On Higher-Order Logical Grounds*

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Abstract

Existential claims are widely held to be grounded in their true instances. However, this principle is shown to be problematic by arguments due to Kit Fine. Stephan Krämer has given an especially simple form of such an argument using propositional quantifiers. This note shows that even if a *schematic* principle of existential grounds for propositional quantifiers has to be restricted, this does not immediately apply to a corresponding non-schematic principle in higher-order logic.

1 Introduction

In the recent literature on metaphysics, the label *ground* is used for a relation among truths, which may be explained by saying that one truth grounds another truth if the latter holds (partly) in virtue of the former. Logical connectives are widely taken to lead to instances of grounding. However, Fine (2010) already notes that such instances quickly lead to inconsistency with plausible auxiliary assumptions, and Krämer (2013) shows that some instances involving propositional quantifiers lead to inconsistency with minimal auxiliary assumptions. The central principle in Krämer's argument is a schematic principle of existential grounds, capturing the informal idea that existential claims are grounded in their true instances. There is, however, an alternative regimentation of this informal idea which avoids the schematicity of Krämer's principle, using the resources of higher-order logic. As it turns out, this non-schematic regimentation is not inconsistent without more substantial additional assumptions. Here, the prospects of endorsing such a non-schematic principle of existential grounds are explored.

2 Krämer's Argument

Existential claims are often considered to be grounded in their true instances. So, using the symbolism of first-order logic, if Fa is true, then Fa grounds that something is F , i.e., $\exists xFx$. Using $\prec$ as a binary sentential connective for ground, this can be stated as follows:

$$Fa \rightarrow (Fa \prec \exists xFx)$$

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(As usual, quotation marks around formal expressions will be omitted when *mentioning*, e.g., the connective ‘ $\prec$ ’. However, formal expressions will also sometimes be *used*, as in ‘ Fa grounds ...’. Context will disambiguate between use and mention where this is important.)

Fine (2010) notes that such a principle of existential grounding leads to inconsistency with plausible assumptions when one considers the existence of facts, propositions or sentences. Krämer (2013) shows that an especially simple form of this problem arises using propositional quantifiers, i.e., quantifiers binding variables in sentential position. In Krämer’s setting, variables $p, q, r, \dots$ are used in sentential position, and allowed to be bound by quantifiers $\forall$ and $\exists$. A formal discussion of such quantifiers in the context of modal logic can be found in Fine (1970). In such a language, there are existential claims of the form $\exists p\varphi$, which may be instantiated using any formula ψ . In the following, such an instance will be written as $\varphi[\psi/p]$; this stands for the formula obtained by uniformly replacing the free occurrences of p in φ by ψ . In this setting, the general principle that existential propositionally quantified claims are grounded in their true instances can be formulated by stating that $\varphi[\psi/p]$ is true only if $\varphi[\psi/p]$ grounds $\exists p\varphi$:

$$(\exists S) \quad \varphi[\psi/p] \rightarrow (\varphi[\psi/p] \prec \exists p\varphi)$$

Krämer’s version of Fine’s puzzle arises from the instance of $(\exists S)$ obtained by letting φ be p and letting ψ be $\exists p p$. $p[\exists p p/p]$ is simply $\exists p p$, so this instance may be stated as follows:

$$\exists p p \rightarrow (\exists p p \prec \exists p p)$$

Since the antecedent of the conditional is true – there is surely some truth – it follows that there being some truth grounds itself:

$$\exists p p \prec \exists p p$$

This contradicts the claim that grounding is asymmetric and so irreflexive, which is widely endorsed; see, e.g., Correia and Schnieder (2012, p. 17).

Such problems are seriously worrying for grounding theorists: The recent notion of ground in metaphysics was explicitly proposed in an exploratory spirit, to be rejected if it turns out to be incoherent (see Rosen, 2010, p. 114). Grounding theorists have therefore considered radical measures in order to save principles of ground such as $(\exists S)$, which serve as paradigmatic examples used to motivate the intelligibility and importance of ground. E.g., Correia (2014, pp. 54–55) and Woods (2018) accept the conclusion of the argument, and give up the irreflexivity of ground, while Fine (2010) considers a number of different responses, some of which require rejecting classical inferences. Krämer (2013) notes that many of the responses which Fine considers are not applicable to the very simple form of the problem arising from $(\exists S)$, and suggests that restricting $(\exists S)$ to predicative instances (instances in which ψ does not itself involve propositional quantifiers) is the only plausible response available, apart from giving up irreflexivity.

One of the responses considered by Fine (2010, p. 108) is to weaken the existential grounding principle: instead of claiming that an existential claim is grounded in *every* true instance, one may claim that any true existential claim is grounded in *some* true instance. Yet, as Fine (2010, p. 109) notes, an analogous weakening is far less plausible in the case of the universal quantifier:

it seems that all of the instances of a true universal claim are required for its truth, and so should be among its grounds. Krämer's argument is easily adapted to universal quantifiers, paralleling variant arguments in Fine (2010). In general, the discussion below in terms of existential quantifiers could as well be carried out in terms of universal quantifiers. Weakening the existential grounding principle in this way may thus safely be set aside as a response to the problems of logical ground for present purposes.

3 Existential Grounds in Higher-Order Logic

Those who want to hold on to the irreflexivity of ground, the existence of a truth ($\exists pp$), and modus ponens as a truth-preserving form of inference have no choice but to reject some instances of ($\exists S$). But there is nevertheless an unrestricted principle of existential propositional grounding they can endorse. This section formulates such a principle; the following two sections explore its tenability.

In higher-order logic, quantifiers are naturally understood as second-order properties, as proposed already by Frege (1884). To formalize this idea, it is helpful to introduce λ -operators. In the case of first-order logic, with x an individual variable and φ a formula, $\lambda x\varphi$ can be understood as a unary predicate expressing the property of being an x such that φ . Using E as a higher-order predicate for the higher-order property of being an instantiated first-order property, $\exists x\varphi$ may then be understood to stand for $E\lambda x\varphi$, expressing that the property of being an x such that φ is instantiated. Propositional quantifiers may be understood similarly: $\lambda p\varphi$ expresses the property of being a proposition p such that φ . Using E again for the property of being an instantiated property (now of propositions), $\exists p\varphi$ may be understood to stand for $E\lambda p\varphi$, expressing that the property of being a proposition p such that φ is instantiated.

In higher-order logic, any kind of predicate can be replaced by a suitable variable. So, let X be a variable occupying the position of predicates like $\lambda p\varphi$ which take sentences as arguments, which we may also think of as sentential operators. Informally, we may consider X to stand for any property of propositions. Using quantificational terminology rather than schematicity, talk of existential claims may be captured by considering EX , for various properties X , rather than $\exists p\varphi$ for various formulas φ . As an instance of EX , we may consider Xp , for any proposition p . Thus a non-schematic principle of existential grounding can be formulated as stating that for any X and p , Xp is true only if Xp grounds that X is instantiated, i.e., EX . The general principle may thus be formulated quantificationally using the following single sentence:

$$(\exists Q) \quad \forall X \forall p (Xp \rightarrow (Xp \prec EX))$$

What happens to Krämer's argument in this setting? The relevant instance may now be obtained by universal instantiation. In the above case, the problematic existential claim was $\exists pp$, which can now be read as $E\lambda pp$, and this was the problematic instantiating proposition as well. For brevity, let T abbreviate λpp (*being true*). The relevant instance of ($\exists Q$) is thus obtained by universal instantiation, instantiating X with T (*truth*) and p with ET (*there being a truth*). This leads to the following conditional:

$$T(ET) \rightarrow (T(ET) \prec ET)$$

Analogous to before, the antecedent $T(ET)$ is surely true (it is surely true that there is a truth), whence:

$$T(ET) \prec ET$$

But in contrast to the earlier conclusion, this is no counterexample to the irreflexivity of ground, as long as $T(ET) \neq ET$. So, while the schematic principle $(\exists S)$ leads straightforwardly to violations of the irreflexivity of ground, this problem does not immediately arise from $(\exists Q)$, even though the latter is also a very natural and general principle expressing the idea that existential claims are grounded in their true instances.

4 Ground and Truth

Although $T(ET) \prec ET$ is not immediately a violation of the irreflexivity of ground, such a violation threatens to reappear when general principles governing λ are considered. This section sets out the problem, and the next section tentatively sketches a possible answer.

The relevant principles governing λ concern the relationship between a statement $(\lambda p\varphi)\psi$ ascribing a property expressed by a λ -term $\lambda p\varphi$ to a proposition ψ , and $\varphi[\psi/p]$ (which, recall, is the formula obtained by uniformly replacing the free occurrences of p in φ by ψ). An attractive principle of higher-order logic is β -conversion, the principle that these are the same proposition:

$$(\beta C) \quad \varphi[\psi/p] = (\lambda p\varphi)\psi$$

To illustrate this with an example, the claim that q is true or q is false may plausibly be the same as the claim that q has the property of being true or false:

$$q \vee \neg q = (\lambda p p \vee \neg p)q$$

An alternative option is proposed by Fine (2012, section 1.9), which is that in such a case, the former *grounds* the latter (provided they are true). Generally, Fine proposes that $\varphi[\psi/p]$ only if $\varphi[\psi/p]$ grounds that ψ is a p such that φ . Call this the principle of β -grounding:

$$(\beta G) \quad \varphi[\psi/p] \rightarrow (\varphi[\psi/p] \prec (\lambda p\varphi)\psi)$$

Given the irreflexivity of ground, not both of these principles can be true: a truth $\varphi[\psi/p]$ cannot both ground and be identical to $(\lambda p\varphi)\psi$. Grounding theorists who find (βC) and (βG) attractive thus face a choice. But for present purposes, it does not matter how the choice is made, since *both* options reintroduce the problem of existential grounds.

To see why, consider the instances of β -conversion and β -grounding where the λ -term is T , i.e., $\lambda p p$, and ψ is ET :

$$ET = T(ET)$$

$$ET \rightarrow (ET \prec T(ET))$$

Both of these are in tension with the instance of $(\exists Q)$ reached above, which was $T(ET) \prec ET$. In the first case, of $ET = T(ET)$, a violation of the irreflexivity of

ground arises immediately. In the second case, as the antecedent is true (there is some truth), one obtains $ET \prec T(ET)$. Thus ET and $T(ET)$ end up grounding each other. This violates the asymmetry of ground, and with the widely made assumption that (the relevant notion of) ground is transitive, it also violates the irreflexivity of ground.

To endorse $(\exists Q)$, one must therefore reject *both* β -conversion and β -grounding. Merely rejecting these principles is theoretically unsatisfactory. A convincing defense of $(\exists Q)$ therefore requires an alternative account of the relationship between $\varphi[\psi/p]$ and $(\lambda p\varphi)\psi$.

One alternative option would be to generalize from $T(ET) \prec ET$ to the reversal of the grounding direction in β -grounding. In the case of truth, this is captured by the following quantified principle:

$$\forall p(p \rightarrow (Tp \prec p))$$

However, given the transitivity and irreflexivity of ground, this means that every truth leads to an infinite regress, with any truth p being grounded in Tp , which itself is grounded in TTp , and so on. As noted by Fine (2010, p. 107), this is unattractive.

A more appealing alternative option is to hold that a truth which ascribes truth is identical to the truth that it is true, and so does not ground it, while any other truth is grounded in the truth that it is true, and so is distinct from it. This idea is captured by the following two principles:

$$(T1) \quad \forall p((p \wedge \exists q(p = Tq)) \rightarrow (Tp = p))$$

$$(T2) \quad \forall p((p \wedge \neg \exists q(p = Tq)) \rightarrow (Tp \prec p))$$

This disjunctive account ties things up nicely from a formal perspective, but no philosophical motivation has been given for it. The next section explores such a motivation by showing how it might fall out of an appropriate picture of propositional granularity.

Before moving on, it is worth noting a connection to the original presentation of the problem of existential grounds by Fine (2010): In one of his versions, Fine relies on a principle which captures the idea that any truth grounds the truth that it is true (see *Propositional Grounding* on p. 101). Krämer's simple formulation of the problem dispenses with the need to appeal to such a principle. In a sense, the move from a schematic to a quantified principle of existential grounds thus reintroduces some of the possible responses available in Fine's original formulation.

5 Logical Structure

In order to motivate the specific principles $(T1)$ and $(T2)$, it is helpful to consider first a more general issue. The view about grounding which emerged from $(\exists Q)$ led to a curious disparity between logical operations like quantification, conjunction, etc. on the one hand, and truth on the other hand: while the truth-conditional dependencies of the former are supposed to be reflected in grounding relationships between the relevant propositions, this is supposedly not (always) so in the case of the latter. What accounts for the difference?

An argument going back to Russell (1903, Appendix B) and Myhill (1958) can be used to shed light on the matter. (For recent discussion of this argument, see Uzquiano (2015), Hodes (2015, pp. 384–386), Dorr (2016, pp. 63–64) and Goodman (2017).) This argument shows that in higher-order logic, relations of syntactic constituency among complex sentences and predicates can demonstrably not be reflected in relations of constituency among the properties and propositions they express, since one can prove the following claim in classical higher-order logic with λ -terms:

$$\exists X \exists Y \exists p ((Xp = Yp) \wedge X \neq Y)$$

Thus, from a proposition expressed by a predication Xp , one cannot always recover a property and proposition such that the former is uniquely attributed to the latter by the proposition.

This argument does rely on an *extensional* version of β -conversion:

$$(\beta C_e) \quad \varphi[\psi/p] \leftrightarrow (\lambda p \varphi)\psi$$

But this is highly plausible even for those who reject β -conversion itself. E.g., even if stating that Earth is a blue planet does not amount to stating that Earth is a blue and Earth is a planet, surely Earth is a blue planet just in case Earth is blue and Earth is a planet.

The argument due to Russell and Myhill suggests a possible explanation of why quantifiers and Boolean connectives may give rise to universal grounding principles, without T (i.e., $\lambda p p$) doing so as well: Maybe logical structure, including the use of quantifiers and Boolean connectives, can be recovered from propositions, while this is not always so for other properties such as T . Here, for a unary property F to be *recoverable* can be understood as the claim that for any property X and propositions p and q , Fp is Xq only if X is F :

$$(FR) \quad \forall X \forall p \forall q (Fp = Xq \rightarrow F = X)$$

E.g., the instance for F being $\neg$ captures the idea that negation is recoverable:

$$(\neg R) \quad \forall X \forall p \forall q (\neg p = Xq \rightarrow \neg = X)$$

Analogous polyadic and higher-order principles are straightforward to formulate, in order to capture the cases of other logical properties like quantifiers, conjunction, and so on.

In general, the idea would be that there is a metaphysical division between logical and non-logical properties, with quantifiers and Boolean connectives being logical, and the properties expressed by λ -terms like T failing to be logical, such that all and only the logical properties can be recovered from propositions formed using them, in the sense of the general recovery principle just stated. Moreover, such non-recoverable properties may not give rise to general grounding principles.

Why should there be a difference between logical operators and defined operators like T when it comes to satisfying the recovery principles? First, there is no reason to expect an inconsistency to arise from $(\neg R)$, the recoverability of negation. But the matter is different when we consider a schematic principle for arbitrary λ -terms:

$$\forall X \forall p \forall q ((\lambda r \varphi)p = Xq \rightarrow (\lambda r \varphi) = X)$$

Russell and Myhill’s argument shows that the instance for φ being $\exists Y((r = Ys) \wedge \neg Yr)$ is inconsistent, where s may be any proposition. The kind of recovery principles we may have for logical connectives is therefore unavailable for arbitrary operators specified using λ -terms.

Here is a simple picture of propositions which fleshes out what has emerged so far: Consider propositions to be composed of two kinds of contents, a logical content which encodes the qualitative information of the proposition, and a structural content which includes the main property to be recovered, if such a property exists. We have seen that we neither want to endorse that T is always recoverable from Tp (since we would then expect any truth p to ground Tp), nor that the structural content of Tp is always the same as that of p (since we would then expect any p to be identical to Tp). A natural remaining option is that the application of properties specified using λ terms obliterates structural content, but does not introduce any structural content itself. Thus, no property $\lambda r\varphi$ will be recoverable.

This is in fact not such a strange proposal. For λp is a variable-binder, and so functions fundamentally differently from logical and non-logical constants, as it is non-compositional in the sense that propositions φ and ψ may be identical without $\lambda p\varphi$ being $\lambda p\psi$ (see Rabern (2012) and Yli-Vakkuri (2013) for detailed discussion of this). The semantic machinery in play when interpreting occurrences of $\lambda p\varphi$ is thus radically different from what is involved in interpreting occurrences of logical operators, and this may account for the difference in the contributions the relevant properties make to the structural content of the relevant propositions.

All of this is of course highly speculative, and I am not endorsing any such view here. Indeed, it is not clear whether developing the present picture in natural ways will actually lead to a consistent theory of propositional granularity and grounding at all. The suggestions made here are merely supposed to illustrate how the surprising commitments concerning grounding we have ended up with may in principle arise as natural consequences of interesting views about the individuation of propositions: Why should conjuncts ground conjunctions, but truths not ground their truth ascriptions? The answer is clear on the present picture: the relevant difference is that $\wedge$ is recoverable, whereas T is not. Furthermore, the picture can explain the earlier suggestion captured by (T1) and (T2) that Tp grounds a truth p if p is not itself a truth-ascription, and that $Tp = p$ otherwise: If p has structural content, e.g., in virtue of being a conjunction, then Tp strips away this structural content. The remaining purely qualitative proposition is naturally taken to ground any conjunctive proposition with the same logical content. And once the structural content has been stripped off, nothing remains to be stripped off, so since logical content is preserved, TTp is always just Tp .

6 Restricting the Existential Grounding Schema

Although the discussion so far indicates how higher-order grounding theorists may endorse an unrestricted principle of propositionally quantified existential grounding, it admittedly does not provide any straightforward answer to the problem posed by Krämer’s argument. The schematic principle ($\exists S$) still has to be restricted to rule out problematic instances, and no straightforward restric-

tion emerges from the present discussion of the non-schematic principle $(\exists Q)$. However, some systematic conclusions can be drawn.

First, recall that given $(\exists Q)$, we need to reject the principle of β -conversion (βC) . Above, this was illustrated using T , but the observation can be generalized: Let φ and ψ be formulas. Universal instantiation allows one to infer from $(\exists Q)$ the following:

$$(\lambda p\varphi)\psi \rightarrow ((\lambda p\varphi)\psi \prec E\lambda p\varphi)$$

From this, one obtains the following with Leibniz's Law, the relevant instance of (βC) , and the fact that $\exists p\varphi$ abbreviates $E\lambda p\varphi$:

$$\varphi[\psi/p] \rightarrow (\varphi[\psi/p] \prec \exists p\varphi)$$

And this is just the instance of $(\exists S)$ for the arbitrary given φ and ψ . Thus the required restrictions on $(\exists S)$ are constrained by the restrictions to (βC) , in the sense that on the assumption of $(\exists Q)$, every true instance of (βC) leads to a true instance of $(\exists S)$.

Second, the required restrictions on (βC) may arise naturally from principles of individuation of propositions, such as the recovery principles discussed in the previous section. As discussed above, the guiding idea was that logical properties are recoverable, as illustrated by the recoverability of negation $(\neg R)$. This is straightforwardly inconsistent with the following instance of (βC) , where φ is p and ψ is $\neg p$:

$$\forall p(T\neg p = \neg p)$$

From this it follows with $(\neg R)$ that $T\neg p = \neg p$ only if $T = \neg$, which is of course false. Certain restrictions on β -conversion can thus be derived from the recoverability principles for logical properties.

These observations do not amount to a full specification of which instances of the schematic principle $(\exists S)$ have to be rejected and which can be accepted. But they do indicate systematic grounds for accepting and rejecting such instances. Returning to the starting point of the problem of Krämer (2013), it is also worth noting that these systematic grounds are unconnected to restrictions of predicativity, and so show, *contra* Krämer, that there are options other than barring impredicative instances in response to his problem.

7 Conclusion

This paper has tentatively developed a new way of responding to the problems of existential and universal grounds: Violations of the asymmetry of ground can be avoided if the schematic principle $(\exists S)$ is replaced by the higher-order quantified principle $(\exists Q)$. In fact, the proposed solution not only blocks the known problems of ground of Fine (2010) and Krämer (2013), but also the problem of quantificational grounds of Fritz (unpublished).

The main cost of the proposed solution is its requirement for a novel theory concerning the relationship between the propositions expressed by pairs of statements of the form $\varphi[\psi/p]$ and $(\lambda p\varphi)\psi$. The discussion of logical structure in section 5 pointed to one direction this could take, but much remains to be done to show that this can be developed into a systematic theory.

A limitation of the present proposal is that it cannot deal with a variant of the argument of Fritz (unpublished). This variant derives an inconsistency from the natural principle of so-called *immediate* grounds of arbitrary conjunctions, according to which the immediate grounds of the conjunction of any truths are just those truths. Using plural quantifiers, this principle can be stated using a single sentence. There is therefore no way of addressing this problem along the present lines, by moving from a schematic to a quantified grounding principle. If the notion of forming the conjunction from an arbitrary plurality of truths makes sense without being understood as an abbreviation of some more complicated statement, then it should obey the principle of immediate grounds just stated. In this case, one may conclude that the notion of ground is not in good standing after all, present efforts to save it notwithstanding.¹

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