Mean-Field Stochastic Control with Elephant Memory in Finite and Infinite Time Horizon

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21 June 2019

The final version of this paper will be published in Stochastics

Abstract

Our purpose of this paper is to study stochastic control problems for systems driven by mean-field stochastic differential equations with elephant memory, in the sense that the system (like the elephants) never forgets its history. We study both the finite horizon case and the infinite time horizon case.

- In the finite horizon case, results about existence and uniqueness of solutions of such a system are given. Moreover, we prove sufficient as well as necessary stochastic maximum principles for the optimal control of such systems. We apply our results to solve a mean-field linear quadratic control problem.
- For infinite horizon, we derive sufficient and necessary maximum principles. As an illustration, we solve an optimal consumption problem from a cash flow modelled by an elephant memory mean-field system.

MSC(2010): 60H05, 60H20, 60J75, 93E20, 91G80, 91B70.

Keywords: Mean-field stochastic differential equation; memory; stochastic maximum principle; partial information; backward stochastic differential equation.

1 Introduction

In this paper we study optimal control of stochastic systems with memory. There are many ways of modelling such systems. Examples include systems with delay or Volterra integral

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This research was carried out with support of the Norwegian Research Council, within the research project Challenges in Stochastic Control, Information and Applications (STOCONINF), project number 250768/F20.

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equations. See e.g. Agram *et al* [9]. Here we are interested in stochastic differential equations (SDEs) where the coefficients of the system depend upon the whole past. In this case we say that the system has *elephant memory*, inspired by the folklore that an elephant never forgets. In addition we allow the dynamics of the system to depend on both the current and previous laws of the state. Specifically, we assume that the state $X(t)$ at time t satisfies the following equation

$$\begin{cases} dX(t) &= b(t, X(t), X_t, M(t), M_t)dt + \sigma(t, X(t), X_t, M(t), M_t)dB(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, \zeta) \tilde{N}(dt, d\zeta); t \geq 0, \\ X(0) &= x_0, \end{cases} \quad (1.1) \quad \{\text{eq1.1}\}$$

where $X_t := \{X(t-s)\}_{0 \leq s \leq t}$ is the path of X up to time t , $M(t) = \mathcal{L}(X(t))$ is the law of $X(t)$, and $M_t := \{M(t-s)\}_{0 \leq s \leq t}$ is the path of the law process.

We call equation (1.1) a mean-field stochastic differential equation (MF-SDE) with elephant memory. For more information on mean-field SDEs without memory we refer to e.g. Carmona and Delarue [10],[11] and the references therein.

A historical process $X_t := \{X(s)\}_{0 \leq s \leq t}$ was studied by Dynkin [14], but in a different framework. Different types of systems with memories were discussed in the seminal work of Mohammed [21]. A stochastic version of Pontryagin's maximum principle for systems with delay (discrete/distributed) has been derived by Chen and Wu [12], Dahl *et al* [13] and Øksendal *et al* [23].

The above mentioned works deal only with the finite horizon case. We refer to Agram *et al* [1], [3] for the infinite time horizon setting.

Systems with discrete delay and mean-field have been studied by Meng and Shen [20], Agram and Røse [8], but the mean-field terms considered there are of a special kind, namely the expectation of a function of the state, i.e. $\mathbb{E}[\varphi(X(t-\delta))]$ for some bounded function φ and δ is a positive delay constant.

In this paper we consider a more general situation, where the dynamics of the state $X(t)$ at time t depends on both the history of the state, the law for the random variable $X(t)$ and the history of this law, as we have seen in (1.1). Moreover, we consider both the finite horizon case (Section 3) and the infinite horizon case (Section 4).

Since the system is not Markovian, it is not obvious how to derive the dynamic programming approach, but one can still get the HJB equation by using the minimal backward stochastic differential equation (BSDE). This has been studied by Fuhrman and Pham [16] by using the control randomization method, considering measures defined on the Wasserstein metric space of probability measures with finite second moment and using Lions lifting techniques for differentiating the function of the measure.

In our paper, we use the Hilbert space of measures constructed in Agram *et al* [5], [6], [7].

In Section 3 we obtain finite horizon maximum principles for the optimal control of such systems. This is related to the paper by Agram and Øksendal in [6], where the memorized paths are defined as $\{X(s)\}_{s \in [t-\delta, t]}$ for a fixed $\delta > 0$. However, in the current paper, we consider as memory the whole trajectory $\{X(s)\}_{s \in [0, t]}$.

In the infinite horizon case in Section 4, we show that by replacing the terminal value of the BSDE for the adjoint processes with a suitable transversality condition at infinity, we can derive stochastic maximum principles also in this case. As an illustration we study an infinite horizon version of an optimal consumption problem with elephant memory.

2 Framework

We now explain our setup in more detail:

Let $B = (B(t))_{t \in [0, T]}$ and $\tilde{N}(dt, d\zeta)$ be a d -dimensional Brownian motion and a compensated Poisson random measure, respectively, defined in a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is assumed to be the $\mathbb{P}$ -augmented filtration generated by B and $\tilde{N}$.

2.1 Sobolev spaces of measures

We now define a *weighted Sobolev spaces of measures*. It is strongly related to the space introduced in Agram and Øksendal [5], [6], but with a different weight, which is more suitable for estimates (see e.g. Lemma 2.4 below):

- Let n be a given integer. Then we define $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^n$ to be the pre-Hilbert space of random measures μ on $\mathbb{R}$ equipped with the norm

$$\|\mu\|_{\tilde{\mathcal{M}}^n}^2 := \mathbb{E}[\int_{\mathbb{R}} |\hat{\mu}(y)|^2 (1 + |y|)^{-n} dy],$$

where $\hat{\mu}$ is the Fourier transform of the measure μ , i.e.

$$\hat{\mu}(y) := \int_{\mathbb{R}} e^{-ixy} d\mu(x); \quad y \in \mathbb{R}.$$

- For simplicity of notation, we will in the following fix

$$n \geq 2$$

and we let $\mathcal{M} = \mathcal{M}^n$ denote the completion of $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^n$ and we let $\mathcal{M}_0$ denote the set of deterministic elements of $\mathcal{M}$.

- Let $\tilde{\mathcal{M}}_t$ be the pre-Hilbert space of all paths $\bar{\mu} = \{\mu(s)\}_{s \in [0, t]}$ of processes $\mu(\cdot)$ with $\mu(s) \in \tilde{\mathcal{M}}^n = \tilde{\mathcal{M}}$ for each $s \in [0, t]$ equipped with the norm

$$\|\bar{\mu}\|_{\tilde{\mathcal{M}}_t}^2 := \int_0^t \|\mu(s)\|_{\tilde{\mathcal{M}}}^2 ds.$$

- We denote by $\tilde{\mathcal{M}}_{0,t}$ the set of all deterministic elements of $\tilde{\mathcal{M}}_t$ and by $\mathcal{M}_t$ and $\mathcal{M}_{0,t}$ their completions respectively.

- If $\bar{x} \in \mathbb{R}^{[0,\infty)}$ (the set of all functions from $[0, \infty)$ into $\mathbb{R}$), we define $\bar{x}_t \in \mathbb{R}^{[0,\infty)}$ by

$$\begin{aligned}\bar{x}_t(s) &= \bar{x}(t-s); & s \in [0, t], \\ \bar{x}_t(s) &= 0; & s > t.\end{aligned}$$

- If $\bar{x} \in \mathbb{R}^{[0,\infty)}$, we define $\bar{x}^t \in \mathbb{R}^{[0,\infty)}$ by

$$\begin{aligned}\bar{x}^t(s) &= \bar{x}(t+s); & s \in [0, t], \\ \bar{x}^t(s) &= 0; & s > t.\end{aligned}\tag{2.1} \quad \{\text{fs}\}$$

The following results is essential for our approach:

{Lemma 4}

Lemma 2.1 Assume that $n \geq 4$.

- (i) Let $X^{(1)}$ and $X^{(2)}$ be two 1-dimensional random variables in $L^2(\mathbb{P})$.

Then there exists a constant C_0 not depending on $X^{(1)}$ and $X^{(2)}$ such that

$$\|\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})\|_{\mathcal{M}_0}^2 \leq C_0 \mathbb{E}[(X^{(1)} - X^{(2)})^2].$$

- (ii) Let $\{X^{(1)}(t)\}_{t \geq 0}$, $\{X^{(2)}(t)\}_{t \geq 0}$ be two processes such that

$$\mathbb{E}[\int_0^T X^{(i)2}(s)ds] < \infty \text{ for } i = 1, 2.$$

Then, for all t ,

$$\|\mathcal{L}(X_t^{(1)}) - \mathcal{L}(X_t^{(2)})\|_{\mathcal{M}_{0,t}}^2 \leq C_0 \mathbb{E}[\int_0^t (X^{(1)}(t-s) - X^{(2)}(t-s))^2 ds].$$

Proof. By definition of the norms and standard properties of the complex exponential function, we have

$$\begin{aligned}& \|\mathcal{L}(X^{(1)}) - \mathcal{L}(X^{(2)})\|_{\mathcal{M}_0}^2 \\&= \int_{\mathbb{R}} |\widehat{\mathcal{L}}(X^{(1)})(y) - \widehat{\mathcal{L}}(X^{(2)})(y)|^2 (1 + |y|)^{-n} dy \\&= \int_{\mathbb{R}} |\int_{\mathbb{R}} e^{-ixy} d\mathcal{L}(X^{(1)})(x) - \int_{\mathbb{R}} e^{-ixy} d\mathcal{L}(X^{(2)})(x)|^2 (1 + |y|)^{-n} dy \\&= \int_{\mathbb{R}} |\mathbb{E}[e^{-iyX^{(1)}} - e^{-iyX^{(2)}}]|^2 (1 + |y|)^{-n} dy \\&\leq \int_{\mathbb{R}} \mathbb{E}[|e^{-iyX^{(1)}} - e^{-iyX^{(2)}}|^2] (1 + |y|)^{-n} dy \\&\leq \int_{\mathbb{R}} y^2 (1 + |y|)^{-n} dy \mathbb{E}[|X^{(1)} - X^{(2)}|^2] \\&\leq C_0 \mathbb{E}[(X^{(1)} - X^{(2)})^2],\end{aligned}$$

where

$$C_0 = \int_{\mathbb{R}} y^2 (1 + |y|)^{-n} dy < \infty \text{ since } n \geq 4.\tag{2.2}$$

Similarly we get that

$$\begin{aligned}\|\mathcal{L}(X_t^{(1)}) - \mathcal{L}(X_t^{(2)})\|_{\mathcal{M}_{0,t}}^2 &\leq \int_0^t \|\mathcal{L}(X^{(1)}(t-s)) - \mathcal{L}(X^{(2)}(t-s))\|_{\mathcal{M}_0}^2 ds \\&\leq C_0 \mathbb{E}[\int_0^t (X^{(1)}(t-s) - X^{(2)}(t-s))^2 ds].\end{aligned}$$

□

Remark 2.2 If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then by the Fourier inversion formula and the Fubini theorem

$$\begin{aligned} \left| \int_{\mathbb{R}} f(x) d\mu(x) \right|^2 &= \left| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{2\pi} \widehat{f}(y) e^{ixy} dy \right) d\mu(x) \right|^2 = \left| \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{2\pi} e^{ixy} d\mu(x) \right) \widehat{f}(y) dy \right|^2 \\ &= \frac{1}{2\pi} \left| \int_{\mathbb{R}} \widehat{\mu}(-y) \widehat{f}(y) dy \right|^2. \end{aligned} \quad (2.3) \quad \{\text{eq1.2}\}$$

If we assume in addition that $f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then we know that

$$\int_{\mathbb{R}} \left| \widehat{f}(y) \right|^2 (1 + |y|)^2 dy \leq C(\|\widehat{f}\|_{L^2(\mathbb{R})}^2 + \|\widehat{f}'\|_{L^2(\mathbb{R})}^2) =: C_2(f) < \infty. \quad (2.4)$$

and hence by (2.3) we get the following result:

Lemma 2.3 Suppose $f, f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then

$$|E[f(X(t))]|^2 \leq \frac{1}{2\pi} C_2(f) \|\mu\|_{\mathcal{M}_0^2}^2.$$

Proof.

$$\begin{aligned} |E[f(X(t))]|^2 &= \left| \int_{\mathbb{R}} f(x) d\mu(x) \right|^2 \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}} \widehat{\mu}(-y) \widehat{f}(y) dy \right|^2 \\ &\leq \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\mu}(y)|^2 (1 + |y|)^{-2} dy \int_{\mathbb{R}} |\widehat{f}(y)|^2 (1 + |y|)^2 dy \\ &= \frac{1}{2\pi} C_2(f) \|\mu\|_{\mathcal{M}_0^2}^2. \end{aligned}$$

□

This is a useful estimate, because if $\mu := \mathcal{L}_{X(t)}$ where $X(t)$ solves a MF-SDE of the type (1.1), then we always have $\|\mu\|_{\mathcal{M}_0^2} < \infty$.

Applying the previous result to $\mu := \mu_1 - \mu_2$ where $\mu_i = \mathcal{L}(X_i(t)); i = 1, 2$, we the following Lipschitz estimate. This is useful when we want to verify the Lipschitz condition (ii) in Section 3.1 in specific MF-SDEs with memory.

Lemma 2.4 Let $X_1(t), X_2(t)$ be two solutions of a MF-SDE, with corresponding laws μ_1, μ_2 at time t . Then if $f, f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, the following Lipschitz continuity holds:

$$|E[f(X_1(t))] - E[f(X_2(t))]|^2 \leq \frac{1}{2\pi} C(f) \|\mu_1 - \mu_2\|_{\mathcal{M}_0^2}^2. \quad (2.5)$$

Definition 2.5 (Law process) From now on we use the notation

$$M_t := M(t) := \mathcal{L}(X(t)); \quad 0 \leq t \leq T$$

for the law process $\mathcal{L}(X(t))$ of $X(t)$ with respect to $\mathbb{P}$.

We recall the following result, which is proved in Agram and Øksendal [5], Lemma 2.3/Lemma 6:

Lemma 2.6 *If $X(t)$ is an Itô-Lévy process as in (1.1), then the map $t \mapsto M(t) : [0, T] \rightarrow \mathcal{M}_0^4$ is absolutely continuous. Hence the derivative*

{2.6}

$$M'(t) := \frac{d}{dt}M(t)$$

exists for a.a. t . and we have

$$M(t) = M(0) + \int_0^t M'(s)ds; \quad t \geq 0.$$

We will also use the following spaces:

- C^d stands for the space of $\mathbb{R}^d$ -valued continuous functions defined over the time interval $[0, T]$.
- Given a finite time horizon $T > 0$, for $1 \leq p < +\infty$, let $S^p[0, T]$ denote the space of $\mathbb{R}^d$ -valued $\mathbb{F}$ -adapted càdlàg processes $X = (X(t))_{t \in [0, T]}$ such that:

$$\|X\|_{S^p[0, T]}^p := \mathbb{E}[\sup_{t \in [0, T]} |X(t)|^p] < \infty.$$

- We define $\bar{\mathcal{S}}[0, T]$ the space of processes $\bar{x} = \{x(s)\}_{0 \leq s \leq t} : [0, T] \mapsto \mathbb{R}$ such that

$$\|\bar{x}\|_{\bar{\mathcal{S}}[0, T]}^2 := \mathbb{E}[\sup_{s \in [0, t]} x^2(s)ds] < \infty.$$

For finite T we identify functions $\bar{x} : [0, T] \mapsto \mathbb{R}$ with functions $\bar{x} \in \bar{\mathcal{S}}[0, T]$ such that $x(s) = 0$ for $s > T$, and we regard them as functions defined on all $(-\infty, \infty)$ by setting $x(s) = 0$ for $s < 0$.

- Let $\bar{\mathcal{S}}[0, \infty)$ denote the space of processes $\bar{x} = \{x(s)\}_{0 \leq s \leq \infty} : [0, \infty) \mapsto \mathbb{R}$ such that

$$\|\bar{x}\|_{\bar{\mathcal{S}}[0, \infty)}^2 := \mathbb{E}[\int_0^\infty x^2(s)ds] < \infty.$$

- We let $\mathbb{G} := \{\mathcal{G}_t\}_{t \geq 0}$ be a fixed given subfiltration of $\mathbb{F}$ with $\mathcal{G}_t \subseteq \mathcal{F}_t$ for all $t \geq 0$. The sigma-algebra $\mathcal{G}_t$ represents the information available to the controller at time t . By U we denote a nonempty convex subset of $\mathbb{R}^d$ and we denote by $\mathcal{U}_{adm}$ the set of paths U -valued $\mathbb{G}$ -predictable control processes. We consider them as the *admissible* control processes.

2.2 Fréchet derivatives and dual operators

In this subsection we review briefly the Fréchet differentiability and we introduce some dual operators, which will be used when we in the next sections study Pontryagin's maximal principles for our stochastic control problem.

Let $\mathcal{X}, \mathcal{Y}$ be two Banach spaces and let $F : \mathcal{X} \rightarrow \mathcal{Y}$. Then

- We say that F has a directional derivative (or Gâteaux derivative) at $v \in \mathcal{X}$ in the direction $w \in \mathcal{X}$ if

$$D_w F(v) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (F(v + \varepsilon w) - F(v))$$

exists in $\mathcal{Y}$.

- We say that F is Fréchet differentiable at $v \in \mathcal{X}$ if there exists a continuous linear map $A : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\lim_{\substack{h \rightarrow 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|_{\mathcal{X}}} \|F(v + h) - F(v) - A(h)\|_{\mathcal{Y}} = 0,$$

where $A(h) = \langle A, h \rangle$ is the action of the linear operator A on h . In this case we call A the *gradient* (or Fréchet derivative) of F at v and we write

$$A = \nabla_v F.$$

- If F is Fréchet differentiable at v with Fréchet derivative $\nabla_v F$, then F has a directional derivative in all directions $w \in \mathcal{X}$ and

$$D_w F(v) = \nabla_v F(w) = \langle \nabla_v F, w \rangle.$$

In particular, note that if F is a linear operator, then $\nabla_v F = F$ for all v .

In the following we regard any real function $x(\cdot)$ defined on a subset D of $[0, \infty)$ as an element of $\mathbb{R}^{[0, \infty)}$ by setting $x(t) = 0$ for $t \notin D$.

Next, we introduce two useful dual operators.

- For $T \in (0, \infty)$ let $G(t) = G(t, \cdot) : \bar{\mathcal{S}}[0, T] \mapsto \mathbb{R}$ be a bounded linear operator on $\bar{\mathcal{S}}[0, T]$ for each t , uniformly bounded in $t \in [0, T]$. Then the map

$$Y \mapsto \mathbb{E}[\int_0^T \langle G(t), Y_t \rangle dt]; \quad Y \in \bar{\mathcal{S}}[0, T]$$

is a bounded linear functional on the Hilbert space $\bar{\mathcal{S}}[0, T]$. Therefore, by the Riesz representation theorem there exists a unique process denoted by $G^* \in \bar{\mathcal{S}}[0, T]$, such that

$$\mathbb{E}[\int_0^T \langle G(t), Y_t \rangle dt] = \mathbb{E}[\int_0^T G^*(t) Y(t) dt], \tag{2.6} \quad \{\text{eq6.7a}\}$$

for all $Y \in \bar{\mathcal{S}}[0, T]$.

- Proceeding as above, we also see that if $G_{\bar{m}}(t, \cdot) : [0, T] \times \mathcal{M}_{0,t} \mapsto L^1(\mathbb{P})$ is a bounded linear operator on $\mathcal{M}_{0,t}$ for each t , uniformly bounded in t , then the map

$$M(\cdot) \mapsto \int_0^T \langle G_{\bar{m}}(t), M_t \rangle dt; \quad M_t = \mathcal{L}(X_t)$$

is a bounded linear functional on $\mathcal{M}_{0,t}$. Therefore, there exists a unique process denoted by $G_{\bar{m}}^*(t) \in \mathcal{M}_{0,t}$ such that

$$\int_0^T \langle G_{\bar{m}}(t), M_t \rangle dt = \int_0^T G_{\bar{m}}^*(t) M(t) dt, \quad (2.7) \quad \{\text{eq6.9a}\}$$

for all $M \in \mathcal{M}_{0,t}$.

We illustrate these operators by some auxiliary results.

Lemma 2.7 *Consider the case when $G(t, \cdot) : \bar{\mathcal{S}}[0, T] \mapsto \bar{\mathcal{S}}[0, T]$ has the form*

$$G(t, \bar{x}) = \langle F, \bar{x} \rangle p(t), \text{ with } p \in L_0^2.$$

Then

$$G^*(t) := \langle F, p^t \rangle \quad (2.8) \quad \{\text{eq4.8}\}$$

satisfies (2.6), where $p^t := \{p(t+r)\}_{r \in [0,t]}$.

Proof. We must verify that if we define $G^*(t)$ by (2.8), then (2.6) holds. To this end, choose $Y \in \mathcal{S}_{\bar{x}}$ and consider

$$\begin{aligned} \int_0^T \langle F, p^t \rangle Y(t) dt &= \int_0^T \langle F, \{p(t+r)\}_{r \in [0,t]} \rangle Y(t) dt \\ &= \int_0^T \langle F, \{Y(t)p(t+r)\}_{r \in [0,t]} \rangle dt = \left\langle F, \left\{ \int_r^{T+r} Y(u-r)p(u) du \right\}_{r \in [0,t]} \right\rangle \\ &= \left\langle F, \left\{ \int_0^T Y(u-r)p(u) du \right\}_{r \in [0,t]} \right\rangle = \int_0^T \langle F, Y_u \rangle p(u) du \\ &= \int_0^T \langle G(u), Y_u \rangle du. \end{aligned}$$

□

Example 2.1 (i) For example, if $a \in \mathbb{R}^{[0,\infty)}$ is a bounded function with compact support, let $F(\bar{x})$ be the averaging operator defined by

$$F(\bar{x}) = \langle F, \bar{x} \rangle = \int_0^\infty a(r)x(r)dr$$

when $\bar{x} = \{x(s)\}_{s \in [0,\infty)}$, then

$$\langle F, p^t \rangle = \int_0^\infty a(r)p(t+r)dr.$$

(ii) Similarly, if F is evaluation at t_0 , i.e.

$$F(\bar{x}) = x(t_0) \text{ when } \bar{x} = \{x(s)\}_{s \in [0,\infty)},$$

then

$$\langle F, p^t \rangle = p(t+t_0).$$

3 The finite horizon case

In this section we consider the case with a finite time horizon $T < \infty$.

We are interested in the mean-field stochastic control problem with elephant memory, composed of a controlled diffusion equation defining the dynamics which are defined through the following equation:

$$\begin{cases} dX^u(t) &= b(t, X^u(t), X_t^u, M^u(t), M_t^u, u(t))dt + \sigma(t, X^u(t), X_t^u, M^u(t), M_t^u, u(t))dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, X^u(t), X_t^u, M^u(t), M_t^u, u(t), \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X^u(0) &= x_0, \end{cases} \quad (3.1) \quad \{\text{eq3.1a}\}$$

where $x_0 \in \mathbb{R}^d$ is a constant and $u \in \mathcal{U}_{adm}$ (the set of admissible controls) is our control process, and with coefficients $b : [0, T] \times \mathbb{R}^d \times C^d \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times C^d \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \rightarrow \mathbb{R}^{d \times n}$ and $\gamma : [0, T] \times \mathbb{R}^d \times C^d \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \times \mathbb{R}_0^k \rightarrow \mathbb{R}^{k \times n}$ satisfying suitable assumptions (see below). Here and in the following U is the set of possible control values. For given $u \in \mathcal{U}_{adm}$ we define its corresponding performance functional $J(u)$ by

$$J(u) = \mathbb{E}[\int_0^T f(t, X^u(t), X_t^u, M^u(t), M_t^u, u(t))dt + g(X^u(T), M^u(T))], \quad (3.2) \quad \{\text{eq3.4}\}$$

where $f : [0, T] \times \mathbb{R}^d \times C^d \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathcal{M}_0 \rightarrow \mathbb{R}^d$.

We assume that $f(t, x, \bar{x}, m, \bar{m}, u)$ and $g(x, m)$ are $\mathcal{F}_t$ - and $\mathcal{F}_T$ - measurable, respectively.

We consider the following *finite horizon mean-field elephant memory control problem*:

Problem 3.1 Find $\hat{u} \in \mathcal{U}_{adm}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{U}_{adm}} J(u).$$

For simplicity (but without loss of generality), from now on we will consider only the one-dimensional case.

3.1 Existence and uniqueness of the MF-SDE with elephant memory

We begin with the existence and uniqueness results for MF-SDE with elephant memory. Consider the following equation for $X(t) = X^{\bar{u}}(t)$, for fixed $\bar{u} \in \mathcal{U}_{adm}$:

$$\begin{cases} dX(t) &= b(t, X(t), X_t, M(t), M_t)dt + \sigma(t, X(t), X_t, M(t), M_t)dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(0) &= x_0. \end{cases} \quad (3.3) \quad \{\text{sde_fini}\}$$

We make the following assumptions on the coefficients $b : [0, T] \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \rightarrow \mathbb{R}$ and $\gamma : [0, T] \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times \mathbb{R}_0 \rightarrow \mathbb{R}$: Here the drift b , the volatility σ and the jump coefficient γ are supposed to be $\mathbb{F}$ -predictable.

- (i) The coefficients b , σ and γ are Borel measurable.
- (ii) There is a constant C_0 such that, for all $t \in [0, T]$, $\psi, \psi' \in \mathbb{R}$, $\bar{\psi}, \bar{\psi}' \in C$, $m, m' \in \mathcal{M}_0$ and all $\bar{m}, \bar{m}' \in \mathcal{M}_{0,t}$, the following holds for $h = b$ and for $h = \sigma$:

$$\left\{ \begin{array}{l} h \text{ is adapted and } |h(t, \psi, \bar{\psi}, m, \bar{m})| \leq C_0, \\ h(t, \cdot, \cdot, \cdot, \cdot) \text{ is Lipschitz uniformly with respect to } t, \\ |h(t, \psi, \bar{\psi}, m, \bar{m}) - h(t, \psi', \bar{\psi}', m', \bar{m}')|^2 \leq C_0(|\psi - \psi'|^2 + \sup_{0 \leq s \leq t} |\psi(s) - \psi'(s)|^2 \\ + ||M(t) - M(t)||_{\mathcal{M}_0}^2 + ||M_t - M'_t||_{\mathcal{M}_{0,t}}^2). \end{array} \right.$$

Similarly, we assume that γ is predictable and

$$\int_{\mathbb{R}_0} |\gamma(t, \psi, \bar{\psi}, m, \bar{m}, \zeta)| \nu(d\zeta) \leq C_0,$$

$$\int_{\mathbb{R}_0} |\gamma(t, \psi, \bar{\psi}, m, \bar{m}, \zeta) - \gamma(t, \psi', \bar{\psi}', m', \bar{m}', \zeta)|^2 \nu(d\zeta) \leq C_0(|\psi - \psi'| + \sup_{0 \leq s \leq t} |\psi(s) - \psi'(s)|^2 + ||M(t) - M(t)||_{\mathcal{M}_0}^2 + ||M_t - M'_t||_{\mathcal{M}_{0,t}}^2).$$

Theorem 3.2 *Under the assumptions (i) – (ii) our elephant memory MF-SDE*

{thm-ex}

$$\left\{ \begin{array}{l} dX(t) = b(t, X(t), X_t, M(t), M_t)dt + \sigma(t, X(t), X_t, M(t), M_t)dB(t) \\ \quad + \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \\ X(0) = x_0. \end{array} \right.$$

for any initial condition $x_0 \in \mathbb{R}$ admits a unique solution $X \in S^p[0, T]$.

We recall the following inequality which will be useful for our proof.

Lemma 3.3 (Kunita's inequality [18]) *Suppose $p \geq 2$ and*

$$X(t) = x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} \gamma(s, \zeta) \tilde{N}(ds, d\zeta).$$

Then there exists a positive constant $C_{p,T}$, (depending only on p, T) such that the following inequality holds

$$\begin{aligned} \mathbb{E}[\sup_{0 \leq t \leq T} |X(t)|^p] &\leq C_{p,T}(|x_0|^p + \mathbb{E}[\int_0^t \{|b(s)|^p + |\sigma(s)|^p \\ &\quad + \int_{\mathbb{R}_0} |\gamma(s, \zeta)|^p \nu(d\zeta) + (\int_{\mathbb{R}_0} |\gamma(s, \zeta)|^2 \nu(d\zeta))^{\frac{p}{2}}\} ds]). \end{aligned}$$

Proof of Theorem 3.2.

Existence. For the convenience of the reader, but without loss of generality of the method, we assume that $b = \sigma = 0$, but we can get the same result by using Kunita's inequality above for $b \neq 0$ and $\sigma \neq 0$. In the following we denote by C_p the constant that may change from line to line.

Choose arbitrary $X^0(t)$ with corresponding $X_t^0, M^0(t), M_t^0$ and consider inductively the equation

$$\begin{cases} X(0) &:= x_0, \\ X^{n+1}(t) &= x_0 + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X^n(s), X_s^n, M^n(s), M_s^n, \zeta) \tilde{N}(ds, d\zeta), \quad t \in [0, T], \quad n \geq 0. \end{cases}$$

It is clear that $X^n(t) \in S^p[0, T]$, for all $n \geq 0$. Let $\bar{X}^n := X^{n+1} - X^n$. Then, by the Kunita's inequality (for $b = \sigma = 0$), the following estimation holds for all $p \geq 2$:

$$\begin{aligned} \mathbb{E}[|\bar{X}^n(s)|^p] &\leq C_p(\mathbb{E}[\int_0^t \int_{\mathbb{R}_0} |\gamma(s, X^n(s), X_s^n, M^n(s), M_s^n, \zeta) \\ &\quad - \gamma(s, X^{n-1}(s), X_s^{n-1}, M^{n-1}(s), M_s^{n-1}, \zeta)|^p \nu(d\zeta) ds] \\ &\quad + \mathbb{E}[\int_0^t (\int_{\mathbb{R}_0}^2 |\gamma(s, X^n(s), X_s^n, M^n(s), M_s^n, \zeta) \\ &\quad - \gamma(s, X^{n-1}(s), X_s^{n-1}, M^{n-1}(s), M_s^{n-1}, \zeta)|^2 \nu(d\zeta)) \frac{p}{2} ds]), \quad t \in [0, T], \quad n \geq 1. \end{aligned}$$

Applying the Lipschitz assumption (ii), we get

$$\begin{aligned} \mathbb{E}[\sup_{s \leq t} |\bar{X}^n(s)|^p] &\leq C_p \mathbb{E}[\sup_{s \leq t} |\bar{X}^{n-1}(s)| + \|M^n(s) - M^{n-1}(s)\|_{\mathcal{M}_0} \\ &\quad + \int_0^t (\sup_{r \leq s} |\bar{X}^{n-1}(r)| + \|M^n(r) - M^{n-1}(r)\|_{\mathcal{M}_0})^2 ds]^{p/2} \\ &\leq C_p \mathbb{E}[\sup_{s \leq t} |\bar{X}^{n-1}(s)|^p], \quad t \in [0, T], \quad n \geq 1. \end{aligned}$$

Hence, from a standard argument we see that there is some $X \in \bigcap_{p \geq 1} S^p[0, T]$, such that

$$\mathbb{E}[\sup_{t \in [0, T]} |X^n(t) - X(t)|^p] \xrightarrow{n \rightarrow \infty} 0, \quad \text{for all } p \geq 2.$$

Finally, taking the limit in the Picard iteration as $n \rightarrow +\infty$, yields

$$X(t) = x_0 + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X(s), X_s, M(s), M_s, u_s, \zeta) \tilde{N}(ds, d\zeta), \quad t \in [0, T].$$

Uniqueness. The proof of uniqueness is obtained by the estimate of the difference of two solutions, and it is carried out similarly to the argument above. $\square$

3.2 Stochastic maximum principles

We now turn to the problem of optimal control of the mean-field equation (3.1) with performance functional (3.2). Because of the mean-field terms, it is natural to consider the two-dimensional system $(X(t), M(t))$, where the dynamics for $M(t)$ is the following:

$$\begin{cases} dM(t) = \beta(M(t))dt, \\ M(0) \in \mathcal{M}_0, \end{cases}$$

where we have put $\beta(M(t)) = M'(t)$. See Lemma 2.6.

Let $\mathcal{R}$ denote the set of Borel measurable functions $r : \mathbb{R}_0 \rightarrow \mathbb{R}$.

Define the Hamiltonian $H : [0, T] \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times C_a([0, T], \mathcal{M}_0) \rightarrow \mathbb{R}$, as follows

$$H(t, x, \bar{x}, m, \bar{m}, u, p^0, q^0, r^0, p^1) := f(t, x, \bar{x}, m, \bar{m}, u) + p^0 b(t, x, \bar{x}, m, \bar{m}, u) \\ + q^0 \sigma(t, x, \bar{x}, m, \bar{m}, u) + \int_{\mathbb{R}_0} r^0(\zeta) \gamma(t, x, \bar{x}, m, \bar{m}, u, \zeta) \nu(d\zeta) + \langle p^1, \beta(m) \rangle; \quad t \in [0, T],$$

and $H(t, x, \bar{x}, m, \bar{m}, u, p^0, q^0, r^0, p^1) = 0$ for all $t > T$.

We assume that all the coefficients f, b, σ, γ and g are continuously differentiable (C^1) with respect to x, u , and admit Fréchet derivatives with respect to $\bar{x}, m, \bar{m}$. Then the same holds for the Hamiltonian H .

We define the adjoint processes $(p^0, q^0, r^0), (p^1, q^1, r^1)$ as the solution of the following finite horizon backward stochastic differential equations (BSDEs):

$$\begin{cases} dp^0(t) &= - \left\{ \frac{\partial H}{\partial x}(t) + \mathbb{E}[\nabla_{\bar{x}}^* H(t) | \mathcal{F}_t] \right\} dt + q^0(t) dB(t) \\ &\quad + \int_{\mathbb{R}_0} r^0(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p^0(t) &= \frac{\partial g}{\partial x}(T); \quad t \geq T, \\ q^0(t) &= r^0(t, \cdot) = 0; \quad t > T, \end{cases} \quad (3.4) \quad \{\text{bsde0}\}$$

$$\begin{cases} dp^1(t) &= - (\nabla_m H(t) + \mathbb{E}[\nabla_{\bar{m}}^* H(t) | \mathcal{F}_t]) dt + q^1(t) dB(t) \\ &\quad + \int_{\mathbb{R}_0} r^1(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ p^1(t) &= \nabla_m g(T); \quad t \geq T, \\ q^1(t) &= r^1(t, \cdot) = 0; \quad t > T, \end{cases} \quad (3.5) \quad \{\text{eqp1}\}$$

where $g(T) = g(X(T), M(T))$ and

$$H(t) = H(t, x, \bar{x}, m, \bar{m}, u, p^0, q^0, r^0, p^1)_{x=X(t), \bar{x}=X_t, m=M(t), \bar{m}=M_t, u=u(t), p^0=p^0(t), q^0=q^0(t), r^0=r^0(t, \zeta), p^1=p^1(t)}.$$

In the next section, we will give an example on how to calculate this adjoint operator in particular cases.

We are now able to give a sufficient (a verification theorem) and a necessary maximum principle.

We do not give the proof of the following result, since it is similar to the proof in the infinite horizon case, which will be discussed in Section 4.

Theorem 3.4 (Sufficient conditions of optimality) *Let $\hat{u} \in \mathcal{U}_{\text{adm}}$ with corresponding solutions $\hat{X}$ and $(\hat{p}^0, \hat{q}^0, \hat{r}^0)$ and $(\hat{p}^1, \hat{q}^1, \hat{r}^1)$ of the forward and the backward stochastic differential equations (3.3), (3.4) and (3.5) respectively. Suppose that*

1. (Concavity) *The Hamiltonian is such that*

$$(x, \bar{x}, m, \bar{m}, u) \mapsto H(t, x, \bar{x}, m, \bar{m}, u, \hat{p}^0(t), \hat{q}^0(t), \hat{r}^0(t, \zeta), \hat{p}^1(t), \omega),$$

and the terminal condition

$$(x, m) \mapsto g(x, m, \omega),$$

are concave $\mathbb{P}$ -a.s. for each t .

2. (Maximum condition)

$$\begin{aligned} & \mathbb{E}[H(t, \hat{X}(t), \hat{X}_t, \hat{M}(t), \hat{M}_t, \hat{u}(t), \hat{p}^0(t), \hat{q}^0(t), \hat{r}^0(t, \cdot), \hat{p}^1(t)) | \mathcal{G}_t] \\ &= \sup_{u \in \mathcal{U}_{adm}} \mathbb{E}[H(t, \hat{X}(t), \hat{X}_t, \hat{M}(t), \hat{M}_t, u, \hat{p}^0(t), \hat{q}^0(t), \hat{r}^0(t, \cdot), \hat{p}^1(t)) | \mathcal{G}_t], \end{aligned} \quad (3.6)$$

$\mathbb{P}$ -a.s. for each $t \in [0, T]$.

Then $\hat{u}$ is an optimal control for Problem 3.1.

Next we consider a converse, in the sense that we look for necessary conditions of optimality. To this end, we make the following assumptions:

- *Assumption A1.*

Whenever $u \in \mathcal{U}_{adm}$, and $\pi \in \mathcal{U}_{adm}$ is bounded, there exists $\epsilon > 0$ such that for $\lambda \in (-\epsilon, \epsilon)$ we have

$$u + \lambda\pi \in \mathcal{U}_{adm}.$$

- *Assumption A2.*

For each $t_0 \in [0, T]$ and each bounded $\mathcal{G}_{t_0}$ -measurable random variables α , the process

$$\pi(t) = \alpha \mathbf{1}_{(t_0, T]}(t)$$

belongs to $\mathcal{U}_{adm}$.

- *Assumption A3.*

In general, if $K^u = (K^u(t))_{t \in [0, T]}$ is a process depending on u , and if $\pi \in \mathcal{U}$ we define the operator $D = D_\pi$ on K by

$$DK^u(t) := D_\pi K^u(t) = \frac{d}{d\lambda} K^{u+\lambda\pi}(t) |_{\lambda=0},$$

whenever the derivative exists. In particular, we define the *derivative process* $Z = Z_\pi = (Z(t))_{t \in [0, T]}$ by

$$Z(t) = DX^u(t) := \frac{d}{d\lambda} X^{u+\lambda\pi}(t) |_{\lambda=0}.$$

We assume that for all bounded $\pi \in \mathcal{U}_{adm}$ the derivative process $Z(t) = Z_\pi(t)$ exists and satisfies the equation

$$\left\{ \begin{aligned} dZ(t) &= \left[\frac{\partial b}{\partial x}(t) Z(t) + \langle \nabla_{\bar{x}} b(t), Z_t \rangle + \langle \nabla_m b(t), DM(t) \rangle \right. \\ &\quad \left. + \langle \nabla_{\bar{m}} b(t), DM_t \rangle + \frac{\partial b}{\partial u}(t) \pi(t) \right] dt \\ &\quad + \left[\frac{\partial \sigma}{\partial x}(t) Z(t) + \langle \nabla_{\bar{x}} \sigma(t), Z_t \rangle + \langle \nabla_m \sigma(t), DM(t) \rangle \right. \\ &\quad \left. + \langle \nabla_{\bar{m}} \sigma(t), DM_t \rangle + \frac{\partial \sigma}{\partial u}(t) \pi(t) \right] dB(t) \\ &\quad + \int_{\mathbb{R}_0} \left[\frac{\partial \gamma}{\partial x}(t, \zeta) Z(t) + \langle \nabla_{\bar{x}} \gamma(t, \zeta), Z_t \rangle + \langle \nabla_m \gamma(t, \zeta), DM(t) \rangle \right. \\ &\quad \left. + \langle \nabla_{\bar{m}} \gamma(t, \zeta), DM_t \rangle + \frac{\partial \gamma}{\partial u}(t, \zeta) \pi(t) \right] \tilde{N}(dt, d\zeta); \quad t \in [0, T], \\ Z(0) &= 0. \end{aligned} \right. \quad (3.7) \quad \{\text{eq3.25}\}$$

Remark 3.5 Using the Itô formula we see that Assumption A3 holds under reasonable smoothness conditions on the coefficients of the equation. A proof for a similar system is given in Lemma 12 in Agram and Øksendal [5]. We omit the details.

We do not give the proof of the following result, since it is similar to the proof in the infinite horizon case, which will be discussed in Section 4.

Theorem 3.6 Let $\hat{u} \in \mathcal{U}_{adm}$ with corresponding solutions $\hat{X}$ and $(\hat{p}^0, \hat{q}^0, \hat{r}^0)$ and $(\hat{p}^1, \hat{q}^1, \hat{r}^1)$ of the forward and the backward stochastic differential equations (3.3), (3.4) and (3.5) respectively with corresponding derivative process $\hat{Z}$ given by (3.7).

Then the following are equivalent:

•

$$\frac{d}{d\lambda} J(\hat{u} + \lambda\pi)|_{\lambda=0} = 0 \text{ for all bounded } \pi \in \mathcal{U}_{adm}. \quad (3.8) \quad \{\text{eq3.26}\}$$

•

$$\mathbb{E}[\frac{\partial H}{\partial u}(t, \hat{X}(t), \hat{X}_t, \hat{M}(t), \hat{M}_t, u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))|_{u=\hat{u}} | \mathcal{G}_t] = 0. \quad (3.9) \quad \{\text{eq3.27}\}$$

3.3 Example: A mean-field LQ control problem

As an example, consider the following optimization problem which is to maximize the performance functional

$$J(u) = \mathbb{E}[-\frac{1}{2}X^2(T) - \frac{1}{2}\int_0^T u^2(t)dt],$$

where $X(t)$ is subject to

$$\begin{cases} dX(t) &= \mathbb{E}[X(t)](b_0 + u(t))dt + \sigma_0 \mathbb{E}[X(t)]dB(t) \\ &+ \int_{\mathbb{R}_0} \gamma_0(\zeta) \mathbb{E}[X(t)]\tilde{N}(dt, d\zeta), \\ X(0) &= x_0 \in \mathbb{R}, \end{cases} \quad (3.10) \quad \{\text{eq4.20}\}$$

for some given constants b_0, σ_0 and $\gamma_0(\zeta) > -1$ a.s. ν .

We associate to this problem the Hamiltonian

$$\begin{aligned} H(t, m, u, p^0, q^0, r^0, p^1) &= -\frac{1}{2}u^2 + F(m)(b_0 + u)p^0 + F(m)\sigma_0 q^0 \\ &+ \int_{\mathbb{R}_0} F(m)\gamma_0(\zeta)r^0(\zeta)\nu(d\zeta) + \langle p^1, \beta(m) \rangle. \end{aligned} \quad (3.11) \quad \{\text{eq4.21}\}$$

Here

$$\begin{aligned} b(t, X(t), X_t, M(t), M_t) &= F(M(t))(b_0 + u(t)), \\ \sigma(t, X(t), X_t, M(t), M_t) &= F(M(t))\sigma_0, \\ \gamma(t, X(t), X_t, M(t), M_t, \zeta) &= F(M(t))\gamma_0(\zeta)r(\zeta)\nu(d\zeta), \end{aligned}$$

where the operator F is defined by

$$F(m) = \int_{\mathbb{R}} x dm(x); \quad m \in \mathcal{M}_0,$$

so that

$$F(M(t)) = \int_{\mathbb{R}} x dM(t)(x) = \mathbb{E}[X(t)] \text{ when } M(t) = \mathcal{L}(X(t)).$$

Note that, since H does not depend on $x, \bar{x}, \bar{m}$, we have

$$\frac{\partial H}{\partial x}(t) = \nabla_{\bar{x}} H(t) = \nabla_{\bar{m}} H(t) = 0.$$

And, since $m \mapsto F(m)$ and $m \mapsto \beta(m)$ are linear, we have

$$\nabla_m H(t) = F(\cdot)(b_0 + u)p^0(t) + F(\cdot)\sigma_0 q^0(t) + \int_{\mathbb{R}_0} F(\cdot)\gamma_0(\zeta)r^0(\zeta)\nu(d\zeta) + \langle p^1, \beta(\cdot) \rangle.$$

Hence, the adjoint equation for (p^0, q^0, r^0) is

$$\begin{cases} dp^0(t) &= q^0(t)dB(t) + \int_{\mathbb{R}_0} r^0(t, \zeta)\tilde{N}(dt, d\zeta); & 0 \leq t \leq T, \\ p^0(T) &= -X(T), \end{cases} \quad (3.12) \quad \{\text{1bsde}\}$$

and the adjoint equation for (p^1, q^1, r^1) is

$$\begin{cases} dp^1(t) &= -[F(\cdot)(b_0 + u)p^0(t) + F(\cdot)\sigma_0 q^0(t) + F(\cdot)\gamma_0(\zeta)r^0(\zeta)\nu(d\zeta) + \langle p^1, \beta(\cdot) \rangle]dt \\ &+ q^1(t)dB(t) + \int_{\mathbb{R}_0} r^1(t, \zeta)\tilde{N}(dt, d\zeta); & t \in [0, T], \\ p^1(T) &= 0. \end{cases}$$

The map $u \mapsto H(u)$ is maximal when $\frac{\partial H}{\partial u} = 0$, i.e., when

$$u = \hat{u}(t) = \mathbb{E}[\hat{X}(t)]\hat{p}^0(t) = -\mathbb{E}[\hat{X}(t)]\mathbb{E}[\hat{X}(T)|\mathcal{F}_t]. \quad (3.13) \quad \{\text{eq3.14}\}$$

Substituting this into (3.10) we get that $Y(t) := \mathbb{E}[\hat{X}(t)]$ satisfies the following Riccati equation

$$\begin{cases} Y'(t) &= b_0 Y(t) - Y^2(t)Y(T); & 0 \leq t \leq T, \\ Y(0) &= x_0. \end{cases} \quad (3.14) \quad \{\text{eq4.31}\}$$

Solving this Riccati equation, we find an explicit expression for $Y(t)$ in terms of $Y(T)$ and hence by putting $t = T$ also an explicit expression for $Y(T)$, and then we find $Y(t)$ for all $t \in [0, T]$.

Equation (3.14) has the solution:

$$Y(t) = \mathbb{E}[\hat{X}(t)] = \frac{b_0 x_0 \exp(b_0 t)}{(b_0 - x_0 \mathbb{E}[\hat{X}(T)])(1 + \exp(b_0 t))}.$$

Consequently,

$$Y(T) = \mathbb{E}[\hat{X}(T)] = \frac{b_0 x_0 \exp(b_0 T)}{(b_0 - x_0 \mathbb{E}[\hat{X}(T)])(1 + \exp(b_0 T))}.$$

Then we see that we also know $\mathbb{E}[\hat{X}(T)|\mathcal{F}_t]$ by the equation

$$\begin{aligned} \mathbf{K}(t) &= \mathbb{E}[\hat{X}(T)|\mathcal{F}_t] \\ &= \mathbf{K}(0) + \int_0^t Y(s)\sigma_0 dB(s) + \int_0^t \int_{\mathbb{R}_0} Y(s)\gamma_0(\zeta)\tilde{N}(ds, d\zeta). \end{aligned}$$

Thus we have proved the following:

Theorem 3.7 *The optimal control $\hat{u}$ of the mean-field LQ problem is given by*

$$\hat{u}(t) = -\mathbb{E}[\hat{X}(t)]\mathbb{E}[\hat{X}(T)|\mathcal{F}_t],$$

with $\mathbb{E}[\hat{X}(t)]$ and $\mathbb{E}[\hat{X}(T)|\mathcal{F}_t]$ given above.

4 The infinite horizon case

We now study the case when the time horizon is $[0, \infty)$. Consider the equation

$$\begin{cases} dX(t) = b(t, X(t), X_t, M(t), M_t, u(t))dt + \sigma(t, X(t), X_t, M(t), M_t, u(t))dB(t) \\ \quad + \int_{\mathbb{R}_0} \gamma(t, X(t), X_t, M(t), M_t, u(t), \zeta) \tilde{N}(dt, d\zeta); t \in [0, \infty), \\ X(0) = x_0, \end{cases} \quad (4.1) \quad \{\mathbf{F}\}$$

where $x_0 \in \mathbb{R}$ is the initial condition, $u \in \mathcal{U}_{adm}$, and the coefficients $b : [0, \infty) \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \rightarrow \mathbb{R}$, $\sigma : [0, \infty) \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \rightarrow \mathbb{R}$ and $\gamma : [0, \infty) \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \times \mathbb{R}_0 \rightarrow \mathbb{R}$ are $\mathcal{F}_t$ -measurable. Here C stands for the space of $\mathbb{R}$ -valued continuous functions defined over the time interval $[0, \infty)$. We assume that

$$\mathbb{E}[\int_0^\infty |X(s)|^2 ds] < \infty.$$

For given $u \in \mathcal{U}_{adm}$, we define its corresponding performance functional by

$$J(u) = \mathbb{E}[\int_0^\infty f(t, X(t), X_t, M(t), M_t, u(t))dt], \quad (4.2) \quad \{\mathbf{P}\}$$

where the reward function $f : [0, \infty) \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \rightarrow \mathbb{R}$ is assumed to satisfy the condition

$$\mathbb{E}[\int_0^\infty |f(t, X(t), X_t, M(t), M_t, u(t))|^2 dt] < \infty, \quad \text{for all } u \in \mathcal{U}_{adm}.$$

We consider the following *infinite horizon mean-field elephant memory control problem*:

Problem 4.1 Find $\hat{u} \in \mathcal{U}_{adm}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{U}_{adm}} J(u).$$

Define the Hamiltonian $H : [0, \infty) \times \mathbb{R} \times C \times \mathcal{M}_0 \times \mathcal{M}_{0,t} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \times C_a([0, T], \mathcal{M}_0) \rightarrow \mathbb{R}$, by

$$\begin{aligned} H(t, x, \bar{x}, m, \bar{m}, u, p^0, q^0, r^0, p^1) &:= f(t, x, \bar{x}, m, \bar{m}, u) + p^0 b(t, x, \bar{x}, m, \bar{m}, u) \\ &+ q^0 \sigma(t, x, \bar{x}, m, \bar{m}, u) + \int_{\mathbb{R}_0} r^0(\zeta) \gamma(t, x, \bar{x}, m, \bar{m}, u, \zeta) \nu(d\zeta) + \langle p^1, m' \rangle. \end{aligned} \quad (4.3) \quad \{\text{eq3.6}\}$$

In the following we assume that all the coefficients f, b, σ and γ are continuously differentiable (C^1) with respect to x and admit Fréchet derivatives with respect to $\bar{x}, m, \bar{m}$ and u .

Associated to the control $\hat{u}$ we define the following infinite horizon BSDE for the adjoint processes $(\hat{p}^0, \hat{q}^0, \hat{r}^0), (\hat{p}^1, \hat{q}^1, \hat{r}^1)$:

$$\begin{aligned} dp^0(t) &= -\left\{ \frac{\partial H}{\partial x}(t) + \mathbb{E}[\nabla_{\bar{x}}^* H(t) | \mathcal{F}_t] \right\} dt \\ &\quad + q^0(t) dB(t) + \int_{\mathbb{R}_0} r^0(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \geq 0, \end{aligned} \quad (4.4) \quad \{\text{eq4.11}\}$$

$$\begin{aligned} dp^1(t) &= -[\nabla_m H(t) + \mathbb{E}[\nabla_{\bar{m}}^* H(t) | \mathcal{F}_t]] dt + q^1(t) dB(t) \\ &\quad + \int_{\mathbb{R}_0} r^1(t, \zeta) \tilde{N}(dt, d\zeta); \quad t \geq 0. \end{aligned} \quad (4.5) \quad \{\text{eq4.11}'\}$$

Remark 4.2 Note that without further conditions there are infinitely many solutions $(\hat{p}^0, \hat{q}^0, \hat{r}^0)$ and $(\hat{p}^1, \hat{q}^1, \hat{r}^1)$ of these equations.

4.1 Sufficient infinite horizon maximum principle

In this subsection, we give sufficient conditions which ensure the existence of an optimal control in the infinite horizon case.

Theorem 4.3 (Sufficient condition of optimality) *Let $\hat{u} \in \mathcal{U}_{adm}$ with corresponding solution $\hat{X}$ of the forward stochastic differential equation (4.1). Assume that $(\hat{p}^0, \hat{q}^0, \hat{r}^0)$ and $(\hat{p}^1, \hat{q}^1, \hat{r}^1)$ is some solution of the associated backward stochastic differential equations (4.4) and (4.5) respectively. Suppose the following holds:*

1. (Concavity) *The function*

$$(x, \bar{x}, m, \bar{m}, u) \mapsto H(t, x, \bar{x}, m, \bar{m}, u, \hat{p}^0, \hat{q}^0, \hat{r}^0, \hat{p}^1),$$

is concave $\mathbb{P}$ -a.s. for each $t, \hat{p}^0, \hat{q}^0, \hat{r}^0, \hat{p}^1$.

2. (Maximum condition)

$$\mathbb{E}[\hat{H}(t)|\mathcal{G}_t] = \sup_{u \in \mathcal{U}_{adm}} \mathbb{E}[H(t)|\mathcal{G}_t], \quad (4.6) \quad \{\text{maxQ}\}$$

$\mathbb{P}$ -a.s. for each $t \geq 0$.

3. (Transversality condition) *For all $u \in \mathcal{U}_{adm}$ with corresponding solution $X^u = X$ we have*

$$\lim_{T \rightarrow \infty} \mathbb{E}[\hat{p}^0(T)(X(T) - \hat{X}(T))] + \lim_{T \rightarrow \infty} \mathbb{E}[\hat{p}^1(T)(M(T) - \hat{M}(T))] \geq 0, \quad (4.7) \quad \{\text{tcond}_1\}$$

Then $\hat{u}$ is an optimal control for Problem 4.1.

Proof. Choose arbitrary $u \in \mathcal{U}_{adm}$. We want to show that $J(u) \leq J(\hat{u})$, i.e.,

$$A := J(u) - J(\hat{u}) = \mathbb{E}[\int_0^\infty \{f(t) - \hat{f}(t)\} dt] \leq 0, \quad (4.8) \quad \{\text{J2}\}$$

where we have used the simplified notation $\hat{f}(t) := f(t, \hat{X}(t), \hat{X}_t, \hat{M}(t), \hat{M}_t, \hat{u}(t))$ and so on. By concavity of the Hamiltonian (4.3), we have

$$\begin{aligned} A &= \mathbb{E}[\int_0^\infty \{H(t) - \hat{H}(t) - \hat{p}^0(t)\tilde{b}(t) - \hat{q}^0(t)\tilde{\sigma}(t) - \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta)\} dt] \\ &\leq \mathbb{E}[\int_0^\infty \{\frac{\partial \hat{H}}{\partial x}(t)\tilde{X}(t) + \langle \nabla_{\bar{x}} \hat{H}(t), \tilde{X}_t \rangle + \langle \nabla_m \hat{H}(t), \tilde{M}(t) \rangle + \langle \nabla_{\bar{m}} \hat{H}(t), \tilde{M}_t \rangle + \frac{\partial \hat{H}}{\partial u}(t)\tilde{u}(t) \\ &\quad - \hat{p}^0(t)\tilde{b}(t) - \hat{q}^0(t)\tilde{\sigma}(t) - \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta)\} dt], \end{aligned} \quad (4.9) \quad \{\text{eq3.13}\}$$

where $\tilde{b}(t) = b(t) - \hat{b}(t)$, etc.

For fixed $T \geq 0$, define an increasing sequence of stopping times τ_n , as follows

$$\begin{aligned} \tau_n(\cdot) &:= T \wedge \inf\{t \geq 0 : \int_0^t ((\hat{p}^0(s)\tilde{\sigma}(s))^2 + (\hat{q}^0(s)\tilde{X}(s))^2 \\ &\quad + \int_{\mathbb{R}_0} \{(\hat{r}^0(s, \zeta)\tilde{X}(s))^2 + (\hat{p}^0(s)\tilde{\gamma}(s, \zeta))^2\} \nu(d\zeta) ds \geq n\}, n \in \mathbb{N}, \end{aligned}$$

it clearly holds that $\tau_n \rightarrow T$ $\mathbb{P}$ -a.s. By the Itô formula applied to $\hat{p}^0(\tau_n)\tilde{X}(\tau_n)$, we get

$$\begin{aligned}\mathbb{E}[\hat{p}^0(T)\tilde{X}(T)] &= \lim_{n \rightarrow \infty} \mathbb{E}[\hat{p}^0(\tau_n)\tilde{X}(\tau_n)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[\int_0^{\tau_n} \hat{p}^0(t)d\tilde{X}(t) + \int_0^{\tau_n} \tilde{X}(t)d\hat{p}^0(t) + \int_0^{\tau_n} \hat{q}^0(t)\tilde{\sigma}(t)dt \\ &\quad + \int_0^{\tau_n} \int_{\mathbb{R}_0} \hat{r}(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta)dt] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[\int_0^{\tau_n} \{\hat{p}^0(t)\tilde{b}(t) - \tilde{X}(t)(\frac{\partial \hat{H}}{\partial x}(t) + \nabla_x^* \hat{H}(t)) \\ &\quad + \hat{q}^0(t)\tilde{\sigma}(t) + \int_{\mathbb{R}_0} \hat{r}^0(t, \zeta)\tilde{\gamma}(t, \zeta)\nu(d\zeta)\}dt].\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\mathbb{E}[\langle \hat{p}_1^1(T), \tilde{M}(T) \rangle] \\ &= \mathbb{E}[\int_0^T \langle \hat{p}_1^1(t), d\tilde{M}(t) \rangle + \int_0^T \tilde{M}(t)d\hat{p}_1^1(t)] \\ &= \mathbb{E}[\int_0^T \langle \hat{p}_1^1(t), \tilde{M}'(t) \rangle dt - \int_0^T \{\langle \nabla_m \hat{H}_1(t), \tilde{M}(t) \rangle - \nabla_m^* \hat{H}_1(t)\tilde{M}(t)\}dt].\end{aligned}$$

In the above we have used that the expectation of the martingale terms, i.e. the $dB(t)$ - and $\tilde{N}(dt, d\zeta)$ -integrals, have mean zero. Taking the limit superior and using the transversality conditions (4.7) combined with (4.9) we obtain, using that u and $\hat{u}$ are $\mathbb{G}$ -adapted,

$$\begin{aligned}A &\leq -\overline{\lim}_{T \rightarrow \infty} \mathbb{E}[\hat{p}^0(T)\tilde{X}(T)] - \overline{\lim}_{T \rightarrow \infty} \mathbb{E}[\hat{p}_1^1(T)\tilde{M}(T)] + \mathbb{E}[\int_0^T \frac{\partial \hat{H}}{\partial u}(t)_{u=\hat{u}(t)}\tilde{u}(t)dt] \\ &\leq \mathbb{E}[\int_0^T \frac{\partial \hat{H}}{\partial u}(t)_{u=\hat{u}(t)}\tilde{u}(t)dt] \leq 0,\end{aligned}$$

since $u \mapsto \mathbb{E}[\hat{H}(t, u)|\mathcal{G}_t]$ is maximal at $u = \hat{u}(t)$. That completes the proof. $\square$

4.2 Necessary maximum principle under partial information

We now consider the converse, i.e. we look for necessary conditions of optimality. The following result is the infinite horizon version of Theorem 3.6:

Theorem 4.4 *Assume that Assumptions A1-A3 of Section 3.2 hold but now with $t \in [0, \infty)$. Let $u \in \mathcal{U}_{adm}$ with corresponding solutions X and (p^0, q^0, r^0) and (p^1, q^1, r^1) of the forward and the backward stochastic differential equations (4.1) and (4.4) and (4.5) respectively with corresponding derivative process Z given by (3.7) but now with the time horizon $[0, \infty)$.*

Moreover, assume that the following transversality condition holds:

$$\overline{\lim}_{T \rightarrow \infty} \mathbb{E}[p^0(T)Z(T)] = \overline{\lim}_{T \rightarrow \infty} \mathbb{E}[\langle p^1(T), DM(T) \rangle] = 0; \quad \text{for all bounded } \pi \in \mathcal{U}_{adm}. \quad (4.10) \quad \{\text{trv_c_n}\}$$

Then the following are equivalent:

•

$$\frac{d}{d\lambda} J(u + \lambda\pi)|_{\lambda=0} = 0 \text{ for all bounded } \pi \in \mathcal{U}_{adm}. \quad (4.11) \quad \{\text{eq4.18}\}$$

$$\mathbb{E}[\frac{\partial H}{\partial u}(t, u)|\mathcal{G}_t] = 0. \quad (4.12) \quad \{\text{eq4.19}\}$$

Proof. Assume that (4.11) holds. Then

$$\begin{aligned} 0 &= \frac{d}{d\lambda} J(u + \lambda\pi)|_{\lambda=0} \\ &= \mathbb{E}[\int_0^\infty \{ \frac{\partial f}{\partial x}(t)Z(t) + \langle \nabla_{\bar{x}}f(t), Z_t \rangle + \langle \nabla_m f(t), DM(t) \rangle \\ &\quad + \langle \nabla_{\bar{m}}f(t), DM_t \rangle + \frac{\partial f}{\partial u}(t)\pi(t) \} dt]. \end{aligned} \quad (4.13) \quad \{\text{eq4.12}\}$$

By the definition of the Hamiltonian (4.3), we have

$$\nabla f(t) = \nabla H(t) - \nabla b(t)p^0(t) - \nabla \sigma(t)q^0(t) - \int_{\mathbb{R}_0} \nabla \gamma(t, \zeta)r^0(t, \zeta)\nu(d\zeta),$$

where $\nabla = (\frac{\partial}{\partial x}, \nabla_{\bar{x}}, \nabla_m, \nabla_{\bar{m}}, \frac{\partial}{\partial u})$.

Define a sequence of stopping times by

$$\tau_n(\cdot) := T \wedge \inf\{t \geq 0 : \int_0^t \{(p^0(s))^2 + (q^0(s))^2 + \int_{\mathbb{R}_0} (r^0(s, \zeta))^2 \nu(d\zeta) + \pi^2(s)\} ds \geq n\}, n \in \mathbb{N}.$$

Clearly $\tau_n \rightarrow T$ $\mathbb{P}$ -a.s. as $n \rightarrow \infty$. Applying the Itô formula, we get

$$\begin{aligned} &\mathbb{E}[p^0(T)Z(T)] + \mathbb{E}[\langle p^1(T), DM(T) \rangle] = \lim_{n \rightarrow \infty} (\mathbb{E}[p^0(\tau_n)Z(\tau_n)] + \mathbb{E}[\langle p^1(\tau_n), DM(\tau_n) \rangle]) \\ &= \mathbb{E}\left[\int_0^{\tau_n} p^0(t) \left\{ \frac{\partial b}{\partial x}(t)Z(t) + \langle \nabla_{\bar{x}}b(t), Z_t \rangle + \langle \nabla_m b(t), DM(t) \rangle + \langle \nabla_{\bar{m}}b(t), DM_t \rangle + \frac{\partial b}{\partial u}(t)\pi(t) \right. \right. \\ &\quad - \left(\frac{\partial H}{\partial x}(t) + \mathbb{E}[\nabla_{\bar{x}}^* H(t)|\mathcal{F}_t] \right) Z(t) \\ &\quad + q(t) \left(\frac{\partial \sigma}{\partial x}(t)Z(t) + \langle \nabla_{\bar{x}}\sigma(t), Z_t \rangle + \langle \nabla_m \sigma(t), DM(t) \rangle + \langle \nabla_{\bar{m}}\sigma(t), DM_t \rangle + \frac{\partial \sigma}{\partial u}(t)\pi(t) \right) \\ &\quad + \int_{\mathbb{R}_0} r(t, \zeta) \left(\frac{\partial \gamma}{\partial x}(t, \zeta)Z(t) + \langle \nabla_{\bar{x}}\gamma(t, \zeta), Z_t \rangle + \langle \nabla_m \gamma(t, \zeta), DM(t) \rangle \right. \\ &\quad \left. \left. + \langle \nabla_{\bar{m}}\gamma(t, \zeta), DM_t \rangle + \frac{\partial \gamma}{\partial u}(t, \zeta)\pi(t) \right) \nu(d\zeta) \right\} dt \Big] \\ &\quad + \mathbb{E}\left[\int_0^{\tau_n} \{ \langle p^1(t), DM'((t)) \rangle - \langle \nabla_m H(t), DM((t)) \rangle - \nabla_{\bar{m}}^* H(t)DM((t)) \} dt \right]. \end{aligned}$$

Taking the limit superior, combining this with (4.13) and using the transversality condition (4.10), we get

$$0 = \overline{\lim}_{T \rightarrow \infty} \mathbb{E}[p^0(T)Z(T)] + \overline{\lim}_{T \rightarrow \infty} \mathbb{E}[\langle p^1(T), DM(T) \rangle] = \mathbb{E}[\int_0^\infty \frac{\partial H}{\partial u}(t)\pi(t)dt].$$

Now choose $\pi(t) = \alpha \mathbf{1}_{(t_0, T]}(t)$, where $\alpha = \alpha(\omega)$ is bounded and $\mathcal{G}_{t_0}$ -measurable and $t_0 \in [0, T)$. Then we deduce that

$$\mathbb{E}[\int_{t_0}^\infty \frac{\partial H}{\partial u}(t)\alpha dt] = 0.$$

Differentiating with respect to t_0 we obtain

$$\mathbb{E}[\frac{\partial H}{\partial u}(t_0)\alpha] = 0.$$

Since this holds for all such α , we conclude that

$$\mathbb{E}[\frac{\partial H}{\partial u}(t_0)|\mathcal{G}_{t_0}] = 0, \quad \text{which is (4.12).}$$

This argument can be reversed, to prove that (4.12) $\implies$ (4.11). We omit the details. $\square$

5 Optimal consumption from an elephant memory cash flow

To illustrate our results, let us consider an example of an infinite horizon optimal consumption problem, where the wealth process of the investor $X = (X^u(t))_{t \geq 0}$ is given by the following dynamics:

$$\begin{cases} dX^u(t) &= \{ \langle F, X_t^u \rangle - u(t) \} dt + \beta X^u(t) dB(t); t \geq 0, \\ X^u(0) &= x_0 > 0, \end{cases}$$

where $u(t) \geq 0$ denotes the consumption rate (our control), $\beta > 0$ (constant) denotes the volatility and $F(\cdot) : L_0(\mathbb{R}) \mapsto \mathbb{R}$ is a bounded linear operator on the whole memory path $X_t^u = \{X^u(t-s)\}_{0 \leq s \leq t}$ of X up to time t . Thus the term $\langle F, X_t^u \rangle$ represents a drift term in the dynamics depending on the whole history of the process. A specific example is given below.

We define $\mathcal{U}_{adm}$ to be the set of nonnegative adapted processes u such that

$$\mathbb{E} \left[\int_0^\infty |X^u(t)|^2 dt \right] < \infty$$

For $u \in \mathcal{U}_{adm}$ we also require that u satisfies the following *budget constraint*:

The expected total discounted consumption is bounded by the initial capital x_0 , i.e.:

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} u(t) dt \right] \leq x_0, \quad (5.1) \quad \{\text{eq5.2}\}$$

where $\rho > 0$ is a given discount exponent. Consider the following problem:

Problem 5.1 Find $\hat{u} \in \mathcal{U}_{adm}$ such that

$$J(\hat{u}) = \sup_{u \in \mathcal{U}_{adm}} J(u), \quad (5.2)$$

where the performance functional $J(u)$ is the total discounted logarithmic utility of the consumption u , i.e.

$$J(u) = \mathbb{E} \left[\int_0^\infty e^{-\delta t} \ln(u(t)) dt \right]; u \in \mathcal{U}_{adm}, \quad (5.3)$$

for some constant $\delta > 0$.

The Hamiltonian in this case takes the form

$$H(t, x, \bar{x}, u, p^0, q^0) = e^{-\delta t} \ln(u) + p^0[\langle F, \bar{x} \rangle - u] + q^0 \beta x,$$

and the adjoint process pair $(p^0(t), q^0(t))$ is a solution of the corresponding adjoint BSDE

$$dp^0(t) = -\{ \beta q^0(t) + \mathbb{E}[\nabla_{\bar{x}}^* H(t) | \mathcal{F}_t] \} dt + q^0(t) dB(t); t \in [0, \infty). \quad (5.4) \quad \{\text{eq5.5}\}$$

Note that by Lemma 2.7 we have

$$\nabla_x^* H(t) = \langle F, (p^0)^t \rangle.$$

For example, let us from now on assume that $F(\cdot)$ is a weighted average operator of the form

$$\langle F, \bar{x} \rangle = \int_0^t e^{-\rho r} x(r) dr. \quad (5.5) \quad \{\text{average}\}$$

Then we get

$$\nabla_x^* H(t) = \int_0^\infty e^{-\rho r} p^0(t+r) dr,$$

and the state equation becomes

$$\begin{cases} dX^u(t) &= \left\{ \int_0^t e^{-\rho r} X^u(t-r) dr - u(t) \right\} dt + \beta X^u(t) dB(t); t \geq 0, \\ X^u(0) &= x_0 > 0. \end{cases}$$

The adjoint BSDE (5.1) will take the form

$$dp^0(t) = -\left\{ \beta q^0(t) + \mathbb{E}[\int_0^\infty e^{-\rho r} p^0(t+r) dr | \mathcal{F}_t] \right\} dt + q^0(t) dB(t); t \in [0, \infty). \quad (5.6) \quad \{\text{eq5.2}\}$$

Maximising the Hamiltonian with respect to u gives the following equation for a possible optimal consumption rate $u = \hat{u}$:

$$e^{-\delta t} \frac{1}{\hat{u}(t)} - \hat{p}^0(t) = 0,$$

i.e.

$$\hat{u}(t) = \frac{e^{-\delta t}}{\hat{p}^0(t)}. \quad (5.7) \quad \{\text{eq5.3}\}$$

With this choice $u = \hat{u}$ the equations above get the form

$$\begin{cases} d\hat{X}(t) &= \left\{ \int_0^t e^{-\rho r} \hat{X}(t-r) dr - \frac{e^{-\delta t}}{\hat{p}^0(t)} \right\} dt + \beta \hat{X}(t) dB(t); t \geq 0, \\ \hat{X}(0) &= x_0, \end{cases}$$

and

$$d\hat{p}^0(t) = -\left\{ \beta \hat{q}^0(t) + \mathbb{E}[\int_0^\infty e^{-\rho r} \hat{p}^0(t+r) dr | \mathcal{F}_t] \right\} dt + \hat{q}^0(t) dB(t); t \in [0, \infty), \quad (5.8) \quad \{\text{eq5.2}\}$$

We want to find a solution $(\hat{p}^0, \hat{q}^0)$ of this infinite horizon BSDE such that the transversality condition holds, i.e.

$$\overline{\lim}_{T \rightarrow \infty} \mathbb{E}[\hat{p}^0(T)(X^u(T) - \hat{X}(T))] \geq 0$$

for all admissible controls u .

Remark 5.2 This problem may be regarded as an infinite horizon version of a stochastic control problem for a Volterra equation, without memory. To see this, note that by a change of variable and a change of the order of integration the equation (5.6) can be written

$$\begin{aligned} X(t) &= x_0 + \int_0^t \left(\int_0^s e^{-\rho r} X(t-r) dr \right) ds - \int_0^t u(s) ds + \int_0^t \beta X(s) dB(s) \\ &= x_0 + \int_0^t \frac{1}{\rho} (1 - e^{\rho(s-t)}) X(s) ds - \int_0^t u(s) ds + \int_0^t \beta X(s) dB(s), \end{aligned} \quad (5.9) \quad \{\text{eq5.8}\}$$

which is a stochastic Volterra equation of the type studied in [4] and [9].

Let us try to assume that $\hat{q}^0 = 0$ and hence that $\hat{p}^0$ is deterministic. Then the equation for $\hat{p}^0(t)$ reduces to the integral equation

$$d\hat{p}^0(t) = - \left(\int_0^\infty e^{-\rho r} \hat{p}^0(t+r) dr \right) dt. \quad (5.10)$$

By a similar procedure as in (5.9) above we obtain that this equation can be transformed to the equation

$$d\hat{p}^0(t) = -\frac{1}{\rho} (1 - e^{-\rho t}) \hat{p}^0(t) dt, \quad (5.11)$$

which has the general solution

$$\hat{p}^0(t) = \hat{p}^0(0) \exp\left(-\frac{t}{\rho} + \frac{1 - e^{-\rho t}}{\rho^2}\right); \quad t \geq 0, \quad (5.12) \quad \{\text{eq5.11}\}$$

for some constant $\hat{p}^0(0)$.

Substituted into (5.6) this gives

$$\hat{u}(t) = \frac{1}{\hat{p}^0(0)} \exp\left(\left(\frac{1}{\rho} - \delta\right)t - \frac{1 - e^{-\rho t}}{\rho^2}\right). \quad (5.13) \quad \{\text{eq5.12}\}$$

The problem is to find $\hat{p}^0(0)$ such that the following two conditions hold:

$$\overline{\lim}_{T \rightarrow \infty} \hat{p}^0(T) \mathbb{E}[\hat{X}(T)] = 0 \quad (5.14) \quad \{\text{eq5.13}\}$$

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} u(t) dt \right] \leq x_0 \quad (\text{the budget constraint}). \quad (5.15) \quad \{\text{eq5.14}\}$$

Define $y_0(t)$ to be the solution of the integral equation

$$y_0(t) = x_0 + \int_0^t \frac{1}{\rho} (1 - e^{\rho(s-t)}) y_0(s) ds; \quad t \geq 0, \quad (5.16)$$

and let $\lambda_0 > 0$ be the top Lyapunov exponent of y_0 . See e.g. Kunita [18] and Mang and Sheng [20] for more information about Lyapunov exponents. Then, since clearly

$$y_0(t) \geq \mathbb{E}[\hat{X}(t)] \text{ for all } t \geq 0,$$

we see by (5.12) that if

$$\rho < \frac{1}{\lambda_0}, \quad (5.17)$$

then

$$\overline{\lim}_{T \rightarrow \infty} \hat{p}^0(T) \mathbb{E}[\hat{X}(T)] = 0, \quad (5.18)$$

and hence (5.14) holds for any choice of $\hat{p}^0(0)$. By (5.13) the budget constraint (5.15) gives

$$\hat{p}^0(0) \geq \frac{1}{x_0} \int_0^\infty \exp\left(\left(\frac{1}{\rho} - \delta - \rho\right)t - \frac{1 - e^{-\rho t}}{\rho^2}\right) dt. \quad (5.19) \quad \{\text{eq5.22}\}$$

The admissible value of $\hat{p}^0(0)$ that gives the maximal consumption is therefore, by (5.13),

$$\hat{p}^0(0) = \frac{1}{x_0} \int_0^\infty \exp\left(\left(\frac{1}{\rho} - \delta - \rho\right)t - \frac{1 - e^{-\rho t}}{\rho^2}\right) dt. \quad (5.20) \quad \{\text{eq5.23}\}$$

We summarise what we have proved as follows:

Theorem 5.3 *Assume that*

$$\rho < \frac{1}{\lambda_0}.$$

Then the optimal consumption rate $\hat{u}(t)$ for Problem 5.1, with F defined by (5.5), is given by (5.13), where $\hat{p}^0(0)$ is given by (5.20).

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