > 0$, then the smooth affine 4-fold $\mathcal{Y}(n, \alpha_i, p)$ in (I.13) becomes $\mathbb{A}^1$ -contractible after a finite suspension with the projective line $\mathbb{P}^1$ pointed at infinity:*

$$\mathcal{Y}(n, \alpha_i, p) \wedge \mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} *.$$

Relation to other works

Asok-Doran produced families of non-isomorphic smooth quasi-affine but not affine n -dimensional varieties for $n \geq 4$ that are $\mathbb{A}^1$ -contractible [AD07]. Their main technique, tracing back to classical contractibility results for PL and smooth manifolds, is to form quotients of affine spaces by free actions of unipotent groups. Inspired by Asok's lectures [Aso11a] and connections to the Zariski cancellation problem, [DF18] and [HKØ16] established $\mathbb{A}^1$ -contractibility results for smooth affine Koras-Russell 3-folds of the first and second kind. In this paper we open a systematic study of such examples by utilizing the geometric technique of affine modifications to craft accessible and widely applicable proofs of results pertaining to $\mathbb{A}^1$ -contractibility. This represents a concise version of the theory that strengthens the results in [DF18] and [HKØ16], presents new insights, and opens new lines of inquiry.

Outline

In Section I.2 we review background on affine algebraic geometry and $\mathbb{A}^1$ -homotopy theory. We review and give examples of affine modifications. Related to $\mathbb{A}^1$ -contractibility we study $\mathbb{A}^1$ -chain connectedness — an algebro-geometric

analogue of path connectedness introduced by Asok-Morel [AM11] — which loosely speaking means that any two points can be connected by the images of a chain of maps from the affine line. There are many elementary open problems related to these notions; we record a few sample questions which we believe warrant further investigation.

Section I.3 improves on the stable $\mathbb{A}^1$ -contractibility results in [HKØ16] by producing several new families of examples. This is achieved by explicit calculations of Bloch's higher Chow groups [Blo86] or equivalently of motivic cohomology in the sense of Suslin-Voevodsky [SV00], [Voe02]. Using the techniques of stable motivic homotopy theory allow us to conclude $\mathbb{A}^1$ -contractibility after a finite number of suspensions with $\mathbb{P}^1$. For an elucidation of these techniques and further references we refer to [HKØ16].

Our main results are shown in Section I.4. The proof of Theorem I.1.3 proceeds by induction on the number of singular fibers of (I.7). For the induction step we invoke the geometric construction of an affine modification along a divisor in the sense of Kaliman-Zaidenberg [KZ99], a notion tracing back to [Dav67] and [Zar43]. Moreover, by homotopy purity for closed embeddings [MV99, Theorem 3.2.23] and excision over open neighborhoods of the origin, we reduce the proof to the special case of deformed Koras-Russell 3-folds of the first kind (I.4). As alluded to above an explicit calculation shows $\mathcal{X}(n, \alpha_i, p) \rightarrow \mathrm{Spec}(F)$ induces an isomorphism on higher Chow groups, and there is an $\mathbb{A}^1$ -weak equivalence:

$$\mathcal{X}(n, \alpha_i, p) \wedge \mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} *.$$
 (I.14)

Improving upon (I.14) requires a deeper understanding of the geometry of the 3-fold $\mathcal{X}(n, \alpha_i, p)$. To that end, we study the $\mathbb{G}_a$ -action on $\mathcal{X}(n, \alpha_i, p)$ corresponding to the locally nilpotent derivation

$$\partial = x^n \frac{\partial}{\partial y} + (\alpha_1 y^{\alpha_1 - 1} + x \frac{\partial}{\partial y} p(x, y, t)) \frac{\partial}{\partial z}$$

on the coordinate ring of $\mathcal{X}(n, \alpha_i, p)$, with fixed point locus the affine line $\{x = y = t = 0\} \cong \mathbb{A}_z^1$. The geometric quotient $\mathcal{X}(n, \alpha_i, p) \rightarrow \mathcal{X}(n, \alpha_i, p)/\mathbb{G}_a$ yields an $\mathbb{A}^1$ -bundle $\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 \rightarrow \mathfrak{S}(\alpha_1, \alpha_2)$ in the category of algebraic spaces [Knu71]. We note there exists a factorization

$$\begin{array}{ccc} \mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 & \xrightarrow{\pi|} & \mathbb{A}_{x,t}^2 \setminus \{(0, 0)\} \\ & \searrow & \nearrow \\ & \mathfrak{S}(\alpha_1, \alpha_2), & \end{array}$$
 (I.15)

where $\pi|$ is the restriction of $\pi := \mathrm{pr}_{x,t} : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}_{x,t}^2$ to $\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1$. To construct (I.15) we form a cyclic Galois cover of $\mathbb{A}_{x,t}^2$ of order α_1 and consequently of $\mathcal{X}(n, \alpha_i, p)$ by pullback via π . The maps arise as geometric quotients for μ_{α_1} -equivariant maps by gluing copies of $\mathbb{A}^2 \setminus \{(0, 0)\}$ by the identity away from a family of cuspidal curves. Here $\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 \rightarrow \mathfrak{S}(\alpha_1, \alpha_2)$ is an étale locally

trivial $\mathbb{A}^1$ -bundle. We have $\mathfrak{S}(\alpha_1, \alpha_2) \cong \mathfrak{S}(\alpha_1, 1)$, and both of the projection maps for the smooth quasi-affine 4-fold

$$(\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1) \times_{\mathfrak{S}(\alpha_1, \alpha_2)} (\mathcal{X}(n, \alpha_1, 1, p) \setminus \mathbb{A}_z^1) \quad (\text{I.16})$$

are Zariski locally trivial $\mathbb{A}^1$ -bundles, and hence $\mathbb{A}^1$ -weak equivalences. The fiber product in (I.16) is formed in algebraic spaces over the punctured affine plane $\mathbb{A}_{x,t}^2 \setminus \{(0, 0)\}$. Furthermore, the projection $\text{pr}_x : \mathcal{X}(n, \alpha_1, 1, p) \rightarrow \mathbb{A}_x^1$ is a trivial $\mathbb{A}^2$ -bundle. Hence $\mathcal{X}(n, \alpha_1, 1, p) \cong \mathbb{A}_{x,y,z}^3$ and we obtain:

$$\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 \sim_{\mathbb{A}^1} \mathcal{X}(n, \alpha_1, 1, p) \setminus \mathbb{A}_z^1 \sim_{\mathbb{A}^1} \mathbb{A}_{x,t}^2 \setminus \{(0, 0)\}.$$

Combined with an explicit calculation of cohomology with Milnor-Witt K -theory sheaves using (I.14) we conclude $\pi|$ is an $\mathbb{A}^1$ -weak equivalence.

The projection π gives rise to a commutative diagram of cofiber sequences:

$$\begin{array}{ccccc} \mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 & \longrightarrow & \mathcal{X}(n, \alpha_i, p) & \longrightarrow & \mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1) \sim_{\mathbb{A}^1} (\mathbb{A}_z^1)_+ \wedge (\mathbb{P}^1)^{\wedge 2} \\ \pi| \downarrow & & \pi \downarrow & & \sim_{\mathbb{A}^1} \downarrow \\ \mathbb{A}_{x,t}^2 \setminus \{(0, 0)\} & \longrightarrow & \mathbb{A}_{x,t}^2 & \longrightarrow & \mathbb{A}_{x,t}^2 / \mathbb{A}_{x,t}^2 \setminus \{(0, 0)\} \sim_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2}. \end{array} \quad (\text{I.17})$$

The rightmost vertical map in (I.17) is an $\mathbb{A}^1$ -weak equivalence according to homotopy purity for closed embeddings [MV99, Theorem 3.2.23]. By continuing (I.17) into a long cofiber sequence we find that the simplicial suspension of $\mathcal{X}(n, \alpha_i, p)$ is $\mathbb{A}^1$ -contractible:

$$\Sigma_s \mathcal{X}(n, \alpha_i, p) := \mathcal{X}(n, \alpha_i, p) \wedge S^1 \sim_{\mathbb{A}^1} *. \quad (\text{I.18})$$

This finishes the proof of Theorem I.1.3 since $\mathbb{P}^1 \sim_{\mathbb{A}^1} \mathbb{G}_m \wedge S^1$ by the standard open covering of the projective line.

Throughout the paper we work over an algebraically closed field F of characteristic zero. We write Sch_F and Sm_F for the categories of separated and separated smooth schemes of finite type over F .

I.2 Background in affine algebraic geometry and $\mathbb{A}^1$ -homotopy theory

I.2.1 Affine modifications

Our main technique for generating examples of $\mathbb{A}^1$ -contractible smooth affine schemes involves affine modifications. This construction is ubiquitous, e.g., every affine birational map between integral affine schemes of finite type arises from an affine modification see [KZ99, Theorem 1.1]. An affine modification is a birational construction which also makes sense for non-affine schemes [Dub05]:

Definition I.2.1. Suppose $(\mathcal{Z} \subset \mathcal{D} \subset \mathcal{X})$ be a triple consisting of an effective Cartier divisor $\mathcal{D}$ on an integral scheme $\mathcal{X}$ with sheaf of rational functions $\mathcal{K}_{\mathcal{X}}$

and a closed subscheme $\mathcal{Z}$ with ideal sheaf $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}(-\mathcal{D})$. Then the affine modification of $\mathcal{X}$ along $\mathcal{D}$ with center in $\mathcal{Z}$ is the affine $\mathcal{X}$ -scheme

$$\pi_{\mathcal{D}, \mathcal{Z}} : \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) = \operatorname{Spec}(\mathcal{O}_{\mathcal{X}}[\mathcal{I}_{\mathcal{Z}}/\mathcal{D}]) \rightarrow \mathcal{X},$$

where $\mathcal{O}_{\mathcal{X}}[\mathcal{I}_{\mathcal{Z}}/\mathcal{D}]$ denotes the quotient of the Rees algebra

$$\mathcal{R}(\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D})) = \bigoplus_{n \geq 0} (\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}))^n t^n \subset \mathcal{K}_{\mathcal{X}}[t]$$

of the fractional ideal $\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}) \subset \mathcal{K}_{\mathcal{X}}$ by the ideal generated by $1 - t$.

We note there is a canonical open immersion

$$j : \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \hookrightarrow \operatorname{Proj}_{\mathcal{X}}(\mathcal{R}(\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}))) \simeq \operatorname{Proj}_{\mathcal{X}}(\mathcal{R}(\mathcal{I}_{\mathcal{Z}})).$$

Via j , the affine modification $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$ coincides with the complement of the proper transform of $\mathcal{D}$

$$\mathcal{D}^{pr} := \operatorname{Proj}_{\mathcal{X}}(\mathcal{R}(\mathcal{I}_{\mathcal{Z}} \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{D}))/t) \subset \operatorname{Bl}_{\mathcal{I}_{\mathcal{Z}}} \mathcal{X}$$

in the blow-up

$$\sigma_{\mathcal{Z}} : \operatorname{Bl}_{\mathcal{Z}} \mathcal{X} = \operatorname{Proj}_{\mathcal{X}}(\mathcal{R}(\mathcal{I}_{\mathcal{Z}})) \rightarrow \mathcal{X}.$$

The proper transform $\mathcal{D}^{pr}$ of $\mathcal{D}$ is in general different from its strict transform $\mathcal{D}'$, i.e., the closure of $\sigma_{\mathcal{Z}}^{-1}(\mathcal{D} \setminus \mathcal{Z})$ in $\operatorname{Bl}_{\mathcal{Z}} \mathcal{X}$. However, the said proper and strict transforms coincide in the case when $\mathcal{Z}$ is a local complete intersection in $\mathcal{D}$ [Dub05, Proposition 1.10].

Letting $\mathcal{E} := \sigma_{\mathcal{Z}}^{-1}(\mathcal{Z})$ be the exceptional divisor of the blow-up $\sigma_{\mathcal{Z}}$, we note that $\sigma_{\mathcal{Z}}$ restricts to an isomorphism $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \setminus \mathcal{E} \xrightarrow{\cong} \mathcal{X} \setminus \mathcal{D}$. We refer to

$$\mathcal{E}_{\mathcal{Z}/\mathcal{D}} := \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \cap \mathcal{E} = \mathcal{E} \setminus \mathcal{D}^{pr}$$

as the exceptional fiber of the affine modification $\pi_{\mathcal{D}, \mathcal{Z}} : \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \rightarrow \mathcal{X}$.

For an integral affine scheme $\mathcal{X} = \operatorname{Spec}(A)$ and $f \neq 0$ an element in an ideal $\mathcal{I} \subset A$, the affine modification of $\mathcal{X}$ along the principal divisor $\operatorname{div}(f)$ with center $V(\mathcal{I})$ is isomorphic to the spectrum of the sub- A -algebra

$$A[\mathcal{I}/f] \cong A[t\mathcal{I}]/(1 - tf) \cong \{a/f^k \mid a \in \mathcal{I}^k, k \geq 0\} \subset A_f.$$

as defined in [KZ99]. We denote the exceptional divisor of an affine modification of an integral affine scheme by $\mathcal{E}_{\mathcal{I}/f}$.

Example I.2.2. (See [Dav67].) Suppose $f, a_1, \dots, a_n$ is a regular sequence generating the ideal of the closed subscheme $\mathcal{Z} \subset \mathbb{A}^m = \operatorname{Spec}(F[x_1, \dots, x_m])$. Then $\tilde{\mathbb{A}}^n(\operatorname{div}(f), \mathcal{Z})$ is isomorphic to the subscheme of $\mathbb{A}^{m+n}$ defined by the equations:

$$\{a_i(x_1, \dots, x_m) - t_i f(x_1, \dots, x_m) = 0\}_{1 \leq i \leq n} \subset \mathbb{A}^{m+n} = \operatorname{Spec}(F[t_1, \dots, t_n, x_1, \dots, x_m]).$$

Example I.2.3. The following smooth schemes from Section I.1 are examples of affine modifications. Here we identify the coordinate ring of $\mathcal{X}(n, \alpha_i, p)$ with the subalgebra $F[x, y, t, x^{-n}(y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t))]$ of $F[x^{\pm 1}, y, t]$, i.e., $\mathcal{X}(n, \alpha_i, p)$ is the affine modification $\text{pr}_{x,y,t} : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}_{x,y,t}^3$ of $\text{Spec}(F[x, y, t])$ along the principal divisor $\text{div}(x^n)$ with center $\mathcal{Z}$ defined by the ideal $(x^n, y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t))$. That is, $\mathcal{X}(n, \alpha_i, p)$ is isomorphic to the complement of the proper transform of the divisor $n\{x = 0\}$ in the blowup of $\mathbb{A}^3$ with center $\mathcal{Z}$. Similar reasoning applies to the other examples in I.1.

$\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$	$\mathcal{X}$	divisor $\mathcal{D}$	ideal defining $\mathcal{Z}$
(I.4) $\mathcal{X}(n, \alpha_i, p)$	$\mathbb{A}^3$	$n\{x = 0\}$	$(x^n, y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t))$
(I.8) $\mathcal{X}_m(n_i, \alpha_j, \beta_j, a)$	$\mathbb{A}^3$	$\sum_i n_i \text{div}(l_i(x))$	$(\prod_i l_i(x)^{n_i}, \sum_i ((\prod_{j \neq i} l_j(x))(y^{\alpha_i} + t^{\beta_i})) + \prod_i l_i(x))$
(I.7) $\mathcal{X}(s, R)$	$\mathbb{A}^3$	$\text{div}(s(x))$	$(s(x), R(x, y, t))$
(I.11) $\mathcal{Y}(m, n_i, \alpha_i, p)$	(I.4)	$n_2\{x = 0\}$	$(x^{n_2}, y - z^m)$
(I.13) $\mathcal{Y}(n, \alpha_i, p)$	$\mathbb{A}^4$	$n\{x = 0\}$	$(x^n, y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3} + xp(x, y_1, y_2, y_3))$

Table I.1: Affine modifications

Given a closed embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$ between smooth schemes, the (affine) deformation space $D(\mathcal{X}, \mathcal{Z})$ of $\mathcal{Z}$ in $\mathcal{X}$ [Ful98, Chapter 5], [Ros96] is the complement of the proper transform of $\mathcal{X} \times \{0\}$ in the blow-up $\text{Bl}_{\mathcal{Z} \times \{0\}}(\mathcal{X} \times \mathbb{A}^1) \rightarrow \mathcal{X} \times \mathbb{A}^1$ of $\mathcal{X} \times \mathbb{A}^1$ along $\mathcal{Z} \times \{0\}$. In other words, $D(\mathcal{X}, \mathcal{Z})$ is the affine modification of $\mathcal{X} \times \mathbb{A}^1$ along the divisor $\mathcal{D} = \mathcal{X} \times \{0\}$ with center in $\mathcal{Z} = \mathcal{Z} \times \{0\}$. When $\mathcal{X}$ is affine, this construction is called the hyperbolic modification of $\mathcal{X}$ with center $\mathcal{Z}$ (see [Zai99]).

I.2.2 Exotic affine spaces

Recall that a smooth affine d -dimensional $\mathbb{C}$ -variety $\mathcal{X}$ is called exotic if $\mathcal{X}$ is non-isomorphic to $\mathbb{A}_{\mathbb{C}}^d$ and its underlying smooth manifold is diffeomorphic to $\mathbb{R}^{2d}$ [Zai99, Definition 3.1]. In dimension $d \geq 3$, the h -cobordism and Lefschetz hyperplane section theorems imply that smooth affine contractible $\mathbb{C}$ -varieties are diffeomorphic to $\mathbb{R}^{2d}$ by the Dimca-Ramanujam theorem [Zai99, §3]. The following beautiful result due to Kaliman [Kal02] is a useful test for detecting exotic 3-folds. It was extended to all fields of characteristic zero by Daigle-Kaliman [DK09, Theorem 4.2] using a Lefschetz principle.

Theorem I.2.4. *If the general fibers of a regular function $\phi: \mathbb{A}^3 \rightarrow \mathbb{A}^1$ are isomorphic to the affine plane $\mathbb{A}^2$, then all the closed fibers of ϕ are isomorphic to $\mathbb{A}^2$.*

Example I.2.5. Theorem I.2.4 does not apply to the 4-fold $\mathcal{Y}(n, \alpha_i, p)$ defined in (I.13) (see [KZ00]).

Kaliman-Zaidenberg noted the following version of Miyanishi's characterization of $\mathbb{A}_{\mathbb{C}}^3$ [KZ00, p.1650].

Theorem I.2.6. *A smooth contractible affine complex 3-fold $\mathcal{X}$ is isomorphic to $\mathbb{A}_{\mathbb{C}}^3$ if and only if there exists a regular function $\phi : \mathcal{X} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ such that every closed fiber has at most isolated singularities and the general fibers are isomorphic to $\mathbb{A}_{\mathbb{C}}^2$.*

I.2.3 Topological contractibility of affine modifications

In [KZ99, Proposition 3.1, Theorem 3.1] Kaliman-Zaidenberg give useful criteria for when an affine modification induces isomorphisms on fundamental groups and singular homology. When combined with the classical Hurewicz and Whitehead theorems (see e.g., [Hat02, Theorems 4.5, 4.23]), one obtains the following version of [KZ99, Corollary 3.1, Remark 3.1].

Theorem I.2.7. *Suppose $(\mathcal{Z} = V(\mathcal{I}) \subset \mathcal{D}_f = \text{div}(f) \subset \mathcal{X} = \text{Spec}(A))$ is a triple in $\text{Sch}_{\mathbb{C}}$, where $\mathcal{X}$ is an affine variety and $\mathcal{I}$ is generated by a regular sequence $f, a_1, \dots, a_n$. Let $\pi : \tilde{\mathcal{X}}(\mathcal{D}_f, \mathcal{Z}) \rightarrow \mathcal{X}$ be the affine modification of $\mathcal{X}$ along $\mathcal{D}$ with center $\mathcal{Z}$ and suppose that:*

- (1) *the supports of $\mathcal{D}_f$ and the exceptional divisor $\mathcal{E}_{\mathcal{I}/f}$ are irreducible,*
- (2) *$\pi^*(\text{supp } \mathcal{D}_f) = \mathcal{E}_{\mathcal{I}/f}$,*
- (3) *$\text{supp } \mathcal{D}_f$ and $\mathcal{E}_{\mathcal{I}/f}$ are topological manifolds,*
- (4) *$\mathcal{Z} \hookrightarrow \mathcal{D}_f$ is a homotopy equivalence.*

Then $\tilde{\mathcal{X}}(\mathcal{D}_f, \mathcal{Z})$ is contractible if and only if $\mathcal{X}$ is contractible.

Remark I.2.8. Assumption (2) implies $\sigma_{\mathcal{I}}(\mathcal{E}_{\mathcal{I}/f})$ intersects the smooth part of $\mathcal{D}_f$ nontrivially. Theorem I.2.7 holds more generally for finite decompositions into irreducible components $\mathcal{D}_f = \sum_{i=1}^m \mathcal{D}_i$ and $\mathcal{E}_{\mathcal{I}/f} = \sum_{i=1}^m \mathcal{E}_i$ such that (2)-(4) hold componentwise.

Example I.2.9. The hypersurface $\mathcal{S} := \{x^2z = y^2 + x\} \subset \mathbb{A}^3$ is the affine modification of $\mathbb{A}^2$ along the principal divisor $\mathcal{D}_f$ given by $f = x^2$ with center defined by $(x^2, y^2 + x)$. It is not topologically contractible since the first homology group $H_1(\mathcal{S}) \cong \mathbb{Z}/2\mathbb{Z}$. This follows from the Thom isomorphism and the commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
 H_2(\mathcal{S}) & \longrightarrow & H_0(\mathcal{E}_{\mathcal{I}/f}) = \mathbb{Z} & \longrightarrow & H_1(\mathcal{S} \setminus \mathcal{E}_{\mathcal{I}/f}) & \longrightarrow & H_1(\mathcal{S}) \longrightarrow 0 \\
 \downarrow & & \downarrow \cdot 2 & & \downarrow \cong & & \downarrow \\
 H_2(\mathbb{A}^2) = 0 & \longrightarrow & H_0(\mathcal{D}_f) = \mathbb{Z} & \xrightarrow{\cong} & H_1(\mathbb{A}^2 \setminus \mathcal{D}_f) & \longrightarrow & H_1(\mathbb{A}^2) = 0 \longrightarrow 0.
 \end{array} \tag{I.19}$$

This example does not contradict Theorem I.2.7 because $\pi^*(\text{supp } \mathcal{D}_f) = 2\mathcal{E}_{\mathcal{I}/f} \neq \mathcal{E}_{\mathcal{I}/f}$.

Example I.2.10. Theorem I.2.4 and Theorem I.2.7 show the following examples are exotic 3-folds.

- Every deformed Koras-Russell 3-fold $\mathcal{X}(n, \alpha_i, p)$ given by (I.4) is topologically contractible. The projection $\text{pr}_x : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}_{\mathbb{C}}^1$ (I.4) has zero fiber $\text{pr}_x^{-1}(0) = \Gamma_{\alpha_1, \alpha_2} \times \mathbb{A}_z^1 \not\cong \mathbb{A}_{\mathbb{C}}^2$ (I.6) with non-isolated singularities. Hence $\mathcal{X}(n, \alpha_i, p)$ is not isomorphic to $\mathbb{A}^3$.
- By Remark I.2.8, $\mathcal{X}_m(n_i, \alpha_j, \beta_j, a)$ defined in (I.8) is topologically contractible. The projection $\text{pr}_x : \mathcal{X}_m(n_i, \alpha_j, \beta_j, a) \rightarrow \mathbb{A}_x^1$ has general fiber $\mathbb{A}^2$ while $\text{pr}_x^{-1}(x_j) \cong \mathbb{A}^1 \times \Gamma_{\alpha_j, \beta_j}$.
- As noted in Remark I.1.2 a Koras-Russell fiber bundle is isomorphic to $\mathbb{A}^3$ if and only if for every zero of $s(x)$ the curve $\{a(y, t) = 0\}$ is isomorphic to $\mathbb{A}^1$.
- $\mathcal{Y}(m, n_i, \alpha_i, p)$ defined in (I.11) is topologically contractible because it is an affine modification of some topologically contractible deformed Koras-Russell 3-fold. Here $\text{pr}_x : \mathcal{Y}(m, n_i, \alpha_i, p) \rightarrow \mathbb{A}_x^1$ has general fiber $\mathbb{A}^2$ while $\text{pr}_x^{-1}(0) \cong \mathbb{A}^1 \times \Gamma_{m\alpha_1, \alpha_2}$.

We note every vector bundle on such an exotic 3-fold is trivial by a result of Murthy [Mur02, Theorem 3.6].

I.2.4 Locally nilpotent derivations

The Makar-Limanov invariant of an affine algebraic variety $\mathcal{X}$ is the subring $\text{ML}(\mathcal{X})$ of $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ comprised of regular functions invariant under any $\mathbb{G}_a$ -action [Mak96]. For a commutative F -domain A , let $\text{LND}(A)$ denote the locally nilpotent derivations $\partial : A \rightarrow A$. With this notation we have:

$$\text{ML}(\mathcal{X}) = \bigcap_{\partial \in \text{LND}(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))} \ker(\partial).$$

Moreover, the Derksen invariant $\text{D}(\mathcal{X})$ of $\mathcal{X}$ is the subalgebra of $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ generated by the kernels of all nonzero locally nilpotent derivations $\partial \in \text{LND}(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$ (see [Fre06] for a survey of these invariants).

Example I.2.11.

- $\text{ML}(\mathbb{A}^n) = F$ for $n \geq 1$ and $\text{D}(\mathbb{A}^n) = F[x_1, \dots, x_n]$ for $n \geq 2$ while $\text{D}(\mathbb{A}^1) = 0$.
- The same arguments as in [KM07, Theorem 9.1] for the Russell cubic, i.e., the hypersurface

$$\mathcal{R} := \{x^2z = y^2 + t^3 + x\} \subset \mathbb{A}^4, \quad (\text{I.20})$$

show more generally that $\mathrm{ML}(\mathcal{X}(s, R)) = F[x]$ and $\mathrm{D}(\mathcal{X}(s, R)) = F[x, z, t]$ for the Koras-Russell fiber bundles introduced in Definition I.1.1.

Lemma I.2.12. *For $m \neq m' \in \mathbb{Z}$ the smooth affine 3-folds $\mathcal{X}_m$ and $\mathcal{X}_{m'}$ with degenerate fibers (I.8) are non-isomorphic.*

Proof. If there exists an isomorphism $\phi : \mathcal{X}_m \rightarrow \mathcal{X}_{m'}$ we obtain a commutative diagram:

$$\begin{array}{ccc} \mathcal{X}_m & \xrightarrow{\phi} & \mathcal{X}_{m'} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathrm{ML}(\mathcal{X}_m)) & \xrightarrow{\phi|} & \mathrm{Spec}(\mathrm{ML}(\mathcal{X}_{m'})). \end{array} \quad (\text{I.21})$$

Here the vertical maps, induced by $\mathrm{ML}(\mathcal{X}_m) \subset \mathcal{O}_{\mathcal{X}_m}$ and $\mathrm{ML}(\mathcal{X}_{m'}) \subset \mathcal{O}_{\mathcal{X}_{m'}}$ can be identified with the respective projection maps to $\mathbb{A}_x^1$ because $\mathrm{ML}(\mathcal{X}_m) = \mathrm{ML}(\mathcal{X}_{m'}) = F[x]$ according to Example I.2.11. Since $m \neq m'$ we conclude by counting the number of closed fibers non-isomorphic to $\mathbb{A}^2$. $\blacksquare$

I.2.5 $\mathbb{A}^1$ -homotopy theory

Following Morel-Voevodsky [MV99] (see e.g., [Dun+07] for an introduction) we view schemes over F as analogous to topological spaces with $\mathbb{A}^1$ playing the role of the unit interval. Recall that $\mathcal{X}$ is $\mathbb{A}^1$ -contractible if the canonical map $\mathcal{X} \rightarrow \mathrm{Spec}(F)$ is an $\mathbb{A}^1$ -weak equivalence in the sense of [MV99]. Affine spaces are readily $\mathbb{A}^1$ -contractible. We will make repeatedly use of the following example.

Example I.2.13. The cuspidal curve $\Gamma_{\alpha_1, \alpha_2}$ is $\mathbb{A}^1$ -contractible. Since $\mathcal{O}_{\Gamma_{\alpha_1, \alpha_2}}$ is isomorphic to $F[w^{\alpha_1}, w^{\alpha_2}]$, the normalization of the cuspidal curve $\mathbb{A}_w^1 \rightarrow \Gamma_{\alpha_1, \alpha_2}$ is given by $w \mapsto (w^{\alpha_2}, w^{\alpha_1})$. This map is an $\mathbb{A}^1$ -weak equivalence, even an isomorphism of representable presheaves on Sm_F (see [AD07, Example 2.1]).

The following result elucidates the role of affine modifications in $\mathbb{A}^1$ -homotopy theory.

Theorem I.2.14. *Suppose $(\mathcal{Z} \subset \mathcal{D} \subset \mathcal{X})$ is a triple in Sm_F as in Definition I.2.1 with affine modification $\pi : \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \rightarrow \mathcal{X}$ such that the following holds:*

- (1) *the supports of $\mathcal{D}$ and of the exceptional divisor $\mathcal{E}_{\mathcal{Z}/\mathcal{D}}$ are irreducible,*
- (2) *$\mathcal{Z} \subset \mathcal{D}$ is an $\mathbb{A}^1$ -weak equivalence.*

Then there is a naturally induced $\mathbb{A}^1$ -weak equivalence between simplicial suspensions:

$$\Sigma_s \pi : \Sigma_s \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \xrightarrow{\sim \mathbb{A}^1} \Sigma_s \mathcal{X}. \quad (\text{I.22})$$

Furthermore, if $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$ is $\mathbb{A}^1$ -contractible, then $\mathcal{X}$ is $\mathbb{A}^1$ -contractible.

Proof. Since $(\mathcal{Z} \subset \mathcal{D} \subset \mathcal{X})$ is a smooth triple, the affine modification $\tilde{\mathcal{X}} := \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$ is isomorphic to the complement of the proper transform $\mathcal{D}'$ of $\mathcal{D}$ in the blow-up $\sigma : \text{Bl}_{\mathcal{Z}}\mathcal{X} \rightarrow \mathcal{X}$ of $\mathcal{X}$ along $\mathcal{Z}$, and $\pi : \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ coincides via this isomorphism with the restriction of σ . The exceptional divisor $\mathcal{E}$ of σ is isomorphic to the projective bundle of lines in the normal bundle $\mathcal{N}_{\mathcal{Z}/\mathcal{X}}$ of $\mathcal{Z}$ in $\mathcal{X}$, on which σ restricts to the structure morphism $p : \mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \rightarrow \mathcal{Z}$. The normal bundle $\mathcal{N}_{\mathcal{E}/\tilde{\mathcal{X}}}$ of $\mathcal{E}$ in $\text{Bl}_{\mathcal{Z}}\mathcal{X}$ is equal to the tautological line sub-bundle $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})}(-1)$ of $p^*\mathcal{N}_{\mathcal{Z}/\mathcal{X}}$. From the exact sequence $0 \rightarrow \mathcal{N}_{\mathcal{Z}/\mathcal{D}} \rightarrow \mathcal{N}_{\mathcal{Z}/\mathcal{X}} \rightarrow \mathcal{N}_{\mathcal{D}/\mathcal{X}}|_{\mathcal{Z}} \rightarrow 0$, we obtain a closed embedding $\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{D}}) \hookrightarrow \mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})$ as a hyperplane sub-bundle given by the zero locus of the global section of $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})}(1) \otimes p^*\mathcal{N}_{\mathcal{D}/\mathcal{X}}|_{\mathcal{Z}}$ deduced from the composition $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})}(-1) \rightarrow p^*\mathcal{N}_{\mathcal{Z}/\mathcal{X}} \rightarrow p^*\mathcal{N}_{\mathcal{D}/\mathcal{X}}|_{\mathcal{Z}}$.

It follows from the setup that $\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{D}}) \subset \mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})$ is equal to the intersection of the proper transform $\mathcal{D}'$ of $\mathcal{D}$ with $\mathcal{E}$. The restriction $q : \mathcal{E}_{\mathcal{Z}/\mathcal{D}} = \mathcal{E} \setminus \mathcal{D}' = \mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}}) \setminus \mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{D}}) \rightarrow \mathcal{Z}$ of π is then a locally trivial $\mathbb{A}^d$ -bundle, $d = \text{codim}_{\mathcal{D}}(\mathcal{Z})$; in particular, it is an $\mathbb{A}^1$ -weak equivalence. Moreover, since $\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})$ is the zero locus of a section of $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})}(1) \otimes p^*\mathcal{N}_{\mathcal{D}/\mathcal{X}}|_{\mathcal{Z}}$, it follows that the restriction of $\mathcal{N}_{\mathcal{E}/\tilde{\mathcal{X}}} = \mathcal{O}_{\mathbb{P}(\mathcal{N}_{\mathcal{Z}/\mathcal{X}})}(-1)$ to $\mathcal{E}_{\mathcal{Z}/\mathcal{D}}$ is equal to $q^*\mathcal{N}_{\mathcal{D}/\mathcal{X}}|_{\mathcal{Z}}$.

Now consider the following diagram of cofiber sequences:

$$\begin{array}{ccccc} \tilde{\mathcal{X}} \setminus \mathcal{E}_{\mathcal{Z}/\mathcal{D}} & \longrightarrow & \tilde{\mathcal{X}} & \longrightarrow & \tilde{\mathcal{X}}/(\tilde{\mathcal{X}} \setminus \mathcal{E}_{\mathcal{Z}/\mathcal{D}}) \\ \pi| \downarrow & & \pi \downarrow & & \bar{\pi} \downarrow \\ \mathcal{X} \setminus \mathcal{D} & \longrightarrow & \mathcal{X} & \longrightarrow & \mathcal{X}/(\mathcal{X} \setminus \mathcal{D}). \end{array} \quad (\text{I.23})$$

Since $\sigma^{-1}(\mathcal{D}) = \mathcal{D}' \cup \mathcal{E}$, the leftmost vertical map $\pi| : \tilde{\mathcal{X}} \setminus \mathcal{E}_{\mathcal{Z}/\mathcal{D}} \rightarrow \mathcal{X} \setminus \mathcal{D}$ is an isomorphism. By construction, the restriction of π to $\mathcal{E}_{\mathcal{Z}/\mathcal{D}}$ factors as the composition of $q : \mathcal{E}_{\mathcal{Z}/\mathcal{D}} = \mathcal{E} \setminus \mathcal{D}' \rightarrow \mathcal{Z}$ with $i : \mathcal{Z} \hookrightarrow \mathcal{D}$. On the other hand, by homotopy purity for closed embeddings [MV99, Theorem 3.2.23], we have $\mathbb{A}^1$ -weak equivalences:

$$\tilde{\mathcal{X}}/(\tilde{\mathcal{X}} \setminus \mathcal{E}_{\mathcal{Z}/\mathcal{D}}) \xrightarrow{\sim \mathbb{A}^1} \text{Th}(\mathcal{N}_{\mathcal{E}_{\mathcal{Z}/\mathcal{D}}/\tilde{\mathcal{X}}}) = \text{Th}(q^*\mathcal{N}_{\mathcal{D}/\mathcal{X}}|_{\mathcal{Z}}) = \text{Th}((i \circ q)^*\mathcal{N}_{\mathcal{D}/\mathcal{X}})$$

and

$$\mathcal{X}/(\mathcal{X} \setminus \mathcal{D}) \xrightarrow{\sim \mathbb{A}^1} \text{Th}(\mathcal{N}_{\mathcal{D}/\mathcal{X}}).$$

Furthermore the map $\bar{\pi} : \tilde{\mathcal{X}}/(\tilde{\mathcal{X}} \setminus \mathcal{E}_{\mathcal{Z}/\mathcal{D}}) \rightarrow \mathcal{X}/(\mathcal{X} \setminus \mathcal{D})$ induced by π coincides via these equivalences with the one induced by the composition:

$$\mathcal{E}_{\mathcal{Z}/\mathcal{D}} = \mathcal{E} \setminus \mathcal{D}' \xrightarrow{q} \mathcal{Z} \xrightarrow{i} \mathcal{D}.$$

Here $q : \mathcal{E}_{\mathcal{Z}/\mathcal{D}} \rightarrow \mathcal{Z}$ is an $\mathbb{A}^1$ -weak equivalence. By assumption (2) we conclude that $\bar{\pi}$ is an $\mathbb{A}^1$ -weak equivalence. Since $\Sigma_s(\tilde{\mathcal{X}} \setminus \mathcal{E}_{\mathcal{Z}/\mathcal{D}}) \rightarrow \Sigma_s(\mathcal{X} \setminus \mathcal{D})$ is an $\mathbb{A}^1$ -weak equivalence, it follows that $\Sigma_s \pi : \Sigma_s \tilde{\mathcal{X}} \rightarrow \Sigma_s \mathcal{X}$ is an $\mathbb{A}^1$ -weak equivalence by simplicially suspending the homotopy cofiber sequences in (I.23).

Finally, assuming that $\tilde{\mathcal{X}}$ is $\mathbb{A}^1$ -contractible, applying the very weak five lemma for pointed model categories [DF18, Lemma 2.1] to (I.23) implies $\mathcal{X}$ is $\mathbb{A}^1$ -contractible. $\blacksquare$

Remark I.2.15. Theorem I.2.14 holds more generally for finite decompositions into irreducible components $\mathcal{D} = \sum_{i=1}^m \mathcal{D}_i$ and $\mathcal{E}_{\mathcal{Z}/\mathcal{D}} = \sum_{i=1}^m \mathcal{E}_i$ such that (1) and (2) hold componentwise. For an analogue of Theorem I.2.14 in the setting of motives we refer to [Aso11b, Proposition 4].

Example I.2.16.

- Let $\mathcal{R}$ be the Russell cubic defined in (I.20). The affine modification of $\mathcal{R}$ along the divisor $\mathcal{D} := \text{div}(x)$ with center $\mathcal{Z}$ given by the ideal $(x, t+1, y-1)$ is the hypersurface in $\mathbb{A}^4$ given by the equation:

$$\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) := \{xz = x^2t^3 - 3xt + 3t + y^2x + 2y + 1\}. \quad (\text{I.24})$$

Theorem I.2.6 shows $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$ is isomorphic to $\mathbb{A}^3$, and hence $\mathbb{A}^1$ -contractible. However, we cannot use Theorem I.2.14 to conclude $\mathcal{R}$ is $\mathbb{A}^1$ -contractible because $(\mathcal{Z} \subset \mathcal{D} \subset \mathcal{R})$ is not a smooth triple.

- Let $\mathcal{X}$ be the hypersurface in $\mathbb{A}^6 = \text{Spec}(F[x, y, z, t, u, v])$ defined by:

$$\{uv = x^2z - (y^2 + t^3 + x)\}.$$

Setting $\mathcal{D} := \text{div}(u) \cong \mathcal{R} \times \mathbb{A}_v^1$ and $\mathcal{Z} := \{u = x = y = t = 0\} \cong \mathbb{A}_{v,z}^2$ the affine modification

$$\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) := \{v = x^2uz - y^2u - t^3u^2 - x\} \subset \mathbb{A}^6$$

is isomorphic to $\mathbb{A}^5$. Since $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$, $\mathcal{D}$, and $\mathcal{Z}$ are $\mathbb{A}^1$ -contractible and the triple $(\mathcal{Z} \subset \mathcal{D} \subset \mathcal{X})$ is smooth, Theorem I.2.14 implies $\mathcal{X}$ is $\mathbb{A}^1$ -contractible. It is an open question whether or not $\mathcal{X}$ is isomorphic to $\mathbb{A}^5$.

- We suggest a generalization of the previous example by constructing a potentially nontrivial family of $\mathbb{A}^1$ -contractible smooth affine schemes of higher dimensions. For $m \geq 5$ and pairwise coprime integers α_i we define hypersurfaces in $\mathbb{A}^{m+1} = \text{Spec}(F[u, v, x, z, y_1, \dots, y_{m-3}])$ by:

$$\mathcal{X}(n, \alpha_1, \dots, \alpha_{m-3}) := \{uv = x^n z + y_1^{\alpha_1} + \dots + y_{m-3}^{\alpha_{m-3}} + x\}. \quad (\text{I.25})$$

Then $\mathcal{Z} = V(u, x, y_1, \dots, y_{m-1}) \subset \mathcal{D}_u = \text{div}(u) \subset \mathcal{X}(n, \alpha_i)$ is a smooth triple in Sm_F satisfying the assumptions in Theorem I.2.14, and:

$$\widetilde{\mathcal{X}(n, \alpha_1, \dots, \alpha_{m-3})}(\mathcal{D}, \mathcal{Z}) \cong \mathbb{A}^m \sim_{\mathbb{A}^1} *.$$

It is unknown whether or not $\mathcal{X}(n, \alpha_1, \dots, \alpha_{m-3})$ is isomorphic to an affine space.

Recall from Subsection I.2.1 the deformation space $D(\mathcal{X}, \mathcal{Z})$ of a closed embedding $\mathcal{Z} \hookrightarrow \mathcal{X}$ between smooth schemes. The composite map

$$\rho = \text{pr}_2 \circ \pi : D(\mathcal{X}, \mathcal{Z}) = \widetilde{\mathcal{X} \times \mathbb{A}^1(\mathcal{D}, \mathcal{Z})} \rightarrow \mathbb{A}^1$$

is flat. Its restriction over $\mathbb{A}^1 \setminus \{0\}$ is isomorphic to the projection on the second factor of $\mathcal{X} \times (\mathbb{A}^1 \setminus \{0\})$, whereas its fiber over $\{0\}$ is canonically isomorphic to the normal bundle $\mathcal{N}_{\mathcal{Z}/\mathcal{X}}$ of $\mathcal{Z}$ in $\mathcal{X}$.

Corollary I.2.17. *Let $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a closed embedding between $\mathbb{A}^1$ -contractible irreducible smooth schemes. Then the simplicial suspension $\Sigma_s D(\mathcal{X}, \mathcal{Z})$ of $D(\mathcal{X}, \mathcal{Z})$ is $\mathbb{A}^1$ -contractible.*

Proof. Since $\mathcal{Z}$ and $\mathcal{X}$ are both $\mathbb{A}^1$ -contractible, the inclusion $i \times \text{id} : \mathcal{Z} = \mathcal{Z} \times \{0\} \hookrightarrow \mathcal{D} = \mathcal{X} \times \{0\}$ is an $\mathbb{A}^1$ -weak equivalence. Theorem I.2.14 implies $\Sigma_s \pi : \Sigma_s D(\mathcal{X}, \mathcal{Z}) \rightarrow \Sigma_s \mathcal{X}$ is an $\mathbb{A}^1$ -weak equivalence. We are done since $\mathcal{X} \times \mathbb{A}^1$ is $\mathbb{A}^1$ -contractible. $\blacksquare$

Example I.2.18. As examples of deformation spaces we look at the origin, a special line and a particular affine plane in the Koras-Russell 3-fold $\mathcal{X} = \mathcal{X}(n, \alpha_i, 1) := \{x^n z = y^{\alpha_1} + t^{\alpha_2} + x\} \subset \mathbb{A}^4$ (I.1). As a matter of notation we write $\mathbb{A}^4 \times \mathbb{A}^1 = \text{Spec}(F[x, y, z, t][u])$.

- $\mathcal{Z} = \{(0, 0, 0, 0)\}$. The ideal of $\mathcal{Z} \times \{0\}$ in $\mathcal{X} \times \mathbb{A}^1$ is globally generated by the regular sequence u, y, z, t . The deformation space $D(\mathcal{X}, \mathcal{Z})$ is thus isomorphic to the smooth 4-fold

$$\{x^n u z_1 = (u y_1)^{\alpha_1} + (u t_1)^{\alpha_2} + x\} \subset \mathbb{A}^4 \times \mathbb{A}^1 = \text{Spec}(F[x, y_1, z_1, t_1][u]),$$

and the modification map $\pi : D(\mathcal{X}, \mathcal{Z}) \rightarrow \mathcal{X} \times \mathbb{A}^1$ is given by $(x, y_1, z_1, t_1, u) \mapsto (x, u y_1, u z_1, u t_1, u)$.

- $\mathcal{Z} = \{x = y = t = 0\}$. The ideal of $\mathcal{Z} \times \{0\}$ in $\mathcal{X} \times \mathbb{A}^1$ is globally generated by u, x, y, t . We find that the deformation space $D(\mathcal{X}, \mathcal{Z})$ is isomorphic to the smooth 4-fold

$$\{x_1^n u^{n-1} z = u^{\alpha_1-1} y_1^{\alpha_1} + u^{\alpha_2-1} t_1^{\alpha_2} + x_1\} \subset \mathbb{A}^4 \times \mathbb{A}^1 = \text{Spec}(F[x_1, y_1, z, t_1][u]),$$

with the modification map $\pi : D(\mathcal{X}, \mathcal{Z}) \rightarrow \mathcal{X} \times \mathbb{A}^1$ given by $(x_1, y_1, z, t_1, u) \mapsto (u x_1, u y_1, z, u t_1, u)$.

- $\mathcal{Z} = \{z = 0\}$. The ideal of $\mathcal{Z} \times \{0\}$ in $\mathcal{X} \times \mathbb{A}^1$ is globally generated by the regular sequence u, z . The deformation space $D(\mathcal{X}, \mathcal{Z})$ is isomorphic to the smooth 4-fold

$$\{x^n u z_1 = y^{\alpha_1} + t^{\alpha_2} + x\} \subset \mathbb{A}^4 \times \mathbb{A}^1 = \text{Spec}(F[x, y, z_1, t][u]),$$

and the modification map $\pi : D(\mathcal{X}, \mathcal{Z}) \rightarrow \mathcal{X} \times \mathbb{A}^1$ is given by $(x, y, z_1, t, u) \mapsto (x, y, u z_1, t, u)$.

In each case, $D(\mathcal{X}, \mathcal{Z})$ is smooth and the flat map $\rho : D(\mathcal{X}, \mathcal{Z}) \rightarrow \mathbb{A}^1$ coincides with the restriction of $\text{pr}_u : \mathbb{A}^4 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. Here ρ restricts to a trivial bundle $\mathcal{X} \times (\mathbb{A}^1 \setminus \{0\})$ over $(\mathbb{A}^1 \setminus \{0\})$ and $\rho^{-1}(0)$ is isomorphic to $\mathbb{A}^3$. Furthermore, since $\mathcal{X}$ is $\mathbb{A}^1$ -contractible, so is the simplicial suspension $\Sigma_s D(\mathcal{X}, \mathcal{Z})$. We do not know in any of these cases whether $D(\mathcal{X}, \mathcal{Z})$ is $\mathbb{A}^1$ -contractible. When the base field F is $\mathbb{C}$, it follows from Theorem I.2.7 that $D(\mathcal{X}, \mathcal{Z})$ is topologically contractible.

I.2.6 $\mathbb{A}^1$ -chain connectedness

Recall that an F -scheme $\mathcal{X} \in \text{Sch}_F$ is $\mathbb{A}^1$ -chain connected if for every finitely generated separable field extension E/F the set of $\mathbb{A}^1$ -equivalence classes of E -points $\mathcal{X}(E)/\sim$ consists of exactly one element. Here, rational points $x, y \in \mathcal{X}(E)$ are $\mathbb{A}^1$ -equivalent if there exist rational points $x = x_0, x_1, \dots, x_{n-1}, x_n = y \in \mathcal{X}(E)$ and an elementary $\mathbb{A}^1$ -equivalence $f_i : \mathbb{A}_E^1 \rightarrow \mathcal{X}$ between x_{i-1} and x_i for $1 \leq i \leq n$ (f_i is a map of schemes such that $f_i(0) = x_{i-1}$ and $f_i(1) = x_i$). Equivalently, $\pi_0(\text{Sing}^{\mathbb{A}^1}(\mathcal{X})(E)) = *$ for the $\mathbb{A}^1$ -singular functor of [MV99] and every E/F as above.

Remark I.2.19. Our definition of $\mathbb{A}^1$ -chain connectedness is that of [Aso11a, Definition 2.2.2], [AM11, Definition 2.2.2] generalized to Sch_F . All $\mathbb{A}^1$ -chain connected smooth schemes are $\mathbb{A}^1$ -connected [AM11, Proposition 2.2.7]. The converse implication is false: It follows from [BS17] that any anisotropic simply connected group whose underlying variety is rational (e.g. $SU(2)$ over $\mathbb{R}$), is $\mathbb{A}^1$ -connected but not $\mathbb{A}^1$ -chain connected.

Question I.2.20. *Is every $\mathbb{A}^1$ -contractible smooth scheme also $\mathbb{A}^1$ -chain connected?*

We give some supporting examples for Question I.2.20 based on the affine modifications in I.1. Antieau [Aso11a, Proposition 2.4.6] observed the following result for the Russell cubic $\mathcal{R}$ in (I.20).

Example I.2.21. Every Koras-Russell 3-fold of the first kind $\mathcal{X}(n, \alpha_i, 1)$ in (I.1) is $\mathbb{A}^1$ -chain connected.

Proof. Let E/F be a finitely generated separable field extension. For rational points $0, (y_0, t_0) \in \Gamma_{\alpha_1, \alpha_2}(E)$ on the cuspidal curve $\{y^{\alpha_1} + t^{\alpha_2} = 0\}$ there is an elementary $\mathbb{A}^1$ -equivalence $\mathbb{A}^1 \rightarrow \Gamma_{\alpha_1, \alpha_2}$ given by:

$$s \mapsto (y_0 s^{\alpha_2}, t_0 s^{\alpha_1}).$$

Thus the fibers $\mathbb{A}_{y,t}^2$ and $\Gamma_{\alpha_1, \alpha_2} \times \mathbb{A}_z^1$ of the projection $\text{pr}_x : \mathcal{X}(n, \alpha_i, 1) \rightarrow \mathbb{A}_x^1$ are $\mathbb{A}^1$ -chain connected.

To construct elementary $\mathbb{A}^1$ -equivalences between points in different fibers we choose polynomials $y(s), t(s) \in F[s]$ such that s^n divides $y(s)^{\alpha_1} + t(s)^{\alpha_2} + s$. For a rational point $a \in \mathbb{A}^1(E)$ the map $\mathbb{A}^1 \rightarrow \mathcal{X}(n, \alpha_i, 1)$ defined by

$$s \mapsto (x(s), z(s), y(s), t(s)) = (as, \frac{y(as)^{\alpha_1} + t(as)^{\alpha_2} + as}{(as)^n}, y(as), t(as))$$

connects a point in the 0-fiber with a point in the $a \neq 0$ -fiber of pr_x . $\blacksquare$

Remark I.2.22. For the Russell cubic $\mathcal{R}$ we can choose the elementary $\mathbb{A}^1$ -equivalence $\mathbb{A}^1 \rightarrow \mathcal{R}$ defined by:

$$s \mapsto (x(s), z(s), y(s), t(s)) = (as, -2 - as, as + 1, -as - 1).$$

Example I.2.23. Every Brieskorn-Pham surface $\mathcal{S}_{\alpha_i}$ in (I.12) is $\mathbb{A}^1$ -chain connected.

Proof. For $(a_1, a_2, a_3) \in \mathcal{S}_{a_1, a_2, a_3}(E)$ and E/F as above, the map $\mathbb{A}^1 \rightarrow \mathcal{S}_{a_1, a_2, a_3}$ defined by

$$s \mapsto (a_1 s^{\alpha_2 \alpha_3}, a_2 s^{\alpha_1 \alpha_3}, a_3 s^{\alpha_1 \alpha_2})$$

is an elementary $\mathbb{A}^1$ -equivalence between the rational points $0, (a_1, a_2, a_3) \in \mathcal{S}_{a_1, a_2, a_3}(E)$. $\blacksquare$

Example I.2.24. The 4-fold $\mathcal{Y}(2, (2, 3, 5), 1) := \{x^2 z = y_1^2 + y_2^3 + y_3^5 + x\} \subset \mathbb{A}^5$ in (I.13) is $\mathbb{A}^1$ -chain connected. The fibers $\mathbb{A}^3$ and $\mathcal{S}_{2,3,5}$ of the projection map $\text{pr}_x : \mathcal{Y}(2, (2, 3, 5), 1) \rightarrow \mathbb{A}_x^1$ are $\mathbb{A}^1$ -chain connected (see Example I.2.23). An elementary $\mathbb{A}^1$ -equivalence between the 0-fiber and the $a \neq 0$ -fiber is given by:

$$s \mapsto (x(s), z(s), y_1(s), y_2(s), y_3(s)) = (as, -2 - as, as + 1, -as - 1, 0).$$

Example I.2.25. For coprime integers $k > l \geq 2$, the smooth tom Dieck-Petrie surface

$$\mathcal{V}_{k,l} := \left\{ \frac{(xz + 1)^k - (yz + 1)^l - z}{z} = 0 \right\} \subset \mathbb{A}^3 \quad (\text{I.26})$$

is the affine modification of $\mathcal{Z} = (1, 1) \subset \mathcal{D} = \{x^k - y^l = 0\} \subset \mathcal{X} = \mathbb{A}^2$. It is topologically contractible by Theorem I.2.7, and stably $\mathbb{A}^1$ -contractible by Theorem I.3.2 and [HKØ16, Proposition 3.2]. However, $\mathcal{V}_{k,l}$ has logarithmic Kodaira dimension $\bar{\kappa}(\mathcal{V}_{k,l}) = 1$, and thus it is not $\mathbb{A}^1$ -chain connected (see Subsection I.2.7). This example shows that the affine modification construction does not preserve $\mathbb{A}^1$ -chain connectedness. It is an open question whether or not $\mathcal{V}_{k,l}$ is $\mathbb{A}^1$ -contractible.

I.2.7 $\mathbb{A}^1$ -contractible smooth affine surfaces

Let F be an algebraically closed field of characteristic 0. $\mathbb{A}^1$ -contractible smooth affine curves over F are isomorphic to the affine line $\mathbb{A}^1$ by [AD07, Claim 5.7]. Let $\mathcal{S}$ be an $\mathbb{A}^1$ -contractible smooth affine surface over F . By [Aso11a, Lemma 2.3.8] $\mathbb{A}^1$ -contractibility implies topological contractibility. From [Fuj82] it follows that $\mathcal{S}$ has logarithmic Kodaira dimension $\bar{\kappa}(\mathcal{S})$ equal to $-\infty$, 1, or 2. In case $\bar{\kappa}(\mathcal{S}) = -\infty$, the birational classification results due to Miyanishi, Sugie and Fujita show that $\mathcal{S}$ contains a nonempty open subset isomorphic to $\mathbb{A}^1 \times \mathcal{C}$, where $\mathcal{C}$ is a smooth affine curve [MS80]. By Miyanishi's algebraic characterization of the affine plane we conclude $\mathcal{S}$ is isomorphic to $\mathbb{A}^2$ [Miy75]. This reduces the question whether $\mathbb{A}^2$ is the only smooth affine $\mathbb{A}^1$ -contractible surface to the following.

Question I.2.26. *Does every $\mathbb{A}^1$ -contractible smooth surface over F have negative logarithmic Kodaira dimension? (An affirmative answer to Question I.2.20 would show this is the case.)*

Remark I.2.27. The Zariski cancellation problem for $\mathcal{X}$ a smooth affine d -dimensional scheme asks whether $\mathcal{X} \times \mathbb{A}^1 \cong \mathbb{A}^{d+1}$ implies $\mathcal{X} \cong \mathbb{A}^d$. This has been settled affirmatively for curves by Abhyankar, Eakin, and Heinzer [AHE72], and for surfaces by Fujita [Fuj79], Miyanishi and Sugie [MS80]. When $d \geq 3$ this remains an open problem over fields of characteristic zero, while there exists counterexamples over fields of positive characteristic; we refer to [Gup15] for a survey. Note that if $\mathcal{X}$ is stably isomorphic to an affine space then it is $\mathbb{A}^1$ -contractible, e.g., the Russell cubic $\mathcal{R}$ is a potential counterexample to Zariski cancellation. Indeed, the latter was a motivation for [DF18] and [HKØ16].

All $\mathbb{A}^1$ -chain connected smooth schemes in the sense of Asok-Morel [AM11] are log-uniruled and hence of negative logarithmic Kodaira dimension [DK15]. This implies the following classification result.

Proposition I.2.28. *An $\mathbb{A}^1$ -contractible and $\mathbb{A}^1$ -chain connected smooth affine surface over an algebraically closed field of characteristic 0 is isomorphic to the affine plane $\mathbb{A}^2$.*

A landmark result due to Ramanujam [Ram71] tells us that any smooth complex surface whose underlying analytic space is contractible and simply connected at infinity is isomorphic to $\mathbb{A}^2$.¹

Question I.2.29. *Is the underlying analytic space of every $\mathbb{A}^1$ -contractible smooth affine surface simply connected at infinity? (An affirmative answer to Question I.2.26 combined with the characterization of the affine plane due to Miyanishi and Sugie would imply that any smooth $\mathbb{A}^1$ -contractible affine surface is isomorphic to $\mathbb{A}^2$, and hence topologically simply connected at infinity.)*

Remark I.2.30. There exist non $\mathbb{A}^1$ -chain connected topologically contractible smooth affine surfaces non-isomorphic to $\mathbb{A}^2$, e.g., the Ramanujam surface [Ram71] and the tom Dieck-Petrie surfaces (see Example I.2.25). Proposition 4.1 in [AD08] shows that all vector bundles on these surfaces are trivial, i.e., algebraic K -groups cannot disprove $\mathbb{A}^1$ -contractibility of these surfaces.

I.3 Higher Chow groups calculations and stable $\mathbb{A}^1$ -contractibility

In the following we will make use of a criterion for stable $\mathbb{A}^1$ -contractibility shown in [HKØ16, §4].

Theorem I.3.1. *Suppose $\mathcal{X}$ is a smooth scheme equipped with a rational point $x \in \mathcal{X}(F)$. If the natural map $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$ induces an isomorphism on higher*

¹An open manifold $\mathcal{M}$ is said to be simply connected at infinity if for every compact subset $\mathcal{C} \subset \mathcal{M}$, there is a compact subset $\mathcal{C}'$ such that $\mathcal{C} \subset \mathcal{C}' \subset \mathcal{M}$ and $\mathcal{M} \setminus \mathcal{C}'$ is connected and simply connected.

Chow groups for any smooth affine scheme $\mathcal{Y}$, then $\mathcal{X}$ is stably $\mathbb{A}^1$ -contractible in the sense that it becomes $\mathbb{A}^1$ -contractible after a finite suspension with the projective line pointed at infinity:

$$(\mathcal{X}, x) \wedge \mathbb{P}^1 \wedge \cdots \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} *.$$

In the proof of the following result we make repeatedly use of the facts that higher Chow groups are contravariantly functorial with respect to flat maps [Blo86, Proposition 1.7] in addition to satisfying homotopy invariance [Blo86, Theorem 2.1] and localization [Blo86], [Blo94].

Theorem I.3.2. *Let $(\mathcal{Z} \subset \mathcal{D} \subset \mathcal{X})$ be a triple as in Definition I.2.1, where $\mathcal{X} = \text{Spec}(A)$ is an affine scheme, $\mathcal{D} = \text{div}(f)$, and $\mathcal{Z}$ is defined by an ideal $I = (f, a_1, \dots, a_r)$ generated by a regular sequence.*

(i) *If for every smooth affine scheme $\mathcal{Y}$, the maps $\text{pr}_{\mathcal{Y}} : \mathcal{Z} \times \mathcal{Y} \rightarrow \mathcal{Y}$, $\text{pr}_{\mathcal{Y}} : \mathcal{D} \times \mathcal{Y} \rightarrow \mathcal{Y}$, and $(f, \text{id}_{\mathcal{Y}}) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ induce isomorphisms on higher Chow groups, then so does $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \times \mathcal{Y} \rightarrow \mathcal{Y}$. In addition, if the affine modification $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$ is smooth, then it is stably $\mathbb{A}^1$ -contractible.*

(ii) *If for every smooth affine scheme $\mathcal{Y}$, the maps $\text{pr}_{\mathcal{Y}} : \mathcal{Z} \times \mathcal{Y} \rightarrow \mathcal{Y}$, $\text{pr}_{\mathcal{Y}} : \mathcal{D} \times \mathcal{Y} \rightarrow \mathcal{Y}$, and $(f \circ \pi, \text{id}_{\mathcal{Y}}) : \tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z}) \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ induce isomorphisms on higher Chow groups, then so does $\mathcal{X} \times \mathcal{Y} \rightarrow \mathcal{Y}$. In addition, if $\mathcal{X}$ is smooth, then it is stably $\mathbb{A}^1$ -contractible.*

Proof. Let $\tilde{\mathcal{X}}$ be shorthand for $\tilde{\mathcal{X}}(\mathcal{D}, \mathcal{Z})$, and set $\mathcal{E} := \mathcal{E}_{\mathcal{Z}/\mathcal{D}}$. To prove (i), note that $\mathcal{E}$ and $\mathcal{U} = \tilde{\mathcal{X}} \setminus \mathcal{E}$ give rise to a commutative diagram of exact localization sequences for higher Chow groups:

$$\begin{array}{ccccccccc} \text{CH}_*(\mathbb{G}_m \times \mathcal{Y}, i+1) & \longrightarrow & \text{CH}_*(\mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathbb{A}^1 \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathbb{G}_m \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{Y}, i-1) \\ (1) \downarrow & & (2) \downarrow & & (3) \downarrow & & (4) \downarrow & & (5) \downarrow \\ \text{CH}_*(\mathcal{U} \times \mathcal{Y}, i+1) & \longrightarrow & \text{CH}_*(\mathcal{E} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\tilde{\mathcal{X}} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{U} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{E} \times \mathcal{Y}, i-1). \end{array} \quad (\text{I.27})$$

The vertical maps are induced by $(f \circ \pi, \text{pr}_{\mathcal{Y}}) : \tilde{\mathcal{X}} \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$. Here $\pi|_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{Z}$ is a locally trivial $\mathbb{A}^d$ -bundle, $d = \text{codim}_{\mathcal{D}}(\mathcal{Z})$, by the assumption on $\mathcal{Z}$. Hence, the map $\text{pr}_{\mathcal{Y}} : \mathcal{E} \times \mathcal{Y} \rightarrow \mathcal{Y}$ induces an isomorphism on higher Chow groups by hypothesis and homotopy invariance of higher Chow groups. It follows that (2) and (5) in (I.27) are isomorphisms. Furthermore, $\mathcal{U} \cong \mathcal{X} \setminus \mathcal{D}$ and since the maps $\text{pr}_{\mathcal{Y}} : \mathcal{D} \times \mathcal{Y} \rightarrow \mathcal{Y}$ and $(f, \text{id}_{\mathcal{Y}}) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ induce isomorphisms on higher Chow groups by assumption, applying the five lemma to (I.28) shows (1) and (4) in (I.27) are isomorphisms:

$$\begin{array}{ccccccccc} \text{CH}_*(\mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathbb{A}^1 \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathbb{G}_m \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{Y}, i-1) & \longrightarrow & \text{CH}_*(\mathbb{A}^1 \times \mathcal{Y}, i-1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{CH}_*(\mathcal{D} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{X} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{U} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{D} \times \mathcal{Y}, i-1) & \longrightarrow & \text{CH}_*(\mathcal{X} \times \mathcal{Y}, i-1). \end{array} \quad (\text{I.28})$$

I. $\mathbb{A}^1$ -contractibility of affine modifications

It follows that (3) in (I.27) is an isomorphism, and we conclude by homotopy invariance. Theorem I.3.1 implies the assertion of stable $\mathbb{A}^1$ -contractibility.

Part (ii) is proved similarly by replacing $\tilde{\mathcal{X}}$ with $\mathcal{X}$ and interchanging the roles of $\mathcal{E}$ and $\mathcal{D}$. $\blacksquare$

Corollary I.3.3. *The deformed Koras-Russel 3-folds of the first kind in (I.4) are stably $\mathbb{A}^1$ -contractible.*

Proof. Recall that $\mathcal{X}(n, \alpha_i, p)$ in (I.4) is the affine modification of $\mathbb{A}^3$ along $\mathcal{D} := \text{div}(x^n)$ with center $\mathcal{Z}$ given by the ideal $(x^n, y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t))$, cf. I.1. We identify the reduced scheme $\mathcal{Z}_{\text{red}}$ with the cuspidal curve $\Gamma_{\alpha_1, \alpha_2} = \{y^{\alpha_1} + t^{\alpha_2} = 0\} \subset \mathbb{A}^2$, so that $\text{CH}_*(\mathcal{Z}, i) \cong \text{CH}_*(\Gamma_{\alpha_1, \alpha_2}, i)$ (see [Blo94]). If $\mathcal{Y}$ is a smooth affine scheme, then $\mathcal{Z} \times \mathcal{Y} \rightarrow \mathcal{Y}$ induces an isomorphism in higher Chow groups [HKØ16, Proposition 3.2]. The same conclusion holds for the maps $\mathcal{D} \times \mathcal{Y} \rightarrow \mathcal{Y}$ and $(\text{pr}_x, \text{id}_{\mathcal{Y}}) : \mathcal{X} \times \mathcal{Y} = \mathbb{A}^3 \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ by homotopy invariance. Applying Theorem I.3.2 finishes the proof. $\blacksquare$

Corollary I.3.4. *The iterated Koras-Russell 3-folds in (I.11) are stably $\mathbb{A}^1$ -contractible.*

Proof. Recall that $\mathcal{Y}(m, n_i, \alpha_i, p)$ in (I.11) is the affine modification $\mathcal{X}(n, \alpha_i, p)(\mathcal{D} = \text{div}(x^{n_2}), \mathcal{Z} = V(\mathcal{I}))$ where $\mathcal{I} = (x^{n_2}, y - z^m)$, cf. I.1. Here $\mathcal{Z}_{\text{red}}$ is isomorphic to $\Gamma_{m\alpha_1, \alpha_2}$ and $\mathcal{D}_{\text{red}}$ is isomorphic to the cylinder on $\Gamma_{\alpha_1, \alpha_2}$. For any smooth affine scheme $\mathcal{Y}$, [HKØ16, Proposition 3.2] shows the maps $\mathcal{D} \times \mathcal{Y} \rightarrow \mathcal{Y}$ and $\mathcal{Z} \times \mathcal{Y} \rightarrow \mathcal{Y}$ induce isomorphisms on higher Chow groups. The same holds for $(\text{pr}_x, \text{id}_{\mathcal{Y}}) : \mathcal{X}(n, \alpha_i, p) \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ by Corollary I.3.3. To conclude we apply Theorem I.3.2. $\blacksquare$

We show stable $\mathbb{A}^1$ -contractibility of 4-folds introduced by Kaliman-Zaidenberg in [KZ00]. These can be distinguished from $\mathbb{A}^4$ by the Makar-Limanov invariant.

Corollary I.3.5. *For $m \geq 2$ and coprime integers $k > l \geq 3$, the 4-fold defined by*

$$\left\{ u^m v = \frac{(xz + 1)^k - (yz + 1)^l - z}{z} \right\} \subset \mathbb{A}^5 \quad (\text{I.29})$$

is stably $\mathbb{A}^1$ -contractible.

Proof. The 4-fold in (I.29) is the affine modification of $\mathbb{A}_{u,x,y,z}^4$ along $\text{div}(u^2)$ with center $\mathcal{Z}$ given by:

$$\left(u^m, \frac{(xz + 1)^k - (yz + 1)^l - z}{z} \right).$$

Here $\mathcal{Z}$ is isomorphic to a tom Dieck-Petrie surface, i.e., an affine modification of $\mathbb{A}^2$ (see Example I.2.25). Applying Theorem I.3.2 to these affine modifications yields the claim. $\blacksquare$

Remark I.3.6. Setting $m = 2$, $k = 4$, and $l = 2$ the hypersurface in (I.29) is given by:

$$\{u^2v = x^4z^3 + 4x^3z^2 + 6x^2z + 4x - y^3z^2 - 3y^2z - 3y - 1\}. \quad (\text{I.30})$$

Its affine modification along $\text{div}(u^2)$ with center given by $I = (u^2, x - 1, y - 1, z)$ is isomorphic to $\mathbb{A}^4$. If the tom Dieck-Petrie surfaces are $\mathbb{A}^1$ -contractible, then (I.30) is $\mathbb{A}^1$ -contractible by Theorem I.2.14.

We give two more examples of stably $\mathbb{A}^1$ -contractible smooth affine 4-folds starting with Theorem I.1.6.

Corollary I.3.7. *If $\alpha_3 = m\alpha_1\alpha_2 + 1$, $m > 0$, then $\mathcal{Y}(n, \alpha_i, p)$ in (I.13) is stably $\mathbb{A}^1$ -contractible.*

Proof. Recall $\mathcal{Y} := \mathcal{Y}(n, \alpha_i, p)$ is the affine modification of $\mathbb{A}^4$ along $\text{div}(x^n)$ and with reduced center isomorphic to the surface $\mathcal{S} = \{y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3} = 0\} \subset \mathbb{A}^3$, cf. I.1. By Theorem I.3.2 it suffices to show $\mathcal{S} \times \mathcal{Y} \rightarrow \mathcal{Y}$ induces an isomorphism on higher Chow groups for any smooth affine scheme $\mathcal{Y}$. The map $(\text{pr}_{y_3}, \text{id}_{\mathcal{Y}}) : \mathcal{S} \times \mathcal{Y} \rightarrow \mathbb{A}^1 \times \mathcal{Y}$ fits into the commutative diagram of exact localization sequences:

$$\begin{array}{ccccccccc} \text{CH}_*(\mathbb{G}_m \times \mathcal{Y}, i+1) & \longrightarrow & \text{CH}_*(\mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathbb{A}^1 \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathbb{G}_m \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{Y}, i-1) \\ (1) \downarrow & & (2) \downarrow & & (3) \downarrow & & (4) \downarrow & & (5) \downarrow \\ \text{CH}_*(\mathcal{U}, i+1) & \longrightarrow & \text{CH}_*(\Gamma \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{S} \times \mathcal{Y}, i) & \longrightarrow & \text{CH}_*(\mathcal{U}, i) & \longrightarrow & \text{CH}_*(\Gamma \times \mathcal{Y}, i-1). \end{array} \quad (\text{I.31})$$

Here $\Gamma := \Gamma_{\alpha_1, \alpha_2}$ and $\mathcal{U} := (\mathcal{S} \times \mathcal{Y}) \setminus \{y_3 = 0\}$. By [HKØ16, Proposition 3.2] the maps (2) and (5) in (I.31) are isomorphisms. We note there is an isomorphism $\mathcal{U} \cong \mathbb{A}^2 \setminus \Gamma_{\alpha_1, \alpha_2} = \text{Spec}(F[u, v]) \setminus \Gamma_{\alpha_1, \alpha_2}$ via the map

$$(x, y, z) \mapsto (u(-u^{\alpha_1} - v^{\alpha_2})^{m\alpha_2}, v(-u^{\alpha_1} - v^{\alpha_2})^{m\alpha_1}, -u^{\alpha_1} - v^{\alpha_2}),$$

with inverse:

$$(u, v) \mapsto \left(\frac{x}{z^{m\alpha_2}}, \frac{y}{z^{m\alpha_1}} \right).$$

The five lemma and [HKØ16, Proposition 3.2] show (1) and (4) in (I.31) are isomorphisms, cf. the proof of Theorem I.3.2. Another application of the five lemma yields that (3) in (I.31) is also an isomorphism. $\blacksquare$

Corollary I.3.8. *The smooth affine 4-fold $\mathcal{Y}(n, (2, 3, 5), p)$ in (I.13) is stably $\mathbb{A}^1$ -contractible.*

Proof. As in Corollary I.3.7 it suffices to check that $\mathcal{S}_1 \times \mathcal{Y} \rightarrow \mathcal{Y}$ induces an isomorphism on higher Chow groups, where $\mathcal{Y}$ is an arbitrary smooth affine scheme and $\mathcal{S}_1 := \{y_1^2 + y_2^3 + y_3^5 = 0\} \subset \mathbb{A}^3$. In effect, we form the affine modifications:

$$\mathcal{S}_4 \xrightarrow{\pi_3} \mathcal{S}_3 \xrightarrow{\pi_2} \mathcal{S}_2 \xrightarrow{\pi_1} \mathcal{S}_1. \quad (\text{I.32})$$

Here $\mathcal{S}_2 := \{y_1^2 + y_2^3 y_3 + y_3^3\} \subset \mathbb{A}^3$ is the affine modification of $\mathcal{S}_1$ along $\text{div}(y_3)$ with center given by (y_3, y_1, y_2) , $\mathcal{S}_3 := \{y_1^2 + y_2^2 y_3 + y_3^3 y_2 = 0\} \subset \mathbb{A}^3$ is the affine modification of $\mathcal{S}_2$ along $\text{div}(y_2)$ with center given by (y_2, y_1, y_3) , and $\mathcal{S}_4 := \{y_1^2 + y_2^2 y_3 + y_2 = 0\} \subset \mathbb{A}^3$ is the affine modification of $\mathcal{S}_3$ along $\text{div}(y_3^3)$ with center given by (y_3^3, y_1, y_2) . By Theorem I.3.2, $\mathcal{S}_1 \times \mathcal{Y} \rightarrow \mathcal{Y}$ induces an isomorphism on higher Chow groups if and only if the same holds for $\mathcal{S}_4 \times \mathcal{Y} \rightarrow \mathcal{Y}$. Now $\mathcal{S}_4$ is the affine modification of $\mathbb{A}^2$ along $\text{div}(y_2^2)$ with center given by $(y_2^2, -y_1^2 - y_2)$, and we are done by Theorem I.3.2. $\blacksquare$

I.4 Unstable $\mathbb{A}^1$ -contractibility

I.4.1 Koras-Russell fiber bundles

For the convenience of the reader we recall the notion of a Koras-Russell fiber bundle, see Definition I.1.1 in Section I.1.

For a polynomial $s(x) \in F[x]$ of positive degree and $R(x, y, t) \in F[x, y, t]$, we define the closed subscheme $\mathcal{X}(s, R)$ of $\mathbb{A}^1 \times \mathbb{A}^3 = \text{Spec}(F[x][y, z, t])$ by the equation $\{s(x)z = R(x, y, t)\}$. The projection map $\rho = \text{pr}_x : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ in (I.7) is called a Koras-Russell fiber bundle if

- (a) $\mathcal{X}(s, R)$ is a smooth scheme, and
- (b) For every zero $x_0 \in \mathbb{A}^1(F)$ of $s(x)$, the zero locus in $\mathbb{A}^2 = \text{Spec}(F[y, t])$ of the polynomial $R(x_0, y, t)$ is an integral rational plane curve $\mathcal{C}$ with a unique place at infinity and at most unibranch singularities. Equivalently, there exists a polynomial $a \in F[y, t]$ and $b(x, y, t) \in F[x, y, t]$ such that

$$R(x, y, t) = a(y, t) + (x - x_0)b(x, y, t),$$

such that $\mathcal{C} = \{a(y, t) = 0\}$.

By definition, $\mathcal{X}(s, R)$ can be realized as the affine modification of $\mathbb{A}^1 \times \mathbb{A}^2 = \text{Spec}(F[x][y, t])$ along the principal divisor $\text{div}(s)$ with center at the closed subscheme defined by the ideal $(s(x), R(x, y, t))$. The map $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ restricts to the trivial $\mathbb{A}^2$ -bundle with total space $\text{Spec}(F[x]_{s(x)}[y, t])$ over the principal open subset $\{s(x) \neq 0\}$ of $\mathbb{A}_x^1$. For a zero $x_0 \in \mathbb{A}^1(F)$ of $s(x)$, we have:

- (c) If the curve $\mathcal{C} := \{a(y, t) = 0\}$ is smooth, then it is isomorphic to $\mathbb{A}^1$ and $\rho^{-1}(x_0) \cong \mathbb{A}^2$.
- (d) If $\mathcal{C}$ is singular the Lin-Zaidenberg theorem [LZ83], [NR87], [NR88] implies there exists an automorphism φ^* of $F[y, t]$ such that $\varphi^*(a) = y^{\alpha_1} + t^{\alpha_2}$ for coprime integers $\alpha_i \geq 2$. Moreover, the fiber $\rho^{-1}(x_0)$ is singular and isomorphic to $\mathcal{C} \times \mathbb{A}^1$.

Example I.4.1. Every deformed Koras-Russell 3-fold $\mathcal{X}(n, \alpha_i, p)$ of the first kind (I.4) is an example of a Koras-Russell fiber bundle. The unique singular fiber of $\rho : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}_x^1$ is $\rho^{-1}(0) \cong \Gamma_{\alpha_1, \alpha_2} \times \mathbb{A}_z^1$.

The following key geometric result reveals that Koras-Russell fiber bundles are deformed Koras-Russell 3-folds Zariski-locally.

Lemma I.4.2. *Suppose $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ is a Koras-Russell fiber bundle and let $x_0 \in \mathbb{A}^1(F)$.*

- (1) *If $\rho^{-1}(x_0)$ is smooth there exists an open neighborhood $\mathcal{U}$ of x_0 such that $\rho : \mathcal{X}(s, R)|_{\mathcal{U}} \rightarrow \mathcal{U}$ is a trivial $\mathbb{A}^2$ -bundle.*
- (2) *If $\rho^{-1}(x_0)$ is singular there exist an automorphism φ of $\mathbb{A}^1$ mapping x_0 to the origin 0, a deformed Koras-Russell 3-fold $\mathcal{X}(n, \alpha_i, p)$ of the first kind (I.4), an open neighborhood $\mathcal{U}$ of x_0 , and an isomorphism $\Psi : \mathcal{X}(s, R)|_{\mathcal{U}} \rightarrow \mathcal{X}(n, \alpha_i, p)|_{\varphi(\mathcal{U})}$ rendering the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{X}(s, R)|_{\mathcal{U}} & \xrightarrow{\Psi} & \mathcal{X}(n, \alpha_i, p)|_{\varphi(\mathcal{U})} \\ \rho \downarrow & & \downarrow \text{pr}_x \\ \mathcal{U} & \xrightarrow{\varphi} & \varphi(\mathcal{U}). \end{array}$$

Proof. (1) The assumption implies $\rho^{-1}(x_0) \cong \mathbb{A}^2$. Since the generic fiber of ρ is isomorphic to $\mathbb{A}^2$ over the function field $F(x)$ of $\mathbb{A}^1$, letting $B = \text{Spec}(\mathcal{O}_{\mathbb{A}^1, x_0})$ we deduce from [Sat83, Theorem 1] that $\text{pr}_B : \mathcal{X}(s, R) \times_{\mathbb{A}^1} B \rightarrow B$ is a trivial $\mathbb{A}^2$ -bundle. This implies the existence of $\mathcal{U}$ with the claimed properties.

(2) Since ρ restricts to a trivial $\mathbb{A}^2$ -bundle over the principal open subset $\mathcal{D}_s \subset \mathbb{A}^1$, where $s \neq 0$, x_0 is a zero of $s(x)$. Up to a coordinate change, we may assume $x_0 = 0$ and write $s(x) = x^n \tilde{s}(x)$, where $n \geq 1$ and $\tilde{s}(0) \neq 0$. This yields $R(x, y, t) = a(y, t) + xb(x, y, t)$ and $\rho^{-1}(0) = \{a(y, t) = 0\} \times \text{Spec}(F[z])$. Since by hypothesis $\mathcal{C} = \{a(y, t) = 0\}$ is a singular curve, the Lin-Zaidenberg theorem shows there exists an F -automorphism φ^* of $F[y, t]$ such that $\varphi^*(a) = y^{\alpha_1} + t^{\alpha_2}$ for coprime integers $\alpha_i \geq 2$. This automorphism φ^* extends to an $F[x]$ -automorphism Φ^* of $F[x][y, t]$ defined by $\Phi^*(\sum a_{ij}(x)y^i t^j) = \sum a_{ij}(x)\varphi^*(y^i t^j)$. Letting $p(x, y, t) = \Phi^*(b(x, y, t))$, we have by construction:

$$\Phi^*(R(x, y, t)) = \Phi^*(a(y, t) + xb(x, y, t)) = y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t).$$

So Φ^* maps $s(x)$ identically to itself and it maps the ideal $\mathcal{I} = (s(x), R(x, y, t))$ isomorphically to $\mathcal{J} = (s(x), y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t))$. By the universal property of affine modifications [KZ99, Proposition 2.1], the corresponding automorphism Φ of $\mathbb{A}^3 = \text{Spec}(F[x, y, t])$ lifts to an $\mathbb{A}_x^1$ -isomorphism between $\mathcal{X}(s, R)$ and the Koras-Russell fiber bundle:

$$\rho : \widetilde{\mathcal{X}(s, R)} := \{s(x)z = y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t)\} \rightarrow \mathbb{A}_x^1.$$

We define the closed subscheme $\mathcal{X}(n, \alpha_i, p)$ of $\mathbb{A}^4$ by the equation

$$\{x^n z = y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t)\},$$

and let $\mathcal{U} := \mathcal{D}_{\tilde{s}} \subset \mathbb{A}^1$ be the principal open subset where $\tilde{s} \neq 0$. Then $\widetilde{\mathcal{X}(s, R)}|_{\mathcal{U}}$ and $\mathcal{X}(n, \alpha_i, p)|_{\mathcal{U}}$ are isomorphic as $\mathcal{U}$ -schemes since $\tilde{s}(x)z$ is a coordinate function on $F[x]_{\tilde{s}(x)}[y, z, t]$ over $F[x]_{\tilde{s}(x)}$. To conclude $\mathcal{X}(n, \alpha_i, p)$ is a deformed Koras-Russell 3-fold as in Example I.4.1, it remains to show $n \geq 2$. Since $\mathcal{X}(s, R)|_{\mathcal{U}}$ is smooth, $\mathcal{X}(n, \alpha_i, p)$ is smooth in an open neighborhood of $\text{pr}_x^{-1}(0)$, which implies $n \geq 2$ by the Jacobian criterion. ■

Corollary I.4.3. *Suppose $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ is a Koras-Russell fiber bundle, and let $\mathcal{U}_0 \subset \mathbb{A}^1$ be the complement of the finitely many points for which the scheme-theoretic fiber of ρ is singular. Then $\rho : \mathcal{X}(s, R)|_{\mathcal{U}_0} \rightarrow \mathcal{U}_0$ is a trivial $\mathbb{A}^2$ -bundle.*

Proof. By Lemma I.4.2 (1), $\rho : \mathcal{X}(s, R)|_{\mathcal{U}_0} \rightarrow \mathcal{U}_0$ is a Zariski locally trivial $\mathbb{A}^2$ -bundle, hence isomorphic to a rank 2 vector bundle by [BCW77, Theorem 4.4], which is in fact trivial since $\mathcal{U}_0$ is an open subset of $\mathbb{A}^1$. ■

Example I.4.4. For coprime integers $\alpha_i \geq 2$ we define $\mathcal{X}(s, R) \subset \mathbb{A}^4$ by the equation:

$$\{x^2(x-1)z = (x-1)(y^{\alpha_1} + t^{\alpha_2}) + x(y+t^2) + x(x-1)\}.$$

This is a smooth scheme by the Jacobian criterion. We have $R(0, y, t) = -(y^{\alpha_1} + t^{\alpha_2})$, while for $x = 1$:

$$R(x, y, t) = (y + t^2) + (x-1)(y^{\alpha_1} + t^{\alpha_2} + y + t^2 + x).$$

Here $\{y^{\alpha_1} + t^{\alpha_2} = 0\}$ and $\{y + t^2 = 0\}$ satisfy hypothesis (1) of Definition I.1.1; hence $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ is a Koras-Russell fiber bundle. Furthermore, ρ restricts to a trivial $\mathbb{A}^2$ -bundle over $\mathbb{A}^1 \setminus \{0\}$ by Corollary I.4.3.

Lemma I.4.5. *Suppose $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ is a Koras-Russell fiber bundle, and $\rho^{-1}(x_0) \cong \mathcal{C} \times \mathbb{A}^1$ is singular for the rational point $x_0 \in \mathbb{A}^1(F)$. Let $\pi : \widetilde{\mathcal{X}(s, R)} \rightarrow \mathcal{X}(s, R)$ be the affine modification of $\mathcal{X}(s, R)$ along the divisor $\{x = x_0\}$ with center $\{c\} \times \mathbb{A}^1 \subset \rho^{-1}(x_0)$ over a smooth point c on the curve $\mathcal{C}$.*

- (1) *Then $\tilde{\rho} = \rho \circ \pi : \widetilde{\mathcal{X}(s, R)} \rightarrow \mathbb{A}_x^1$ is a Koras-Russell fiber bundle with $\tilde{\rho}^{-1}(x_0) \cong \mathbb{A}^2$.*
- (2) *By restricting π to $\widetilde{\mathcal{X}(s, R)} \setminus \tilde{\rho}^{-1}(x_0)$ we obtain an isomorphism and a commutative diagram:*

$$\begin{array}{ccc} \widetilde{\mathcal{X}(s, R)} \setminus \tilde{\rho}^{-1}(x_0) & \xrightarrow{\cong} & \mathcal{X}(s, R) \setminus \rho^{-1}(x_0) \\ & \searrow \tilde{\rho} & \swarrow \rho \\ & \mathbb{A}^1. & \end{array}$$

Proof. (1) Up to a change of coordinates, we may assume $x_0 = 0$, $s(x) = x^n \tilde{s}(x)$, $n \geq 2$, $\tilde{s}(0) \neq 0$, and:

$$R(x, y, t) = y^{\alpha_1} - t^{\alpha_2} + xb(x, y, t).$$

Furthermore, we may assume the center is the line over the point $c = (1, 1)$ on the curve $\mathcal{C} = \{y^{\alpha_1} = t^{\alpha_2}\}$. Since $\mathcal{X}(s, R)$ and the center $\{c\} \times \mathbb{A}^1$ of the affine modification $\pi : \widetilde{\mathcal{X}(s, R)} \rightarrow \mathcal{X}(s, R)$ are smooth, so is $\widetilde{\mathcal{X}(s, R)}$. It remains to check $\tilde{\rho}^{-1}(0) \cong \mathbb{A}^2$. Setting $y = xy_1 + 1$ and $t = xt_1 + 1$ yields:

$$y^{\alpha_1} - t^{\alpha_2} + xb(x, y, t) = x((\alpha_1 y_1 - \alpha_2 t_1) + b(0, 1, 1)) + x^2 \tilde{b}(x, y_1, t_1).$$

Thus the equation defining $\widetilde{\mathcal{X}(s, R)}$ in $\mathbb{A}^4$ takes the form:

$$\{x^{n-1} \tilde{s}(x)z = x^{-1}R(x, xy_1 + 1, xt_1 + 1) = \alpha_1 y_1 - \alpha_2 t_1 + b(0, 1, 1) + x \tilde{b}(x, y_1, t_1)\}.$$

This implies $\tilde{\rho}^{-1}(0) \cong \{\alpha_1 y_1 - \alpha_2 t_1 + b(0, 1, 1) = 0\} \times \mathbb{A}^1$ is the fiber product of an affine line in $\mathbb{A}^2$ by $\mathbb{A}^1$, i.e., it is isomorphic to $\mathbb{A}^2$. Part (2) follows readily. ■

I.4.2 $\mathbb{A}^1$ -contractibility of deformed Koras-Russell 3-folds after simplicial suspension

In this section we refine the results and techniques in [DF18, §3.2].

Lemma I.4.6. *If $\mathcal{X}(n, \alpha_i, p)$ is a deformed Koras-Russell 3-fold of the first kind (I.4), then:*

$$\mathcal{X}(n, \alpha_i, p) \setminus (\{x = y = t = 0\} \cong \mathbb{A}_z^1) \sim_{\mathbb{A}^1} \mathbb{A}^2 \setminus \{(0, 0)\}.$$

Proof. Using a coordinate change we may rewrite the defining equation for $\mathcal{X}(n, \alpha_i, p)$ as:

$$\mathcal{X}(\alpha_2) := \{x^n z = y^{\alpha_1} - t^{\alpha_2} + xp(x, y, t)\}, \alpha_2 \geq 1.$$

Let $\pi := \text{pr}_{x,t} : \mathcal{X}(\alpha_2) \rightarrow \mathbb{A}^2$ and define $\varphi_{\alpha_2} : \mathbb{A}^2 = \text{Spec}(F[u, v]) \rightarrow \mathbb{A}^2$ by $(u, v) \mapsto (x, t) = (u^{\alpha_1} - v^{\alpha_2}, v)$. Here φ_{α_2} is a cyclic Galois cover, fully ramified over the line $\{t = 0\} \subset \mathbb{A}^2$ and étale elsewhere. The fiber product $\mathcal{X}(\alpha_2) \times_{\mathbb{A}_{x,t}^2} \mathbb{A}_{u,v}^2$ is isomorphic to the closed subscheme of $\mathbb{A}^4 = \text{Spec}(F[u, v][y, z])$ defined by:

$$\tilde{\mathcal{X}}(\alpha_2) := \{(u^{\alpha_1} - v^{\alpha_2})^n z = y^{\alpha_1} - v^{\alpha_2} + (u^{\alpha_1} - v^{\alpha_2})p(u^{\alpha_1} - v^{\alpha_2}, y, v)\}.$$

The Galois group μ_{α_1} of the cover φ_{α_2} of $\tilde{\mathcal{X}}(\alpha_2)$ is generated by $(u, v, y, z) \mapsto (\varepsilon u, v, y, z)$, where $\varepsilon \in F^*$ is a primitive α_1 -th root of unity. The projection $\tilde{\pi} := \text{pr}_{u,v} : \tilde{\mathcal{X}}(\alpha_2) \rightarrow \mathbb{A}^2$ restricts to a trivial $\mathbb{A}^1$ -bundle over the complement of the irreducible curve $\Gamma_{\alpha_1, \alpha_2} := \{u^{\alpha_1} - v^{\alpha_2} = 0\}$. The fiber $\tilde{\pi}^{-1}((0, 0)) \cong \text{Spec}(F[y]/(y^{\alpha_1}))$ of $\tilde{\pi}$ over $(0, 0)$ is supported on the line $\widetilde{\mathbb{A}}_z^1 := \{u = v = y = 0\}$, while $\tilde{\pi}^{-1}(\Gamma_{\alpha_1, \alpha_2})$ is isomorphic to:

$$\text{Spec}(F[u, v][y, z]/(u^{\alpha_1} - v^{\alpha_2}, y^{\alpha_1} - u^{\alpha_1})).$$

The normalization map $\nu : \widetilde{\Gamma_{\alpha_1, \alpha_2}} = \text{Spec}(F[w]) \rightarrow \Gamma_{\alpha_1, \alpha_2}$, $w \mapsto (w^{\alpha_2}, w^{\alpha_1})$, restricts to an isomorphism between $\widetilde{\Gamma_{\alpha_1, \alpha_2}} \setminus \{(0, 0)\} = \mathbb{A}_*^1$ and $\Gamma_{\alpha_1, \alpha_2} \setminus \{(0, 0)\}$. Moreover, $\widetilde{\pi}^{-1}(\Gamma_{\alpha_1, \alpha_2} \setminus \{(0, 0)\})$ splits into a disjoint union of α_1 irreducible components $\mathcal{D}_{\varepsilon^i} := \{u^{\alpha_1} - v^{\alpha_2} = y - \varepsilon^i u\} \setminus \{u = v = y = 0\}$, $0 \leq i \leq \alpha_1 - 1$, all isomorphic to $\mathbb{A}_*^1 \times \text{Spec}(F[z])$. It follows that $\widetilde{\pi}$ restricts to a smooth affine map

$$\widetilde{\pi}_i : \widetilde{\mathcal{X}}_i := \widetilde{\mathcal{X}}(\alpha_2) \setminus \widetilde{\mathbb{A}}_z^1 \cup \bigcup_{j \neq i} \mathcal{D}_{\varepsilon^j} \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\},$$

and a locally trivial $\mathbb{A}^1$ -bundle since all its fibers are isomorphic to $\mathbb{A}^1$ [KM78, Theorem 1].

We let $\widetilde{\delta} : \widetilde{\mathcal{S}} \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\}$ denote the smooth scheme obtained by gluing the α_1 copies

$$\widetilde{\mathcal{S}}_{\varepsilon^i} := \text{Spec}(F[u_{\varepsilon^i}, v]) \setminus \{(0, 0)\}, 0 \leq i \leq \alpha_1 - 1,$$

of $\mathbb{A}^2 \setminus \{(0, 0)\}$ by the identity map away from the cuspidal curves $\{u_{\varepsilon^i}^{\alpha_1} = v^{\alpha_2}\}$. With this definition,

$$\widetilde{\pi} : \widetilde{\mathcal{X}}(\alpha_2) \setminus \widetilde{\mathbb{A}}_z^1 \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\}$$

factors through the locally trivial $\mathbb{A}^1$ -bundle:

$$\widetilde{\rho}_{\alpha_2} : \widetilde{\mathcal{X}}(\alpha_2) \setminus \widetilde{\mathbb{A}}_z^1 \rightarrow \widetilde{\mathcal{S}}.$$

The action of the Galois group μ_{α_1} lifts readily to an action on $\widetilde{\mathcal{S}}$ defined by sending $(u_{\varepsilon^i}, v) \in \widetilde{\mathcal{S}}_{\varepsilon^i}$ to $(\varepsilon u_{\varepsilon^{i+1}}, v) \in \widetilde{\mathcal{S}}_{\varepsilon^{i+1}}$. This yields μ_{α_1} -equivariant maps $\widetilde{\rho}_{\alpha_2} : \widetilde{\mathcal{X}}(\alpha_2) \setminus \widetilde{\mathbb{A}}_z^1 \rightarrow \widetilde{\mathcal{S}}$ and $\widetilde{\delta} : \widetilde{\mathcal{S}} \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\}$.

As algebraic spaces, the geometric quotients of $\widetilde{\mathcal{X}}(\alpha_2) \setminus \widetilde{\mathbb{A}}_z^1$ and $\mathbb{A}^2 \setminus \{(0, 0)\}$ under the μ_{α_1} -actions are isomorphic to $\mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1$ respectively $\mathbb{A}^2 \setminus (\{0, 0\}) = \text{Spec}(F[x, t]) \setminus \{(0, 0)\}$. The geometric quotient for the μ_{α_1} -action on $\widetilde{\mathcal{S}}$ is an algebraic space $\mathcal{S}(\alpha_1, \alpha_2)$ (see e.g., [Ryd13]). The map $\delta : \mathcal{S}(\alpha_1, \alpha_2) \rightarrow \mathbb{A}^2 \setminus (\{0, 0\})$ induced by $\widetilde{\delta}$ has the following properties:

- δ restricts to an isomorphism away from the punctured affine line $\{x = 0\} \cong \text{Spec}(F[t^{\pm 1}])$.
- The restriction of δ to $\delta^{-1}(\{x = 0\})$ is an étale cyclic Galois cover of order α_1 isomorphic to the projection $\text{pr}_t : \text{Spec}(F[t^{\pm 1}][u]/(u^{\alpha_1} - t^{\alpha_2})) \rightarrow \text{Spec}(F[t^{\pm 1}])$.

Likewise, $\widetilde{\rho}_{\alpha_2} : \widetilde{\mathcal{X}}(\alpha_2) \setminus \widetilde{\mathbb{A}}_z^1 \rightarrow \widetilde{\mathcal{S}}$ descends to an étale locally trivial $\mathbb{A}^1$ -bundle $\rho_{\alpha_2} : \mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1 \rightarrow \mathcal{S}(\alpha_1, \alpha_2)$. While the ramified Galois cover $\varphi_{\alpha_2} : \mathbb{A}^2 \rightarrow \mathbb{A}^2$ depends on α_1 and α_2 , the algebraic space $\delta : \mathcal{S}(\alpha_1, \alpha_2) \rightarrow \mathbb{A}^2 \setminus (\{0, 0\})$ depends only on α_1 as explained in [DF18, Lemma 3.2] and [DFJ14, §1.1]. When $\alpha_2 \geq 2$, $\mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1$ and $\mathcal{X}(1) \setminus \mathbb{A}_z^1$ are étale locally trivial $\mathbb{A}^1$ -bundles over the algebraic

space $\mathcal{S}(\alpha_1, \alpha_2) = \mathcal{S}(\alpha_1, 1) = \mathcal{S}$; in general, this is not a scheme (see [DFJ14, Remark 1.1]). As a consequence, the fiber product

$$W := (\mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1) \times_{\mathcal{S}} (\mathcal{X}(1) \setminus \mathbb{A}_z^1)$$

is simultaneously an étale locally trivial $\mathbb{A}^1$ -bundle over $\mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1$ and $\mathcal{X}(1) \setminus \mathbb{A}_z^1$ via the first and second projection, respectively. Since $\mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1$ is a scheme for $\alpha_2 \geq 1$, these bundles are locally trivial in the Zariski topology. It follows that $\text{pr}_1 : W \rightarrow \mathcal{X}(\alpha_2) \setminus \mathbb{A}_z^1$ and $\text{pr}_2 : W \rightarrow \mathcal{X}(1) \setminus \mathbb{A}_z^1$ are $\mathbb{A}^1$ -weak equivalences.

To complete the proof it remains to show $\mathcal{X}(1) \setminus \mathbb{A}_z^1$ is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{A}^2 \setminus \{(0, 0)\}$. When $\alpha_2 = 1$, the fiber $\text{pr}_x^{-1}(0)$ of $\text{pr}_x : \mathcal{X}(1) \rightarrow \mathbb{A}_x^1$ is isomorphic to $\mathbb{A}^2 = \text{Spec}(F[y, z])$, and $\text{pr}_x : \mathcal{X}(1) \rightarrow \mathbb{A}_x^1$ is a trivial $\mathbb{A}^2$ -bundle by virtue of [Sat83]. Furthermore, since $\mathbb{A}_z^1 \subset \text{pr}_x^{-1}(0)$ is the line $\{y = 0\}$, the chain of inclusions $\mathbb{A}_z^1 \subset \text{pr}_x^{-1}(0) \subset \mathcal{X}(1)$ identifies with the linear subspaces $\{x = \alpha = 0\} \subset \{x = 0\} \subset \mathbb{A}^1 \times \mathbb{A}^2$. The projection $\text{pr}_{x, \alpha} : \mathbb{A}^1 \times \mathbb{A}^2 \setminus \{x = \alpha = 0\} \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\}$ yields the desired $\mathbb{A}^1$ -weak equivalence $\mathcal{X}(1) \setminus \mathbb{A}_z^1 \sim_{\mathbb{A}^1} \mathbb{A}^2 \setminus \{(0, 0)\}$. $\blacksquare$

The proof of the next result combines the techniques used in [DF18] and [HKØ16]. In outline, we show a certain connecting map for a cofiber sequence involving $\mathbb{P}^1 \wedge \mathbb{P}^1$ is an $\mathbb{A}^1$ -weak equivalence by using Milnor-Witt K -theory sheaves to reduce the proof to stable $\mathbb{A}^1$ -contractibility shown in Theorem I.3.1.

Theorem I.4.7. *If $\mathcal{X}(n, \alpha_i, p)$ is a deformed Koras-Russell 3-fold of the first kind (I.4), then:*

$$\Sigma_s \mathcal{X}(n, \alpha_i, p) \sim_{\mathbb{A}^1} *.$$

Proof. Recall $\mathcal{X}(n, \alpha_i, p) := \{x^n z = y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t)\} \subset \mathbb{A}^4$ and $\mathbb{A}_z^1 \cong \{x = y = t = 0\} \subset \mathcal{X}(n, \alpha_i, p)$. Lemma I.4.6 shows $\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1$ is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{A}^2 \setminus \{(0, 0)\}$. Since $\mathbb{A}_z^1$ has trivial normal bundle in $\mathcal{X}(n, \alpha_i, p)$, homotopy purity [MV99, Theorem 3.2.23] implies $\mathbb{A}^1$ -weak equivalences:

$$\mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1) \sim_{\mathbb{A}^1} (\mathbb{A}_z^1)_+ \wedge (\mathbb{P}^1)^{\wedge 2} \sim_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2}.$$

Since $\Sigma_s \mathbb{A}^2 \setminus \{(0, 0)\}$ is $\mathbb{A}^1$ -weakly equivalent to $S^2 \wedge (\mathbb{G}_m)^{\wedge 2} \sim_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2}$, the cofiber sequence

$$\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 \rightarrow \mathcal{X}(n, \alpha_i, p) \rightarrow \mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1) \quad (\text{I.33})$$

induces via simplicial suspension the connecting map:

$$\begin{aligned} \mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1) &\sim_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2} \\ \xrightarrow{\partial} \Sigma_s \mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1 &\sim_{\mathbb{A}^1} \Sigma_s \mathbb{A}^2 \setminus \{(0, 0)\} \sim_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2}. \end{aligned} \quad (\text{I.34})$$

To conclude that $\Sigma_s \mathcal{X}(n, \alpha_i, p)$ is $\mathbb{A}^1$ -contractible, it suffices to show (I.34) is an $\mathbb{A}^1$ -weak equivalence. There are isomorphisms for Milnor-Witt K -theory sheaves

$$H^1(\mathcal{X}(n, \alpha_i, p) \setminus \mathbb{A}_z^1, \mathbf{K}_2^{\text{MW}}) \cong H^1(\mathbb{A}^2 \setminus \{(0, 0)\}, \mathbf{K}_2^{\text{MW}}) \cong \mathbf{K}_0^{\text{MW}}(F) \cong H^2((\mathbb{P}^1)^{\wedge 2}, \mathbf{K}_2^{\text{MW}}),$$

I. $\mathbb{A}^1$ -contractibility of affine modifications

so that the long exact sequence in cohomology associated to (I.33) takes the form:

$$\cdots \rightarrow H^1(\mathcal{X}(n, \alpha_i, p), \mathbf{K}_2^{\text{MW}}) \rightarrow \mathbf{K}_0^{\text{MW}}(F) \xrightarrow{\alpha} \mathbf{K}_0^{\text{MW}}(F) \rightarrow H^2(\mathcal{X}(n, \alpha_i, p), \mathbf{K}_2^{\text{MW}}) \rightarrow \cdots$$

The $\mathbf{K}_0^{\text{MW}}(F)$ -linear map α is multiplication by the element of $\mathbf{K}_0^{\text{MW}}(F)$ corresponding to ∂ in (I.34) under the isomorphism $[(\mathbb{P}^1)^{\wedge 2}, (\mathbb{P}^1)^{\wedge 2}]_{\mathbb{A}^1} \cong \mathbf{K}_0^{\text{MW}}(F)$. We claim α is an isomorphism, in fact:

$$H^1(\mathcal{X}(n, \alpha_i, p), \mathbf{K}_2^{\text{MW}}) = H^2(\mathcal{X}(n, \alpha_i, p), \mathbf{K}_2^{\text{MW}}) = 0.$$

Since $H^{i+m}((\mathbb{P}^1)^{\wedge m}, \mathbf{K}_{2+m}^{\text{MW}}) = 0$ for $i \geq 1$, the fact that there is a cofiber sequence

$$(\mathbb{P}^1)^{\wedge m} \rightarrow \mathcal{X}(n, \alpha_i, p)_+ \wedge (\mathbb{P}^1)^{\wedge m} \rightarrow \mathcal{X}(n, \alpha_i, p) \wedge (\mathbb{P}^1)^{\wedge m}$$

and

$$H^i(\mathcal{X}(n, \alpha_i, p), \mathbf{K}_2^{\text{MW}}) \cong H^{i+m}(\mathcal{X}(n, \alpha_i, p)_+ \wedge (\mathbb{P}^1)^{\wedge m}, \mathbf{K}_{2+m}^{\text{MW}})$$

for $i, m \geq 1$ [Fas13], it suffices to show $\mathcal{X}(n, \alpha_i, p) \wedge (\mathbb{P}^1)^{\wedge m} \sim_{\mathbb{A}^1} *$ for $m \gg 0$. This follows from Theorem I.3.1 since for every smooth affine scheme $\mathcal{Y}$ the pullback map

$$\text{CH}_*(\mathcal{Y}) \rightarrow \text{CH}_*(\mathcal{X}(n, \alpha_i, p) \times \mathcal{Y})$$

is an isomorphism, owing to Corollary I.3.3. ■

Let $\sigma = \text{pr}_{x,y,t} : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}^3$ be the morphism expressing $\mathcal{X}(n, \alpha_i, p)$ as the affine modification of $\mathbb{A}^3$ along $\text{div}(x^n)$ with center at the closed subscheme defined by the ideal $(x^n, y^{\alpha_1} + t^{\alpha_2} + xp(x, y, t))$. Let

$$\pi : \widetilde{\mathcal{X}(n, \alpha_i, p)} \rightarrow \mathcal{X}(n, \alpha_i, p)$$

be the affine modification of $\mathcal{X}(n, \alpha_i, p)$ along $\text{div}(x)$ with center $\tilde{Z} : \{q\} \times \mathbb{A}^1 \subset \rho^{-1}(0)$ for a smooth point q on the cuspidal curve $\Gamma_{\alpha_1, \alpha_2}$. Corollary I.4.3 and Lemma I.4.5 imply the composite

$$\tilde{\rho} = \rho \circ \pi : \widetilde{\mathcal{X}(n, \alpha_i, p)} \rightarrow \mathbb{A}_x^1$$

is a trivial $\mathbb{A}^2$ -bundle. This yields a commutative diagram of horizontal cofiber sequences:

$$\begin{array}{ccccc} \widetilde{\mathcal{X}(n, \alpha_i, p)} \setminus \tilde{\rho}^{-1}(0) & \longrightarrow & \widetilde{\mathcal{X}(n, \alpha_i, p)} & \longrightarrow & \widetilde{\mathcal{X}(n, \alpha_i, p)} / (\widetilde{\mathcal{X}(n, \alpha_i, p)} \setminus \tilde{\rho}^{-1}(0)) \sim_{\mathbb{A}^1} \mathbb{A}_+^2 \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} \mathbb{P}^1 \\ \downarrow \cong & & \downarrow \pi & & \downarrow \bar{\pi} \\ \mathcal{X}(n, \alpha_i, p) \setminus \rho^{-1}(0) & \longrightarrow & \mathcal{X}(n, \alpha_i, p) & \longrightarrow & \mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \rho^{-1}(0)) \\ \downarrow \cong & & \downarrow \sigma & & \downarrow \bar{\sigma} \\ \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \times \mathbb{A}^2 / (\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^2) \sim_{\mathbb{A}^1} \mathbb{A}_+^2 \wedge \mathbb{P}^1 \sim_{\mathbb{A}^1} \mathbb{P}^1. \end{array}$$

Since $\widetilde{\mathcal{X}(n, \alpha_i, p)} \cong \mathbb{A}^1 \times \mathbb{A}^2$ is $\mathbb{A}^1$ -contractible and $\Sigma_s \mathcal{X}(n, \alpha_i, p)$ is $\mathbb{A}^1$ -contractible by Theorem I.4.7, we obtain the induced $\mathbb{A}^1$ -weak equivalences

$$\Sigma_s \pi : \Sigma_s \widetilde{\mathcal{X}(n, \alpha_i, p)} \xrightarrow{\sim \mathbb{A}^1} \Sigma_s \mathcal{X}(n, \alpha_i, p), \quad (\text{I.35})$$

and:

$$\Sigma_s \sigma : \Sigma_s \mathcal{X}(n, \alpha_i, p) \xrightarrow{\sim \mathbb{A}^1} \Sigma_s (\mathbb{A}^1 \times \mathbb{A}^2). \quad (\text{I.36})$$

Corollary I.4.8. *There are naturally induced $\mathbb{A}^1$ -weak equivalences*

$$\Sigma_s \bar{\pi} : \Sigma_s \mathbb{P}^1 \xrightarrow{\sim \mathbb{A}^1} \Sigma_s (\mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \rho^{-1}(0))),$$

and:

$$\Sigma_s \bar{\sigma} : \Sigma_s (\mathcal{X}(n, \alpha_i, p) / (\mathcal{X}(n, \alpha_i, p) \setminus \rho^{-1}(0))) \xrightarrow{\sim \mathbb{A}^1} \Sigma_s \mathbb{P}^1.$$

I.4.3 $\mathbb{A}^1$ -contractibility of Koras-Russell fiber bundles after simplicial suspension

In this section we prove Theorem I.1.3 stated in the introduction.

Theorem I.4.9. *Let $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$ be a Koras-Russell fiber bundle. Then $\Sigma_s \mathcal{X}(s, R)$ is $\mathbb{A}^1$ -contractible.*

Proof. We proceed by induction on the number of singular fibers of $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$. If there are no singular fibers, then ρ is a trivial $\mathbb{A}^2$ -bundle by Corollary I.4.3, so that $\mathcal{X}(s, R)$ is trivially $\mathbb{A}^1$ -contractible. Suppose $x_0 \in \mathbb{A}^1(F)$ is a point with singular fiber $\rho^{-1}(x_0)$. As in Lemma I.4.5 the affine modification

$$\pi : \widetilde{\mathcal{X}(s, R)} \rightarrow \mathcal{X}(s, R)$$

yields a Koras-Russell fiber bundle with one less singular fiber than $\rho : \mathcal{X}(s, R) \rightarrow \mathbb{A}_x^1$, namely:

$$\tilde{\rho} = \rho \circ \pi : \widetilde{\mathcal{X}(s, R)} \rightarrow \mathbb{A}_x^1.$$

It suffices to show that if $\Sigma_s \widetilde{\mathcal{X}(s, R)}$ is $\mathbb{A}^1$ -contractible, then so is $\Sigma_s \mathcal{X}(s, R)$. By Lemma I.4.2(2), there exists an open neighborhood $\mathcal{U}$ of 0, a deformed Koras-Russell 3-fold of the first kind $\text{pr}_x : \mathcal{X}(n, \alpha_i, p) \rightarrow \mathbb{A}_x^1$, and an isomorphism of $\mathcal{U}$ -schemes $\Psi : \mathcal{X}(s, R)|_{\mathcal{U}} \xrightarrow{\cong} \mathcal{X}(n, \alpha_i, p)|_{\mathcal{U}}$. We form the affine modification

$$\pi' : \widetilde{\mathcal{X}(n, \alpha_i, p)} \rightarrow \mathcal{X}(n, \alpha_i, p)$$

along the principal divisor $\text{pr}_x^{-1}(x_0)$ with center in the image of Ψ restricted to the center of π , and set $\widetilde{\text{pr}_x} = \text{pr}_x \circ \pi' : \widetilde{\mathcal{X}(n, \alpha_i, p)} \rightarrow \mathbb{A}_x^1$. By [KZ99, Proposition 2.1] Ψ lifts to an isomorphism of $\mathcal{U}$ -schemes:

$$\tilde{\Psi} : \widetilde{\mathcal{X}(s, R)}|_{\mathcal{U}} \xrightarrow{\cong} \widetilde{\mathcal{X}(n, \alpha_i, p)}|_{\mathcal{U}}.$$

Invoking excision, Ψ induces an $\mathbb{A}^1$ -weak equivalence

$$\theta : \mathrm{Th} := \mathcal{X}(s, R)/(\mathcal{X}(s, R) \setminus \rho^{-1}(x_0)) \xrightarrow{\sim \mathbb{A}^1} \mathrm{Th}' := \mathcal{X}(n, \alpha_i, p)/(\mathcal{X}(n, \alpha_i, p) \setminus \mathrm{pr}_x^{-1}(x_0))$$

between the homotopy cofibers of the open inclusions $\mathcal{X}(s, R) \setminus \rho^{-1}(x_0) \hookrightarrow \mathcal{X}(s, R)$ and $\mathcal{X}(n, \alpha_i, p) \setminus \mathrm{pr}_x^{-1}(x_0) \hookrightarrow \mathcal{X}(n, \alpha_i, p)$, respectively. Similarly, $\widetilde{\Psi}$ induces an $\mathbb{A}^1$ -weak equivalence

$$\widetilde{\theta} : \widetilde{\mathrm{Th}} := \widetilde{\mathcal{X}(s, R)}/(\widetilde{\mathcal{X}(s, R)} \setminus \widetilde{\rho}^{-1}(x_0)) \xrightarrow{\sim \mathbb{A}^1} \widetilde{\mathrm{Th}}' := \widetilde{\mathcal{X}(n, \alpha_i, p)}/(\widetilde{\mathcal{X}(n, \alpha_i, p)} \setminus \widetilde{\mathrm{pr}}_x^{-1}(x_0))$$

between the homotopy cofibers of the open inclusions $\widetilde{\mathcal{X}(s, R)} \setminus \widetilde{\rho}^{-1}(x_0) \hookrightarrow \widetilde{\mathcal{X}(s, R)}$ and $\widetilde{\mathcal{X}(n, \alpha_i, p)} \setminus \widetilde{\mathrm{pr}}_x^{-1}(x_0) \hookrightarrow \widetilde{\mathcal{X}(n, \alpha_i, p)}$, respectively. Letting $\bar{\pi} : \widetilde{\mathrm{Th}} \rightarrow \mathrm{Th}$ and $\bar{\pi}' : \widetilde{\mathrm{Th}}' \rightarrow \mathrm{Th}'$ denote the natural maps induced by π and π' , respectively, we obtain the commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathrm{Th}} & \xrightarrow{\widetilde{\theta}} & \widetilde{\mathrm{Th}}' \\ \bar{\pi} \downarrow & & \downarrow \bar{\pi}' \\ \mathrm{Th} & \xrightarrow{\theta} & \mathrm{Th}' \end{array}$$

The map π' gives rise to a commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Sigma_s \widetilde{\mathcal{X}(n, \alpha_i, p)} \setminus \widetilde{\mathrm{pr}}_x^{-1}(x_0) & \longrightarrow & \Sigma_s \widetilde{\mathcal{X}(n, \alpha_i, p)} & \longrightarrow & \Sigma_s \widetilde{\mathrm{Th}}' \longrightarrow \cdots \\ & & \cong \downarrow & & \downarrow \Sigma_s \pi' & & \downarrow \Sigma_s \bar{\pi}' \\ \cdots & \longrightarrow & \Sigma_s \mathcal{X}(n, \alpha_i, p) \setminus \mathrm{pr}_x^{-1}(x_0) & \longrightarrow & \Sigma_s \mathcal{X}(n, \alpha_i, p) & \longrightarrow & \Sigma_s \mathrm{Th}' \longrightarrow \cdots \end{array}$$

Corollary I.4.3 shows $\widetilde{\mathrm{pr}}_x : \widetilde{\mathcal{X}(n, \alpha_i, p)} \rightarrow \mathbb{A}_x^1$ is a trivial $\mathbb{A}^2$ -bundle, so that $\Sigma_s \widetilde{\mathcal{X}(n, \alpha_i, p)}$ is $\mathbb{A}^1$ -contractible. Here $\Sigma_s \bar{\pi}' : \Sigma_s \widetilde{\mathrm{Th}}' \rightarrow \Sigma_s \mathrm{Th}'$ and hence $\Sigma_s \bar{\pi} : \Sigma_s \widetilde{\mathrm{Th}} \rightarrow \Sigma_s \mathrm{Th}$ are $\mathbb{A}^1$ -weak equivalences by Corollary I.4.8. In the commutative diagram of cofiber sequences

$$\begin{array}{ccccc} \Sigma_s \widetilde{\mathcal{X}(s, R)} \setminus \widetilde{\rho}^{-1}(x_0) & \longrightarrow & \Sigma_s \widetilde{\mathcal{X}(s, R)} & \longrightarrow & \Sigma_s \widetilde{\mathrm{Th}} \\ \cong \downarrow & & \downarrow \Sigma_s \pi & & \downarrow \Sigma_s \bar{\pi} \\ \Sigma_s \mathcal{X}(s, R) \setminus \rho^{-1}(x_0) & \longrightarrow & \Sigma_s \mathcal{X}(s, R) & \longrightarrow & \Sigma_s \mathrm{Th}, \end{array}$$

$\Sigma_s \widetilde{\mathcal{X}(s, R)}$ is $\mathbb{A}^1$ -contractible by the induction hypothesis. The very weak five lemma in pointed model categories [DF18, Lemma 2.1] implies $\Sigma_s \pi$ is an $\mathbb{A}^1$ -weak equivalence, and hence $\Sigma_s \mathcal{X}(s, R) \sim_{\mathbb{A}^1} *$. $\blacksquare$

I.4.4 $\mathbb{A}^1$ -contractibility of iterated Koras-Russell 3-folds after simplicial suspension

In this section we prove Theorem I.1.5 stated in the introduction.

Theorem I.4.10. *If $\mathcal{Y}(m, n_i, \alpha_i, p)$ is an iterated Koras-Russell 3-fold of the first kind (I.11), then:*

$$\Sigma_s \mathcal{Y}(m, n_i, \alpha_i, p) \sim_{\mathbb{A}^1} *.$$

Proof. Setting $\mathbb{A}_y^1 := \{x = z = t = 0\} \subset \mathcal{Y}(m, n_i, \alpha_i, p)$ gives us the connecting map

$$\mathcal{Y}(m, n_i, \alpha_i, p) / (\mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1) \sim_{\mathbb{A}^1} (\mathbb{P}^1)^{\wedge 2} \xrightarrow{\partial} \Sigma_s \mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1 \quad (\text{I.37})$$

associated to the cofiber sequence:

$$\mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1 \rightarrow \mathcal{Y}(m, n_i, \alpha_i, p) \rightarrow \mathcal{Y}(m, n_i, \alpha_i, p) / (\mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1).$$

To conclude $\Sigma_s \mathcal{Y}(m, n_i, \alpha_i, p) \sim_{\mathbb{A}^1} *$ it suffices to show ∂ in (I.37) is an $\mathbb{A}^1$ -weak equivalence. First we observe that $\mathcal{Y}(m, n_i, \alpha_i, p) \wedge (\mathbb{P}^1)^{\wedge n} \sim_{\mathbb{A}^1} *$ for $n \gg 0$ by Corollary I.3.4. Following the proof of Theorem I.4.7 we show $\mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1$ is $\mathbb{A}^1$ -weakly equivalent to $\mathbb{A}^2 \setminus \{(0, 0)\}$.

Set $\pi := \text{pr}_{x,t} : \mathcal{Y}(m, n_i, \alpha_i, p) \rightarrow \mathbb{A}^2$ and define $\varphi : \mathbb{A}^2 = \text{Spec}(F[u, v]) \rightarrow \mathbb{A}^2$ by $(u, v) \mapsto (x, t) = (-u^{\alpha_1 m} - v^{\alpha_2}, v)$; this is a cyclic Galois cover of order $m\alpha_1$ which is fully ramified over $\{t = 0\} \subset \mathbb{A}^2$ and étale elsewhere. The fiber product $\mathcal{Y}(m, n_i, \alpha_i, p) \times_{\mathbb{A}_{x,t}^2} \mathbb{A}_{u,v}^2$ is isomorphic to the closed subscheme

$$\begin{aligned} \widetilde{\mathcal{Y}(m, n_i, \alpha_i, p)} &:= \{(-u^{\alpha_1 m} - v^{\alpha_2})^{n_1} z \\ &= ((-u^{\alpha_1 m} - v^{\alpha_2})^{n_2} y + z^m)^{\alpha_1} + v^{\alpha_2} + (-u^{\alpha_1 m} - v^{\alpha_2})p(-u^{\alpha_1 m} - v^{\alpha_2}, y, v)\}, \end{aligned}$$

of $\mathbb{A}_{u,v,y,z}^4$. The Galois group $\mu_{m\alpha_1}$ of the cover φ of $\widetilde{\mathcal{Y}(m, n_i, \alpha_i, p)}$ is generated by $(u, v, y, z) \mapsto (\varepsilon u, v, y, z)$, where $\varepsilon \in F^*$ is a primitive $\alpha_1 m$ -th root of unity. Moreover, the projection map

$$\tilde{\pi} := \text{pr}_{u,v} : \widetilde{\mathcal{Y}(m, n_i, \alpha_i, p)} \rightarrow \mathbb{A}^2$$

restricts to a trivial $\mathbb{A}^1$ -bundle over the complement in $\mathbb{A}^2$ of the irreducible cuspidal curve $\Gamma_{\alpha_1 m, \alpha_2} = \{u^{\alpha_1 m} + v^{\alpha_2} = 0\}$. The fiber $\tilde{\pi}^{-1}((0, 0)) \cong \text{Spec}(F[z]/(z^{m\alpha_1}))$ of $\tilde{\pi}$ over the origin $\{(0, 0)\}$ is supported on the line $\widetilde{\mathbb{A}_z^1} := \{u = v = z = 0\} \subset \widetilde{\mathcal{Y}(m, n_i, \alpha_i, p)}$, and $\tilde{\pi}^{-1}(\Gamma_{\alpha_1 m, \alpha_2})$ is isomorphic to:

$$\text{Spec}(F[u, v][y, z]/(-u^{\alpha_1 m} - v^{\alpha_2}, z^{m\alpha_1} - u^{m\alpha_1})).$$

The normalization map $\nu : \widetilde{\Gamma_{\alpha_1 m, \alpha_2}} = \mathbb{A}_w^1 \rightarrow \Gamma_{\alpha_1 m, \alpha_2}$, $w \mapsto (w^{\alpha_2}, w^{m\alpha_1})$, restricts to an isomorphism between $\widetilde{\Gamma_{\alpha_1 m, \alpha_2}} \setminus \{(0, 0)\} = \mathbb{A}_*^1$ and $\Gamma_{\alpha_1 m, \alpha_2} \setminus \{(0, 0)\}$, while $\tilde{\pi}^{-1}(\Gamma_{\alpha_1 m, \alpha_2} \setminus \{(0, 0)\})$ splits into a disjoint union of $\alpha_1 m$ irreducible components

$$\mathcal{D}_{\varepsilon^i} : \{u^{m\alpha_1} - v^{\alpha_2} = z - \varepsilon^i u = 0\} \setminus \{u = v = z = 0\}, 0 \leq i \leq \alpha_1 m - 1,$$

all isomorphic to $\mathbb{A}_*^1 \times \text{Spec}(F[y])$. As in the proof of Lemma I.4.6 we conclude the map

$$\tilde{\pi} : \widetilde{\mathcal{Y}(m, n_i, \alpha_i, p)} \setminus \widetilde{\mathbb{A}_z^1} \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\}$$

factors through a locally trivial $\mathbb{A}^1$ -bundle

$$\tilde{\rho} : \mathcal{Y}(\widetilde{m, n_i, \alpha_i, p}) \setminus \widetilde{\mathbb{A}_z^1} \rightarrow \tilde{\mathcal{S}}$$

over a smooth scheme $\tilde{\delta} : \tilde{\mathcal{S}} \rightarrow \mathbb{A}^2 \setminus \{(0, 0)\}$. This is achieved by gluing together the $m\alpha_1$ -copies $\tilde{\mathcal{S}}_{\varepsilon^i} = \text{Spec}(F[u_{\varepsilon^i}, v]) \setminus \{(0, 0)\}$ of $\mathbb{A}^2 \setminus \{(0, 0)\}$, $0 \leq i \leq m\alpha_1 - 1$, using the identity map away from the curves $\{u_{\varepsilon^i}^{m\alpha_1} - v^{\alpha_2} = 0\}$. Furthermore, $\tilde{\rho}$ descends to an étale locally trivial $\mathbb{A}^1$ -bundle

$$\rho : \mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1 \rightarrow \mathcal{S}(m\alpha_1, \alpha_2)$$

over the algebraic space constructed in Lemma I.4.6. Since $\mathcal{S}(m\alpha_1, \alpha_2) = \mathcal{S}(m\alpha_1, 1)$, we conclude as in the proof of Lemma I.4.6 that there exists a scheme W which is simultaneously the total space of Zariski locally trivial $\mathbb{A}^1$ -bundles over $\mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1$ and $\mathbb{A}^2 \setminus \{(0, 0)\} \times \mathbb{A}^1$. This construction furnishes the desired $\mathbb{A}^1$ -weak equivalence between $\mathcal{Y}(m, n_i, \alpha_i, p) \setminus \mathbb{A}_y^1$ and the punctured affine plane $\mathbb{A}^2 \setminus \{(0, 0)\}$. $\blacksquare$

I.4.5 A family of $\mathbb{A}^1$ -contractible 3-folds

In conclusion we note a family of $\mathbb{A}^1$ -contractible 3-folds with an $\mathbb{A}^1$ -contractible total space. Let us fix $n \geq 4$, coprime integers $\alpha_1, \alpha_2 \geq 2$, and set $\mathcal{Y} := \text{Spec}(F[a_2, \dots, a_{n-1}]) \simeq \mathbb{A}_F^{n-2}$. With these conventions we define the subscheme $\mathfrak{X} \subset \mathcal{Y} \times \mathbb{A}_F^4$ by the equation:

$$\mathfrak{X} := \{x^n z = y^{\alpha_1} + t^{\alpha_2} + x + x^2 + \sum_{i=2}^{n-1} a_i x^{i+1}\}. \quad (\text{I.38})$$

Here $\text{pr}_{\mathcal{Y}} : \mathfrak{X} \rightarrow \mathcal{Y}$ is a smooth family whose fibers are $\mathbb{A}^1$ -contractible and non-isomorphic to $\mathbb{A}^3$ over the corresponding residue fields [DF18]. Furthermore, the fibers of $\text{pr}_{\mathcal{Y}}$ over the F -rational points of $\mathcal{Y}$ are pairwise non-isomorphic F -schemes which are all $\mathbb{A}^1$ -stably isomorphic. We claim the total space $\mathfrak{X}$ of this family is $\mathbb{A}^1$ -contractible. Indeed, rewriting the defining equation (I.38) for $\mathfrak{X}$ in the form

$$\{x^3(-a_2 - \sum_{i=3}^{n-1} a_i x^{i-2} + x^{n-3} z) = y^{\alpha_1} + t^{\alpha_2} + x(1+x)\},$$

and noting that $Z = -a_2 - \sum_{i=3}^{n-1} a_i x^{i-2} + x^{n-3} z$ is a variable of $F[x][a_2, a_3, \dots, a_{n-1}, z]$ in the sense that $F[x][a_2, a_3, \dots, a_{n-1}, z] = F[x][Z, a_3, \dots, a_{n-1}, z]$, we see that $\mathfrak{X}$ is isomorphic to the product of $\mathbb{A}_F^{n-2} = \text{Spec}(F[a_3, \dots, a_{n-1}, z])$ with the deformed Koras-Russell threefold:

$$\mathcal{X}(3, \alpha_1, \alpha_2, (1+x)) := \{x^3 Z = y^{\alpha_1} + t^{\alpha_2} + x(1+x) \subset \text{Spec}(F[x, Z, y, t])\}.$$

Since $\mathcal{X}(3, \alpha_1, \alpha_2, (1+x))$ is $\mathbb{A}^1$ -contractible [DF18], so is $\mathfrak{X}$ by homotopy invariance.

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