A high order time discretization of the solution of the non-linear filtering problem

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Abstract The solution of the continuous time filtering problem can be represented as a ratio of two expectations of certain functionals of the signal process that are parametrized by the observation path. We introduce a class of discretization schemes of these functionals of arbitrary order. The result generalizes the classical work of Picard, who introduced first order discretizations to the filtering functionals. For a given time interval partition, we construct discretization schemes with convergence rates that are proportional with the m -power of the mesh of the partition for arbitrary $m \in \mathbb{N}$. The result paves the way for constructing high order numerical approximation for the solution of the filtering problem.

Keywords Non-linear filtering · Kallianpur-Striebel's formula · high order time discretization.

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1 Introduction

Partially observed dynamical systems are ubiquitous in a multitude of real-life phenomena. The dynamical system is typically modelled by a continuous time stochastic process called the signal process X . The signal process cannot be measured directly, but only via a related process Y , called the observation process. The filtering problem is that of estimating the current state of the dynamical system at the current time given the observation data accumulated up to that time. Mathematically the problem entails computing the conditional distribution of the signal process X_t , denoted by π_t , given $\mathcal{Y}_t$, the σ -algebra generated by Y . In a few special cases, π_t can be expressed in closed form as a functional of the observation path. For example, the celebrated Kalman-Bucy filter does this in the linear case. In general, an explicit formula for π_t is not available and inferences can only be made by numerical approximations of π_t . As expected the problem has attracted a lot of attention in the last fifty years (see Chapter 8 of [2] for a survey of existing numerical methods for approximating π_t).

The basis of this class of numerical methods is the representation of π_t given by the Kallianpur–Striebel formula (see (2.2) below). In the case when the signal process is modelled by the solution of a stochastic differential equation (SDE) and the observation process is a function of the signal perturbed by white noise (see Section 2 below for further details), the formula entails the computation of expectations of functionals of the solution of the signal SDE that are parametrized by the observation path. The numerical approximation of π_t requires three procedures:

- the discretization of the functionals (corresponding to a partition of the interval $[0, t]$).
- the approximation of the law of the signal with a discrete measure.
- the control of the computational effort.

The first step is typically achieved by the discretization scheme introduced by Picard in [16]. This offers a first order approximation for the functionals appearing in formula (2.2). More precisely, the L^1 -rate of convergence of the approximation is proportional with the mesh of the partition of the time interval $[0, t]$ (see Theorem 21.5 in [4]). The second and the third step are achieved by a combination of an Euler approximation of the solution of the SDE, a Monte Carlo step that gives a sample from the law of the Euler approximation and a re-sampling step that acts as a variance reduction method and keeps the computational effort in control. There are a variety of algorithms that follow this template. Further details can be found, for instance, in

Part VII of [5]. It is worth pointing out that once the functional discretization and the Euler approximation have been applied, the problem can be reduced to one where the signal evolves and is observed in discrete time. The discrete version of the filtering problem is popular both with practitioners and with theoreticians. The majority of the existing theoretical results and the numerical algorithms are constructed and analyzed in the discrete framework. For more details, the interested reader can consult the comprehensive theoretical monograph [7] and the reference therein and the equally comprehensive methodological volume [8] and the references therein with some updates in Part VII of [5].

The first order discretization introduced by Picard creates a bottleneck: There exist higher order schemes for approximating the law of the signal that can be used, but which won't bring any substantial improvements because of this. For example, in the recent paper [6], the authors employ high order cubature methods to approximate the law of the signal with only minimal improvements due to the low order discretization of the required functionals. The aim of this paper is to address this issue. More precisely, we introduce a class of high order discretizations of the functionals. As we shall see, we prove that the L^1 -rate of convergence of the approximations is proportional with the m -power of the mesh of the partition of the time interval $[0, t]$. For details, see Theorem 2.3 below. In a work in progress, this discretization procedure is employed to produce a second order particle filter. It is hoped that this discretization will be used in conjunction with other high order approximations of the law of the signal, in particular with cubature methods. We are not aware of any other similar high order discretization schemes.

The paper is organized as follows: In Section 2 we introduce some basic definitions and state the main result of the paper, Theorem 2.3. Section 3 is devoted to prove our main result. We start by proving several auxiliary results on iterated stochastic integrals and on the integrability of the likelihood function and its discretizations. These lead to the two main results of the section, Proposition 3.12 and Proposition 3.13, from which we will deduce our main result. In Section 4 we address the most technical aspects of the paper. We first introduce some technical tools on Malliavin calculus (subsection 4.1), the Stroock-Taylor formula (subsection 4.2) and backward martingales (subsection 4.3). Then, with the aid of these tools, we prove in subsection 4.4 the estimates on the conditional expectation with respect to $\mathcal{Y}_t$ that are essential in proving Proposition 3.12.

2 Basic framework and statement of the main result

Let $(\Omega, \mathcal{F}, P)$ be a probability space together with a filtration $(\mathcal{F}_t)_{t \geq 0}$ which satisfies the usual conditions. On $(\Omega, \mathcal{F}, P)$ we consider a $d_X \times d_Y$ -dimensional partially observed system (X, Y) satisfying

$$\begin{aligned} X_t &= X_0 + \int_0^t f(X_s) ds + \int_0^t \sigma(X_s) dV_s, \\ Y_t &= \int_0^t h(X_s) ds + W_t, \end{aligned}$$

where V is a standard $\mathcal{F}_t$ -adapted d_V -dimensional Brownian motion and W is a standard $\mathcal{F}_t$ -adapted d_W -dimensional Brownian motion, independent of each other. We also assume that X_0 is a random variable independent of V and W and denote by π_0 its law. We assume that $f = (f_i)_{i=1,\dots,d_X} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_X}$ and $\sigma = (\sigma_{i,j})_{i=1,\dots,d_X, j=1,\dots,d_V} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_X \times d_V}$ are globally Lipschitz continuous. In addition, we assume that $h = (h_i)_{i=1,\dots,d_Y} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_Y}$ is measurable and has linear growth.

Let $\{\mathcal{B}_t\}_{t \geq 0}$ be the usual augmentation of the filtration generated by the process Y , that is, $\mathcal{B}_t = \sigma(Y_s, s \in [0, t]) \vee \mathcal{N}$, where $\mathcal{N}$ are all the P -null sets of $(\Omega, \mathcal{F}, P)$. We are interested in determining π_t , the conditional law of the signal X at time t given the information accumulated from observing Y in the interval $[0, t]$. More precisely, for any Borel measurable and bounded function φ , we want to compute $\pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{B}_t]$. By an application of Girsanov's theorem (see, for example, Chapter 3 in [2]) one can construct a new probability measure $\tilde{P}$, absolutely continuous with respect to P , under which Y becomes a Brownian motion independent of X and the law of X remains unchanged. The Radon-Nikodym derivative of $\tilde{P}$ with respect to P is given by the process $Z(X, Y) = (Z_t(X, Y))_{t \geq 0}$ given by

$$Z_t(X, Y) = \exp \left(\sum_{i=1}^{d_Y} \int_0^t h_i(X_s) dY_s^i - \frac{1}{2} \sum_{i=1}^{d_Y} \int_0^t h_i^2(X_s) ds \right), \quad t \geq 0, \quad (2.1)$$

which is an $\mathcal{F}_t$ -adapted martingale under $\tilde{P}$ under the assumptions introduced above. We will denote by $\tilde{\mathbb{E}}$ to be the expectation with respect to $\tilde{P}$. In the following we will make use of the measure valued process $\rho = (\rho_t)_{t \geq 0}$, defined by the formula $\rho_t(\varphi) = \tilde{\mathbb{E}}[\varphi(X_t) Z_t | \mathcal{B}_t]$, for any bounded Borel measurable function φ . The processes π and ρ are connected through the Kallianpur-Striebel's formula:

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)} = \frac{\tilde{\mathbb{E}} \left[\varphi(X_t) \exp \left(\sum_{i=1}^{d_Y} \int_0^t h_i(X_s) dY_s^i - \frac{1}{2} \sum_{i=1}^{d_Y} \int_0^t h_i^2(X_s) ds \right) \middle| \mathcal{B}_t \right]}{\tilde{\mathbb{E}} \left[\exp \left(\sum_{i=1}^{d_Y} \int_0^t h_i(X_s) dY_s^i - \frac{1}{2} \sum_{i=1}^{d_Y} \int_0^t h_i^2(X_s) ds \right) \middle| \mathcal{B}_t \right]}, \quad (2.2)$$

P -a.s., where 1 is the constant function. As a result, ρ is called the unnormalized conditional distribution of the signal. For further details on the filtering framework, see [2].

It follows from (2.2) that $\pi_t(\varphi)$ is a ratio of two conditional expectations of functionals of the signal that depend on the stochastic integrals with respect to the process Y . In the following we will introduce a class of time discretization schemes for these conditional expectations which, in turn, will generate time discretisation schemes π_t (of any order). This is the main result of the paper and is stated Theorem 2.3 below.

We first introduce some useful notation and definitions. We denote by:

- $\mathcal{B}_b$ the space of bounded Borel-measurable functions.
- $\mathcal{B}_P$ the space of Borel-measurable functions with polynomial growth.
- C_b^k the space of continuously differentiable functions up to order $k \in \mathbb{Z}_+$ with bounded derivatives of order greater or equal to one.
- C_P^k the space of continuously differentiable functions up to order $k \in \mathbb{Z}_+$ such that the function and its derivatives have at most polynomial growth.

- $L^p(\Omega, \mathcal{F}, \tilde{P})$ the space of p -integrable random variables (with respect to $\tilde{P}$) and denote by $\|\cdot\|_p$ the corresponding norm on $L^p(\Omega, \mathcal{F}, \tilde{P})$, i.e., for $\xi \in L^p(\Omega, \mathcal{F}, \tilde{P})$, $\|\xi\|_p \triangleq \tilde{\mathbb{E}}[|\xi|^p]^{1/p}$.

Let $\mathcal{M}(0, \dots, d_V)$ denote the set of all multi-indices with values in $\{0, \dots, d_V\}$, including the multi-index of length zero, which we will denote by $\emptyset$. For $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathcal{M}(0, \dots, d_V)$ define the following operations

$$\begin{aligned} |\alpha| &\triangleq k, \\ |\alpha|_0 &\triangleq \#\{j : \alpha_j = 0, j = 1, \dots, k\}, \\ \alpha - &\triangleq (\alpha_1, \dots, \alpha_{k-1}), \\ -\alpha &\triangleq (\alpha_2, \dots, \alpha_k), \end{aligned}$$

where $|\emptyset| = 0$, $-\emptyset = \emptyset - \emptyset = \emptyset$. If $|\alpha| = k$, we say that the multi-index α has order k . Given two multi-indices $\alpha, \beta \in \mathcal{M}(0, \dots, d_V)$ we denote its concatenation by $\alpha * \beta$. For $n, m \in \mathbb{N}_0$, we will also consider the following subsets of multi-indices

$$\begin{aligned} \mathcal{M}_{n,m}(0, \dots, d_V) &\triangleq \{\alpha \in \mathcal{M}(0, \dots, d_V) : 0 \leq |\alpha| \leq m\}, \\ \mathcal{M}_m(0, \dots, d_V) &\triangleq \mathcal{M}_{m,m}(0, \dots, d_V) = \{\alpha \in \mathcal{M}(0, \dots, d_V) : |\alpha| = m\}, \end{aligned}$$

that is, the subsets of multi-indices of order less or equal than m and of order m , respectively.

To simplify the notation, it is convenient to add an additional component to the Brownian motion V . Let $V_s^0 = s$, for all $s \geq 0$ and consider the $(d_V + 1)$ -dimensional process $V = (V^i)_{i=0}^{d_V}$. We will consider the filtration $\{\mathcal{F}_t^{0,V}\}_{t \geq 0}$ defined to be the usual augmentation of the filtration generated by the process V and initially enlarged with the random variable X_0 . Moreover, for fixed t , we will also consider the filtration $\{\mathcal{H}_s^t \triangleq \mathcal{F}_s^{0,V} \vee \mathcal{G}_t\}_{s \geq 0}$. For $\alpha \in \mathcal{M}(0, \dots, d_V)$, denote by $I_\alpha(h)_{s,t}$ the following Itô iterated integral

$$I_\alpha(h)_{s,t} = \begin{cases} h_t & \text{if } |\alpha| = 0 \\ \int_s^t I_{\alpha-}(h)_{s,u} dV_u^{\alpha_{|\alpha|}} & \text{if } |\alpha| \geq 1 \end{cases},$$

where $h = \{h_s\}_{s \geq 0}$ is an $\mathcal{H}_s^t$ -adapted process (satisfying appropriate integrability conditions). We introduce the differential operators $L^0, L^r, r = 1, \dots, d_V$ defined by

$$\begin{aligned} L^0 g(x) &\triangleq \sum_{k=1}^{d_X} f^k(x) \frac{\partial g}{\partial x^k}(x) + \frac{1}{2} \sum_{k,l=1}^{d_X} \sum_{r=1}^{d_V} \sigma_{k,r}(x) \sigma_{l,r}(x) \frac{\partial^2 g}{\partial x^k \partial x^l}(x), \\ L^r g(x) &\triangleq \sum_{k=1}^{d_X} \sigma_{k,r}(x) \frac{\partial g}{\partial x^k}(x), \quad r = 1, \dots, d_V, \end{aligned}$$

where $g : \mathbb{R}^{d_X} \rightarrow \mathbb{R}$ belongs to $C_P^2(\mathbb{R}^{d_X}; \mathbb{R})$. For $\alpha \in \mathcal{M}(0, \dots, d_V)$, with $\alpha = (\alpha_1, \dots, \alpha_k)$, and the differential operator L^α is defined by

$$L^\alpha g = L^{\alpha_1} \circ L^{\alpha_2} \circ \dots \circ L^{\alpha_k} g,$$

and, by convention $L^\emptyset g = g$. Finally, let $\tau \triangleq \{0 = t_0 < \dots < t_i < \dots < t_n = t\}$ be a partition of $[0, t]$. Associated to τ we define the following elements

$$\begin{aligned}\delta_i &\triangleq t_i - t_{i-1}, \quad i = 1, \dots, n, \\ \delta &\triangleq \max_{i=1, \dots, n} \delta_i, \\ \delta_{\min} &\triangleq \min_{i=1, \dots, n} \delta_i, \\ \tau(s) &\triangleq t_{i-1}, \quad s \in [t_{i-1}, t_i], i = 1, \dots, n, \\ \eta(s) &\triangleq t_i, \quad s \in [t_{i-1}, t_i], i = 1, \dots, n.\end{aligned}$$

We will only consider partitions satisfying the following condition

$$\delta \leq C\delta_{\min}, \quad (2.3)$$

for some finite constant $C \geq 1$. We denote by $\Pi(t)$ the set of all partitions of $[0, t]$ satisfying (2.3) and such that δ converges to zero when n tends to infinity. We denote by $\Pi(t, \delta_0)$ the set of all partitions of $[0, t]$ satisfying (2.3), such that δ converges to zero when n tends to infinity and $\delta < \delta_0$.

Remark 2.1 Under the assumption (2.3) one has that

$$n \leq t\delta_{\min}^{-1} \leq Ct\delta^{-1}. \quad (2.4)$$

To simplify the notation, we will add an additional component to the Brownian motion Y . Let Y^0 be the process $Y_s^0 = s$, for all $s \geq 0$ and consider the $(d_Y + 1)$ -dimensional process $Y = (Y^i)_{i=0}^{d_Y}$. Then the martingale $Z = (Z_t)_{t \geq 0}$ defined in (2.1) can be written as $Z_t = \exp(\xi_t)$, $t \geq 0$, where

$$\xi_t = \sum_{i=0}^{d_Y} \int_0^t h_i(X_s) dY_s^i, \quad t \geq 0,$$

and $h_0 = -\frac{1}{2} \sum_{i=1}^{d_Y} h_i^2$. For $\tau \in \Pi(t)$ and $m \in \mathbb{N}$ we consider the processes

$$\begin{aligned}\xi_t^{\tau, m} &\triangleq \sum_{j=0}^{n-1} \xi_t^{\tau, m}(j) \triangleq \sum_{j=0}^{n-1} \sum_{i=0}^{d_Y} \sum_{\alpha \in \mathcal{M}_{0, m-1}(0, \dots, d_Y)} L^\alpha h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} I_\alpha(1)_{t_j, s} dY_s^i \\ &= \sum_{i=0}^{d_Y} \int_0^t \left\{ \sum_{\alpha \in \mathcal{M}_{0, m-1}(0, \dots, d_Y)} L^\alpha h_i(X_{\tau(s)}) I_\alpha(1)_{\tau(s), s} \right\} dY_s^i.\end{aligned}$$

For $m > 2$, we can write

$$\xi_t^{\tau, m} = \xi_t^{\tau, 2} + \sum_{j=0}^{n-1} \mu^{\tau, m}(j),$$

where

$$\mu^{\tau, m}(j) \triangleq \sum_{i=0}^{d_Y} \sum_{\alpha \in \mathcal{M}_{2, m-1}(0, \dots, d_Y)} L^\alpha h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} I_\alpha(1)_{t_j, s} dY_s^i. \quad (2.5)$$

The processes $\xi^{\tau,m}$ are obtained by replacing $h_i(X_s)$ in the formula for the process ξ with the truncation of degree $(m-1)$ of the corresponding stochastic Taylor expansion of $h_i(X_s)$. They are used to produce discretization schemes of order 1 and 2 for $\pi_t(\varphi)$. They *cannot* be used to produce discretization schemes of order $m > 2$ as they don't have finite exponential moments (required to define the discretization schemes). More precisely, the quantities $\mu^{\tau,m}(j)$ do not have finite exponential moments because of the high order iterated integral involved. For this, we need to introduce a truncation of $\mu^{\tau,m}(j)$ resulting in a (partial) taming procedure to the stochastic Taylor expansion of $h_i(X)$. We define the processes

$$\bar{\xi}_t^{\tau,m} \triangleq \sum_{j=0}^{n-1} \bar{\xi}_t^{\tau,m}(j),$$

where

$$\bar{\xi}_t^{\tau,i}(j) = \begin{cases} \xi_t^{\tau,i}(j) & \text{if } i = 1, 2 \\ \xi_t^{\tau,2}(j) + \Gamma_{m,\delta_j}(\mu^{\tau,m}(j)) & \text{if } i > 2 \end{cases}, \quad j = 0, \dots, n-1$$

with the truncation function Γ being defined as

$$\Gamma_{q,\delta}(z) = \frac{z}{1 + (z/\delta)^{2q}}, \quad z \in \mathbb{R}$$

for some $\delta > 0$ and $q \in \mathbb{N}$. Finally, for $\tau \in \Pi(t)$ and $m \in \mathbb{N}$ consider the processes $Z^{\tau,m} = (Z_t^{\tau,m})_{t \geq 0}$ given by

$$Z_t^{\tau,m} = \exp(\bar{\xi}_t^{\tau,m}). \quad (2.6)$$

For any Borel measurable function φ such that $\varphi(X_t)Z_t^{\tau,m} \in L^1(\Omega, \mathcal{F}, \tilde{P})$ define the m -th order discretizations

$$\rho_t^{\tau,m}(\varphi) \triangleq \tilde{\mathbb{E}}[\varphi(X_t)Z_t^{\tau,m} | \mathcal{Y}_t],$$

and

$$\pi_t^{\tau,m}(\varphi) \triangleq \rho_t^{\tau,m}(\varphi) / \rho_t^{\tau,m}(1),$$

of ρ_t and π_t , respectively.

Let $m \in \mathbb{N}$, our main assumption is the following:

Assumption 2.2 ($\mathbf{H}(m)$) We have that:

- $f = (f_i)_{i=1,\dots,d_X} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_X} \in \mathcal{B}_b \cap C_b^{2 \vee (2m-1)}$,
- $\sigma = (\sigma_{i,j})_{i=1,\dots,d_X, j=1,\dots,d_V} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_X \times d_V} \in \mathcal{B}_b \cap C_b^{2m}$,
- $h = (h_i)_{i=0,\dots,d_Y} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_Y+1} \in \mathcal{B}_b \cap C_b^{2m+1}$,
- X_0 has moments of all orders.

Note that if assumption $\mathbf{H}(m)$ holds for some $m \in \mathbb{N}$, then it also holds for any $n \leq m$.

Theorem 2.3 *Let assumption $\mathbf{H}(m)$ be satisfied. Then, there exists constants $\delta_0, C > 0$ not depending on the choice of the partition $\tau \in \Pi(t, \delta_0)$, such that*

$$\tilde{\mathbb{E}} \left[\left| \rho_t(\varphi) - \rho_t^{\tau, m}(\varphi) \right|^2 \right] \leq C \delta^{2m},$$

for $\varphi \in C_P^{m+1}$. Moreover, if $\sup_{\tau \in \Pi(t, \delta_0)} \tilde{\mathbb{E}} \left[\left| \pi_t^{\tau, m}(\varphi) \right|^{2+\varepsilon} \right] < \infty$, for some $\varepsilon > 0$, then

$$\mathbb{E} \left[\left| \pi_t(\varphi) - \pi_t^{\tau, m}(\varphi) \right| \right] \leq \bar{C} \delta^m,$$

where $\bar{C}$ is another constant independent of $\tau \in \Pi(t, \delta_0)$.

Remark 2.4 The assumption $\sup_{\tau \in \Pi(t, \delta_0)} \tilde{\mathbb{E}} \left[\left| \pi_t^{\tau, m}(\varphi) \right|^{2+\varepsilon} \right] < \infty$ for some $\varepsilon > 0$ is satisfied if φ is bounded. If φ is unbounded one could reason as in Lemma 3.10 to justify this assumption.

Remark 2.5

i. In the case $m = 1$ we can consider any partition $\tau \in \Pi(t)$. For $m \geq 2$, we must consider partitions τ with mesh δ smaller than

$$\delta_0 = \frac{1}{2 \|Lh\|_\infty \sqrt{d_Y d_V}}, \quad (2.7)$$

where

$$\|Lh\|_\infty \triangleq \max_{i=1, \dots, d_Y} \max_{r=1, \dots, d_V} \|L^r h^i\|_\infty.$$

ii. The functional discretization given in (2.6) is recursive. More precisely, if $\tau' \in \Pi(t+s)$ is a partition that includes t as an intermediate point, for example

$$\tau' \triangleq \{0 = t_0 < \dots < t_k = t < t_{k+1} < \dots < t_n = t+s\},$$

with $0 < k < n$, then

$$Z_{t+s}^{\tau', m} = Z_t^{\tau, m} \prod_{j=k}^{n-1} \exp(\tilde{\xi}_t^{\tau, m}(j)).$$

This property is essential for implementation purposes as at every discretization time we only need to use the previous functional discretization and the term corresponding to the next interval to obtain the new functional discretization.

iii. The discretization introduced by Picard in [16] corresponds to the case $m = 1$. In this case, $\rho_t^{\tau, m}$ can be explicitly written as

$$\rho_t^{\tau, m}(\varphi) \triangleq \tilde{\mathbb{E}} \left[\varphi(X_t) \exp \left(\sum_{j=0}^{n-1} \sum_{i=0}^{d_Y} h_i(X_{t_j}) (Y_{t_{j+1}}^i - Y_{t_j}^i) \right) \middle| \mathcal{H}_t \right], \quad (2.8)$$

This discretization scheme leads to a wealth of numerical methods that can be used to approximate π_t . Among them, particle methods¹ are algorithms which approximate π_t with discrete random measures of the form $\sum_i a_i(t) \delta_{v_i(t)}$, in other words with

¹ Also known as *particle filters* or *sequential Monte Carlo methods*.

empirical distributions associated with sets of randomly located particles of stochastic masses $a_1(t), a_2(t), \dots$, which have stochastic positions $v_1(t), v_2(t), \dots$. These methods are currently among the most successful and versatile for numerically solving the filtering problem. Based on (2.8), the garden-variety particle filter uses particles that evolve according to the signal equation (or, rather, the Euler approximation of the signal) and carry exponential weights. These weights are proportional with

$$\exp \left(\sum_{i=0}^{d_Y} h_i(v_{t_n}^i) (Y_{t_{n+1}}^i - Y_{t_n}^i) \right),$$

where v^j is the process modelling the trajectory of the particle and t_n is the update time. The method also involves a variance reduction procedure (for further details, see for example Chapter 9 in [2]). Alternatively one can use a cubature method to approximate the law of the signal, see [6]. In both cases, higher order approximations of the signal can be used, but this would not improve the rate of convergence of the method as Picard's discretisation has an error of order 1. The remedy is to exploit the result in this paper and use a higher order discretisation. The second author is working on a particle filter that uses the second order discretisation presented in this paper.

Remark 2.6 There is an alternative methodology to approximate $\rho_t(\varphi)$. To explain this let us assume that $d_Y = d_X = 1$. Using integration by parts and the independence of X and Y under $\tilde{P}$, we deduce that

$$\rho_t(\varphi) = \tilde{\mathbb{E}} \left[\varphi(X_t) e^{h(X_t)Y_t} \exp \left(- \int_0^t Y_s L^1 h(X_s) dV_s - \int_0^t \left(Y_s L^0 h(X_s) + \frac{1}{2} h^2(X_s) \right) ds \right) \middle| \mathcal{Y}_t \right]$$

One then uses the change of probability measure

$$\frac{dQ}{d\tilde{P}} \bigg|_{\mathcal{F}_t} = \hat{Z}_t := \exp \left(- \int_0^t Y_s L^1 h(X_s) dV_s - \frac{1}{2} \int_0^t (Y_s L^1 h(X_s))^2 ds \right)$$

to obtain that

$$\rho_t(\varphi) = \mathbb{E}^Q \left[\varphi(X_t) e^{h(X_t)Y_t + S_t} \middle| \mathcal{Y}_t \right]$$

where

$$\begin{aligned} dX_t &= (f(X_t) + Y_t L^1 h(X_t) \sigma(X_t)) dt + \sigma(X_t) dV_t \\ dS_t &= \left(\frac{1}{2} (Y_t L^1 h(X_t))^2 - Y_t L^0 h(X_t) - \frac{1}{2} h^2(X_t) \right) dt \end{aligned}$$

One then can attempt to approximate $\rho_t(\varphi)$ by using a standard discretisation scheme (possibly high order one) that will produce a pair $(\tilde{X}, \tilde{S})$ of approximating processes for (X, S) and then compute $\mathbb{E}^Q \left[\varphi(\tilde{X}_t) e^{h(\tilde{X}_t)Y_t + \tilde{S}_t} \middle| \mathcal{Y}_t \right]$ for example via a (sequential) Monte Carlo method. The main difficulty with this approach is that the equation for X now depends on the time dependent path Y , which is not smooth/Lipschitz in the time variable and therefore the rate of convergence of any discretisation of X will be affected by this, see for instance [19] and [11].

3 Proof of the main result

We start by recalling and introducing some basic results on iterated integrals and martingale representations. Throughout the rest of the paper we will be assuming that $\mathbf{H}(m)$ holds, without recalling it in each result statement. Moreover, C will denote a constant that usually depends on d_V , d_X , d_Y , f , σ , h and possibly other parameters but NOT on the partition τ . As we are interested in showing a rate of convergence for our approximations, the particular form of dependence of C with respect to these parameters is not relevant and, hence, omitted. Of course, the choice of the constant C may change from line to line.

Remark 3.1 Some immediate consequences of assumption $\mathbf{H}(m)$ are the following:

1. The signal process X has moments of all orders and for any $p \geq 1$, we have

$$\tilde{\mathbb{E}} \left[\sup_{s \in [0, t]} |X_s^i|^p \right] < \infty,$$

for all $i \in \{1, \dots, d_X\}$.

2. If $\Upsilon : \mathbb{R}^{d_X} \rightarrow \mathbb{R}$ is a function with polynomial growth we have

$$\tilde{\mathbb{E}} \left[\sup_{s \in [0, t]} |\Upsilon(X_s)|^p \right] < \infty,$$

in particular,

$$\tilde{\mathbb{E}} \left[\sup_{s \in [0, t]} |L^\alpha h_i(X_s)|^p \right] < \infty,$$

for $i = 0, \dots, d_Y$ and $\alpha \in \mathcal{M}_{0, m}(0, \dots, d_V)$.

3. The processes ξ_t and $\xi_t^{\tau, m}$, $m \in \mathbb{N}$, as defined above have finite moments of all orders.

Remark 3.2 Consider the truncation function

$$\Gamma_{q, \delta}(z) = \frac{z}{1 + (z/\delta)^{2q}},$$

where $q \in \mathbb{N}$ and $\delta > 0$.

1. For any $z \in \mathbb{R}$, $|\Gamma_{q, \delta}(z)| \leq \delta$. To check this observe that if $|z| \leq \delta$ we have that $1 + (z/\delta)^{2q} \geq 1$ and then

$$|\Gamma_{q, \delta}(z)| = \frac{|z|}{1 + (z/\delta)^{2q}} \leq |z| \leq \delta.$$

On the other hand, if $|z| > \delta$ we have that $|z/\delta|^{-1} + |z/\delta|^{2q-1} > 1$ and then

$$|\Gamma_{q, \delta}(z)| = \frac{|z|}{1 + |z|^{2q} \delta^{-2q}} = \frac{1}{|z|^{-1} + |z|^{2q-1} \delta^{-2q}} = \frac{\delta}{|z/\delta|^{-1} + |z/\delta|^{2q-1}} \leq \delta.$$

2. Moreover, if we define

$$\mathcal{E}_{q,\delta}(z) \triangleq \Gamma_{q,\delta}(z) - z, \quad (3.1)$$

we get that

$$\begin{aligned} |\mathcal{E}_{q,\delta}(z)| &= |\Gamma_{q,\delta}(z) - z| = \left| \frac{z}{1 + (z/\delta)^{2q}} - z \right| = \frac{|z|^{2q+1} \delta^{-2q}}{1 + (z/\delta)^{2q}} = \frac{|z|^{2q+1}}{\delta^{2q} + z^{2q}}, \\ &= \delta^{-2q} \frac{|z|^{2q+1}}{1 + (z/\delta)^{2q}} \leq \delta^{-2q} |z|^{2q+1}, \quad \forall z \in \mathbb{R}. \end{aligned} \quad (3.2)$$

3. Finally, note that for $m \geq 3$ we can write

$$\begin{aligned} \bar{\xi}_t^{\tau,m} &\triangleq \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \Gamma_{m,\delta_j}(\mu^{\tau,m}(j)) \\ &= \xi_t^{\tau,m} + \sum_{j=0}^{n-1} \mathcal{E}_{m,\delta_j}(\mu^{\tau,m}(j)) \end{aligned} \quad (3.3)$$

3.1 Iterated integrals

In this section we introduce Itô-Taylor expansions and some estimates related to iterate integrals. Itô-Taylor expansions are usually done with particular subsets of multi-indices, the so called hierarchical sets. We call a subset $\mathcal{A} \subset \mathcal{M}(0, \dots, d_V)$ a hierarchical set if $\mathcal{A}$ is nonempty, $\sup_{\alpha \in \mathcal{A}} |\alpha| < \infty$, and

$$-\alpha \in \mathcal{A} \quad \text{if} \quad \alpha \in \mathcal{A} \setminus \{\emptyset\},$$

where $\emptyset$ is the multi-index of length zero. For any given hierarchical set $\mathcal{A}$ we define the remainder set $\mathcal{R}(\mathcal{A})$ of $\mathcal{A}$ by

$$\mathcal{R}(\mathcal{A}) \triangleq \{\alpha \in \mathcal{M}(0, \dots, d_V) \setminus \mathcal{A} : -\alpha \in \mathcal{A}\}.$$

One can check that

$$\mathcal{M}_{0,m}(0, \dots, d_V) = \{\alpha \in \mathcal{M}(0, \dots, d_V) : 0 \leq |\alpha| \leq m\},$$

is a hierarchical set and that its associated remainder set is given by

$$\mathcal{R}(\mathcal{M}_{0,m}(0, \dots, d_V)) = \mathcal{M}_{m+1}(0, \dots, d_V) = \{\alpha \in \mathcal{M}(S) : |\alpha| = m+1\}.$$

The following two results are well known and can be found in Kloeden and Platen [10], Theorem 5.5.1 and Lemma 5.7.5, respectively.

Theorem 3.3 *Let ρ_1 and ρ_2 be two stopping times with $0 \leq \rho_1 \leq \rho_2 \leq t$, a.s., let $\mathcal{A} \subset \mathcal{M}(0, \dots, d_V)$ be a hierarchical set and $g : \mathbb{R}^{d_X} \rightarrow \mathbb{R}$. Then, the Itô-Taylor expansion*

$$g(X_{\rho_2}) = \sum_{\alpha \in \mathcal{A}} L^\alpha g(X_{\rho_1}) I_\alpha(1)_{\rho_1, \rho_2} + \sum_{\alpha \in \mathcal{R}(\mathcal{A})} I_\alpha(L^\alpha g(X))_{\rho_1, \rho_2}, \quad (3.4)$$

holds, provided all of the derivatives of g, f and σ and all of the iterated Itô integrals appearing in (3.4) exist.

Example 3.4 For the hierarchichal set

$$\mathcal{A} = \mathcal{M}_{0,0}(0, \dots, d_V) = \{\alpha \in \mathcal{M}(0, \dots, d_V) : |\alpha| = 0\} = \emptyset,$$

we have

$$\mathcal{R}(\mathcal{A}) = \mathcal{M}_1(0, \dots, d_V) = \{\alpha \in \mathcal{M}(0, \dots, d_V) : |\alpha| = 1\},$$

and Theorem 3.3 coincides with the Itô formula, that is,

$$g(X_{\rho_2}) = g(X_{\rho_1}) + \int_{\rho_1}^{\rho_2} L^0 g(X_s) ds + \sum_{r=1}^{d_V} \int_{\rho_1}^{\rho_2} L^r g(X_s) dV_s^r.$$

For the hierarchichal set

$$\mathcal{A} = \mathcal{M}_{0,1}(0, \dots, d_V) = \{\alpha \in \mathcal{M}(0, \dots, d_V) : 0 \leq |\alpha| \leq 1\},$$

we have

$$\mathcal{R}(\mathcal{A}) = \mathcal{M}_2(0, \dots, d_V) = \{\alpha \in \mathcal{M}(0, \dots, d_V) : |\alpha| = 2\},$$

and Theorem 3.3 reads as

$$\begin{aligned} g(X_{\rho_2}) &= g(X_{\rho_1}) + L^0 g(X_{\rho_1})(\rho_2 - \rho_1) + \sum_{r=1}^{d_V} L^r g(X_{\rho_1})(V_{\rho_2}^r - V_{\rho_1}^r) \\ &\quad + \sum_{r_1, r_2=0}^{d_V} \int_{\rho_1}^{\rho_2} \int_{\rho_1}^s L^{(r_1, r_2)} g(X_u) dV_u^{r_1} dV_s^{r_2}. \end{aligned}$$

Lemma 3.5 Let $\alpha \in \mathcal{M}(0, \dots, d_V)$, let $\theta = \{\theta_s\}_{0 \leq s \leq t}$ be an $\mathcal{H}_s^t$ -adapted process, let $p \geq 1$ and let ρ_1 and ρ_2 be two stopping times with $0 \leq \rho_1 \leq \rho_2 \leq t$ and ρ_2 being $\mathcal{H}_{\rho_1}^t$ -measurable. Then,

$$\mathbb{E} \left[\left| I_\alpha(\theta)_{\rho_1, \rho_2} \right|^{2p} \middle| \mathcal{H}_{\rho_1}^t \right] \leq CR(\theta)(\rho_2 - \rho_1)^{p(|\alpha| + |\alpha|_0)},$$

where

$$R(\theta) = \mathbb{E} \left[\sup_{s \in [\rho_1, \rho_2]} |\theta_s|^{2p} \middle| \mathcal{H}_{\rho_1}^t \right].$$

The following lemma gives a basic estimate on the difference between the log likelihood functional ξ_t and its m -th order discretization. Its proof relies on Theorem 3.3 and Lemma 3.5.

Lemma 3.6 We have that

$$\xi_t - \xi_t^{\tau, m} = \sum_{i=0}^{d_Y} \int_0^t \left\{ \sum_{\alpha \in \mathcal{M}_m(0, \dots, d_V)} I_\alpha(L^\alpha h_i(X))_{\tau(s), s} \right\} dY_s^i, \quad (3.5)$$

and

$$\mathbb{E} \left[\left| \xi_t - \xi_t^{\tau, m} \right|^{2p} \right] \leq C \delta^{pm}.$$

Proof By Remark 3.1, we can apply Theorem 3.3 and get equation 3.5. Applying the Itô isometry and Jensen's inequality (or Jensen's inequality directly if $i = 0$), we obtain the following bound

$$\begin{aligned}\tilde{\mathbb{E}} \left[|\xi_t - \xi_t^{\tau, m}|^{2p} \right] &= \tilde{\mathbb{E}} \left[\left| \sum_{i=0}^{d_Y} \int_0^t \left\{ \sum_{\alpha \in \mathcal{M}_m(0, \dots, d_V)} I_\alpha (L^\alpha h_i(X))_{\tau(s), s} \right\} dY_s^i \right|^{2p} \right] \\ &\leq C \sum_{\alpha \in \mathcal{M}_m(0, \dots, d_V)} \int_0^t \tilde{\mathbb{E}} \left[|I_\alpha (L^\alpha h_i(X))_{\tau(s), s}|^{2p} \right] ds.\end{aligned}$$

Let $\alpha \in \mathcal{M}_m(0, \dots, d_V)$, by Lemma 3.5 and Remark 3.1 we get that

$$\tilde{\mathbb{E}} \left[|I_\alpha (L^\alpha h_i(X))_{\tau(s), s}|^{2p} \right] \leq C \tilde{\mathbb{E}} \left[\sup_{u \in [\tau(s), s]} |L^\alpha h_i(X.)|^{2p} \right] (s - \tau(s))^{p(|\alpha| + |\alpha|_0)} \leq C \delta^{pm},$$

where in the last inequality we have used that $|\alpha| + |\alpha|_0 \geq m$ for $\alpha \in \mathcal{M}_m(0, \dots, d_V)$. From the previous inequality the result follows easily.

Lemma 3.7 *Let $p \geq 1$ and integers $q \geq 1$ and $m \geq 3$. Then, we have that*

$$\tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \mathcal{E}_{q, \delta_j}(\mu^{\tau, m}(j)) \right|^{2p} \right] \leq C \delta^{p(2q+1)},$$

where $\mathcal{E}_{q, \delta_j}(z)$ is defined in (3.1) and $\mu^{\tau, m}(j)$ is defined in (2.5).

Proof We have that

$$\tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \mathcal{E}_{q, \delta_j}(\mu^{\tau, m}(j)) \right|^{2p} \right] \leq C n^{2p-1} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left| \mathcal{E}_{q, \delta_j}(\mu^{\tau, m}(j)) \right|^{2p} \right].$$

Moreover, using similar arguments as in Lemma 3.6, for any $r \geq 1$, we have that

$$\begin{aligned}\tilde{\mathbb{E}} \left[|\mu^{\tau, m}(j)|^r \right] &\leq C \delta^{\frac{r}{2}(1 + |\alpha| + |\alpha|_0)} \\ &\leq C \delta^{\frac{3}{2}r},\end{aligned}$$

because as $\alpha \in \mathcal{M}_{2, m-1}(0, \dots, d_V)$, $m \geq 3$ then $|\alpha| \in [2, m-1]$ and $|\alpha|_0 \in [0, m-1]$. Then, using equation (3.2) and Remark 2.1 we get that

$$\begin{aligned}n^{2p-1} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left| \mathcal{E}_{q, \delta_j}(\mu^{\tau, m}(j)) \right|^{2p} \right] &\leq n^{2p-1} \sum_{j=0}^{n-1} \delta^{-4pq} \tilde{\mathbb{E}} \left[|\mu^{\tau, m}(j)|^{2p(2q+1)} \right] \\ &\leq C n \delta^{-(2p-1)} \delta^{-4pq} \delta^{3p(2q+1)} \\ &\leq C \delta^{p(2q+1)}.\end{aligned}$$

Lemma 3.8 *Let $\theta = \{\theta_s\}_{0 \leq s \leq t}$ and $\Psi = \{\Psi_s\}_{0 \leq s \leq t}$ be two $\mathcal{H}_s^t$ -adapted process. Then:*

1. For $\alpha \in \mathcal{M}(0, \dots, d_V)$ and $0 \leq s_1 \leq s_2 \leq s_3 \leq t$, we have that

$$\tilde{\mathbb{E}}[I_\alpha(\theta)_{s_2, s_3} | \mathcal{H}_{s_1}^t] = 1_{\{|\alpha| = |\alpha|_0\}} I_\alpha(\tilde{\mathbb{E}}[\theta | \mathcal{H}_{s_1}^t])_{s_2, s_3}.$$

2. For $\alpha \in \mathcal{M}(0, \dots, d_V)$ with $|\alpha| \neq |\alpha|_0$, $r \in \{1, \dots, d_V\}$, $0 \leq s_1 \leq s_2 \leq t$ and $0 \leq s_3 \leq s_4 \leq t$ we have that

$$\begin{aligned} \tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right) I_{\alpha-}(\theta)_{s_1, s_2} \middle| \mathcal{Y}_t\right] &= 1_{\{\alpha_{|\alpha|} = 0\}} \int_{s_1}^{s_2} \tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right) I_{\alpha-}(\theta)_{s_1, u} \middle| \mathcal{Y}_t\right] du \\ &\quad + 1_{\{\alpha_{|\alpha|} = r\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_4} \tilde{\mathbb{E}}[\Psi_u I_{\alpha-}(\theta)_{s_1, u} | \mathcal{Y}_t] du. \end{aligned}$$

3. For $\alpha \in \mathcal{M}(0, \dots, d_V)$ with $|\alpha| \geq 2$, $\alpha_{|\alpha|} \neq 0$, $\alpha_{|\alpha|-1} \neq 0$, $r_1, r_2 \in \{1, \dots, d_V\}$, $0 \leq s_1 \leq s_2 \leq t$ and $0 \leq s_3 \leq s_4 \leq s_5 \leq s_6 \leq t$ we have that

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_6} \int_{s_3}^{s_5} \Psi_{s_4} dV_{s_4}^{r_1} dV_{s_5}^{r_2}\right) I_{\alpha}(\theta)_{s_1, s_2} \middle| \mathcal{Y}_t\right] \\ &= 1_{\{\alpha_{|\alpha|} = r_2\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_6} \tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_5} \Psi_{s_4} dV_{s_4}^{r_1}\right) I_{\alpha-}(\theta)_{s_1, u} \middle| \mathcal{Y}_t\right] du \\ &= 1_{\{\alpha_{|\alpha|} = r_2, \alpha_{|\alpha|-1} = r_1\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_6} \int_{s_1 \vee s_3}^u \tilde{\mathbb{E}}[\Psi_v I_{(\alpha)-}(\theta)_{s_1, v} | \mathcal{Y}_t] dv du \end{aligned}$$

Proof

1. If $|\alpha| \neq |\alpha|_0$, then the iterated integral $I_\alpha(\theta)_{s_2, s_3}$ contains a Brownian differential dV^r and it vanishes when we take the conditional expectation with respect to $\mathcal{H}_{s_1}^t$. If $|\alpha| = |\alpha|_0$, all the differentials in the iterated integral $I_\alpha(\theta)_{s_2, s_3}$ are Lebesgue differentials and we can write the conditional expectation inside the inner integral.

2. Note that if $\alpha_{|\alpha|} = 0$ we can write

$$\tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right) \int_{s_2}^{s_3} I_{\alpha-}(\theta)_{s_1, u} du \middle| \mathcal{Y}_t\right] = \tilde{\mathbb{E}}\left[\int_{s_2}^{s_3} \left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right) I_{\alpha-}(\theta)_{s_2, u} du \middle| \mathcal{Y}_t\right],$$

because we can push the random variable $\left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right)$ inside the Lebesgue integral.

If $\alpha_{|\alpha|} \neq 0$ we can write

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right) \int_{s_1}^{s_2} I_{\alpha-}(\theta)_{s_1, u} dV_u^{\alpha_{|\alpha|}} \middle| \mathcal{Y}_t\right] \\ &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[\left(\int_{s_3}^{s_4} \Psi_s dV_s^r\right) \int_{s_1}^{s_2} I_{\alpha-}(\theta)_{s_1, u} dV_u^{\alpha_{|\alpha|}} \middle| \mathcal{H}_0^t\right] \middle| \mathcal{Y}_t\right], \\ &= 1_{\{\alpha_{|\alpha|} = r\}} \int_{s_1 \vee s_3}^{s_2 \wedge s_4} \tilde{\mathbb{E}}[\Psi_u I_{\alpha-}(\theta)_{s_1, u} | \mathcal{Y}_t] du \end{aligned}$$

where we have just applied the $\mathcal{H}_s^t$ -semimartingale covariation formula.

3. Same reasoning as for statement 2.

3.2 Integrability of the likelihood functional and its discretizations

In this section we state some integrability results for the likelihood functional and its discretizations. The first result is on the integrability of the likelihood functional. It follows from the basic fact that any Gaussian distribution has exponential moments of all orders.

Lemma 3.9 *Assume that $\mathbf{H}(1)$ holds. Let $p \geq 1$ and τ be any partition. Then, one has that*

$$\tilde{\mathbb{E}}[\exp(p|\xi_t|)] < \infty,$$

and

$$\tilde{\mathbb{E}}[\exp(p\xi_t^{\tau,1})] < \infty.$$

Proof We have that

$$\begin{aligned} \tilde{\mathbb{E}}[\exp(p|\xi_t|)] &= \tilde{\mathbb{E}}\left[\exp\left(\left|p\sum_{i=1}^{d_Y}\int_0^t h_i(X_s)dY_s^i - \frac{p}{2}\sum_{i=1}^{d_Y}\int_0^t h_i^2(X_s)ds\right|\right)\right] \\ &\leq \exp\left(\frac{p}{2}d_Y\|h\|_\infty^2 t\right) \tilde{\mathbb{E}}\left[\exp\left(p\sum_{i=1}^{d_Y}\left|\int_0^t h_i(X_s)dY_s^i\right|\right)\right]. \end{aligned}$$

Recall that if $Z \sim \mathcal{N}(0, \sigma^2)$ under $\tilde{P}$, then

$$\tilde{\mathbb{E}}[e^{p|Z|}] = 2e^{\frac{p^2\sigma^2}{2}}\Phi(p\sigma),$$

where Φ is the cumulative distribution function of a standard normal random variable. As Y is a Brownian motion independent of X under $\tilde{P}$, we have that

$$\begin{aligned} &\tilde{\mathbb{E}}\left[\exp\left(p\sum_{i=1}^{d_Y}\left|\int_0^t h_i(X_s)dY_s^i\right|\right)\right] \\ &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[\exp\left(p\sum_{i=1}^{d_Y}\left|\int_0^t h_i(X_s)dY_s^i\right|\right)\middle|\mathcal{F}_t^{0,Y}\right]\right] \\ &= \tilde{\mathbb{E}}\left[2\exp\left(\frac{p^2}{2}\sum_{i=1}^{d_Y}\int_0^t h_i(X_s)^2 ds\right)\Phi\left(p\left(\sum_{i=1}^{d_Y}\int_0^t h_i(X_s)^2 ds\right)^{1/2}\right)\right] \\ &\leq 2\exp\left(\frac{p^2}{2}d_Y\|h\|_\infty^2 t\right), \end{aligned}$$

and we can conclude that $\tilde{\mathbb{E}}[\exp(p|\xi_t|)] < \infty$. The proof that $\tilde{\mathbb{E}}[\exp(p\xi_t^{\tau,1})] < \infty$ follows by similar arguments.

The following lemma ensures the $L^p(\Omega, \mathcal{F}, \tilde{P})$ integrability of the second order discretization of the likelihood function, provided the discretization is done on a sufficiently fine partition. We give a bound on the mesh of the partition in terms of p , the uniform bounds on the sensor function h and its derivatives and the dimensions of the

noise driving the signal and the observation process. The proof is based on the fact that the square of a centered Gaussian random variable has finite exponential moment of order sufficiently small.

Lemma 3.10 *Assume that $\mathbf{H}(2)$ holds. Let $p \geq 1$ and τ be a partition with mesh size*

$$\delta < \left(p \|Lh\|_\infty \sqrt{d_Y d_V} \right)^{-1},$$

where

$$\|Lh\|_\infty \triangleq \max_{\substack{i=1,\dots,d_Y \\ r=1,\dots,d_V}} \|L^r h_i\|_\infty.$$

Then, one has that

$$\tilde{\mathbb{E}} \left[\exp(p \xi^{\tau,2}) \right] < \infty.$$

Proof We can write $\exp(p \xi^{\tau,2}) \triangleq \prod_{i=1}^4 \left(K_t^{\tau,2,i} \right)^p$, where

$$\begin{aligned} K_t^{\tau,2,1} &\triangleq \exp \left(\sum_{i=1}^{d_Y} \sum_{r=1}^{d_V} \int_0^t L^r h_i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) dY_s^i \right), \\ K_t^{\tau,2,2} &\triangleq \exp \left(\sum_{i=1}^{d_Y} \int_0^t \{ h_i(X_{\tau(s)}) + L^0 h_i(X_{\tau(s)}) (s - \tau(s)) \} dY_s^i \right), \\ K_t^{\tau,2,3} &\triangleq \exp \left(-\frac{1}{2} \sum_{i=1}^{d_Y} \int_0^t \{ h_i^2(X_{\tau(s)}) + L^0(h_i^2)(X_{\tau(s)}) (s - \tau(s)) \} ds \right), \\ K_t^{\tau,2,4} &\triangleq \exp \left(-\frac{1}{2} \sum_{i=1}^{d_Y} \sum_{r=1}^{d_V} \int_0^t L^r(h_i^2)(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) ds \right). \end{aligned}$$

Let $\varepsilon > 0$, then, by Hölder inequality, we have

$$\tilde{\mathbb{E}} \left[\exp(p \xi^{\tau,2}) \right] \leq \tilde{\mathbb{E}} \left[\left| K_t^{\tau,2,1} \right|^{p(1+\varepsilon)} \right]^{\frac{1}{1+\varepsilon}} \tilde{\mathbb{E}} \left[\prod_{i=2}^4 \left| K_t^{\tau,2,i} \right|^{p \frac{1+\varepsilon}{\varepsilon}} \right]^{\frac{\varepsilon}{1+\varepsilon}}.$$

Hence, the result follows by showing that $K_t^{\tau,2,1}$ has finite $p(1+\varepsilon)$ -moment and

$$\tilde{\mathbb{E}} \left[\prod_{i=2}^4 \left| K_t^{\tau,2,i} \right|^{p \frac{1+\varepsilon}{\varepsilon}} \right] < \infty. \quad (3.6)$$

Applying Hölder inequality twice, condition (3.6) follows by showing that $K_t^{\tau,2,i}, i = 2, \dots, 4$ have finite moments of all orders. In what follows, let $q \geq 1$ be a fixed real constant. We start by the easiest term, $K_t^{\tau,2,3}$. We have that

$$\tilde{\mathbb{E}} \left[\left| K_t^{\tau,2,3} \right|^q \right] \leq \exp \left(\frac{qd_Y}{2} t \left(\|h\|_\infty^2 + \delta \|L^0 h^2\|_\infty \right) \right) < \infty,$$

because $\|h\|_\infty^2$ and $\|L^0 h^2\|_\infty = \max_{i=1,\dots,d_Y} \|L^0(h_i^2)\|_\infty$ are finite due to the assumptions on f, σ and h . For the term $K_t^{\tau,2,4}$, we can write

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| K_t^{\tau,2,4} \right|^q \right] &\leq \tilde{\mathbb{E}} \left[\exp \left(\frac{qd_Y d_V}{2} \|L(h^2)\|_\infty \int_0^t |V_s^1 - V_{\tau(s)}^1| ds \right) \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(\frac{qd_Y d_V}{2} \|L(h^2)\|_\infty \left(\int_0^t (s - \tau(s)) ds \right) |V_1^1| \right) \right] \\ &\leq \tilde{\mathbb{E}} \left[\exp \left(\frac{qd_Y d_V}{2} \|L(h^2)\|_\infty t \sqrt{\delta} |V_1^1| \right) \right] < \infty, \end{aligned}$$

because $\|L(h^2)\|_\infty = \max_{\substack{i=1,\dots,d_Y \\ r=1,\dots,d_V}} \|L^r(h_i^2)\|_\infty$ is finite, the law of $V_s^1 - V_{\tau(s)}^1$ coincides with the law of $(s - \tau(s))^{1/2} V_1^1$ by the scaling properties of the Brownian motion and $|V_1^1|$ has exponential moments of any order.

For the term $K_t^{\tau,2,2}$, we first condition with respect to $\mathcal{F}_t^V = \sigma(V_s, 0 \leq s \leq t)$ and use the fact that, conditionally to $\mathcal{F}_t^V$, the stochastic integrals with respect to Y are Gaussian. We get

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| K_t^{\tau,2,2} \right|^q \right] &= \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\exp \left(q \sum_{i=1}^{d_Y} \int_0^t \{h_i(X_{\tau(s)}) + L^0 h_i(X_{\tau(s)}) (s - \tau(s))\} dY_s^i \right) \middle| \mathcal{F}_t^V \right] \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(\frac{q^2}{2} \sum_{i=1}^{d_Y} \int_0^t \{h_i(X_{\tau(s)}) + L^0 h_i(X_{\tau(s)}) (s - \tau(s))\}^2 ds \right) \right] \\ &= \exp \left(q^2 d_Y t \left\{ \|h\|_\infty^2 + \|L^0 h\|_\infty^2 \right\} \right) < \infty. \end{aligned}$$

Finally, the term $K_t^{\tau,2,1}$ is more delicate because, in order to show that has finite $(p + \varepsilon)$ -moment, a relationship between the mesh of the partition δ and $p + \varepsilon$ is needed. Proceeding as with the term $K_t^{\tau,2,2}$, we obtain

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| K_t^{\tau,2,1} \right|^{p(1+\varepsilon)} \right] &= \tilde{\mathbb{E}} \left[\exp \left(p(1+\varepsilon) \sum_{i=1}^{d_Y} \sum_{r=1}^{d_V} \int_0^t L^r h_i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) dY_s^i \right) \right] \\ &= \tilde{\mathbb{E}} \left[\prod_{i=1}^{d_Y} \tilde{\mathbb{E}} \left[\exp \left(\int_0^t p(1+\varepsilon) \sum_{r=1}^{d_V} L^r h_i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) dY_s^i \right) \middle| \mathcal{F}_t^V \right] \right]. \end{aligned}$$

Now, conditionally to $\mathcal{F}_t^V$, the terms in the exponential are centered Gaussian random variables and we get that

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| K_t^{\tau, 2, 1} \right|^{p(1+\varepsilon)} \right] &= \tilde{\mathbb{E}} \left[\prod_{i=1}^{d_V} \exp \left(\frac{p^2(1+\varepsilon)^2}{2} \int_0^t \left(\sum_{r=1}^{d_V} L^r h_i(X_{\tau(s)}) (V_s^r - V_{\tau(s)}^r) \right)^2 ds \right) \right] \\ &\leq \tilde{\mathbb{E}} \left[\prod_{i=1}^{d_V} \exp \left(\frac{p^2(1+\varepsilon)^2 d_V}{2} \int_0^t \left(\sum_{r=1}^{d_V} |L^r h_i(X_{\tau(s)})|^2 (V_s^r - V_{\tau(s)}^r)^2 \right) ds \right) \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(\frac{p^2(1+\varepsilon)^2 d_V d_V \|Lh\|_\infty^2}{2} \sum_{r=1}^{d_V} \int_0^t (V_s^r - V_{\tau(s)}^r)^2 ds \right) \right] \\ &= \tilde{\mathbb{E}} \left[\exp \left(\frac{p^2(1+\varepsilon)^2 d_V d_V \|Lh\|_\infty^2}{2} \int_0^t (V_s^1 - V_{\tau(s)}^1)^2 ds \right) \right]^{d_V}. \end{aligned}$$

So we need to find conditions on $\beta > 0$, such that $\tilde{\mathbb{E}} \left[\exp \left(\beta \int_0^t (V_s^1 - V_{\tau(s)}^1)^2 ds \right) \right] < \infty$. We can write

$$\begin{aligned} \tilde{\mathbb{E}} \left[\exp \left(\beta \int_0^t (V_s^1 - V_{\tau(s)}^1)^2 ds \right) \right] &= \tilde{\mathbb{E}} \left[\exp \left(\beta \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (V_s^1 - V_{t_{j-1}}^1)^2 ds \right) \right] \\ &= \prod_{j=1}^n \tilde{\mathbb{E}} \left[\exp \left(\beta \int_{t_{j-1}}^{t_j} (V_s^1 - V_{t_{j-1}}^1)^2 ds \right) \right] \\ &\triangleq \prod_{j=1}^n \Theta(\beta, \delta_j). \end{aligned}$$

Denote by $M_t \triangleq \sup_{s \in [0, t]} V_s^1$ and recall that the density of M_t is given by

$$f_{M_t}(x) = \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} 1_{(0, \infty)},$$

see Karatzas and Shreve [9], page 96. Moreover, note that for any $A > 0$,

$$\frac{2}{\sqrt{2\pi\sigma^2}} \int_0^\infty \exp \left\{ -A \frac{x^2}{2\sigma^2} \right\} dx = A^{-1/2}.$$

Then, we have that

$$\begin{aligned} \Theta(\beta, \delta_j) &\leq \tilde{\mathbb{E}} \left[\exp(\beta \delta_j M_\delta^2) \right] = \int_0^\infty \frac{2}{\sqrt{2\pi\delta_j}} \exp \left\{ \beta \delta_j x^2 - \frac{x^2}{2\delta_j} \right\} \\ &= \int_0^\infty \frac{2}{\sqrt{2\pi\delta_j}} \exp \left\{ -(1 - 2\beta\delta_j^2) \frac{x^2}{2\delta_j} \right\} = (1 - 2\beta\delta_j^2)^{-1/2} < \infty, \end{aligned}$$

as long as $1 - 2\beta\delta_j^2 > 0$. On the other hand,

$$\begin{aligned} (1 - 2\beta\delta_j^2)^{-1} &= \sum_{k=0}^{\infty} (2\beta\delta_j^2)^k = 1 + 2\beta\delta_j^2 \left(\sum_{k=0}^{\infty} (2\beta\delta_j^2)^k \right) \\ &\leq 1 + 2\beta\delta_j^2 \left(\sum_{k=0}^{\infty} (2\beta\delta^2)^k \right) = 1 + \frac{2\beta\delta_j^2}{1 - 2\beta\delta^2} \\ &\leq \exp \left(\frac{2\beta\delta_j^2}{1 - 2\beta\delta^2} \right), \end{aligned}$$

and, therefore,

$$\begin{aligned} \prod_{j=1}^n \Theta(\beta, \delta_j) &\leq \prod_{j=1}^n \exp \left(\frac{\beta\delta_j^2}{1 - 2\beta\delta^2} \right) \leq \exp \left(\frac{\beta \sum_{j=1}^n \delta_j^2}{1 - 2\beta\delta^2} \right) \\ &\leq \exp \left(\frac{\beta\delta t}{1 - 2\beta\delta^2} \right) < \infty. \end{aligned}$$

As $\beta = \frac{p^2(1+\varepsilon)^2 d_Y d_V \|Lh\|_{\infty}^2}{2}$ and $\varepsilon > 0$ can be made arbitrary small we get the following condition for the partition mesh $\delta < (p \|Lh\|_{\infty} \sqrt{d_Y d_V})^{-1}$.

We complete the section with an application of the previous two lemmas. Note that, in order to control the high order discretizations of the likelihood function, we reduce the problem to the control of the second order discretization via the truncation procedure as described in Remark 3.2.

Corollary 3.11 *Let $\varphi \in \mathcal{B}_p$. One has that:*

1. If **H**(1) holds, then there exists $\varepsilon > 0$ such that

$$\tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_t} \right|^{2+\varepsilon} \right] < \infty, \quad (3.7)$$

and

$$\sup_{\tau \in \Pi(t)} \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_{\tau,1}} \right|^{2+\varepsilon} \right] < \infty. \quad (3.8)$$

2. If **H**(2) holds, then there exists $\varepsilon > 0$ and $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$\sup_{\tau \in \Pi(t, \delta_0)} \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_{\tau,2}} \right|^{2+\varepsilon} \right] < \infty. \quad (3.9)$$

3. If **H**(m) with $m \geq 3$ holds, then there exists $\varepsilon > 0$ and $\delta_0 = \delta_0(\varepsilon) > 0$ such that

$$\sup_{\tau \in \Pi(t, \delta_0)} \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_{\tau,m}} \right|^{2+\varepsilon} \right] < \infty. \quad (3.10)$$

Proof Combining Lemmas 3.9 and 3.10 with Hölder inequality and Remark 3.1 we obtain (3.7), (3.8) and (3.9). Moreover, for $m \geq 3$, note that

$$\begin{aligned}\bar{\xi}_t^{\tau,m} &= \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \Gamma_{m,\delta_j}(\mu^{\tau,m}(j)) \\ &\leq \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \left| \Gamma_{m,\delta_j}(\mu^{\tau,m}(j)) \right| \\ &\leq \xi_t^{\tau,2} + \sum_{j=0}^{n-1} \delta_j = \xi_t^{\tau,2} + t,\end{aligned}$$

and (3.10) follows from (3.9).

3.3 Proof of the Theorem 2.3

In this section we prove the main theorem of the paper. We start by stating and proving two main propositions.

Proposition 3.12 *Let $m \in \mathbb{N}$ and assume that condition $\mathbf{H}(m)$ holds and $\varphi \in C_p^{m+1}$. Then, there exists a constant C independent of the partition $\pi \in \Pi(t)$ such that*

$$\tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[(\xi_t - \bar{\xi}_t^{\tau,m}) \varphi(X_t) e^{\xi_t} \middle| \mathcal{Y}_t \right] \right|^2 \right] \leq C \delta^{2m}.$$

Proof For $m = 1, 2$ we have that $\bar{\xi}_t^{\tau,m} = \xi_t^{\tau,m}$ and we need to prove

$$A_1 \triangleq \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[(\xi_t - \xi_t^{\tau,m}) \varphi(X_t) e^{\xi_t} \middle| \mathcal{Y}_t \right] \right|^2 \right] \leq C \delta^{2m}.$$

For $m \geq 3$ we have that $\bar{\xi}_t^{\tau,m} = \xi_t^{\tau,m} + \sum_{j=0}^{n-1} \mathcal{E}_{m,\delta_j}(\mu^{\tau,m}(j))$. Then, we can write

$$\begin{aligned}& \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[(\xi_t - \bar{\xi}_t^{\tau,m}) \varphi(X_t) e^{\xi_t} \middle| \mathcal{Y}_t \right] \right|^2 \right] \\ & \leq 2 \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[(\xi_t - \xi_t^{\tau,m}) \varphi(X_t) e^{\xi_t} \middle| \mathcal{Y}_t \right] \right|^2 \right] \\ & \quad + 2 \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[\left(\sum_{j=0}^{n-1} \mathcal{E}_{m,\delta_j}(\mu^{\tau,m}(j)) \right) \varphi(X_t) e^{\xi_t} \middle| \mathcal{Y}_t \right] \right|^2 \right] \\ & \triangleq 2A_1 + 2A_2.\end{aligned}$$

For the term A_2 , using Jensen's inequality for conditional expectation, Hölder inequality and Lemma 3.7, we obtain

$$\begin{aligned} A_2 &\leq \tilde{\mathbb{E}} \left[\left| \left(\sum_{j=0}^{n-1} \mathcal{E}_{m,\delta_j}(\mu^{\tau,m}(j)) \right) \varphi(X_t) e^{\xi_t} \right|^2 \right] \\ &\leq \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_t} \right|^{2+\varepsilon} \right]^{\frac{2+\varepsilon}{2}} \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \mathcal{E}_{m,\delta_j}(\mu^{\tau,m}(j)) \right|^{2\frac{2+\varepsilon}{\varepsilon}} \right]^{\frac{\varepsilon}{2+\varepsilon}} \\ &\leq C \left(\delta^{\frac{2+\varepsilon}{\varepsilon}(2m+1)} \right)^{\frac{\varepsilon}{2+\varepsilon}} = C \delta^{2m+1} \leq C \delta^{2m}. \end{aligned}$$

Hence, it remains to prove $A_1 \leq C \delta^{2m}$, for $m \in \mathbb{N}$. By Lemma 3.6 we can write

$$(\xi_t - \xi_t^{\tau,m}) \varphi(X_t) e^{\xi_t} = \varphi(X_t) e^{\xi_t} \left(\sum_{i=0}^{d_Y} \int_0^t \left\{ \sum_{\alpha \in \mathcal{M}_m(0, \dots, d_Y)} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right\} dY_s^i \right),$$

and, therefore, the problem is reduced to prove that

$$A_1(\alpha, i) \triangleq \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \left(\int_0^t \left\{ I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right\} dY_s^i \right) \middle| \mathcal{A}_t \right] \right|^2 \right] \leq C \delta^{2m},$$

for $\alpha \in \mathcal{M}_m(0, \dots, d_Y)$ and $i \in \{0, \dots, d_Y\}$. There are two, essentially different, cases to consider: when $i = 0$, that is, $dY_s^i = ds$, and when $i = \{1, \dots, d_Y\}$, that is, Y^i is a Brownian motion, independent of V , under $\tilde{P}$. In both cases, a crude bound for $A_1(\alpha, i)$, using Jensen and Cauchy-Schwarz's inequalities, of the form

$$A_1(\alpha, i) \leq \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_t} \right|^4 \right]^{1/2} \tilde{\mathbb{E}} \left[\left| \int_0^t \left\{ I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right\} dY_s^i \right|^4 \right]^{1/2},$$

will not suffice to get the desired order of convergence. Note that

$$\tilde{\mathbb{E}} \left[\left| I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right|^4 \right]^{1/2} \leq C \delta^{(|\alpha|+|\alpha|_0)},$$

and, since $|\alpha| = m$, this approach only works when $\alpha = (0, \dots, 0)$, i.e., when $|\alpha|_0 = m$. This case is discussed in Lemma 4.18. The main idea to generate the right order of convergence is to use the Ströck-Taylor formula, Theorem 4.8, to expand $\varphi(X_t) e^{\xi_t}$ as the sum of iterated stochastic integrals with respect to the Brownian motion V . The order of the expansion used will depend on $|\alpha|_0$, the number of zero components in the particular multi-index α , which takes values in $\{0, \dots, m-1\}$. If $|\alpha|_0 = 0$, then we need to expand $\varphi(X_t) e^{\xi_t}$ up to order m , if $|\alpha|_0 = 1$, then we need to expand up to order $m-1$ and so on. Then, we can use the quadratic covariation between these stochastic integrals and the Brownian integrals appearing in $I_\alpha(L^\alpha h_i(X))_{\tau(s),s}$ to reach the order $2m$.

Case $i = 0$: In this case $Y_s^0 = s$, and the approach with the quadratic variation is relatively straightforward. This is done in Lemma 4.19.

Case $i \neq 0$: This case is much more involved. The difficulty is that we cannot use directly the quadratic covariation between the stochastic integrals in the Strock-Taylor expansion and the stochastic integrals in $I_\alpha(L^\alpha h_i(X))_{\tau(s),s}$. This is due to the fact that the latter are integrands in a stochastic integral with respect to the observation process Y , that is, we have expressions of the form

$$\left(I_\beta \left(\tilde{\mathbb{E}} \left[\Theta_{s_1, \dots, s_{|\beta|}}^\beta \middle| \mathcal{H}_0^t \right] \right)_{0,t} \right) \left(\int_0^t \left\{ I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right\} dY_s^i \right), \quad (3.11)$$

where $\Theta_{s_1, \dots, s_{|\beta|}}^\beta$ are the kernels appearing in the Strock-Taylor formula. Moreover, one cannot use the integration by parts formula to change the order of integration in (3.11). The problem is that these iterated integrals are semimartingales with respect to different “incompatible” filtrations. The left factor in (3.11) is a $\mathcal{H}_s^t = \mathcal{F}_s^{0,V} \vee \mathcal{Y}_t$ -semimartingale and the right factor in (3.11) is a $\mathcal{F}_s = \mathcal{F}_s^{0,V} \vee \mathcal{Y}_s$ -semimartingale but not a $\mathcal{H}_s^t$ -semimartingale. To overcome this obstacle, the idea is to rewrite these iterated integrals as sums of iterated integrals over the partition under consideration. One can show that the terms coming from the products of non-overlapping subintervals vanish. Moreover, by integration by parts formula for $\mathcal{F}_s$ -semimartingales we have

$$\begin{aligned} \int_{t_j}^{t_{j+1}} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i &= \left(Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_\alpha(L^\alpha h_i(X))_{t_j, t_{j+1}} \\ &\quad - \int_{t_j}^{t_{j+1}} \left(Y_s^i - Y_{t_j}^i \right) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha_1}, \end{aligned}$$

and, in fact, we can rewrite the right hand side of the previous equality as a well defined $\mathcal{H}_s^t$ -iterated integral and obtain

$$\int_{t_j}^{t_{j+1}} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i = \int_{t_j}^{t_{j+1}} \left(Y_{t_{j+1}}^i - Y_s^i \right) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha_1}.$$

This reasoning allows us to rewrite the expression in (3.11) as sums of products of $\mathcal{H}_s^t$ -semimartingales. This is done for $m \in \{1, 2\}$ in Lemma 4.20 and for $m > 2$ the reasonings are analogous. Now we have expressions that are amenable for computing the quadratic covariation of stochastic integrals with respect to the filtration $\mathcal{H}_s^t$.

For $m = 1$, we only need to consider the case that the multi-index $\alpha = (\alpha_1) \in \{1, \dots, d_V\}$. This is done in Lemma 4.21. In this lemma, computing the quadratic covariation and adding and subtracting the appropriate terms, we show that we can write

$$\begin{aligned} &\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\ &= \sum_{r=1}^{d_V} 1_{\{\alpha_1=r\}} (B_1(r) + B_2(r) + B_3(r)), \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} B_1(r) &\triangleq \tilde{\mathbb{E}} \left[\int_0^t \left(J_s^r - J_{\eta(s)}^r \right) \left(Y_{\eta(s)}^i - Y_s^i \right) L^r h_i(X_s) ds \middle| \mathcal{Y}_t \right], \\ B_2(r) &\triangleq \tilde{\mathbb{E}} \left[\int_0^t J_{\eta(s)}^r \left(Y_{\eta(s)}^i - Y_s^i \right) \left(L^r h_i(X_s) - L^r h_i(X_{\tau(s)}) \right) ds \middle| \mathcal{Y}_t \right], \\ B_3(r) &\triangleq \tilde{\mathbb{E}} \left[\int_0^t J_{\eta(s)}^r L^r h_i(X_{\tau(s)}) \left(Y_{\eta(s)}^i - Y_s^i \right) ds \middle| \mathcal{Y}_t \right], \end{aligned}$$

and $\{J_s^r\}_{0 \leq s \leq t}$, $r = 1, \dots, d_V$ are the kernels in the martingale representation of $\varphi(X_t) e^{\xi_t}$ (or its Ströck-Taylor expansion of order 1). The right bound for the terms $B_1(r)$ and $B_2(r)$ follows from standard arguments, as we can take advantage of the increments inside the integrals. However, we need to require more regularity to h and we also need some Hölder regularity for the kernels J_s^r . In order to prove this regularity for the kernels we need to use the Malliavin calculus and the Clark-Ocone formula, which is the connection between J_s^r and the Malliavin derivative of $\varphi(X_t) e^{\xi_t}$, in particular, we have

$$J_s^r = \tilde{\mathbb{E}} \left[D_s^r \left[\varphi(X_t) e^{\xi_t} \right] \middle| \mathcal{H}_s^t \right].$$

We have to use partial Malliavin calculus because the martingale representation of $\varphi(X_t) e^{\xi_t}$ is with respect to the filtration $\mathcal{H}_s^t = \mathcal{F}_s^{0,V} \vee \mathcal{Y}_t$, that is, the Brownian motion Y up to time t is frozen. These issues are discussed in sections 4.1 and 4.2. The term $B_3(r)$ is more delicate to analyse. We first prove that

$$B_3(r) = \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}}^r L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{Y}_t \right],$$

where β_s^j is deterministic and $|\beta_s^j| \leq \delta$. The next step is to find the right bound for the second moment of $B_3(r)$. In order to do so, first we use again the Clark-Ocone formula to find a more explicit expression for J_s^r in terms of the Malliavin Derivative of $\varphi(X_t) e^{\xi_t}$. Using the product formula for the Malliavin derivative we find that

$$\begin{aligned} \tilde{\mathbb{E}} \left[|B_3(r)|^2 \right] &\leq 2 \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} e^{\xi_t} D_{t_{j+1}}^r \varphi(X_t) L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\ &\quad + 2 \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \varphi(X_t) D_{t_{j+1}}^r e^{\xi_t} L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right]. \end{aligned}$$

Moreover, one can compute

$$D_{t_{j+1}}^r e^{\xi_t} = e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t \alpha_s^{j,k,1} dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t \alpha_s^{j,k,2} ds \right\}$$

where $\alpha_s^{j,k,1}$ and $\alpha_s^{j,k,2}$ are not deterministic but independent of $\mathcal{Y}_t$. Observe that $D_{t_{j+1}}^r e^{\tilde{\zeta}_t}$ can be expressed as a product of $e^{\tilde{\zeta}_t}$ and some integrals with respect to Y and the Lebesgue measure. In addition, we have that

$$e^{2\tilde{\zeta}_t} = M_0^t(2h) \exp \left(\sum_{i=1}^{d_Y} \int_0^t h_i^2(X_u) du \right),$$

where

$$M_s^t(h) = \exp \left(\sum_{i=1}^{d_Y} \int_s^t h_i(X_u) dY_u^i - \frac{1}{2} \sum_{i=1}^{d_Y} \int_s^t h_i^2(X_u) du \right).$$

The crucial point now is to realise that $\{M_s^t(h)\}_{0 \leq s \leq t}$ (as well as the integrals $\int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i$ and $\int_{t_{j+1}}^t \alpha_s^{j,k,1} dY_s^k$) can be seen as a backward martingale with respect to the backward filtration $\{\mathcal{Y}_s^{0,V,t} \triangleq \mathcal{F}_t^{0,V} \vee \mathcal{Y}_s^t\}_{0 \leq s \leq t}$, see Remark 4.12. Therefore, getting good estimates for $\tilde{\mathbb{E}}[|B_3(r)|^2]$ is reduced to getting good estimates for the expectations of products of $\mathcal{Y}_s^{0,V,t}$ -backward martingales. These estimates are obtained by using the quadratic covariation properties of $\mathcal{Y}_s^{0,V,t}$ -backward martingales and this is the goal of the technical subsection 4.3.

For $m = 2$, we need to consider the case in which the multi-index $\alpha = (\alpha_1, \alpha_2)$ contains only one zero and the case in which α does not contain any zero. In the first case, which is treated in Lemma 4.22, we just need to use a Strock-Taylor expansion of order 1 and the reasoning is identical to the case $m = 1$ and $\alpha \in \{1, \dots, d_V\}$. In the second case, which is treated in Lemma 4.23, we need to use a Strock-Taylor expansion of order 2. Here, computing the quadratic covariation and adding and subtracting the appropriate terms, we show that we can write

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\tilde{\zeta}_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\ &= \sum_{r_1, r_2=1}^{d_V} \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} (B_1(r_1, r_2) + B_2(r_1, r_2) + B_3(r_1, r_2) + B_4(r_1, r_2)), \quad (3.13) \end{aligned}$$

where

$$\begin{aligned} B_1(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) \left(\int_{\tau(s)}^s (J_{u,s}^{r_1, r_2} - J_{s,s}^{r_1, r_2}) L^\alpha h_i(X_u) du \right) ds \right) \middle| \mathcal{Y}_t \right], \\ B_2(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) J_{s,s}^{r_1, r_2} \left(\int_{\tau(s)}^s (L^\alpha h_i(X_u) - L^\alpha h_i(X_{\tau(s)})) du \right) ds \right) \middle| \mathcal{Y}_t \right] \\ B_3(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) (J_{s,s}^{r_1, r_2} - J_{\eta(s), \eta(s)}^{r_1, r_2}) L^\alpha h_i(X_{\tau(s)}) \left(\int_{\tau(s)}^s du \right) ds \right) \middle| \mathcal{Y}_t \right] \\ B_4(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) J_{\eta(s), \eta(s)}^{r_1, r_2} L^\alpha h_i(X_{\tau(s)}) \left(\int_{\tau(s)}^s du \right) ds \right) \middle| \mathcal{Y}_t \right]. \end{aligned}$$

The terms $B_1(r_1, r_2)$, $B_2(r_1, r_2)$ and $B_3(r_1, r_2)$ are easy to deal with using standard arguments due to the increments inside the integrals. As before, one needs certain

Hölder regularity in the kernels, which can be proved using the relationship between $J_{s,s}^{r_1,r_2}$ and the second order Malliavin derivative of $\varphi(X_t)e^{\xi_t}$, see Theorem 4.8 and Remark 4.11. The term $B_4(r_1, r_2)$ is the difficult one. We first rewrite $B_4(r_1, r_2)$ as

$$B_4(r_1, r_2) = \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(J_{t_j, t_j}^{r_1, r_2} L^\alpha h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right) \middle| \mathcal{F}_t \right],$$

where β_s^j is deterministic and $|\beta_s^j| \leq \delta^2$. Then, one can proceed as for $B_3(r)$. First, we express the kernel $J_{t_j, t_j}^{r_1, r_2}$ in terms of iterated Malliavin derivatives. Then, we rewrite it as products of $\mathcal{Y}_s^{0, V, t}$ -backward martingales and reduce the problem to estimate the covariations between these martingales. The integral $\int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i$ is crucial to get the right order.

Similar considerations apply for the case $m > 2$. We first split the cases depending on $|\alpha|_0$ the number of zeros in $\alpha = (\alpha_1, \dots, \alpha_m)$, which can take values in $\{0, \dots, m-1\}$. For $|\alpha|_0 \in \{1, \dots, m-1\}$ the procedure is the same as the one used for smaller m 's. In the case $|\alpha|_0 = 0$, computing the quadratic covariation and adding and subtracting the appropriate terms, one can find an analogous expansion for $\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s), s} dY_s^i \middle| \mathcal{F}_t \right]$ as in (3.12) or (3.13). Similarly as before, there is a more difficult term than can be written as

$$B(r_1, \dots, r_m) = \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(J_{t_j, \dots, t_j}^{r_1, \dots, r_m} L^\alpha h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right) \middle| \mathcal{F}_t \right],$$

where β_s^j is deterministic and $|\beta_s^j| \leq \delta^m$. Next, one can again rewrite the kernel $J_{t_j, \dots, t_j}^{r_1, \dots, r_m}$ more explicitly in terms of iterated Malliavin derivatives. Then, we can compute these derivatives and express them as products of $\mathcal{Y}_s^{0, V, t}$ -backward martingales. Finally, we estimate the covariation between these martingales and get the right order thanks to the integral $\int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i$.

Proposition 3.13 *Let $m \in \mathbb{N}$ and assume that condition $\mathbf{H}(m)$ holds and $\varphi \in \mathcal{B}_P$. Then, there exist $\delta_0 > 0$ and constant C independent of any partition $\pi \in \Pi(t, \delta_0)$ such that*

$$\tilde{\mathbb{E}} \left[\left| |\varphi(X_t)| \left(e^{\xi_t} + e^{\bar{\xi}_t^{\tau, m}} \right) (\xi_t - \bar{\xi}_t^{\tau, m})^2 \right|^2 \right] \leq C \delta^{2m}.$$

Proof As $\mathbf{H}(m)$ holds, let $\varepsilon > 0$ such as in Corollary 3.11. By Hölder inequality we have that

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\left| |\varphi(X_t)| \left(e^{\xi_t} + e^{\bar{\xi}_t^{\tau, m}} \right) (\xi_t - \bar{\xi}_t^{\tau, m})^2 \right|^2 \right] \\ & \leq \tilde{\mathbb{E}} \left[\left| |\varphi(X_t)| \left(e^{\xi_t} + e^{\bar{\xi}_t^{\tau, m}} \right) \right|^{2+\varepsilon} \right]^{\frac{2}{2+\varepsilon}} \tilde{\mathbb{E}} \left[|\xi_t - \bar{\xi}_t^{\tau, m}|^{4 \frac{2+\varepsilon}{\varepsilon}} \right]^{\frac{\varepsilon}{2+\varepsilon}}, \end{aligned}$$

Corollary 3.11 yields that there exists $\delta_0 > 0$ such that

$$\sup_{\tau \in \Pi(t, \delta_0)} \tilde{\mathbb{E}} \left[\left| \varphi(X_t) \left(e^{\xi_t} + e^{\bar{\xi}_t^{\tau, m}} \right) \right|^{2+\varepsilon} \right] < \infty. \quad (3.14)$$

On the other hand, by equation (3.3), for any $p \geq 1$ we get that

$$|\xi_t - \bar{\xi}_t^{\tau, m}|^{2p} \leq C \left\{ |\xi_t - \xi_t^{\tau, m}|^{2p} + \left| \sum_{j=0}^{n-1} \mathcal{E}_{m, \delta_j}(\mu^{\tau, m}(j)) \right|^{2p} \right\}.$$

By Lemma 3.6, we obtain

$$\tilde{\mathbb{E}} \left[|\xi_t - \xi_t^{\tau, m}|^{2p} \right] \leq C \delta^{pm},$$

and by Lemma 3.7 with $q = m$ we have that

$$\tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \mathcal{E}_{m, \delta_j}(\mu^{\tau, m}(j)) \right|^{2p} \right] \leq C \delta^{p(2m+1)}$$

Hence, setting $p = 2(2 + \varepsilon)/\varepsilon$, we obtain

$$\tilde{\mathbb{E}} \left[|\xi_t - \bar{\xi}_t^{\tau, m}|^{4(2+\varepsilon)/\varepsilon} \right]^{\varepsilon/(2+\varepsilon)} \leq C \left(\delta^{2m} + \delta^{2(2m+1)} \right) \leq C \delta^{2m}.$$

We are finally ready to put everything together and deduce Theorem 2.3.

Proof (Proof of Theorem 2.3)

To get the desired rate of convergence for the unnormalised conditional distribution $\rho_t^{\tau, m}$, we can write

$$\begin{aligned} \rho_t(\varphi) - \rho_t^{\tau, m}(\varphi) &= \tilde{\mathbb{E}} \left[\varphi(X_t) (\xi_t - \bar{\xi}_t^{\tau, m}) e^{\xi_t} \middle| \mathcal{Y}_t \right] \\ &\quad + \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} - \varphi(X_t) e^{\bar{\xi}_t^{\tau, m}} - \varphi(X_t) (\xi_t - \bar{\xi}_t^{\tau, m}) e^{\xi_t} \middle| \mathcal{Y}_t \right]. \end{aligned}$$

Using the inequality

$$|e^x - e^y - (x - y)e^x| \leq \frac{e^x + e^y}{2} (x - y)^2,$$

we get that

$$\begin{aligned} &\tilde{\mathbb{E}} \left[|\rho_t(\varphi) - \rho_t^{\tau, m}(\varphi)|^2 \right] \\ &\leq C \left\{ \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[(\xi_t - \bar{\xi}_t^{\tau, m}) \varphi(X_t) e^{\xi_t} \middle| \mathcal{Y}_t \right] \right|^2 \right] + \tilde{\mathbb{E}} \left[\left| \varphi(X_t) \frac{e^{\xi_t} + e^{\bar{\xi}_t^{\tau, m}}}{2} (\xi_t - \bar{\xi}_t^{\tau, m})^2 \right|^2 \right] \right\}. \end{aligned}$$

Now, Propositions 3.12 and 3.13 yield

$$\tilde{\mathbb{E}} \left[|\rho_t(\varphi) - \rho_t^{\tau, m}(\varphi)|^2 \right] \leq C \delta^{2m}.$$

To prove the rate for the normalised conditional distribution observe that we can write

$$\pi_t^{\tau,m}(\varphi) - \pi_t(\varphi) = \frac{1}{\rho_t(1)} \frac{\rho_t^{\tau,m}(\varphi)}{\rho_t^{\tau,m}(1)} (\rho_t(1) - \rho_t^{\tau,m}(1)) + \frac{1}{\rho_t(1)} (\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)).$$

Hence,

$$\begin{aligned} & \mathbb{E} [|\pi_t(\varphi) - \pi_t^{\tau,m}(\varphi)|] \\ & \leq C \tilde{\mathbb{E}} \left[\frac{Z_t}{|\rho_t(1)|} \{ |\pi_t^{\tau,m}(\varphi)| |\rho_t(1) - \rho_t^{\tau,m}(1)| + |\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)| \} \right] \\ & = C \tilde{\mathbb{E}} \left[\frac{\tilde{\mathbb{E}}[Z_t | \mathcal{Y}_t]}{|\rho_t(1)|} \{ |\pi_t^{\tau,m}(\varphi)| |\rho_t(1) - \rho_t^{\tau,m}(1)| + |\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)| \} \right] \\ & \leq C \{ \tilde{\mathbb{E}} [|\pi_t^{\tau,m}(\varphi)| |\rho_t(1) - \rho_t^{\tau,m}(1)|] + \tilde{\mathbb{E}} [|\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)|] \} \\ & \leq C \left\{ \tilde{\mathbb{E}} [|\pi_t^{\tau,m}(\varphi)|^2]^{1/2} \tilde{\mathbb{E}} [|\rho_t(1) - \rho_t^{\tau,m}(1)|^2]^{1/2} + \tilde{\mathbb{E}} [|\rho_t^{\tau,m}(\varphi) - \rho_t(\varphi)|^2]^{1/2} \right\}, \end{aligned}$$

where in the last inequality we have applied Hölder inequality. Combining the bounds for the unnormalised distribution and the hypothesis on $\pi_t^{\tau,m}(\varphi)$ we can conclude.

4 Technical Lemmas

We collate in this section the technical lemmas required to prove the main results. We begin with some limited background material on Malliavin Calculus (and partial Malliavin Calculus) with a view to deduce the necessary properties of the functionals to be discretised.

4.1 Malliavin calculus

Let $B = \{B_t\}_{0 \leq t \leq T}$ be a d -dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{S}$ denote the class of smooth random variables, that is, the class of random variables F having the form

$$F = f(B_{t_1}; \dots; B_{t_n}),$$

where the function $f(x^{11}, \dots, x^{d1}; \dots; x^{1n}, \dots, x^{dn})$ belongs to $C_b^\infty(\mathbb{R}^{dn})$ and $t_1, \dots, t_n \in [0, T]$. The Malliavin derivative of a smooth functional F can be defined as the d -dimensional stochastic processes given by

$$(DF)_t^j = \sum_{i=1}^n \frac{\partial f}{\partial x^{ji}}(B_{t_1}; \dots; B_{t_n}) \mathbf{1}_{[0, t_i]}(t),$$

for $0 \leq t \leq T$ and $j = 1, \dots, d$. The derivative DF can be regarded as a random variable taking values in the Hilbert space $H = L^2([0, T]; \mathbb{R}^d)$. Noting the isometry $L^2(\Omega \times [0, T]; \mathbb{R}^d) \simeq L^2(\Omega; H)$ we can identify $(DF)_t^j$ as the value at time t of

the j th component of an $\mathbb{R}^d$ -valued stochastic process. We will also use the notation $D_t^j F$ for $(DF)_t^j$. One can see that the operator D is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$, $p \geq 1$ and we will denote the domain of D in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$. That is, meaning that $\mathbb{D}^{1,p}$ is the closure of smooth random variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{\mathbb{D}^{1,p}} = (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p])^{1/p}.$$

We define the k -th derivative of F , $D^k F$, as the $H^{\otimes k}$ -valued random variable

$$(D^k F)_{s_1, \dots, s_k}^{j_1, \dots, j_k} = \sum_{i_1, \dots, i_k=1}^n \frac{\partial^k f}{\partial x^{j_1 i_1} \dots \partial x^{j_k i_k}}(B_{t_1}; \dots; B_{t_n}) \mathbf{1}_{[0, t_{i_1}]}(s_1) \dots \mathbf{1}_{[0, t_{i_k}]}(s_k),$$

where $s_1, \dots, s_k \in [0, T]$ and $j_1, \dots, j_k = 1, \dots, d$. We will also write $D_{s_1, \dots, s_k}^{j_1, \dots, j_k} F$ for $(D^k F)_{s_1, \dots, s_k}^{j_1, \dots, j_k}$ and notice that it coincides with the iterated derivative $D_{t_1}^{j_1} \dots D_{t_k}^{j_k} F$. For any integer $k \geq 1$ and any real number $p > 1$ we introduce the norm on $\mathcal{S}$ given by

$$\|F\|_{k,p} = \left(\mathbb{E}[|F|^p] + \sum_{j=1}^k \mathbb{E}[\|D^j F\|_{H^{\otimes j}}^p] \right)^{1/p},$$

where

$$\|D^k F\|_{H^{\otimes k}} = \left(\sum_{j_1, \dots, j_k=1}^d \int_{[0, T]^k} |D_{s_1, \dots, s_k}^{j_1, \dots, j_k} F|^2 ds_1 \dots ds_k \right)^{1/2}.$$

We will denote by $\mathbb{D}^{k,p}$ the completion of the family of random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$. We also define the space $\mathbb{D}^{k,\infty} = \bigcap_{p \geq 1} \mathbb{D}^{k,p}$. We have the following chain rule formula for the Malliavin derivative.

Proposition 4.1 *Let $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$ be of class $C_P^1(\mathbb{R}^m)$. Suppose that $F = (F^1, \dots, F^m)$ is a random vector whose components belong to $\mathbb{D}^{1,\infty}$. Then, $\varphi(F) \in \mathbb{D}^{1,\infty}$ and*

$$D_t^j \varphi(F) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x^i}(F) D_t^j F^i,$$

where $t \in [0, T]$ and $j = 1, \dots, d$.

Proof The proof follows the same ideas as in the proof of Proposition 1.2.3 in Nualart [13], where the result is proved for $\varphi \in C_b^1(\mathbb{R}^m)$ and $F \in \mathbb{D}^{1,p}$. One can extend the result to $\varphi \in C_P^1(\mathbb{R}^m)$ by requiring $F \in \mathbb{D}^{1,\infty}$ and using Hölder inequality.

As a corollary of Proposition 4.1 one obtains that the product rule and the binomial formula holds for the Malliavin derivative of products of random variables in $\mathbb{D}^{1,\infty}$. However, Proposition 4.1 does not apply to the exponential function. In order to show that the likelihood functional e^{ξ_t} is smooth in the Malliavin sense we need the following lemma.

Lemma 4.2 *Let $F \in \mathbb{D}^{1,\infty}$ and such that*

$$\mathbb{E}[\exp(p|F|)] < \infty, \quad (4.1)$$

for all $p \geq 1$. Then $G = e^F \in \mathbb{D}^{1,\infty}$ and

$$D_t^j G = G D_t^j F, \quad (4.2)$$

where $t \in [0, T]$ and $j = 1, \dots, d$.

Proof Define $G_n = \sum_{k=0}^n \frac{F^k}{k!}$. As $F \in \mathbb{D}^{1,\infty}$, Proposition 4.1 yields that $G_n \in \mathbb{D}^{1,\infty}$ and

$$DG_n = \sum_{k=1}^n \frac{k F^{k-1}}{k!} DF = G_{n-1} DF.$$

In order to prove that $G \in \mathbb{D}^{1,\infty}$ and that the identity (4.2) is satisfied, it suffices to show that for all $p \geq 1$ one has that G_n converges to G in $L^p(\Omega)$ and

$$\mathbb{E}[\|DG - DG_n\|_H^p] \longrightarrow 0,$$

when n tends to infinity. Note that

$$\begin{aligned} \mathbb{E}[\|DG - DG_n\|_H^p] &= \mathbb{E}\left[\left|G - G_{n-1}\right|^p \left|\int_0^T |D_t F|^2 dt\right|^{p/2}\right] \\ &\leq \mathbb{E}\left[|G - G_{n-1}|^{2p}\right]^{1/2} \mathbb{E}\left[\|DF\|_{L^2([0,T])}^{2p}\right]^{1/2}. \end{aligned}$$

Hence, the problem is reduced to show that G_n converges to G in L^p for all $p \geq 1$. Equivalently, defining

$$G_n^c := G - G_n = \sum_{k=n+1}^{\infty} \frac{F^k}{k!},$$

it suffices to prove that G_n^c converges to 0 in L^p for all $p \geq 1$. Clearly, G_n^c converges to 0 almost surely and, thanks to assumption (4.1), the dominated convergence theorem yields that G_n^c also converges to 0 in $L^p(\Omega)$ for all $p \geq 1$.

We also have the following relationship between the conditional expectation and the Malliavin derivative.

Lemma 4.3 *Let $F \in \mathbb{D}^{1,2}$ and $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ be the P -augmented natural filtration generated by B . Then $\mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}^{1,2}$ and $D_s^j \mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}\left[D_s^j F \middle| \mathcal{F}_t\right] \mathbf{1}_{[0,t]}(s), j = 1, \dots, d$.*

Proof The lemma is a particular case of Proposition 1.2.8 in Nualart [13].

The following is an important result regarding the Malliavin differentiability of the solution of a stochastic differential equation.

Lemma 4.4 *If $X_t \in \mathbb{R}^n$ is the solution to*

$$X_t = x + \int_0^t V_0(X_s) ds + \int_0^t V(X_s) dB_s,$$

where the components of V_0 and V are m -times continuously differentiable with bounded derivatives of order greater or equal than one and $B_t = (B_t^1, \dots, B_t^d)$ is a d -dimensional Brownian motion. Then, $X_t^i \in \mathbb{D}^{m,\infty}$, $t \in [0, T]$, $i = 1, \dots, n$. Furthermore, for any $p \geq 1$ one has that

$$\sup_{r_1, r_2, \dots, r_k \in [0, T]} \mathbb{E} \left[\sup_{t \in [r_1 \vee r_2 \vee \dots \vee r_k, T]} |D_{r_1, r_2, \dots, r_k}^{j_1, j_2, \dots, j_k} X_t^i|^p \right] < \infty,$$

for all $p \geq 1$, $i = 1, \dots, n$, $j_k \in \{1, \dots, d\}$ and $1 \leq k \leq m$.

Proof See Nualart [13], Theorem 2.2.1. and 2.2.2.

Remark 4.5 We will be using a variation of the classical Malliavin calculus known as *partial Malliavin calculus*. This calculus was introduced in Kusuoka and Stroock [12] and Nualart and Zakai [14] with a view towards its application to the stochastic filtering problem, see also Tanaka [18]. The idea is to consider only the Malliavin derivative operator with respect some of the components of the Brownian motion B . In our setting $B=(V, Y)$ is a $d_V + d_Y$ -dimensional Brownian motion under $\tilde{P}$ and the Malliavin differentiation will be only with respect to the Brownian motion V . The main consequence of this approach is that the Malliavin derivative with respect to V commutes with the stochastic integral with respect to Y , that is,

$$D_r \int_0^t \theta_s dY_s = \int_0^t D_r \theta_s dY_s.$$

In the following lemma, $\mathcal{M}_{0,m+1}(1, \dots, d_V)$ will denote the set of multi-indices up to order $m+1$ and with values in $\{1, \dots, d_V\}$.

Lemma 4.6 *Let $m \in \mathbb{N}$ and assume that $H(m)$ holds and $\varphi \in C_p^{m+1}$. Then, the random variable $\varphi(X_t) e^{\xi_t}$ belongs to $\mathbb{D}^{m+1,\infty}$. Moreover,*

$$\sup_{r_1, \dots, r_{|\alpha|} \in [0, t]} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_{|\alpha|}}^{\alpha_1, \dots, \alpha_{|\alpha|}} \left(\varphi(X_t) e^{\xi_t} \right) \right|^p \right] < \infty,$$

for all $p \geq 1$ and $\alpha \in \mathcal{M}_{0,m+1}(1, \dots, d_V)$, where D denotes the partial Malliavin derivative with respect to V .

Proof To ease the notation we are only going to give the proof for $d_V = d_Y = d_X = 1$.

We will also use the notation $D_{r_1, \dots, r_k}^k F = D_{r_1, \dots, r_k}^{\overbrace{1, \dots, 1}^k} F$. Lemma 4.4 yields that $X_t \in \mathbb{D}^{1,\infty}$. Applying iteratively Proposition 4.1 we obtain that $\varphi(X_t) \in \mathbb{D}^{m+1,\infty}$ and $h(X_t) \in \mathbb{D}^{m+1,\infty}$. Taking into account Remark 4.5, we have that $\xi_t \in \mathbb{D}^{m+1,\infty}$. Moreover, thanks

to Lemma 3.9, we can apply iteratively Lemma 4.3 and conclude that $e^{\xi_t} \in \mathbb{D}^{m+1, \infty}$. For any $\alpha \in \mathcal{M}_{m+1}(S_1)$, by Leibniz's rule, we can write

$$\begin{aligned} D_{r_1, \dots, r_{|\alpha|}}^{\alpha_1, \dots, \alpha_{|\alpha|}} \left(\varphi(X_t) e^{\xi_t} \right) &= D_{r_1, \dots, r_{|\alpha|}}^{|\alpha|} \left(\varphi(X_t) e^{\xi_t} \right) \\ &= \sum_{k=0}^{|\alpha|} \binom{|\alpha|}{k} \left(D_{r_1, \dots, r_k}^k \varphi(X_t) \right) \left(D_{r_1, \dots, r_{|\alpha|-k}}^{|\alpha|-k} e^{\xi_t} \right), \end{aligned}$$

and applying Schwartz's inequality one has that

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_{|\alpha|}}^{|\alpha|} \left(\varphi(X_t) e^{\xi_t} \right) \right|^p \right] &\leq C \sum_{k=0}^{|\alpha|} \binom{|\alpha|}{k} \tilde{\mathbb{E}} \left[\left| \left(D_{r_1, \dots, r_k}^k \varphi(X_t) \right) \left(D_{r_1, \dots, r_{|\alpha|-k}}^{|\alpha|-k} e^{\xi_t} \right) \right|^p \right] \\ &\leq C \sum_{k=0}^{|\alpha|} \binom{|\alpha|}{k} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_k}^k \varphi(X_t) \right|^{2p} \right]^{1/2} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_{|\alpha|-k}}^{|\alpha|-k} e^{\xi_t} \right|^{2p} \right]^{1/2}. \end{aligned}$$

Hence, the result follows if we show that

$$\sup_{r_1, \dots, r_k \in [0, t]} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_k}^k \varphi(X_t) \right|^p \right] < \infty, \quad 0 \leq k \leq |\alpha|, \quad (4.3)$$

$$\sup_{r_1, \dots, r_k \in [0, t]} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_k}^k e^{\xi_t} \right|^p \right] < \infty, \quad 0 \leq k \leq |\alpha|, \quad (4.4)$$

for any $p \geq 1$.

▷ *Proof of (4.3):*

If $k = 0$, using that $\mathbf{H}(m)$ holds and $\varphi \in C_p^{m+1}$, we have that $\tilde{\mathbb{E}}[|\varphi(X_t)|^p] < \infty$, by Remark 3.1. If $1 \leq k \leq |\alpha|$, we use Faà di Bruno's formula to obtain an expression for $D_{r_1, \dots, r_k}^k \varphi(X_t)$ in terms of the so called partial Bell polynomials, which are given by

$$B_{k,a}(x_1, \dots, x_k) = \sum_{(j_1, \dots, j_k) \in \Lambda(k,a)} \frac{k!}{j_1! (1!)^{j_1} j_2! (2!)^{j_2} \dots j_k! (k!)^{j_k}} x_1^{j_1} x_2^{j_2} \dots x_k^{j_k},$$

where $1 \leq a \leq k$ and

$$\Lambda(k, a) = \left\{ (j_1, \dots, j_k) \in \mathbb{Z}_+^k : j_1 + 2j_2 + \dots + kj_k = k, j_1 + j_2 + \dots + j_k = a \right\}.$$

In particular, we have that

$$D_{r_1, \dots, r_k}^k \varphi(X_t) = \sum_{a=1}^k \varphi^{(a)}(X_t) B_{k,a} \left(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \dots, D_{r_1, \dots, r_k}^k X_t \right).$$

Hence, for any $p \geq 1$, applying Cauchy-Schwarz inequality we get

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_k}^k \varphi(X_t) \right|^p \right] &\leq C \sum_{a=1}^k \tilde{\mathbb{E}} \left[\left| \varphi^{(a)}(X_t) B_{k,a} \left(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \dots, D_{r_1, \dots, r_k}^k X_t \right) \right|^p \right] \\ &\leq C \sum_{a=1}^k \tilde{\mathbb{E}} \left[\left| \varphi^{(a)}(X_t) \right|^{2p} \right]^{1/2} \tilde{\mathbb{E}} \left[\left| B_{k,a} \left(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \dots, D_{r_1, \dots, r_k}^k X_t \right) \right|^{2p} \right]^{1/2}. \end{aligned}$$

The terms $\tilde{\mathbb{E}} \left[\left| \varphi^{(a)}(X_t) \right|^{2p} \right] < \infty, a = 1, \dots, k$, due to Remark 3.1 combined with that $\mathbf{H}(m)$ holds and $\varphi \in C_p^{m+1}$. On the other hand, using the generalized version of Hölder's inequality, Lemma 4.15, we can bound

$$\tilde{\mathbb{E}} \left[\left| B_{k,a} \left(D_{r_1}^1 X_t, D_{r_1, r_2}^2 X_t, \dots, D_{r_1, \dots, r_k}^k X_t \right) \right|^{2p} \right], \quad 1 \leq a \leq k,$$

by a sum of products of expectations of powers of Malliavin derivatives of X of different orders. Combining this bound with Lemma 4.4 we get that the integrability condition (4.3) is satisfied.

▷ *Proof of (4.4):*

If $k = 0$, we have that $\tilde{\mathbb{E}} \left[\left| e^{\xi_t} \right|^p \right] < \infty$ due to Lemma 3.9. If $1 \leq k \leq |\alpha|$, using again Faà di Bruno's formula we get

$$\begin{aligned} D_{r_1, \dots, r_k}^k e^{\xi_t} &= \sum_{a=1}^k \frac{d^a}{dx^a} e^x \Big|_{x=\xi_t} B_{k,a} \left(D_{r_1}^1 \xi_t, D_{r_1, r_2}^2 \xi_t, \dots, D_{r_1, \dots, r_k}^k \xi_t \right) \\ &= \sum_{a=1}^k \exp(\xi_t) B_{k,a} \left(D_{r_1}^1 \xi_t, D_{r_1, r_2}^2 \xi_t, \dots, D_{r_1, \dots, r_k}^k \xi_t \right). \end{aligned}$$

We can repeat exactly the same arguments as in the proof of (4.3), due to the fact that by Lemma 3.9 e^{ξ_t} has moment of all orders, provided we can show that

$$\sup_{r_1, \dots, r_a \in [0, t]} \tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_a}^a \xi_t \right|^p \right] < \infty, \quad 1 \leq a \leq k, \quad (4.5)$$

for any $p \geq 1$. As noted in Remark 4.5, the Malliavin derivative commute with the stochastic integral with respect to Y and we can write

$$\begin{aligned} D_{r_1, \dots, r_a}^a \xi_t &= D_{r_1, \dots, r_a}^a \left(\int_0^t h(X_s) dY_s - \frac{1}{2} \int_0^t h^2(X_s) ds \right) \\ &= \int_0^t D_{r_1, \dots, r_a}^a h(X_s) dY_s - \frac{1}{2} \int_0^t D_{r_1, \dots, r_a}^a (h^2(X_s)) ds. \end{aligned}$$

Hence, by Burkholder-Davis-Gundy inequality and Jensen's inequality, we get for any $p \geq 1$ that

$$\begin{aligned} &\tilde{\mathbb{E}} \left[\left| D_{r_1, \dots, r_a}^a \xi_t \right|^{2p} \right] \\ &\leq C \left\{ \tilde{\mathbb{E}} \left[\left| \int_0^t D_{r_1, \dots, r_a}^a h(X_s) dY_s \right|^{2p} \right] + \tilde{\mathbb{E}} \left[\left| \int_0^t D_{r_1, \dots, r_a}^a (h^2(X_s)) ds \right|^{2p} \right] \right\} \\ &\leq C \left\{ \tilde{\mathbb{E}} \left[\int_0^t \left| D_{r_1, \dots, r_a}^a h(X_s) \right|^{2p} ds \right] + \tilde{\mathbb{E}} \left[\int_0^t \left| D_{r_1, \dots, r_a}^a (h^2(X_s)) \right|^{2p} ds \right] \right\} \\ &\leq C \{A_1 + A_2\}. \end{aligned}$$

Applying Faà di Bruno formula we can write

$$\begin{aligned} A_1 &\leq C \sum_{l=1}^a \int_0^t \tilde{\mathbb{E}} \left[\left| h^{(l)}(X_s) B_{a,l} (D_{r_1}^1 X_s, D_{r_1, r_2}^2 X_s, \dots, D_{r_1, \dots, r_a}^a X_s) \right|^{2p} \right] ds \\ &\leq C \|h\|_{\infty, a}^q \sum_{l=1}^a \int_0^t \tilde{\mathbb{E}} \left[\left| B_{a,l} (D_{r_1}^1 X_s, D_{r_1, r_2}^2 X_s, \dots, D_{r_1, \dots, r_a}^a X_s) \right|^{2p} \right] ds, \end{aligned}$$

where

$$\|h\|_{\infty, a} \triangleq \sum_{i=0}^{d_Y} \sum_{l=0}^a \left\| h_i^{(l)} \right\|_{\infty} < \infty,$$

because $\mathbf{H}(m)$ holds. Therefore, using the generalized version of Hölder inequality, Lemma 4.15, and Lemma 4.4 we get $A_1 < \infty$. We can repeat the same argument for A_2 and obtain (4.5).

4.2 Martingale representations and Clark-Ocone formula

In this section we recall the Clark-Ocone formula. This formula relates the kernels in the Itô martingale representation of Malliavin differentiable functionals with the Malliavin derivatives of such functionals. We present a truncated version of the well known Stroock-Taylor formula, see Stroock [17], that can be seen as an extension of the Clark-Ocone formula and it will be essential in deducing several conditional expectation estimates (see Section 4.4). We also show that, if the coefficients f and σ of the SDE modeling the signal, the sensor function h and the test function φ are regular enough with bounded derivatives then, the kernels in the Stroock-Taylor formula of order m for $\varphi(X_t) e^{\tilde{\varepsilon}_t}$ satisfy a uniform integrability property. Finally, we show that those kernels also satisfy a Hölder continuity property. Note that all sets of multi-indices in this subsection take values in $\{1, \dots, d_V\}$ instead of $\{0, \dots, d_V\}$.

Theorem 4.7 *Let $F \in L^2(\Omega, \mathcal{H}_t^t, \tilde{\mathbb{P}})$. Then, F admits the following martingale representation*

$$F = \tilde{\mathbb{E}}[F | \mathcal{H}_0^t] + \sum_{r=1}^{d_V} \int_0^t J_s^r dV_s^r,$$

where $J^r = \{J_s^r, s \in [0, t]\}$, $r = 1, \dots, d_V$ are $\mathcal{H}_s^t$ -progressively measurable processes such that

$$\tilde{\mathbb{E}} \left[\int_0^t |J_s^r|^2 ds \right] < \infty, \quad r = 1, \dots, d_V.$$

Moreover, if $F \in \mathbb{D}^{1,2}$ then

$$J_s^r = \tilde{\mathbb{E}}[D_s^r F | \mathcal{H}_s^t], \quad s \in [0, t],$$

which is known as the Clark-Ocone formula.

Proof The proof is similar to that of Lemma 17 in Crisan [4] and the proof of the Clark-Ocone formula can be found in Nualart [13], Proposition 1.3.14.

By applying Theorem 4.7 to the kernels $J^r, r = 1, \dots, d_V$ one can get the following result.

Theorem 4.8 (Stroock-Taylor formula of order m) Assume that $F \in L^2(\Omega, \mathcal{H}_t^t, \tilde{P})$. Then, for $m \in \mathbb{N}$ we can write

$$F = \sum_{\beta \in \mathcal{M}_{0,m-1}(1, \dots, d_V)} I_\beta \left(\tilde{\mathbb{E}} \left[J_{s_1, \dots, s_{|\beta|}}^\beta \middle| \mathcal{H}_0^t \right] \right)_{0,t} + \sum_{\beta \in \mathcal{M}_m(1, \dots, d_V)} I_\beta \left(J_{s_1, \dots, s_{|\beta|}}^\beta \right)_{0,t},$$

where the kernels $J_{s_1, \dots, s_{|\beta|}}^\beta$ for $\beta \in \mathcal{M}_{0,m}(1, \dots, d_V)$ are obtained from the martingale representation of $J_{s_2, \dots, s_{|\beta|}}^{-\beta}$, that is, they satisfy the following relationship

$$J^v \triangleq F, \\ J_{s_2, \dots, s_{|\beta|}}^{-\beta} = \tilde{\mathbb{E}} \left[J_{s_2, \dots, s_{|\beta|}}^{-\beta} \middle| \mathcal{H}_0^t \right] + \sum_{\beta_1=1}^{d_V} \int_0^{s_2} J_{s_1, \dots, s_{|\beta|}}^{\beta_1 * (-\beta)} dV_{s_1}^{\beta_1},$$

where $\beta_1 * (-\beta) = (\beta_1, \beta_2, \dots, \beta_m)$. Moreover, if $F \in \mathbb{D}^{m,2}$ then

$$J_{s_1, \dots, s_{|\beta|}}^\beta = \tilde{\mathbb{E}} \left[D_{s_1, \dots, s_{|\beta|}}^\beta F \middle| \mathcal{H}_{s_1}^t \right], \quad \beta \in \mathcal{M}_{0,m}(1, \dots, d_V).$$

Proof We prove the result by induction. For $m = 1$, the result is precisely Theorem 4.7. We assume that the result holds for $m - 1 \geq 0$ and prove that this implies that it also holds for m . By the induction hypothesis we have that

$$F = \sum_{\beta \in \mathcal{M}_{0,m-2}(1, \dots, d_V)} I_\beta \left(\tilde{\mathbb{E}} \left[J_{s_1, \dots, s_{|\beta|}}^\beta \middle| \mathcal{H}_0^t \right] \right)_{0,t} + \sum_{\beta \in \mathcal{M}_{m-1}(1, \dots, d_V)} I_\beta \left(J_{s_1, \dots, s_{|\beta|}}^\beta \right)_{0,t}.$$

Applying Theorem 4.7 to $J_{s_1, \dots, s_{|\beta|}}^\beta, \beta \in \mathcal{M}_{m-1}(1, \dots, d_V)$ we get

$$\begin{aligned} F &= \sum_{\beta \in \mathcal{M}_{0,m-2}(1, \dots, d_V)} I_\beta \left(\tilde{\mathbb{E}} \left[J_{s_1, \dots, s_{|\beta|}}^\beta \middle| \mathcal{H}_0^t \right] \right)_{0,t} + \sum_{\beta \in \mathcal{M}_{m-1}(1, \dots, d_V)} I_\beta \left(\tilde{\mathbb{E}} \left[J_{s_1, \dots, s_{|\beta|}}^\beta \middle| \mathcal{H}_0^t \right] \right)_{0,t} \\ &+ \sum_{\beta \in \mathcal{M}_{m-1}(1, \dots, d_V)} \sum_{r=1}^{d_V} I_{r*\beta} \left(J_{s, s_1, \dots, s_{|\beta|}}^{r*\beta} \right)_{0,t} \\ &= \sum_{\beta \in \mathcal{M}_{0,m-1}(1, \dots, d_V)} I_\beta \left(\tilde{\mathbb{E}} \left[J_{s_1, \dots, s_{|\beta|}}^\beta \middle| \mathcal{H}_0^t \right] \right)_{0,t} + \sum_{\beta \in \mathcal{M}_m(1, \dots, d_V)} I_\beta \left(J_{s_1, \dots, s_{|\beta|}}^\beta \right)_{0,t}, \end{aligned}$$

where in the last equality we have used that

$$\mathcal{M}_{0,m-1}(1, \dots, d_V) = \mathcal{M}_{0,m-2}(1, \dots, d_V) \dot{\cup} \mathcal{M}_{m-1}(1, \dots, d_V).$$

The Clark-Ocone representation of the kernels also follows from a straightforward induction.

Proposition 4.9 Let $m \in \mathbb{N}$ and assume that $\mathbf{H}(m)$ holds and $\varphi \in C_p^{m+1}$. Then, the kernels $J_{s_1, \dots, s_{|\beta|}}^\beta$, $\beta \in \mathcal{M}_{0, m+1}(1, \dots, d_V)$ appearing in the Stroock-Taylor formula of order $m+1$ for $F = \varphi(X_t) e^{\xi_t}$ satisfy

$$\sup_{0 \leq s_1 < \dots < s_{|\beta|} \leq t} \tilde{\mathbb{E}} \left[\left| J_{s_1, \dots, s_{|\beta|}}^\beta \right|^p \right] < \infty,$$

for $p \geq 1$

Proof It is a straightforward combination of Theorem 4.8, Jensen's inequality for conditional expectations and Lemma 4.6.

Lemma 4.10 Assume that $\mathbf{H}(1)$ holds and $\varphi \in C_p^2$. Then, the kernels $J^r = \{J_s^r, s \in [0, t]\}$, $r = 1, \dots, d_V$ in the martingale representation of $\varphi(X_t) e^{\xi_t}$ satisfy the following Hölder continuity property:

$$\tilde{\mathbb{E}} \left[|J_s^r - J_u^r|^{2p} \right] \leq C(s-u)^p, \quad 0 \leq u \leq s \leq t,$$

for $p \geq 1$.

Proof The idea is to use the Clark-Ocone formula, Theorem 4.7. That is, one has the following representation

$$J_s^r = \tilde{\mathbb{E}} \left[D_s^r \left\{ \varphi(X_t) e^{\xi_t} \right\} \middle| \mathcal{H}_s^t \right], \quad 0 \leq s \leq t,$$

where D_s^r denotes the Malliavin derivative with respect to V^r . Hence, we can write

$$\begin{aligned} J_s^r - J_u^r &= \tilde{\mathbb{E}} \left[D_s^r \left\{ \varphi(X_t) e^{\xi_t} \right\} - D_u^r \left\{ \varphi(X_t) e^{\xi_t} \right\} \middle| \mathcal{H}_s^t \right] \\ &\quad + \tilde{\mathbb{E}} \left[D_u^r \left\{ \varphi(X_t) e^{\xi_t} \right\} \middle| \mathcal{H}_s^t \right] - \tilde{\mathbb{E}} \left[D_u^r \left\{ \varphi(X_t) e^{\xi_t} \right\} \middle| \mathcal{H}_u^t \right] \\ &\triangleq A_1 + A_2. \end{aligned}$$

For the term A_1 , note that we can write

$$\begin{aligned} &D_s^r \left\{ \varphi(X_t) e^{\xi_t} \right\} - D_u^r \left\{ \varphi(X_t) e^{\xi_t} \right\} \\ &= e^{\xi_t} (D_s^r \varphi(X_t) - D_u^r \varphi(X_t)) + \varphi(X_t) (D_s^r e^{\xi_t} - D_u^r e^{\xi_t}) \\ &= e^{\xi_t} \sum_{j=1}^{d_X} \partial_j \varphi(X_t) (D_s^r X_t^j - D_u^r X_t^j) + e^{\xi_t} \varphi(X_t) (D_s^r \xi_t - D_u^r \xi_t), \end{aligned}$$

and

$$\begin{aligned} D_s^r \xi_t - D_u^r \xi_t &= \sum_{i=1}^{d_Y} \sum_{k=1}^{d_X} \int_0^t \partial_k h^i(X_v) (D_s^r X_v^j - D_u^r X_v^j) dY_v^i \\ &\quad + \frac{1}{2} \sum_{i=1}^{d_Y} \sum_{k=1}^{d_X} \int_0^t \partial_k \{h^i(X_v)^2\} (D_s^r X_v^j - D_u^r X_v^j) dv. \end{aligned}$$

Hence, the result follows from the fact that

$$\tilde{\mathbb{E}} \left[\left| D_s^r X_v^j - D_u^r X_v^j \right|^{2p} \right] \leq C (s-u)^p,$$

$D_s^r X_t^j$ satisfies an evolution equation drive by a Brownian motion, see Section 2.2.2 in Nualart [13]. For the term A_2 the result follows from the martingale representation theorem, Theorem 4.7, applied to the random variable $D_u \left\{ \varphi(X_t) e^{\xi_t} \right\} \in L^2(\Omega, \mathcal{H}_t^t, \tilde{P})$ which yields

$$D_u^r \left\{ \varphi(X_t) e^{\xi_t} \right\} = \tilde{\mathbb{E}} \left[D_u^r \left\{ \varphi(X_t) e^{\xi_t} \right\} \middle| \mathcal{H}_0^t \right] + \sum_{r_1=1}^{d_V} \int_0^t G_v^{r_1} dV_v^{r_1},$$

and, hence, $A_2 = \sum_{r_1=1}^{d_V} \int_u^s G_v^{r_1} dV_v^{r_1}$ and

$$\begin{aligned} \tilde{\mathbb{E}} \left[|A_2|^{2p} \right] &\leq C \sum_{r_1=1}^{d_V} \tilde{\mathbb{E}} \left[\left| \int_u^s |G_v^{r_1}|^2 dv \right|^p \right] \\ &\leq C (s-u)^{p-1} \sum_{r_1=1}^{d_V} \int_u^s \tilde{\mathbb{E}} \left[|G_v^{r_1}|^{2p} \right] dv \\ &\leq C (s-u)^p \sum_{r_1=1}^{d_V} \sup_{v \in [0,t]} \tilde{\mathbb{E}} \left[|G_v^{r_1}|^{2p} \right] \\ &\leq C (s-u)^p, \end{aligned}$$

where in the last inequality we have used Proposition 4.9.

Remark 4.11 If $\mathbf{H}(m)$ holds and $\varphi \in C_p^{m+1}$, using the same reasonings as in Lemma 4.10, one can show that the kernels $J^\beta, \beta \in \mathcal{M}_{0,m}(1, \dots, d_V)$ in the Stroock-Taylor formula for $\varphi(X_t) e^{\xi_t}$ satisfy the following Hölder continuity property:

$$\tilde{\mathbb{E}} \left[\left| J_{s_1, \dots, s_i, \dots, s_{|\beta|}}^\beta - J_{s_1, \dots, s_{i-1}, u, s_{i+1}, \dots, s_{|\beta|}}^\beta \right|^{2p} \right] \leq C |s-u|^p,$$

for $p \geq 1, s_{i-1} \leq u \leq s \leq s_{i+1}, i = 2, \dots, m-1$.

4.3 Backward martingales estimates

In this section we start reviewing some basic concepts of backward Itô integration that can be found, for instance, in Pardoux and Protter [15], Bensoussan [3] and Applebaum [1]. Then we compute some technical estimates related to products of backward Itô integrals and backward stochastic exponentials that will be useful in the next section.

We know that under $\tilde{P}$ the observation process Y is a Brownian motion with respect to the filtration $\{\mathcal{Y}_t\}_{t \geq 0}$. For fixed $t \geq 0$, we can consider the process $\overleftarrow{Y} =$

$\{\overleftarrow{Y}_s \triangleq Y_s - Y_t\}_{0 \leq s \leq t}$ which is a Brownian motion with respect to the backward filtration

$$\left\{ \mathcal{Y}_s^t \triangleq \sigma \left(\overleftarrow{Y}_{u,s} \leq u \leq t \right) \vee \mathcal{N} \right\}_{0 \leq s \leq t},$$

where $\mathcal{N}$ are all the P -null sets of $(\Omega, \mathcal{F}, P)$. We can also consider the filtration

$$\left\{ \mathcal{Y}_s^{0,V} \triangleq \mathcal{F}_t^{0,V} \vee \mathcal{Y}_s^t \right\}_{0 \leq s \leq t}$$

and the backward filtration

$$\left\{ \mathcal{Y}_s^{0,V,t} \triangleq \mathcal{F}_t^{0,V} \vee \mathcal{Y}_s^t \right\}_{0 \leq s \leq t}.$$

As X_0 and V are independent of Y under $\tilde{P}$, we also have that Y is a $\mathbb{Y}^{0,V}$ -Brownian motion and $\overleftarrow{Y}$ is $\mathbb{Y}^{0,V,t}$ -Brownian motion.

If $\eta = \{\eta_s^1, \dots, \eta_s^{d_Y}\}_{0 \leq s \leq t}$ is a square integrable measurable process adapted to $\{\mathcal{Y}_s^{0,V,t}\}_{0 \leq s \leq t}$ we can define the backward Itô integral of η with respect to $\overleftarrow{Y}$ by

$$\begin{aligned} \int_s^t \eta_u d\overleftarrow{Y}_u &\triangleq \sum_{i=1}^{d_Y} \int_s^t \eta_u^i d\overleftarrow{Y}_u^i \\ &\triangleq L^2(\tilde{P}) - \lim_{\tau \in \Pi(t), |\tau| \rightarrow 0} \sum_{i=1}^{d_Y} \sum_{j=0}^{n-1} \eta_{t_{j+1}}^i \left(\overleftarrow{Y}_{t_{j+1} \vee s}^i - \overleftarrow{Y}_{t_j \vee s}^i \right), \\ &= L^2(\tilde{P}) - \lim_{\tau \in \Pi(t), |\tau| \rightarrow 0} \sum_{i=1}^{d_Y} \sum_{j=0}^{n-1} \eta_{t_{j+1}}^i \left(Y_{t_{j+1} \vee s}^i - Y_{t_j \vee s}^i \right). \end{aligned}$$

Remark 4.12 If a square integrable process $\theta = \{\theta_u\}_{0 \leq u \leq t}$ is simultaneously adapted to $\{\mathcal{Y}_s^{0,V}\}_{0 \leq s \leq t}$ and $\{\mathcal{Y}_s^{0,V,t}\}_{0 \leq s \leq t}$ both, the Itô and the backward Itô integrals, can be defined over the same interval but, in general, they will be different. However, if θ_u is measurable with respect to $\mathcal{Y}_0^{0,V} = \mathcal{Y}_t^{0,V,t} = \mathcal{F}_t^{0,V}$ for all $0 \leq u \leq t$, then both integrals coincide. In fact, they coincide with the Stratonovich integral, see Pardoux and Protter [15]. This means that in the statement of Lemma 4.16 we can change all backward Itô integrals by Itô integrals and the estimates will hold true. However, in the proof of Lemma 4.16 we use the properties of the backward integral and for that reason we keep the notation of backward integration.

The backward Itô integral is analogous to the Itô integral. In particular, the backward Itô integral has zero expectation and it is a backward martingale with respect to $\{\mathcal{Y}_s^{0,V,t}\}_{0 \leq s \leq t}$, that is

$$\mathbb{E} \left[\int_{s_1}^t \eta_u d\overleftarrow{Y}_u \middle| \mathcal{Y}_{s_2}^{0,V,t} \right] = \int_{s_2}^t \eta_u d\overleftarrow{Y}_u, \quad 0 \leq s_1 < s_2 \leq t.$$

A backward Itô process is a process of the following form

$$Z_s = Z_t + \int_s^t v_u du + \int_s^t \eta_u d\overleftarrow{Y}_u, \quad 0 \leq s \leq t,$$

where v and η are two square integrable, measurable and $\mathcal{Y}_s^{0,V,t}$ -adapted processes of the appropriate dimensions. For backward Itô processes and $f \in C^{1,2}((0,t) \times \mathbb{R}^{dz}; \mathbb{R})$ we have the following Itô formula, see Bensoussan [3],

$$f(s, Z_s) = f(t, Z_t) + \int_s^t \left\{ -\partial_t f(u, Z_u) + v_u Df(u, Z_u) + \frac{1}{2} \text{tr}(D^2 f(u, Z_u) \eta_u \eta_u^T) \right\} du \\ + \int_s^t Df(u, Z_u) \eta_u d\overleftarrow{Y}_u,$$

where D and D^2 stand for the gradient and the Hessian, respectively, with respect to the space variables. As a corollary, one gets the integration by parts formula

$$\left(\int_s^t \eta_u^i d\overleftarrow{Y}_u^i \right) \left(\int_s^t \eta_u^j d\overleftarrow{Y}_u^j \right) = \int_s^t \left(\int_u^t \eta_v^i d\overleftarrow{Y}_v^i \right) \eta_u^j d\overleftarrow{Y}_u^j \\ + \int_s^t \left(\int_u^t \eta_v^j d\overleftarrow{Y}_v^j \right) \eta_u^i d\overleftarrow{Y}_u^i \\ + \mathbf{1}_{\{i=j\}} \int_s^t \eta_u^i \eta_u^j du. \quad (4.6)$$

and

$$\tilde{\mathbb{E}} \left[\left(\int_s^t \eta_u^i d\overleftarrow{Y}_u^i \right) \left(\int_s^t \eta_u^j d\overleftarrow{Y}_u^j \right) \right] = \mathbf{1}_{\{i=j\}} \tilde{\mathbb{E}} \left[\int_s^t \eta_u^i \eta_u^j du \right]. \quad (4.7)$$

Let $\phi \in \mathcal{B}_b(\mathbb{R}^{dx}; \mathbb{R}^{dy})$ be a bounded measurable function and let $M^t(\phi) = \{M_s^t(\phi)\}_{0 \leq s \leq t}$ be the process

$$M_s^t(\phi) = \exp \left(\sum_{i=1}^{dy} \int_s^t \phi_i(X_u) d\overleftarrow{Y}_u^i - \frac{1}{2} \sum_{i=1}^{dy} \int_s^t \phi_i^2(X_u) du \right).$$

It is easy to show that $M^t(\phi)$ is a backward martingale with respect to $\{\mathcal{Y}_s^{0,V,t}\}_{0 \leq s \leq t}$ and applying the backward Itô formula one finds the following formula for the increments of $M^t(\phi)$

$$M_{s_1}^t(\phi) = M_{s_2}^t(\phi) + \sum_{i=1}^{dy} \int_{s_1}^{s_2} M_u^t(\phi) \phi_i(X_u) d\overleftarrow{Y}_u^i. \quad (4.8)$$

Moreover, by the same reasoning as in Lemma 3.9, we have for all $p \geq 1$

$$\tilde{\mathbb{E}}[|M_s^t(\phi)|^p] < \infty. \quad (4.9)$$

Lemma 4.13 *Let $0 \leq s_1 \leq s_2 \leq s_3 \leq t$, Ψ be $\mathcal{Y}_{s_3}^{0,V,t}$ -measurable random variable and $\theta = \{\theta_u\}_{0 \leq u \leq t}$ a square integrable and measurable process such that θ_u is measurable with respect to $\mathcal{Y}_{s_3}^{0,V,t}$ for all $s_2 \leq u \leq s_3$. Then,*

$$\tilde{\mathbb{E}} \left[\Psi M_{s_1}^t(\phi) \int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i \right] = \tilde{\mathbb{E}} \left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \phi_i(X_u) \theta_u du \right].$$

Proof By the backward martingale properties of $M_s^t(\phi)$ and equation (4.8) we can write

$$\begin{aligned}\tilde{\mathbb{E}}\left[\Psi M_{s_1}^t(\phi) \int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right] &= \tilde{\mathbb{E}}\left[\Psi \tilde{\mathbb{E}}[M_{s_1}^t(\phi) | \mathcal{Y}_{s_2}^{0,V,t}] \int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right] \\ &= \tilde{\mathbb{E}}\left[\Psi M_{s_2}^t(\phi) \int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right] \\ &= \tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right] \\ &\quad + \sum_{i_1=1}^{d_Y} \tilde{\mathbb{E}}\left[\Psi \left(\int_{s_2}^{s_3} M_u^t(\phi) \phi_{i_1}(X_u) d\overleftarrow{Y}_u^{i_1}\right) \left(\int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right)\right]\end{aligned}$$

Next, note that

$$\tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right] = \tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \tilde{\mathbb{E}}\left[\int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i \middle| \mathcal{Y}_{s_3}^{0,V,t}\right]\right] = 0,$$

and using equation (4.7) we have

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\Psi \left(\int_{s_2}^{s_3} M_u^t(\phi) \phi_{i_1}(X_u) d\overleftarrow{Y}_u^{i_1}\right) \left(\int_{s_2}^{s_3} \theta_u d\overleftarrow{Y}_u^i\right)\right] \\ &= \mathbf{1}_{\{i_1=i\}} \tilde{\mathbb{E}}\left[\Psi \left(\int_{s_2}^{s_3} M_u^t(\phi) \phi_{i_1}(X_u) \theta_u du\right)\right] \\ &= \mathbf{1}_{\{i_1=i\}} \tilde{\mathbb{E}}\left[\Psi \left(\int_{s_2}^{s_3} \tilde{\mathbb{E}}[M_u^t(\phi) | \mathcal{Y}_{s_3}^{0,V,t}] \phi_{i_1}(X_u) \theta_u du\right)\right] \\ &= \mathbf{1}_{\{i_1=i\}} \tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \left(\int_{s_2}^{s_3} \phi_{i_1}(X_u) \theta_u du\right)\right].\end{aligned}$$

Hence, the result follows.

Lemma 4.14 Let $0 \leq s_1 \leq s_2 \leq s_3 \leq t$, Ψ be $\mathcal{Y}_{s_3}^{0,V,t}$ -measurable random variable and $\theta^1 = \{\theta_u^1\}_{0 \leq u \leq t}$ and $\theta^2 = \{\theta_u^2\}_{0 \leq u \leq t}$ be two square integrable measurable processes such that θ_u^1 and θ_u^2 are also measurable with respect to $\mathcal{Y}_{s_3}^{0,V,t}$ for all $s_2 \leq u \leq s_3$. Then,

$$\begin{aligned}&\tilde{\mathbb{E}}\left[\Psi M_{s_1}^t(\phi) \left(\int_{s_2}^{s_3} \theta_u^1 d\overleftarrow{Y}_u^{i_1}\right) \left(\int_{s_2}^{s_3} \theta_u^2 d\overleftarrow{Y}_u^{i_2}\right)\right] \\ &= \tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \phi_{i_1}(X_u) \theta_u^1 \left(\int_u^{s_3} \phi_{i_2}(X_v) \theta_v^2 dv\right) du\right] \\ &\quad + \tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \phi_{i_2}(X_u) \theta_u^2 \left(\int_u^{s_3} \phi_{i_1}(X_v) \theta_v^1 dv\right) du\right] \\ &\quad + \mathbf{1}_{\{i_1=i_2\}} \tilde{\mathbb{E}}\left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \theta_u^1 \theta_u^2 du\right].\end{aligned}$$

Proof Using the integration by parts formula (4.6) we can write

$$\begin{aligned} \tilde{\mathbb{E}} \left[\Psi M_{s_1}^t(\phi) \left(\int_{s_2}^{s_3} \theta_u^1 dY_u^{\overleftarrow{i_1}} \right) \left(\int_{s_2}^{s_3} \theta_u^2 dY_u^{\overleftarrow{i_2}} \right) \right] &= \tilde{\mathbb{E}} \left[\Psi M_{s_1}^t(\phi) \int_{s_2}^{s_3} \theta_u^1 \left(\int_u^{s_3} \theta_v^2 dY_v^{\overleftarrow{i_2}} \right) dY_u^{\overleftarrow{i_1}} \right] \\ &\quad + \tilde{\mathbb{E}} \left[\Psi M_{s_1}^t(\phi) \int_{s_2}^{s_3} \theta_u^2 \left(\int_u^{s_3} \theta_v^1 dY_v^{\overleftarrow{i_1}} \right) dY_u^{\overleftarrow{i_2}} \right] \\ &\quad + \mathbf{1}_{\{i_1=i_2\}} \tilde{\mathbb{E}} \left[\Psi M_{s_1}^t(\phi) \int_{s_2}^{s_3} \theta_u^1 \theta_u^2 du \right]. \end{aligned}$$

Then, using the same reasonings as in Lemma 4.13 we get that

$$\tilde{\mathbb{E}} \left[\Psi M_{s_1}^t(\phi) \int_{s_2}^{s_3} \theta_u^1 \left(\int_u^{s_3} \theta_v^2 dY_v^{\overleftarrow{i_2}} \right) dY_u^{\overleftarrow{i_1}} \right] = \tilde{\mathbb{E}} \left[\Psi \int_{s_2}^{s_3} M_u^t(\phi) \phi_{i_1}(X_u) \theta_u^1 \left(\int_u^{s_3} \theta_v^2 dY_v^{\overleftarrow{i_2}} \right) du \right]$$

Next, by Fubini's theorem, Lemma 4.13 and Fubini's theorem again we obtain

$$\begin{aligned} &\tilde{\mathbb{E}} \left[\Psi \int_{s_2}^{s_3} M_u^t(\phi) \phi_{i_1}(X_u) \theta_u^1 \left(\int_u^{s_3} \theta_v^2 dY_v^{\overleftarrow{i_2}} \right) du \right] \\ &= \int_{s_2}^{s_3} \tilde{\mathbb{E}} \left[\Psi \phi_{i_1}(X_u) \theta_u^1 M_u^t(\phi) \left(\int_u^{s_3} \theta_v^2 dY_v^{\overleftarrow{i_2}} \right) \right] du \\ &= \int_{s_2}^{s_3} \tilde{\mathbb{E}} \left[\Psi \phi_{i_1}(X_u) \theta_u^1 M_{s_3}^t(\phi) \left(\int_u^{s_3} \phi_{i_2}(X_u) \theta_v^2 dv \right) \right] du \\ &= \tilde{\mathbb{E}} \left[\Psi M_{s_3}^t(\phi) \int_{s_2}^{s_3} \phi_{i_1}(X_u) \theta_u^1 \left(\int_u^{s_3} \phi_{i_2}(X_u) \theta_v^2 dv \right) du \right] \end{aligned}$$

By symmetry we get an analogous expression for the term $\int_{s_2}^{s_3} \theta_u^2 \left(\int_u^{s_3} \theta_v^1 dY_v^{\overleftarrow{i_1}} \right) dY_u^{\overleftarrow{i_2}}$.

Finally, for the last term we only need to take conditional expectation with respect to $\mathcal{G}_{s_3}^{0,V,t}$ and use that $M_s^t(\phi)$ is a $\mathcal{G}_s^{0,V,t}$ -martingale.

The next lemma is a well known generalization of Hölder's inequality.

Lemma 4.15 (Generalized Hölder's inequality) *Let $p_i > 1, i = 1, \dots, m$ such that $\sum_{i=1}^m \frac{1}{p_i} = 1$, and $X_i \in L^{p_i}(\Omega, \mathcal{F}, \tilde{P}), i = 1, \dots, m$. Then,*

$$\tilde{\mathbb{E}} \left[\left| \prod_{i=1}^m X_i \right| \right] \leq \prod_{i=1}^m \tilde{\mathbb{E}} [|X_i|^{p_i}]^{1/p_i} < \infty.$$

Lemma 4.16 *Let $\tau = \{0 = t_0 < t_1 < \dots < t_n = t\}$ be a partition of $[0, t]$, $\phi \in \mathcal{B}_b(\mathbb{R}^{d_X}; \mathbb{R}^{d_Y})$, $\Upsilon \in L^p(\Omega, \mathcal{G}_t^{0,V,t}, \tilde{P})$ for any $p \geq 1$, β_s , be a deterministic processes satisfying*

$$\left| \beta_s \mathbf{1}_{[t_j, t_{j+1}]}(s) \right| \leq \delta^m \quad (4.10)$$

for some $m \in \mathbb{N}$ and $\theta^1, \theta^2, \kappa^1, \kappa^2$, be stochastic processes measurable with respect to $\mathcal{G}_t^{0,V,t}$, such that

$$\begin{aligned} \sup_{s \in [0, t]} \tilde{\mathbb{E}} \left[|\theta_s^j|^p \right] &< \infty, \quad j = 1, 2, \\ \sup_{s \in [0, t]} \tilde{\mathbb{E}} \left[|\kappa_s^j|^p \right] &< \infty, \quad j = 1, 2, \end{aligned} \quad (4.11)$$

for any $p \geq 1$. Then:

1. For $i \in \{1, \dots, d_Y\}$, we have that

$$\left| \tilde{\mathbb{E}} \left[\Upsilon M_0^t(\phi) \left(\int_{t_j}^{t_{j+1}} \beta_s d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^{i_1} \right) \right] \right| \leq C \{ \mathbf{1}_{\{j \neq k\}} \delta^{2m+2} + \mathbf{1}_{\{j=k\}} \delta^{2m+1} \}.$$

2. For $i, i_1 \in \{1, \dots, d_Y\}$, we have that

$$\left| \tilde{\mathbb{E}} \left[\Upsilon M_0^t(\phi) \left(\int_{t_{k+1}}^t \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_j}^{t_{j+1}} \beta_s d\overleftarrow{Y}_s^i \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) \right] \right| \leq C \{ \mathbf{1}_{\{j \neq k\}} \delta^{2m+2} + \mathbf{1}_{\{j=k\}} \delta^{2m+1} \}.$$

3. For $i, i_1, a_1 \in \{1, \dots, d_Y\}$, we have that

$$\left| \tilde{\mathbb{E}} \left[\Upsilon M_0^t(\phi) \left(\int_{t_{j+1}}^t \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{k+1}}^t \kappa_s^1 d\overleftarrow{Y}_s^{a_1} \right) \left(\int_{t_j}^{t_{j+1}} \beta_s d\overleftarrow{Y}_s^i \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) \right] \right| \leq C \{ \mathbf{1}_{\{j \neq k\}} \delta^{2m+2} + \mathbf{1}_{\{j=k\}} \delta^{2m+1} \}.$$

4. For $i, i_1, i_2, a_1, a_2 \in \{1, \dots, d_Y\}$, we have that

$$\left| \tilde{\mathbb{E}} \left[\Upsilon M_0^t(\phi) \prod_{l=1}^2 \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\overleftarrow{Y}_s^{i_l} \right) \left(\int_{t_{k+1}}^t \kappa_s^l d\overleftarrow{Y}_s^{a_l} \right) \right\} \left(\int_{t_j}^{t_{j+1}} \beta_s d\overleftarrow{Y}_s^i \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) \right] \right| \leq C \{ \mathbf{1}_{\{j \neq k\}} \delta^{2m+2} + \mathbf{1}_{\{j=k\}} \delta^{2m+1} \}.$$

Proof The full proof of the lemma is lengthy and depends on applying Lemmas 4.13 and 4.14 repeatedly. We do not present it in full, but only write in detail the proof of the statement (4), the others being similar and easier. Note that by the assumptions on Υ, β^j and θ^j and ϕ the expectations in the statement of the lemma are finite. We start with some preliminary estimations. In what follows C will denote constants that does not depend on δ . For any $p \geq 1$ we have

– Let $0 \leq s \leq t$, then

$$\begin{aligned} \tilde{\mathbb{E}} [|M_s^t(\phi)|^p] &= \tilde{\mathbb{E}} \left[M_s^t(p\phi) \exp \left(\frac{p^2 - p}{2} \sum_{i=1}^{d_Y} \int_s^t \phi_i^2(X_u) du \right) \right] \\ &\leq \exp \left(d_Y \|\phi\|_\infty^2 \frac{p^2 - p}{2} (t - s) \right), \end{aligned}$$

where we have used that for $\phi \in \mathcal{B}_b(\mathbb{R}^{d_X}; \mathbb{R}^{d_Y})$ one has that $M_s^t(\phi)$ is a backward martingale with expectation equal to one. The previous estimate yields that

$$\sup_{s \in [0, t]} \tilde{\mathbb{E}} [|M_s^t(\phi)|^p] \leq \exp \left(d_Y \|\phi\|_\infty^2 \frac{p^2 - p}{2} t \right) \leq C. \quad (4.12)$$

– Let $0 \leq s_1 \leq s_2 \leq t$, then

$$\begin{aligned}
\tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} \theta_u \overleftarrow{Y}_u^i \right|^p \right] &\leq \tilde{\mathbb{E}} \left[\left(\int_{s_1}^{s_2} |\theta_u|^2 du \right)^{p/2} \right] \\
&\leq \tilde{\mathbb{E}} \left[(s_2 - s_1)^{p/2-1} \left(\int_{s_1}^{s_2} |\theta_u|^p du \right) \right] \\
&\leq \sup_{s \in [0, t]} \tilde{\mathbb{E}} [|\theta_s|^p] (s_2 - s_1)^{p/2} \\
&\leq C (s_2 - s_1)^{p/2},
\end{aligned} \tag{4.13}$$

where we have used the Burkholder-Davis-Gundy inequality for backward martingales, Jensen's inequality and Fubini's theorem. The previous estimate yields that

$$\sup_{0 \leq s_1 \leq s_2 \leq t} \tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} \theta_u \overleftarrow{Y}_u^i \right|^p \right] \leq \sup_{s \in [0, t]} \tilde{\mathbb{E}} [|\theta_s|^p] t^{p/2} \leq C. \tag{4.14}$$

Moreover, using Jensen's inequality and Fubini's theorem we have that

$$\begin{aligned}
\tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} \phi_i(X_s) \theta du \right|^p \right] &\leq \|\phi\|_\infty^p \tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} |\theta| du \right|^p \right] \\
&\leq \|\phi\|_\infty^p \tilde{\mathbb{E}} \left[(s_2 - s_1)^{p-1} \int_{s_1}^{s_2} |\theta|^p du \right] \\
&\leq \|\phi\|_\infty^p \sup_{s \in [0, t]} \tilde{\mathbb{E}} [|\theta_s|^p] (s_2 - s_1)^p.
\end{aligned}$$

The previous estimate yields that

$$\tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} \phi_i(X_s) \theta du \right|^p \right] \leq \|\phi\|_\infty^p \sup_{s \in [0, t]} \tilde{\mathbb{E}} [|\theta_s|^p] t^{p/2} \leq C. \tag{4.15}$$

– Let $0 \leq s_1 \leq s_2 \leq t$, then similar reasonings as in the previous point and hypothesis (4.10) give

$$\begin{aligned}
\tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} \beta_u \overleftarrow{Y}_u^i \right|^p \right] &\leq \tilde{\mathbb{E}} \left[\left(\int_{s_1}^{s_2} |\beta_u|^2 du \right)^{p/2} \right] \\
&\leq (s_2 - s_1)^{p/2-1} \int_{s_1}^{s_2} |\beta_u|^p du \\
&\leq t^{p/2-1} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\beta_u|^p du \\
&\leq t^{p/2} \delta^{mp} \leq C
\end{aligned} \tag{4.16}$$

Moreover, if $s_1 = t_j$ and $s_2 = t_{j+1}$ for some $j \in \{0, \dots, n-1\}$ we can conclude using hypothesis (4.10) that

$$\tilde{\mathbb{E}} \left[\left| \int_{s_1}^{s_2} \beta_u \overleftarrow{Y}_u^i \right|^p \right] \leq \delta^{p/2-1} \int_{t_j}^{t_{j+1}} \delta^{mp} du \leq \delta^{(m+\frac{1}{2})p}. \tag{4.17}$$

Finally,

$$\mathbb{E} \left[\left| \int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right|^p \right] \leq \|\phi\|_\infty^p \delta^{(m+1)p} = C \delta^{(m+1)p} \quad (4.18)$$

▷ **Case $j = k$:**

Using Cauchy-Schwarz inequality, Lemma 4.15 and inequalities (4.12), (4.14) and (4.17) we can write

$$\begin{aligned} & \tilde{\mathbb{E}} \left[r M_0^t(\phi) \prod_{l=1}^2 \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\bar{Y}_s^{i_l} \right) \left(\int_{t_{j+1}}^t \kappa_s^l d\bar{Y}_s^{a_l} \right) \right\} \left(\int_{t_j}^{t_{j+1}} \beta_s d\bar{Y}_s^i \right)^2 \right] \\ & \leq \tilde{\mathbb{E}} \left[\left| r M_0^t(\phi) \prod_{l=1}^2 \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\bar{Y}_s^{i_l} \right) \left(\int_{t_{j+1}}^t \kappa_s^l d\bar{Y}_s^{a_l} \right) \right\} \right|^2 \right]^{1/2} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \beta_s d\bar{Y}_s^i \right)^4 \right]^{1/2} \\ & \leq \left(\tilde{\mathbb{E}}[|r|^{12}] \tilde{\mathbb{E}}[|M_0^t(\phi)|^{12}] \prod_{l=1}^2 \tilde{\mathbb{E}} \left[\left| \int_{t_{j+1}}^t \theta_s^l d\bar{Y}_s^{i_l} \right|^{12} \right] \tilde{\mathbb{E}} \left[\left| \int_{t_{j+1}}^t \kappa_s^l d\bar{Y}_s^{a_l} \right|^{12} \right] \right)^{1/12} \delta^{\frac{1}{2}(m+\frac{1}{2})4} \\ & \leq C \delta^{2m+1} \end{aligned}$$

▷ **Case $j < k$:**

Using Lemma 4.13, we can write

$$\begin{aligned} & \tilde{\mathbb{E}} \left[r M_0^t(\phi) \prod_{l=1}^2 \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\bar{Y}_s^{i_l} \right) \left(\int_{t_{k+1}}^t \kappa_s^l d\bar{Y}_s^{a_l} \right) \right\} \left(\int_{t_j}^{t_{j+1}} \beta_s d\bar{Y}_s^i \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\bar{Y}_s^i \right) \right] \\ & = \tilde{\mathbb{E}} \left[r \left(\int_{t_k}^{t_{k+1}} \beta_s d\bar{Y}_s^i \right) \prod_{l=1}^2 \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\bar{Y}_s^{i_l} \right) \left(\int_{t_{k+1}}^t \kappa_s^l d\bar{Y}_s^{a_l} \right) \right\} \right. \\ & \quad \left. \times M_{t_{j+1}}^t(\phi) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \right] \\ & = \tilde{\mathbb{E}} \left[r_1 M_{t_{j+1}}^t(\phi) \prod_{l=1}^2 \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\bar{Y}_s^{i_l} \right) \right\} \right] \\ & \triangleq \sum_{i=1}^9 A_i, \end{aligned}$$

Where

$$r_1 = r \left(\int_{t_k}^{t_{k+1}} \beta_s d\bar{Y}_s^i \right) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \prod_{l=1}^2 \left(\int_{t_{k+1}}^t \kappa_s^l d\bar{Y}_s^{a_l} \right),$$

and

$$\begin{aligned}
A_1 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{j+1}}^{t_k} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_2 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_3 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{k+1}}^t \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_4 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_k}^{t_{k+1}} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{j+1}}^{t_k} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_5 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_k}^{t_{k+1}} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_6 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_k}^{t_{k+1}} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{k+1}}^t \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_7 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{k+1}}^t \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{j+1}}^{t_k} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_8 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{k+1}}^t \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
A_9 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{k+1}}^t \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{k+1}}^t \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right]
\end{aligned}$$

The treatment of some of the terms is completely analogous. We distinguish four subcases:

▷▷ **Subcase 1:** This subcase covers term A_9 . We apply Lemma 4.14 to write

$$\begin{aligned}
A_9 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{k+1}}^t \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{k+1}}^t \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
&= \tilde{\mathbb{E}} \left[\Upsilon_1 M_{t_{k+1}}^t(\phi) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \left(\int_{t_k}^{t_{k+1}} \phi_i(X_s) \beta_s ds \right) \right],
\end{aligned}$$

where

$$\Gamma_1 \triangleq \Upsilon \prod_{l=1}^2 \left(\int_{t_{k+1}}^t \kappa_s^l d\overleftarrow{Y}_s^{a_l} \right) \left(\int_{t_{k+1}}^t \theta_s^l d\overleftarrow{Y}_s^{i_l} \right).$$

Hence, using Lemma 4.15 and inequality (4.14) we have, for $p \geq 1$, that

$$\mathbb{E} [|\Gamma_1|^p] \leq C,$$

and using Lemma 4.15 and inequalities (4.12) and (4.18) we obtain

$$\begin{aligned}
|A_9| &\leq \tilde{\mathbb{E}} \left[|\Gamma_1|^4 \right]^{1/4} \tilde{\mathbb{E}} \left[\left| M_{t_{k+1}}^t(\phi) \right|^4 \right]^{1/4} \mathbb{E} \left[\left| \int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right|^4 \right]^{1/4} \\
&\quad \times \mathbb{E} \left[\left| \int_{t_k}^{t_{k+1}} \phi_i(X_s) \beta_s ds \right|^4 \right]^{1/4} \\
&\leq C \delta^{2m+2}.
\end{aligned}$$

▷▷ **Subcase 2:** The terms A_2, A_4, A_5, A_6 and A_8 are treated analogously. We will write the proof for A_2 . By Lemma 4.14, we can write

$$\begin{aligned} A_2 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M'_{t_{j+1}}(\phi) \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\ &= \tilde{\mathbb{E}} \left[\Gamma_2 M'_{t_{j+1}}(\phi) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) \left(\int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right], \end{aligned}$$

where

$$\Gamma_2 \triangleq \Upsilon \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \prod_{l=1}^2 \left(\int_{t_{k+1}}^t \kappa_s^l d\overleftarrow{Y}_s^{a_l} \right).$$

Hence, using Lemma 4.15 and inequality (4.14) we have, for $p \geq 1$, that

$$\mathbb{E}[|\Gamma_2|^p] \leq C,$$

and using Lemma 4.15 and inequalities (4.12), (4.18), (4.17) and (4.13) we obtain

$$\begin{aligned} |A_2| &\leq \mathbb{E} \left[|\Gamma_2|^5 \right]^{1/5} \mathbb{E} \left[|M'_{t_{j+1}}(\phi)|^5 \right]^{1/5} \mathbb{E} \left[\left| \int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right|^5 \right]^{1/5} \\ &\quad \times \mathbb{E} \left[\left| \int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right|^5 \right]^{1/5} \mathbb{E} \left[\left| \int_{t_k}^{t_{k+1}} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right|^5 \right]^{1/5} \\ &\leq C(t, \phi) \delta^{(m+1)} \delta^{(m+\frac{1}{2})} \delta^{1/2} = C \delta^{2m+2}. \end{aligned}$$

▷▷ **Subcase 3:** The terms A_3 and A_7 are treated analogously. We will write the proof for A_3 . We apply Lemma 4.14 twice to write

$$\begin{aligned} A_3 &= \tilde{\mathbb{E}} \left[\Upsilon_1 M'_{t_{j+1}}(\phi) \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{k+1}}^t \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\ &= \tilde{\mathbb{E}} \left[\Gamma_3 \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) M'_{t_k}(\phi) \left(\int_{t_{j+1}}^{t_k} \phi_{i_1}(X_s) \theta_s^1 ds \right) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \right] \\ &= \tilde{\mathbb{E}} \left[\Gamma_3 \left(\int_{t_{j+1}}^{t_k} \phi_{i_1}(X_s) \theta_s^1 ds \right) M'_{t_{k+1}}(\phi) \left(\int_{t_k}^{t_{k+1}} \phi_i(X_s) \beta_s ds \right) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \right] \end{aligned}$$

where

$$\Gamma_3 = \Upsilon \left(\int_{t_{k+1}}^t \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \prod_{l=1}^2 \left(\int_{t_{k+1}}^t \kappa_s^l d\overleftarrow{Y}_s^{a_l} \right).$$

Using Lemma 4.15 and inequalities (4.14) and (4.15) we have, for $p \geq 1$, that

$$\mathbb{E} \left[\left| \Gamma_3 \left(\int_{t_{j+1}}^{t_k} \phi_{i_1}(X_s) \theta_s^1 ds \right) \right|^p \right] \leq C,$$

and using Lemma 4.15 and inequalities (4.12), (4.18), (4.17) and (4.13)

$$\begin{aligned}
|A_3| &\leq \tilde{\mathbb{E}} \left[\left| \Gamma_3 \left(\int_{t_{j+1}}^{t_k} \phi_{i_1}(X_s) \theta_s^1 ds \right) \right|^4 \right]^{1/4} \tilde{\mathbb{E}} \left[\left| M_{t_{k+1}}^t(\phi) \right|^5 \right]^{1/5} \\
&\quad \times \tilde{\mathbb{E}} \left[\left| \int_{t_k}^{t_{k+1}} \phi_i(X_s) \beta_s ds \right|^4 \right]^{1/4} \tilde{\mathbb{E}} \left[\left| \int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right|^4 \right]^{1/4} \\
&\leq C \delta^{m+1} \delta^{m+1} = C \delta^{2m+1}.
\end{aligned}$$

▷▷ **Subcase 4:** This subcase corresponds to the term A_1 . Applying Lemma 4.14 and Lemma 4.13 we can write

$$\begin{aligned}
A_1 &= \tilde{\mathbb{E}} \left[\Gamma_1 M_{t_{j+1}}^t(\phi) \left(\int_{t_{j+1}}^{t_k} \theta_s^1 d\overleftarrow{Y}_s^{i_1} \right) \left(\int_{t_{j+1}}^{t_k} \theta_s^2 d\overleftarrow{Y}_s^{i_2} \right) \right] \\
&= \tilde{\mathbb{E}} \left[\Gamma_4 \Gamma_5 M_{t_k}^t(\phi) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \right] \\
&= \tilde{\mathbb{E}} \left[\Gamma_4 \Gamma_5 M_{t_{k+1}}^t(\phi) \left(\int_{t_k}^{t_{k+1}} \phi_i(X_s) \beta_s ds \right) \left(\int_{t_j}^{t_{j+1}} \phi_i(X_s) \beta_s ds \right) \right]
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_4 &= \gamma \prod_{l=1}^2 \left(\int_{t_{k+1}}^t \kappa_s^l d\overleftarrow{Y}_s^{a_l} \right) \\
\Gamma_5 &= \mathbf{1}_{\{i_1=i_2\}} \int_{t_{j+1}}^{t_k} \theta_s^1 \theta_s^2 du \\
&\quad + \int_{t_{j+1}}^{t_k} \phi_{i_1}(X_u) \theta_u^1 \left(\int_u^{t_k} \phi_{i_2}(X_v) \theta_v^2 dv \right) du \\
&\quad + \int_{t_{j+1}}^{t_k} \phi_{i_2}(X_u) \theta_u^2 \left(\int_u^{t_k} \phi_{i_1}(X_v) \theta_v^1 dv \right) du
\end{aligned}$$

To finish the proof one follows the same reasonings in the previous subcase taking into account that Γ_4 and Γ_5 have moments of all orders that only depend on t and ϕ .

▷ **Case $j > k$:**

This is completely symmetric to the previous case by swapping the role of the θ 's by the κ 's.

Remark 4.17 Under the assumptions of Lemma 4.16 one can prove that for any $q \in \{1, \dots, m\}$ and $i, i_1, \dots, i_q a_1, \dots, a_q \in \{1, \dots, d_Y\}$, we have that

$$\begin{aligned}
&\left| \tilde{\mathbb{E}} \left[\gamma M_0^t(\phi) \prod_{l=1}^q \left\{ \left(\int_{t_{j+1}}^t \theta_s^l d\overleftarrow{Y}_s^{i_l} \right) \left(\int_{t_{k+1}}^t \kappa_s^l d\overleftarrow{Y}_s^{a_l} \right) \right\} \left(\int_{t_j}^{t_{j+1}} \beta_s d\overleftarrow{Y}_s^i \right) \left(\int_{t_k}^{t_{k+1}} \beta_s d\overleftarrow{Y}_s^i \right) \right] \right| \\
&\leq C \{ \mathbf{1}_{\{j \neq k\}} \delta^{2m+2} + \mathbf{1}_{\{j=k\}} \delta^{2m+1} \}.
\end{aligned}$$

4.4 Conditional expectation estimates

In this subsection we will show the main estimates for conditional expectations with respect to the observation filtration that will allow the proof of our result. Throughout this section we assume that $\varphi \in \mathcal{B}_P$, which ensures that Corollary 3.11 holds, and $m \in \mathbb{N}$.

Lemma 4.18 *Assume that $\mathbf{H}(m)$ holds. For $\alpha \in \mathcal{M}_m(0, \dots, d_Y)$ with $|\alpha|_0 = m$ and $i \in \{0, 1, \dots, d_Y\}$ we have*

$$\tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_0(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right]^2 \right] \leq C \delta^{2m}.$$

Proof Using Jensen inequality, Hölder inequality and Burkholder-Davis-Gundy inequality, if $i \neq 0$, or Jensen inequality, if $i = 0$, we get

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\left| \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \right|^2 \right] \\ & \leq \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \right|^2 \right] \\ & \leq \tilde{\mathbb{E}} \left[\left| \varphi(X_t) e^{\xi_t} \right|^{2+\varepsilon} \right]^{\frac{2}{2+\varepsilon}} \tilde{\mathbb{E}} \left[\left| \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \right|^{2\frac{2+\varepsilon}{\varepsilon}} \right]^{\frac{\varepsilon}{2+\varepsilon}} \\ & \leq C \left\| \varphi(X_t) e^{\xi_t} \right\|_{2+\varepsilon}^2 \tilde{\mathbb{E}} \left[\left| \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} ds \right|^{2\frac{2+\varepsilon}{\varepsilon}} \right]^{\frac{\varepsilon}{2+\varepsilon}}. \end{aligned}$$

Next, using Jensen inequality again, Fubini's Theorem, Assumption $\mathbf{H}(m)$, Lemma 3.5 and that $|\alpha| + |\alpha|_0 = 2m$ we get

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\left| \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} ds \right|^{2\frac{2+\varepsilon}{\varepsilon}} \right] \\ & \leq C \tilde{\mathbb{E}} \left[\left| I_\alpha(L^\alpha h_i(X))_{\tau(s),s} \right|^{2\frac{2+\varepsilon}{\varepsilon}} ds \right] \\ & \leq C \int_0^t \tilde{\mathbb{E}} \left[\sup_{u \in [\tau(s),s]} |L^\alpha h_i(X_u)|^{2\frac{2+\varepsilon}{\varepsilon}} \right] (s - \tau(s))^{\frac{2+\varepsilon}{\varepsilon} \{|\alpha| + |\alpha|_0\}} ds \\ & \leq C \delta^{2m \frac{2+\varepsilon}{\varepsilon}}, \end{aligned}$$

from which follows the result.

Lemma 4.19 *Assume that $\mathbf{H}(m)$ holds. For $\alpha \in \mathcal{M}_m(0, \dots, d_Y)$ with $|\alpha|_0 \neq m$*

$$\tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_0(X))_{\tau(s),s} ds \middle| \mathcal{Y}_t \right]^2 \right] \leq C \delta^{2m}$$

Proof We will give only the proof for the case $m \in \{1, 2\}$. The proof for $m > 2$ follows the same ideas but it is tedious to write down and we leave it to the reader. We split the proof depending on $|\alpha|_0$, the number of zeros in α . If $m = 1$, $|\alpha|_0 \in \{0\}$ and if $m = 2$, $|\alpha|_0 \in \{0, 1\}$. We group the three cases into two: $|\alpha|_0 = m - 1$ and $|\alpha|_0 = 0$, (of course the two overlap when $m = 1$).

Assume that $|\alpha|_0 = m - 1$. Then, using Theorem 4.7 we can write

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} ds \middle| \mathcal{Y}_t \right] \\
&= \int_0^t \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] ds \\
&= \int_0^t \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \middle| \mathcal{H}_0^t \right] I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] ds \\
&\quad + \sum_{r=1}^{d_V} \int_0^t \tilde{\mathbb{E}} \left[\left(\int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] ds \\
&= \int_0^t \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \middle| \mathcal{H}_0^t \right] \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{H}_0^t \right] \middle| \mathcal{Y}_t \right] ds \\
&\quad + \sum_{r=1}^{d_V} \int_0^t \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\left(\int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{H}_0^t \right] \middle| \mathcal{Y}_t \right] ds.
\end{aligned}$$

Moreover, by Lemma 3.8 (1), we get $\tilde{\mathbb{E}} \left[I_\alpha(L^\alpha h_0(X.))_{\tau(s),s} \middle| \mathcal{H}_0^t \right] = 0$ and, by Lemma 3.8 (2), for $r = 1, \dots, d_V$ we have

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\left(\int_0^t J_u^r dV_u^r \right) I_\alpha(L^\alpha h_0(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] \\
&= 1_{\{\alpha_m=0\}} \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\left(\int_{\tau(s)}^s J_v^r dV_v^r \right) I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \middle| \mathcal{Y}_t \right] du \\
&\quad + 1_{\{\alpha_m=r\}} \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[J_u^r I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \middle| \mathcal{Y}_t \right] du.
\end{aligned}$$

Next, using Jensen's inequality, Cauchy-Schwartz inequality, Itô isometry, Lemma 3.5 and Remark 3.1 we get

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\left| \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\left(\int_{\tau(s)}^s J_v^r dV_v^r \right) I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \middle| \mathcal{Y}_t \right] du \right|^2 \right] \\
& \leq C\delta \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\left(\int_{\tau(s)}^s J_v^r dV_v^r \right) I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \middle| \mathcal{Y}_t \right]^2 \right] du \\
& \leq C\delta \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\left| \int_{\tau(s)}^s J_v^r dV_v^r \right|^2 \middle| \mathcal{Y}_t \right] \tilde{\mathbb{E}} \left[\left| I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \right|^2 \middle| \mathcal{Y}_t \right] \right] du \\
& \leq C\delta \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\left| \int_{\tau(s)}^s J_v^r dV_v^r \right|^2 \right] \tilde{\mathbb{E}} \left[\left| I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \right|^2 \right] du \\
& \leq C\delta \int_{\tau(s)}^s \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[|J_v^r|^2 \right] dv (u - \tau(s))^{|\alpha-|+|\alpha-|_0} \tilde{\mathbb{E}} \left[\sup_{v \in [0,t]} |L^\alpha h_i(X_v)|^2 \right] du \\
& \leq C\delta^{|\alpha-|+|\alpha-|_0+3} = C\delta^{m-1+m-2+3} = C\delta^{2m},
\end{aligned}$$

and using similar reasonings we get

$$\begin{aligned}
\tilde{\mathbb{E}} \left[\left| \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[J_u^r I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),u} \middle| \mathcal{Y}_t \right] du \right|^2 \right] & \leq C\delta^{|\alpha-|+|\alpha-|_0+2} \\
& = C\delta^{m-1+m-1+2} = C\delta^{2m},
\end{aligned}$$

and the result for the case $|\alpha|_0 = m - 1$ follows.

The last case is $|\alpha|_0 = 0$ and $m = 2$. Applying Theorem 4.7 we can write

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} ds \middle| \mathcal{Y}_t \right] \\
& = \int_0^t \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \middle| \mathcal{H}_0^t \right] I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] ds \\
& \quad + \sum_{r=1}^{d_V} \int_0^t \tilde{\mathbb{E}} \left[\left(\int_0^t \tilde{\mathbb{E}} \left[J_u^r \middle| \mathcal{H}_0^t \right] dV_u^r \right) I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] ds \\
& \quad + \sum_{r_1, r_2=1}^{d_V} \int_0^t \tilde{\mathbb{E}} \left[\left(\int_0^t \int_0^{u_2} J_{u_1, u_2}^{r_1, r_2} dV_{u_1}^{r_1} dV_{u_2}^{r_2} \right) I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} \middle| \mathcal{Y}_t \right] ds. \\
& \triangleq A_1 + \sum_{r=1}^{d_V} A_2(r) + \sum_{r_1, r_2=1}^{d_V} A_3(r_1, r_2).
\end{aligned}$$

Applying Lemma 3.8 (1), we see that the term A_1 vanishes. Applying Lemma 3.8 (2) and, then, Lemma 3.8 (1), for $r = 1, \dots, d_V$, we can write

$$\begin{aligned} A_2(r) &= \mathbf{1}_{\{\alpha_2=r\}} \int_0^t \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[J_u^r | \mathcal{H}_0^t \right] I_{\alpha_1} (L^\alpha h_i(X))_{\tau(s),u} \middle| \mathcal{Y}_t \right] duds \\ &= \mathbf{1}_{\{\alpha_2=r\}} \int_0^t \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[J_u^r | \mathcal{H}_0^t \right] \tilde{\mathbb{E}} \left[I_{\alpha_1} (L^\alpha h_i(X))_{\tau(s),u} \middle| \mathcal{H}_0^t \right] \middle| \mathcal{Y}_t \right] duds \\ &= \mathbf{1}_{\{\alpha_2=r, |\alpha_1|=|\alpha_1|_0\}} \int_0^t \int_{\tau(s)}^s \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[J_u^r | \mathcal{H}_0^t \right] \right. \\ &\quad \left. \times I_{\alpha_1} \left(\tilde{\mathbb{E}} \left[L^\alpha h_i(X) | \mathcal{H}_0^t \right] \right)_{\tau(s),u} \middle| \mathcal{Y}_t \right] duds \end{aligned}$$

which is equal to zero because $1 = |\alpha_1| \neq |\alpha_1|_0 = 0$. Applying Lemma 3.8 (3), for $r_1, r_2 = 1, \dots, d_V$ we can write

$$\begin{aligned} A_3(r_1, r_2) &= \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^{u_2 \wedge u} \tilde{\mathbb{E}} \left[J_{v,u}^{r_1, r_2} I_{(\alpha-)} (L^\alpha h_i(X))_{\tau(s),v} \middle| \mathcal{Y}_t \right] dv duds \\ &= \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^u \tilde{\mathbb{E}} \left[J_{v,u}^{r_1, r_2} L^\alpha h_i(X_v) \middle| \mathcal{Y}_t \right] dv duds \\ &\leq \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^u \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[|J_{v,u}^{r_1, r_2}|^2 \middle| \mathcal{H}_0^t \right]^{1/2} \right. \\ &\quad \left. \times \tilde{\mathbb{E}} \left[|L^\alpha h_i(X_v)|^2 \middle| \mathcal{H}_0^t \right]^{1/2} \middle| \mathcal{Y}_t \right] dv duds, \end{aligned}$$

Hence, using Jensen's inequality, Cauchy-Schwartz inequality and Remark 3.1 we have

$$\begin{aligned} \tilde{\mathbb{E}} \left[|A_3(r_1, r_2)|^2 \right] &\leq C \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \delta^2 \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^u \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[|J_{v,u}^{r_1, r_2}|^2 \middle| \mathcal{H}_0^t \right] \right. \\ &\quad \left. \times \tilde{\mathbb{E}} \left[|L^\alpha h_i(X_v)|^2 \middle| \mathcal{H}_0^t \right] \right] dv duds \\ &\leq C \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \delta^2 \int_0^t \int_{\tau(s)}^s \int_{\tau(s)}^u \tilde{\mathbb{E}} \left[|J_{v,u}^{r_1, r_2}|^2 \right] dv duds \\ &\leq C \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \delta^4 = C \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \delta^{2m}, \end{aligned}$$

and we can conclude.

Lemma 4.20 *Let $m \in \{1, 2\}$ and assume that $\mathbf{H}(m)$ holds. For $\alpha \in \mathcal{M}_m(0, \dots, d_V)$, $|\alpha|_0 \neq |\alpha|$ and $i \neq 0$, we can write*

$$\begin{aligned} &\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha (L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\ &= \sum_{r=1}^{d_V} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-} (L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha|\alpha} \right) \middle| \mathcal{Y}_t \right], \end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\
&= \sum_{r_1=1}^{d_V} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \tilde{\mathbb{E}}[J_s^{r_1} | \mathcal{H}_0^t] dV_s^{r_1} \right) \right. \\
&\quad \times \left. \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha|\alpha|} \right) \middle| \mathcal{Y}_t \right] \\
&\quad + \sum_{r_1, r_2=1}^{d_V} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \int_0^{s_2} J_{s_1, s_2}^{r_1, r_2} dV_{s_1}^{r_1} dV_{s_2}^{r_2} \right) \right. \\
&\quad \times \left. \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha|\alpha|} \right) \middle| \mathcal{Y}_t \right].
\end{aligned} \tag{4.20}$$

Proof Note that, as $|\alpha|_0 \neq |\alpha|$, by Lemma 3.8 (1) we have that if $0 \leq u \leq v \leq w \leq t$ then

$$\tilde{\mathbb{E}} \left[I_\alpha(L^\alpha h_i(X))_{v,w} \middle| \mathcal{H}_u^t \right] = 0. \tag{4.21}$$

Using Theorem 4.7 we can write

$$\begin{aligned}
& \varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \\
&= \varphi(X_t) e^{\xi_t} \sum_{j_1=0}^{n-1} \int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \\
&= \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \middle| \mathcal{H}_0^t \right] \sum_{j_1=0}^{n-1} \int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \\
&\quad + \sum_{r=1}^{d_V} \sum_{j_1, j_2=0}^{n-1} \left(\int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \right) \left(\int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \right).
\end{aligned}$$

Next, for $j_1 \geq 0$ we get, using equation (4.21) that

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \middle| \mathcal{H}_0^t \right] \int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \middle| \mathcal{H}_0^t \right] \\
&= \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \middle| \mathcal{H}_0^t \right] \int_{t_{j_1}}^{t_{j_1+1}} \tilde{\mathbb{E}} \left[I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} \middle| \mathcal{H}_0^t \right] dY_s^i = 0.
\end{aligned}$$

Moreover, for $r = 1, \dots, d_V$, if $j_2 > j_1$ we get that

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\left(\int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \right) \left(\int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \right) \middle| \mathcal{H}_{t_{j_2}}^t \right] \\
&= \left(\int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \right) \tilde{\mathbb{E}} \left[\int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \middle| \mathcal{H}_{t_{j_2}}^t \right] = 0,
\end{aligned}$$

and if $j_2 < j_1$ we get that

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \right) \left(\int_{t_{j_1}}^{t_{j_1+1}} I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} dY_s^i \right) \middle| \mathcal{H}_{t_{j_1}}^t \right] \\ &= \left(\int_{t_{j_2}}^{t_{j_2+1}} J_s^r dV_s^r \right) \int_{t_{j_1}}^{t_{j_1+1}} \mathbb{E} \left[I_\alpha(L^\alpha h_i(X))_{t_{j_1},s} \middle| \mathcal{H}_{t_{j_1}}^t \right] dY_s^i = 0. \end{aligned}$$

Hence, using the tower property of the conditional expectation we can write

$$\begin{aligned} & \mathbb{E} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\ &= \sum_{r=1}^{d_V} \mathbb{E} \left[\sum_{j=0}^{n-1} \left(\int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left(\int_{t_j}^{t_{j+1}} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \right) \middle| \mathcal{Y}_t \right]. \quad (4.22) \end{aligned}$$

By integration by parts formula for $\mathcal{F}_s^{V,0} \vee \mathcal{Y}_s$ -semimartingales we have

$$\begin{aligned} \int_{t_j}^{t_{j+1}} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i &= \left(Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_\alpha(L^\alpha h_i(X))_{t_j,t_{j+1}} \\ &\quad - \int_{t_j}^{t_{j+1}} \left(Y_s^i - Y_{t_j}^i \right) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha_1}. \end{aligned}$$

Moreover, we can rewrite the right hand side of the previous equality as a well defined $\mathcal{H}_s^t$ -iterated integral and obtain

$$\int_{t_j}^{t_{j+1}} I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i = \int_{t_j}^{t_{j+1}} \left(Y_{t_{j+1}}^i - Y_{t_j}^i \right) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha_1},$$

which combined with equation (4.22) gives equation (4.19). Finally, using Theorem 4.8 with $k=1$ and repeating the same reasonings as before we get equation (4.20).

Lemma 4.21 Assume that $H(1)$ holds and $\varphi \in C_p^2$. For $\alpha \in \mathcal{M}_1(0, \dots, d_V)$ with $|\alpha|_0 \neq 1$ and $i \neq 0$ we have that

$$\mathbb{E} \left[\mathbb{E} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right]^2 \right] \leq C \delta^2.$$

Proof We divide the proof into several steps.

Step 1. First we will find a more convenient expression for

$$\mathbb{E} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right].$$

Recall that $\alpha \in \mathcal{M}_1(0, \dots, d_V)$ with $|\alpha|_0 \neq 1$ coincides with the set of multiindices $\alpha = (\alpha_1)$ with $\alpha_1 \in \{1, \dots, d_V\}$. Using Lemma 4.20, equation (4.19), and taking into

account that $I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} = L^{\alpha_1} h_i(X_s)$, we can write

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{F}_t \right] \\ &= \sum_{r=1}^{d_Y} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) L^{\alpha_1} h_i(X_s) dV_s^{\alpha_1} \right) \middle| \mathcal{F}_t \right] \end{aligned}$$

Next, by Lemma 3.8 (2) we get that

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{F}_t \right] \\ &= \sum_{r=1}^{d_Y} 1_{\{\alpha_1=r\}} \tilde{\mathbb{E}} \left[\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} J_s^r (Y_{t_{j+1}}^i - Y_{t_j}^i) L^{\alpha_1} h_i(X_s) ds \middle| \mathcal{F}_t \right] \\ &= \sum_{r=1}^{d_Y} 1_{\{\alpha_1=r\}} \tilde{\mathbb{E}} \left[\int_0^t J_s^r (Y_{\eta(s)}^i - Y_s^i) L^{\alpha_1} h_i(X_s) ds \middle| \mathcal{F}_t \right] \\ &= \sum_{r=1}^{d_Y} 1_{\{\alpha_1=r\}} (B_1(r) + B_2(r) + B_3(r)), \end{aligned}$$

where

$$\begin{aligned} B_1(r) &\triangleq \tilde{\mathbb{E}} \left[\int_0^t (J_s^r - J_{\eta(s)}^r) (Y_{\eta(s)}^i - Y_s^i) L^r h_i(X_s) ds \middle| \mathcal{F}_t \right], \\ B_2(r) &\triangleq \tilde{\mathbb{E}} \left[\int_0^t J_{\eta(s)}^r (Y_{\eta(s)}^i - Y_s^i) (L^r h_i(X_s) - L^r h_i(X_{\tau(s)})) ds \middle| \mathcal{F}_t \right], \\ B_3(r) &\triangleq \tilde{\mathbb{E}} \left[\int_0^t J_{\eta(s)}^r L^r h_i(X_{\tau(s)}) (Y_{\eta(s)}^i - Y_s^i) ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

Step 2. Next, we prove the result for $B_1(r)$. Applying Jensen inequality, Cauchy-Schwarz inequality, Hölder inequality, Remark 3.1, that Y^i is a Brownian motion under $\tilde{P}$ and Lemma 4.10 we have that

$$\begin{aligned} \tilde{\mathbb{E}} [|B_1(r)|^2] &\leq C \int_0^t \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[(J_s^r - J_{\eta(s)}^r)^2 (Y_{\eta(s)}^i - Y_s^i)^2 \middle| \mathcal{F}_t \right] \tilde{\mathbb{E}} [|L^r h_i(X_s)|^2 \middle| \mathcal{F}_t] \right] ds \\ &\leq C \int_0^t \tilde{\mathbb{E}} \left[(J_s^r - J_{\eta(s)}^r)^2 (Y_{\eta(s)}^i - Y_s^i)^2 \right] ds \\ &\leq C \int_0^t \tilde{\mathbb{E}} \left[(J_s^r - J_{\eta(s)}^r)^{2+\varepsilon} \right]^{2/(2+\varepsilon)} \tilde{\mathbb{E}} \left[(Y_{\eta(s)}^i - Y_s^i)^{2\frac{2+\varepsilon}{\varepsilon}} \right]^{\varepsilon/(2+\varepsilon)} ds \\ &\leq C \delta \int_0^t \tilde{\mathbb{E}} \left[(J_s^r - J_{\eta(s)}^r)^{2+\varepsilon} \right]^{2/(2+\varepsilon)} ds \\ &\leq C \delta^2. \end{aligned}$$

Step 3. Here, we prove the result for $B_2(r)$. Applying Jensen inequality and Cauchy-Schwarz inequality we get

$$\begin{aligned} \tilde{\mathbb{E}} \left[|B_2(r)|^2 \right] &\leq C \int_0^t \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\left| J_{\eta(s)}^r \left(Y_{\eta(s)}^i - Y_s^i \right) \right|^2 \middle| \mathcal{F}_t \right] \tilde{\mathbb{E}} \left[\left| L^r h_i(X_s) - L^r h_i(X_{\tau(s)}) \right|^2 \middle| \mathcal{F}_t \right] \right] ds \\ &\leq C \int_0^t \tilde{\mathbb{E}} \left[\left| J_{\eta(s)}^r \left(Y_{\eta(s)}^i - Y_s^i \right) \right|^2 \right] \tilde{\mathbb{E}} \left[\left| L^r h_i(X_s) - L^r h_i(X_{\tau(s)}) \right|^2 \right] ds. \end{aligned}$$

Applying Hölder inequality and Proposition 4.9 we can conclude that

$$\begin{aligned} \tilde{\mathbb{E}} \left[\left| J_{\eta(s)}^r \left(Y_{\eta(s)}^i - Y_s^i \right) \right|^2 \right] &\leq \tilde{\mathbb{E}} \left[\left| J_{\eta(s)}^r \right|^{2+\varepsilon} \right]^{2/(2+\varepsilon)} \tilde{\mathbb{E}} \left[\left| \left(Y_{\eta(s)}^i - Y_s^i \right) \right|^{2 \frac{2+\varepsilon}{\varepsilon}} \right]^{\varepsilon/(2+\varepsilon)} \\ &\leq \delta \sup_{s \in [0, t]} \tilde{\mathbb{E}} \left[|J_s^r|^{2+\varepsilon} \right]^{2/(2+\varepsilon)} \leq C\delta. \end{aligned}$$

On the other hand, we can write

$$L^r h_i(X_s) - L^r h_i(X_{\tau(s)}) = \int_{\tau(s)}^s L^{(0,r)} h_i(X_u) du + \sum_{r_1=1}^{d_V} \int_{\tau(s)}^s L^{(r_1,r)} h_i(X_u) dV_u^{r_1}.$$

As the worst rate is achieved by the terms with the stochastic integral, it suffices to show that

$$\tilde{\mathbb{E}} \left[\left| \int_{\tau(s)}^s L^{(r_1,r)} h_i(X_u) dV_u^{r_1} \right|^2 \right] \leq C\delta,$$

which easily follows by Itô isometry and Remark 3.1.

Step 4. Finally, we prove the result for $B_3(r)$. We can write

$$\begin{aligned} B_3(r) &= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\int_{t_j}^{t_{j+1}} J_{t_{j+1}}^r L^r h_i(X_{t_j}) \left(Y_{t_{j+1}}^i - Y_s^i \right) ds \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}}^r L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \left(Y_{t_{j+1}}^i - Y_s^i \right) ds \middle| \mathcal{F}_t \right] \\ &= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}}^r L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} (s - t_j) dY_s^i \middle| \mathcal{F}_t \right] \\ &\triangleq \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}}^r L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right] \end{aligned}$$

Moreover,

$$J_{t_{j+1}}^r = \tilde{\mathbb{E}} \left[D_{t_{j+1}}^r \left[\varphi(X_t) e^{\xi_t} \right] \middle| \mathcal{H}_{t_{j+1}}^t \right],$$

by the Clark-Ocone formula. Using the product formula for the Malliavin derivative, we get

$$D_{t_{j+1}}^r \left[\varphi(X_t) e^{\xi_t} \right] = e^{\xi_t} D_{t_{j+1}}^r \varphi(X_t) + \varphi(X_t) D_{t_{j+1}}^r e^{\xi_t}$$

Therefore, using the tower property of the conditional expectation and the previous expression for the Malliavin derivative, we have

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[J_{t_{j+1}}^r L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right] \\
&= \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[D_{t_{j+1}}^r \left[\varphi(X_t) e^{\xi_t} \right] \middle| \mathcal{H}_{t_{j+1}}^t \right] L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right] \\
&= \tilde{\mathbb{E}} \left[e^{\xi_t} D_{t_{j+1}}^r \varphi(X_t) L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right] \\
&\quad + \tilde{\mathbb{E}} \left[\varphi(X_t) D_{t_{j+1}}^r e^{\xi_t} L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right].
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{\mathbb{E}} \left[|B_3(r)|^2 \right] &\leq \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}}^r L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right] \right|^2 \right] \\
&\leq 2 \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} e^{\xi_t} D_{t_{j+1}}^r \varphi(X_t) L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\
&\quad + 2 \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \varphi(X_t) D_{t_{j+1}}^r e^{\xi_t} L^r h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\
&\triangleq 2A_1(r) + 2A_2(r).
\end{aligned}$$

Next, note that

$$\begin{aligned}
D_{t_{j+1}}^r e^{\xi_t} &= e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_0^t D_{t_{j+1}}^r h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_0^t D_{t_{j+1}}^r [h_k^2(X_s)] ds \right\} \\
&= e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^r h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^r [h_k^2(X_s)] ds \right\} \\
&\triangleq e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t \alpha_s^{j,k,1} dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t \alpha_s^{j,k,2} ds \right\},
\end{aligned}$$

where we have used that $D_u^r h_k(X_s) = 0, s < u < t$. In addition, note that

$$e^{2\xi_t} = M_0^t(2h) \exp \left(\sum_{i=1}^{d_Y} \int_0^t h_i^2(X_u) du \right),$$

where

$$M_s^t(h) = \exp \left(\sum_{i=1}^{d_Y} \int_s^t h_i(X_u) dY_u^i - \frac{1}{2} \sum_{i=1}^{d_Y} \int_s^t h_i^2(X_u) du \right),$$

is an exponential martingale. Defining

$$\Gamma(j_1, j_2) \triangleq D_{t_{j_1+1}}^r \varphi(X_t) D_{t_{j_2+1}}^r \varphi(X_t) L^r h_i(X_{t_{j_1}}) L^r h_i(X_{t_{j_2}}) \exp\left(\sum_{i=1}^{d_Y} \int_0^t h_i^2(X_u) du\right),$$

and

$$\Lambda(j_1, j_2) \triangleq \varphi(X_t)^2 L^r h_i(X_{t_{j_1}}) L^r h_i(X_{t_{j_2}}) \exp\left(\sum_{i=1}^{d_Y} \int_0^t h_i^2(X_u) du\right),$$

we can write

$$A_1(r) = \sum_{j_1, j_2=0}^{n-1} \tilde{\mathbb{E}} \left[\Gamma(j_1, j_2) M_0^t(2h) \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^{j_1} dY_s^i \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^{j_2} dY_s^i \right],$$

and

$$A_2(r) = \sum_{k_1, k_2=1}^{d_Y} A_{2,1}(r, k_1, k_2) - \frac{1}{2} A_{2,2}(r, k_1, k_2) - \frac{1}{2} A_{2,3}(r, k_1, k_2) + \frac{1}{4} A_{2,4}(r, k_1, k_2),$$

where

$$\begin{aligned} A_{2,1}(r, k_1, k_2) &\triangleq \sum_{j_1, j_2=0}^{n-1} \tilde{\mathbb{E}} \left[\Lambda(j_1, j_2) M_0^t(2h) \int_{t_{j_1+1}}^t \alpha_s^{j_1, k_1, 1} dY_s^{k_1} \int_{t_{j_2+1}}^t \alpha_s^{j_2, k_2, 1} dY_s^{k_2} \right. \\ &\quad \times \left. \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^{j_1} dY_s^i \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^{j_2} dY_s^i \right], \\ A_{2,2}(r, k_1, k_2) &\triangleq \sum_{j_1, j_2=0}^{n-1} \tilde{\mathbb{E}} \left[\Lambda(j_1, j_2) M_0^t(2h) \int_{t_{j_1+1}}^t \alpha_s^{j_1, k_1, 1} dY_s^{k_1} \int_{t_{j_2+1}}^t \alpha_s^{j_2, k_2, 2} ds \right. \\ &\quad \times \left. \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^{j_1} dY_s^i \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^{j_2} dY_s^i \right], \\ A_{2,3}(r, k_1, k_2) &\triangleq \sum_{j_1, j_2=0}^{n-1} \tilde{\mathbb{E}} \left[\Lambda(j_1, j_2) M_0^t(2h) \int_{t_{j_1+1}}^t \alpha_s^{j_1, k_1, 2} ds \int_{t_{j_2+1}}^t \alpha_s^{j_2, k_2, 1} dY_s^{k_2} \right. \\ &\quad \times \left. \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^{j_1} dY_s^i \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^{j_2} dY_s^i \right], \\ A_{2,4}(r, k_1, k_2) &\triangleq \sum_{j_1, j_2=0}^{n-1} \tilde{\mathbb{E}} \left[\Lambda(j_1, j_2) M_0^t(2h) \int_{t_{j_1+1}}^t \alpha_s^{j_1, k_1, 2} ds \int_{t_{j_2+1}}^t \alpha_s^{j_2, k_2, 2} ds \right. \\ &\quad \times \left. \int_{t_{j_1}}^{t_{j_1+1}} \beta_s^{j_1} dY_s^i \int_{t_{j_2}}^{t_{j_2+1}} \beta_s^{j_2} dY_s^i \right]. \end{aligned}$$

The result follows by applying Lemma 4.16, taking into account Remark 4.12, to the terms $A_1, A_{2,1}, A_{2,2}, A_{2,3}$ and $A_{2,4}$.

Lemma 4.22 Assume that $\mathbf{H}(2)$ holds and $\varphi \in C_p^3$. For $\alpha \in \mathcal{M}_2(0, \dots, d_V)$ with $|\alpha|_0 = 1$ and $i \neq 0$ we have that

$$\tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right]^2 \right] \leq C\delta^4.$$

Proof The proof of this lemma is analogous to the proof of Lemma 4.21. Using Lemma 4.20, we can write

$$\begin{aligned} & \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\ &= \sum_{r=1}^{d_V} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-}(L^\alpha h_i(X.))_{\tau(s),s} dV_s^{\alpha_{| \alpha |}} \right) \middle| \mathcal{Y}_t \right] \\ &\triangleq \sum_{r=1}^{d_V} A(r). \end{aligned}$$

Therefore, by Lemma 3.8 (2), we have that

$$\begin{aligned} & A(r) \\ &= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} J_s^r dV_s^r \right) \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) \left(\int_{t_j}^s L^\alpha h_i(X_u) dV_u^{\alpha_1} \right) dV_s^{\alpha_2} \right) \middle| \mathcal{Y}_t \right] \\ &= \mathbf{1}_{\{\alpha_1=0, \alpha_2=r\}} \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) J_s^r \left(\int_{\tau(s)}^s L^\alpha h_i(X_u) du \right) ds \right) \middle| \mathcal{Y}_t \right] \\ &\quad + \mathbf{1}_{\{\alpha_1=r, \alpha_2=0\}} \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) \left(\int_{\tau(s)}^s J_u^r L^\alpha h_i(X_u) du \right) ds \right) \middle| \mathcal{Y}_t \right] \\ &\triangleq \mathbf{1}_{\{\alpha_1=0, \alpha_2=r\}} A_1(r) + \mathbf{1}_{\{\alpha_1=r, \alpha_2=0\}} A_2(r). \end{aligned}$$

Next, the proof follows by similar reasonings as in Lemma 4.21.

Lemma 4.23 Assume that $\mathbf{H}(2)$ holds and $\varphi \in C_p^3$. For $\alpha \in \mathcal{M}_2(0, \dots, d_V)$ with $|\alpha|_0 = 0$ and $i \neq 0$ we have that

$$\tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X.))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right]^2 \right] \leq C\delta^4.$$

Proof We divide the proof into several steps.

Step 1. Using Lemma 4.20, we can write

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\tilde{\xi}_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\
&= \sum_{r_1=1}^{d_V} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \tilde{\mathbb{E}}[J_s^{r_1} | \mathcal{H}_0^t] dV_s^{r_1} \right) \right. \\
&\quad \times \left. \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha_{[i]}} \right) \middle| \mathcal{Y}_t \right] \\
&\quad + \sum_{r_1, r_2=1}^{d_V} \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \int_0^{s_2} J_{s_1, s_2}^{r_1, r_2} dV_{s_1}^{r_1} dV_{s_2}^{r_2} \right) \right. \\
&\quad \times \left. \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_{t_j}^i) I_{\alpha-}(L^\alpha h_i(X))_{\tau(s),s} dV_s^{\alpha_{[i]}} \right) \middle| \mathcal{Y}_t \right] \\
&\triangleq \sum_{r_1=1}^{d_V} \sum_{j=0}^{n-1} A(r_1, j) + \sum_{r_1, r_2=1}^{d_V} \sum_{j=0}^{n-1} A(r_1, r_2, j).
\end{aligned}$$

Therefore, by Lemma 3.8 (2), we have that

$$\begin{aligned}
& A(r_1, j) \\
&= \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \tilde{\mathbb{E}}[J_s^{r_1} | \mathcal{H}_0^t] dV_s^{r_1} \right) \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_s^i) \left(\int_{t_j}^s L^\alpha h_i(X_u) dV_u^{\alpha_1} \right) dV_s^{\alpha_2} \right) \middle| \mathcal{Y}_t \right] \\
&= \mathbf{1}_{\{\alpha_2=r_1\}} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \tilde{\mathbb{E}}[J_s^{r_1} | \mathcal{H}_0^t] (Y_{t_{j+1}}^i - Y_s^i) \left(\int_{t_j}^s L^\alpha h_i(X_u) dV_u^{\alpha_1} \right) ds \right) \middle| \mathcal{Y}_t \right] \\
&= \mathbf{1}_{\{\alpha_2=r_1\}} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \tilde{\mathbb{E}}[J_s^{r_1} | \mathcal{H}_0^t] (Y_{t_{j+1}}^i - Y_s^i) \tilde{\mathbb{E}} \left[\left(\int_{t_j}^s L^\alpha h_i(X_u) dV_u^{\alpha_1} \right) \middle| \mathcal{H}_{t_j}^t \right] ds \right) \middle| \mathcal{Y}_t \right] \\
&= 0.
\end{aligned}$$

and, by Lemma 3.8 (3) and Lemma 3.8 (2), we obtain

$$\begin{aligned}
& A(r_1, r_2, j) \\
&= \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} \int_0^{s_2} J_{s_1, s_2}^{r_1, r_2} dV_{s_1}^{r_1} dV_{s_2}^{r_2} \right) \right. \\
&\quad \times \left. \left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_s^i) \left(\int_{t_j}^s L^\alpha h_i(X_u) dV_u^{\alpha_1} \right) dV_s^{\alpha_2} \right) \middle| \mathcal{Y}_t \right] \\
&= \mathbf{1}_{\{\alpha_2=r_2\}} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_s^i) \left(\int_0^s J_{s_1, s}^{r_1, r_2} dV_{s_1}^{r_1} \right) \left(\int_{t_j}^s L^\alpha h_i(X_u) dV_u^{\alpha_1} \right) ds \right) \middle| \mathcal{Y}_t \right] \\
&= \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \tilde{\mathbb{E}} \left[\left(\int_{t_j}^{t_{j+1}} (Y_{t_{j+1}}^i - Y_s^i) \left(\int_{t_j}^s J_{u, s}^{r_1, r_2} L^\alpha h_i(X_u) du \right) ds \right) \middle| \mathcal{Y}_t \right].
\end{aligned}$$

Hence, we can write

$$\begin{aligned}
& \tilde{\mathbb{E}} \left[\varphi(X_t) e^{\xi_t} \int_0^t I_\alpha(L^\alpha h_i(X))_{\tau(s),s} dY_s^i \middle| \mathcal{Y}_t \right] \\
&= \sum_{r_1, r_2=1}^{d_Y} \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) \left(\int_{\tau(s)}^s J_{u,s}^{r_1, r_2} L^\alpha h_i(X_u) du \right) ds \right) \middle| \mathcal{Y}_t \right] \\
&= \sum_{r_1, r_2=1}^{d_Y} \mathbf{1}_{\{\alpha_2=r_2, \alpha_1=r_1\}} (B_1(r_1, r_2) + B_2(r_1, r_2) + B_3(r_1, r_2) + B_4(r_1, r_2)),
\end{aligned}$$

where

$$\begin{aligned}
B_1(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) \left(\int_{\tau(s)}^s (J_{u,s}^{r_1, r_2} - J_{s,s}^{r_1, r_2}) L^\alpha h_i(X_u) du \right) ds \right) \middle| \mathcal{Y}_t \right], \\
B_2(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) J_{s,s}^{r_1, r_2} \left(\int_{\tau(s)}^s (L^\alpha h_i(X_u) - L^\alpha h_i(X_{\tau(s)})) du \right) ds \right) \middle| \mathcal{Y}_t \right] \\
B_3(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) (J_{s,s}^{r_1, r_2} - J_{\eta(s), \eta(s)}^{r_1, r_2}) L^\alpha h_i(X_{\tau(s)}) \left(\int_{\tau(s)}^s du \right) ds \right) \middle| \mathcal{Y}_t \right] \\
B_4(r_1, r_2) &= \tilde{\mathbb{E}} \left[\left(\int_0^t (Y_{\eta(s)}^i - Y_s^i) J_{\eta(s), \eta(s)}^{r_1, r_2} L^\alpha h_i(X_{\tau(s)}) \left(\int_{\tau(s)}^s du \right) ds \right) \middle| \mathcal{Y}_t \right]
\end{aligned}$$

Step 2. That the terms $B_1(r_1, r_2)$, $B_2(r_1, r_2)$ and $B_3(r_1, r_2)$ have the right order is deduced analogously to the **Steps 2** and **3** in Lemma 4.21.

Step 3. Finally, we prove the result for $B_4(r_1, r_2)$. We can write

$$\begin{aligned}
B_4(r_1, r_2) &= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[\int_{t_j}^{t_{j+1}} J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i(X_{t_j}) \left(\int_{t_j}^s du \right) (Y_{t_{j+1}}^i - Y_s^i) ds \middle| \mathcal{Y}_t \right] \\
&= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \left(\int_{t_j}^s du \right) (Y_{t_{j+1}}^i - Y_s^i) ds \middle| \mathcal{Y}_t \right] \\
&= \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \frac{(s - t_j)^2}{2} dY_s^i \middle| \mathcal{Y}_t \right] \\
&\triangleq \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{Y}_t \right],
\end{aligned}$$

where $|\beta_s^j| \leq \delta^2$. Moreover,

$$J_{t_{j+1}, t_{j+1}}^{r_1, r_2} = \tilde{\mathbb{E}} \left[D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \left\{ \varphi(X_t) e^{\xi_t} \right\} \middle| \mathcal{H}_{t_j}^t \right],$$

by the Clark-Ocone formula. Using the definition of the iterated Malliavin derivative and the product formula, we get

$$\begin{aligned}
D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \left\{ \varphi(X_t) e^{\xi_t} \right\} &= D_{t_{j+1}}^{r_1} \left(D_{t_{j+1}}^{r_2} \left\{ \varphi(X_t) e^{\xi_t} \right\} \right) \\
&= D_{t_{j+1}}^{r_1} \left(e^{\xi_t} D_{t_{j+1}}^{r_2} \varphi(X_t) + \varphi(X_t) D_{t_{j+1}}^{r_2} e^{\xi_t} \right) \\
&= D_{t_{j+1}}^{r_1} e^{\xi_t} D_{t_{j+1}}^{r_2} \varphi(X_t) + e^{\xi_t} D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \varphi(X_t) \\
&\quad + D_{t_{j+1}}^{r_1} \varphi(X_t) D_{t_{j+1}}^{r_2} e^{\xi_t} + \varphi(X_t) D_{t_{j+1}, t_{j+1}}^{r_1, r_2} e^{\xi_t}
\end{aligned}$$

Reasoning as in Step 4 of Lemma 4.21, we get that

$$\begin{aligned}
\tilde{\mathbb{E}} \left[|B_4(r_1, r_2)|^2 \right] &\leq \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \tilde{\mathbb{E}} \left[J_{t_{j+1}, t_{j+1}}^{r_1, r_2} L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \middle| \mathcal{F}_t \right] \right|^2 \right] \\
&\leq C \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} D_{t_{j+1}}^{r_1} e^{\xi_t} D_{t_{j+1}}^{r_2} \varphi(X_t) L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\
&\quad + C \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} e^{\xi_t} D_{t_{j+1}, t_{j+1}}^{r_1, r_2} \varphi(X_t) L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\
&\quad + C \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} D_{t_{j+1}}^{r_1} \varphi(X_t) D_{t_{j+1}}^{r_2} e^{\xi_t} L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\
&\quad + C \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \varphi(X_t) D_{t_{j+1}, t_{j+1}}^{r_1, r_2} e^{\xi_t} L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right] \\
&\triangleq C \{F_1(r_1, r_2) + F_2(r_1, r_2) + F_3(r_1, r_2) + F_4(r_1, r_2)\}.
\end{aligned}$$

The term $F_2(r_1, r_2)$ is analogous to the term $A_1(r)$ in Lemma 4.21 and the terms $F_2(r_1, r_2)$ and $F_3(r_1, r_2)$ are analogous to the term $A_2(r)$ in Lemma 4.21. For the term $F_4(r_1, r_2)$ we have that

$$\begin{aligned}
D_{t_{j+1}, t_{j+1}}^{r_1, r_2} e^{\xi_t} &= D_{t_{j+1}}^{r_1} \left\{ e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^{r_2} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^{r_2} [h_k^2(X_s)] ds \right\} \right\} \\
&= e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^{r_1} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^{r_1} [h_k^2(X_s)] ds \right\} \\
&\quad \times \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^{r_2} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}}^{r_2} [h_k^2(X_s)] ds \right\} \\
&\quad + e^{\xi_t} \left\{ \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}, t_{j+1}}^{r_1, r_2} h_k(X_s) dY_s^k - \frac{1}{2} \sum_{k=1}^{d_Y} \int_{t_{j+1}}^t D_{t_{j+1}, t_{j+1}}^{r_1, r_2} [h_k^2(X_s)] ds \right\}
\end{aligned}$$

All the terms obtained in the previous expression can be dealt analogously to the terms in Lemma 4.21 except the terms

$$G(j, r_1, r_2) \triangleq e^{\xi_t} \sum_{k_1, k_2=1}^{dy} \left(\int_{t_{j+1}}^t D_{t_{j+1}}^{r_1} h_{k_1}(X_s) dY_s^{k_1} \right) \left(\int_{t_{j+1}}^t D_{t_{j+1}}^{r_2} h_{k_2}(X_s) dY_s^{k_2} \right).$$

Let

$$H \triangleq \tilde{\mathbb{E}} \left[\left| \sum_{j=0}^{n-1} \varphi(X_t) G(j, r_1, r_2) L^{(r_1, r_2)} h_i(X_{t_j}) \int_{t_j}^{t_{j+1}} \beta_s^j dY_s^i \right|^2 \right].$$

Defining

$$\Lambda(j_1, j_2) \triangleq \varphi(X_t)^2 L^{(r_1, r_2)} h_i(X_{t_{j_1}}) L^{(r_1, r_2)} h_i(X_{t_{j_2}}) \exp \left(\sum_{i=1}^{dy} \int_0^t h_i^2(X_u) du \right),$$

we can write

$$\begin{aligned} H = & \sum_{k_1, \dots, k_4=1}^{dy} \sum_{j_1, j_2=0}^{n-1} \tilde{\mathbb{E}} [\Lambda(j_1, j_2) M_0^t(2h) \\ & \times \left(\int_{t_{j_1+1}}^t D_{t_{j_1+1}}^{r_1} h_{k_1}(X_s) dY_s^{k_1} \right) \left(\int_{t_{j_1+1}}^t D_{t_{j_1+1}}^{r_2} h_{k_2}(X_s) dY_s^{k_2} \right) \\ & \times \left(\int_{t_{j_2+1}}^t D_{t_{j_2+1}}^{r_1} h_{k_3}(X_s) dY_s^{k_3} \right) \left(\int_{t_{j_2+1}}^t D_{t_{j_2+1}}^{r_2} h_{k_4}(X_s) dY_s^{k_4} \right) \\ & \times \left(\int_{t_{j_1}}^{t_{j_1+1}} \beta_s^{j_1} dY_s^{i_1} \right) \left(\int_{t_{j_2}}^{t_{j_2+1}} \beta_s^{j_2} dY_s^{i_2} \right)], \end{aligned}$$

and the result follows from Lemma 4.16 and Remark 4.12.

Remark 4.24 Following Remarks 4.11 and 4.17, the results in Lemmas 4.22 and 4.23 can be extended analogously to $m > 2$ and $\alpha \in \mathcal{M}_m(0, \dots, d_V)$ with $|\alpha|_0 \in \{0, \dots, m-1\}$ without any additional difficulties.

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