On very effective hermitian K -theory

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Abstract

We argue that the very effective cover of hermitian K -theory in the sense of motivic homotopy theory is a convenient algebro-geometric generalization of the connective real topological K -theory spectrum. This means the very effective cover acquires the correct Betti realization, its motivic cohomology has the desired structure as a module over the motivic Steenrod algebra, and that its motivic Adams and slice spectral sequences are amenable to calculations.

1 Introduction

Algebraic and hermitian K -theory have been widely studied since the pioneering works on the Grothendieck-Riemann-Roch theorem [1] and on rings with anti-involutions [14]. Both theories are representable in the stable motivic homotopy category $\mathbf{SH}$ of a field of characteristic $\neq 2$, and more generally over regular noetherian base schemes of finite Krull dimension on which 2 is invertible [10], [29]. Fundamental properties imply that the motivic spectra of algebraic K -theory $\mathbf{KGL}$ and hermitian K -theory $\mathbf{KQ}$ are $(2, 1)$ - and $(8, 4)$ -periodic, respectively, with respect to the standard motivic spheres $S^{p,q} := S^{p-q} \wedge \mathbf{G}_m^q$ for $p \geq q$. More precisely, there exist Bott elements in the Grothendieck group $\pi_{2,1}\mathbf{KGL} \cong K_0$ and in the Grothendieck-Witt group $\pi_{8,4}\mathbf{KQ} \cong GW_0$ inducing motivic weak equivalences

$$S^{2,1} \wedge \mathbf{KGL} \xrightarrow{\cong} \mathbf{KGL} \text{ and } S^{8,4} \wedge \mathbf{KQ} \xrightarrow{\cong} \mathbf{KQ}. \quad (1.1)$$

Because of (1.1), $\mathbf{KGL}$ and $\mathbf{KQ}$ are non-connective and should be thought of as “large” motivic spectra. When using K -theoretic invariants to inform the homotopy sheaves of the sphere $\mathbf{1}$ in [25], it is convenient to employ smaller “connective” versions of the motivic K -theory spectra. The geometrical meaning of this notion is still not well understood.

One fascinating aspect of motivic homotopy theory is that it offers different notions of “connectivity” based on:

- Voevodsky’s slice filtration for the localizing triangulated subcategory of effective motivic spectra $\mathbf{SH}^{\text{eff}} \subset \mathbf{SH}$ generated under homotopy colimits by motivic $\mathbf{P}^1$ -suspension spectra of smooth schemes [30, §2].

- Morel's homotopy t -structure [18, §6.2] characterized over perfect fields by the vanishing of homotopy sheaves, and its extension to general base schemes in [11, §2.1].
- Spitzweck-Østvær's very effective slice filtration for the full subcategory $\mathbf{SH}^{\text{veff}} \subset \mathbf{SH}$ generated under homotopy colimits and extensions by motivic $\mathbf{P}^1$ -suspension spectra of smooth schemes [28, Definition 5.5].

The essential difference between the effective and the very effective slice filtrations is that the former records slices with respect to the multiplicative group scheme $\mathbf{G}_m$ and the latter with respect to the projective line $\mathbf{P}^1$. By construction $\mathbf{SH}^{\text{veff}}$ is a subcategory of $\mathbf{SH}^{\text{eff}}$ closed under the tensor product, but it is not closed under simplicial desuspension and hence not a triangulated subcategory of $\mathbf{SH}$. The sphere $\mathbf{1}$, algebraic cobordism $\mathbf{MGL}$, and quotients thereof such as motivic Moore spectra and the effective cover of $\mathbf{KGL}$ are examples of very effective motivic spectra. Both of the slice filtrations interact well with A_∞ - and E_∞ -structures [7], but only the very effective one maps to (the even part of) the topological Postnikov filtration under Betti realization [7, §3.3]. Over perfect fields $\mathbf{SH}^{\text{veff}}$ is the nonnegative part of the t -structure on $\mathbf{SH}^{\text{eff}}$, and the identification of the very effective slices of $\mathbf{KQ}$ up to extensions in [3, Theorem 16] makes a strong case for further investigations of the very effective slice filtration.

In this paper we argue that the very effective cover $\mathbf{kq}$ of hermitian K -theory $\mathbf{KQ}$ is a convenient algebro-geometric generalization of the connective cover $\mathbf{ko}$ of real topological K -theory $\mathbf{KO}$. The question of whether there is a motivic spectrum with similar properties as $\mathbf{ko}$ was first addressed in [13] and [9, Conjecture 5.8] over the fields of complex and real numbers, respectively. As we learned from the referee, work by Mike Hill and Kyle Ormsby concerning such a motivic spectrum was presented during a sectional meeting of the AMS in 2014.

Section 2 begins with some preliminary results on $\mathbf{KGL}$. We identify the effective and very effective covers of $\mathbf{KGL}$ over perfect fields of characteristic $\neq 2$, and similarly for $\mathbf{KGL}/2$ and base schemes over $\text{Spec}(\mathbb{Z}[\frac{1}{2}])$. Proposition 2.11 relates the very effective covers $\mathbf{kq}$ of $\mathbf{KQ}$ and $\mathbf{kgl}$ of $\mathbf{KGL}$ to the motivic Hopf map $\eta: \mathbf{A}^2 \setminus \{0\} \rightarrow \mathbf{P}^1$ via the cofiber sequence

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \rightarrow \mathbf{kgl}. \quad (1.2)$$

Over the complex numbers, (1.2) has been obtained independently by Bachmann. The same result holds for η , $\mathbf{KQ}$, $\mathbf{KGL}$, and base schemes over $\text{Spec}(\mathbb{Z}[\frac{1}{2}])$ by [23, Theorem 3.4], but it is plainly false for the effective covers of $\mathbf{KQ}$ and $\mathbf{KGL}$ by the proof of [24, Corollary 5.1]. By using (1.2) we identify the Betti realization of $\mathbf{kq}$ with $\mathbf{ko}$ and calculate the mod-2 motivic cohomology $\mathbf{M}\mathbb{Z}/2_*\mathbf{kq}$ as $\mathcal{A}^*/\mathcal{A}^*(1)$; the quotient of the mod-2 motivic Steenrod algebra $\mathcal{A}^*$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2_*$ -subalgebra generated by $\mathbf{Sq}^1$ and $\mathbf{Sq}^2$ [12], [32]. By dualizing, the mod-2 motivic homology $\mathbf{M}\mathbb{Z}/2_*\mathbf{kq}$ identifies with $\mathcal{A}_*\square_{\mathcal{A}_*(1)}\mathbf{M}\mathbb{Z}/2_*$ as an $\mathcal{A}_*$ -comodule algebra, and by change-of-rings the $\mathbf{M}\mathbb{Z}/2$ -based Adams spectral sequence for $\mathbf{kq}$ takes the form

$$\text{Ext}_{\mathcal{A}_*(1)}^{*,*}(\mathbf{M}\mathbb{Z}/2_*, \mathbf{M}\mathbb{Z}/2_*) \Rightarrow \mathbf{kq}_{*2,\eta}^\wedge. \quad (1.3)$$

As indicated in the notation, the filtered target groups of the spectral sequence (1.3) are all $(2, \eta)$ -completed. The Ext-algebra over $\mathcal{A}_\star(1)$ appearing in (1.3) is accessible via homological algebra. For explicit calculations with (1.3) we refer to [9] and [13].

Section 3 is concerned with slice calculations. The negative slices of $\mathbf{kq}$ are evidently zero because $\mathbf{kq}$ is an effective motivic spectrum. Over a perfect field of characteristic $\neq 2$ and $i \geq 0$, we show in Theorem 3.2 the calculation

$$s_q \mathbf{kq} = \begin{cases} \Sigma^{2n, 2n} \mathbf{MZ}/2 \vee \Sigma^{2n+2, 2n} \mathbf{MZ}/2 \vee \dots \vee \Sigma^{4n-2, 2n} \mathbf{MZ}/2 \vee \Sigma^{4n, 2n} \mathbf{MZ} & q = 2n, \\ \Sigma^{2n+1, 2n+1} \mathbf{MZ}/2 \vee \Sigma^{2n+3, 2n+1} \mathbf{MZ}/2 \vee \dots \vee \Sigma^{4n+1, 2n+1} \mathbf{MZ}/2 & q = 2n + 1. \end{cases} \quad (1.4)$$

The slices of $\mathbf{kq}$ are considerably “smaller” than those of $\mathbf{KQ}$ [23]. This is a helpful fact which is used in the calculation of the first stable homotopy groups of motivic spheres [25].

An immediate consequence of (1.4) is the explicit form of the slice spectral sequence given by mod-2 motivic cohomology groups $h^\star$ and integral motivic cohomology groups $H^\star$

$$\pi_{p,w} s_q \mathbf{kq} = \begin{cases} h^{2n-p, 2n-w} \oplus \dots \oplus h^{4n-2-p, 2n-w} \oplus H^{4n-p, 2n-w} & q = 2n \\ h^{2n+1-p, 2n+1-w} \oplus \dots \oplus h^{4n+1-p, 2n+1-w} & q = 2n + 1 \end{cases} \Rightarrow \mathbf{kq}_{p,w}. \quad (1.5)$$

In Theorem 3.5 we identify the d_1 -differentials in (1.5) in terms of motivic Steenrod operations. We also calculate the slices and the slice differentials for the η -inversion of $\mathbf{kq}$.

In Section 4 we identify the 0-line of $\mathbf{kq}$ with the Milnor-Witt K -theory over fields of characteristic not 2, and determine the associated graded for the groups on the 1-line of $\mathbf{kq}$.

Throughout the paper we employ the following assumptions and notations.

F, S	perfect field, finite dimensional separated noetherian scheme
$\mathbf{Sm}_S$	smooth schemes of finite type over S
$S^{s,t}, \Omega^{s,t}, \Sigma^{s,t}$	motivic (s, t) -sphere, (s, t) -loop space, (s, t) -suspension
$\mathbf{SH}, \mathbf{SH}^{\text{eff}}$	motivic and effective motivic stable homotopy categories of S
$\mathbf{E}, \mathbf{1} = S^{0,0}$	generic motivic spectrum, the motivic sphere spectrum
$\Lambda, \mathbf{MA}$	ring, motivic Eilenberg-MacLane spectra of a Λ -module A
$\mathbf{KGL}, \mathbf{KQ}, \mathbf{KW}$	algebraic and hermitian K -theory, Witt-theory
$\mathbf{f}_q, \tilde{\mathbf{f}}_q, \mathbf{s}_q$	q th effective cover, very effective cover, and slice

In all results concerning $\mathbf{KQ}$ and $\mathbf{kq}$ we assume that 2 is invertible on the base scheme S , as for $\text{Spec}(\mathbb{Z}[\frac{1}{2}])$, and following [27] we impose the condition that

$$S \text{ is essentially smooth over a Dedekind domain.} \quad (1.6)$$

Applications will mostly concern perfect fields of characteristic $\neq 2$.

2 Connecting connective K -theories

Definition 2.1: Following [28, §5] we let $\mathbf{kq} \rightarrow \mathbf{KQ}$ denote the very effective cover of the hermitian K -theory spectrum $\mathbf{KQ}$ of quadratic forms [10] and let $\mathbf{kgl} \rightarrow \mathbf{KGL}$ denote the

very effective cover of the algebraic K -theory spectrum $\mathbf{KGL}$ [29].

Remark 2.2: Following [3], [28], and working over F we can identify the very effective cover $\tilde{f}_q \mathbf{E}$ of $\mathbf{E}$ with $f_0(\mathbf{E}_{\geq 0})$, the effective cover of the connective cover $\mathbf{E}_{\geq 0}$ of $\mathbf{E}$ with respect to the homotopy t -structure on $\mathbf{SH}$ [18].

Lemma 2.3: *If F admits a complex embedding, the Betti realization of $\mathbf{kgl}$ coincides with the connective cover $\mathbf{ku}$ of the complex topological K -theory spectrum $\mathbf{KU}$ in the topological stable homotopy category.*

Proof. Recall from [28, Proposition 5.12] that $\mathbf{kgl}$ is a homotopy quotient of $\mathbf{MGL}$ under the orientation or Todd genus map, and similarly but easier that $\mathbf{ku}$ is a homotopy quotient of $\mathbf{MU}$. The Betti realization functor [21, Appendix A] preserves homotopy colimits, and sends $\mathbf{MGL}$ to $\mathbf{MU}$. $\square$

Lemma 2.4: *Over F the effective and very effective covers of $\mathbf{KGL}$ coincide in $\mathbf{SH}$.*

Proof. When $\text{char}(F) = 0$ this is shown in [28, Corollary 5.13] by writing the effective cover of $\mathbf{KGL}$ as a homotopy quotient of $\mathbf{MGL}$ (the latter is very effective over any base scheme [28, Theorem 5.7]). If $\text{char}(F) > 0$ we follow the proof of [3, Theorem 16] where the effective cover $f_0 \mathbf{KGL} \rightarrow \mathbf{KGL}$ is shown to be connective. For $t \geq 0$ the presheaf on $\mathbf{Sm}_F$

$$X \mapsto [\Sigma^{s,t} X_+, f_0 \mathbf{KGL}] = K_{s-2t}(X)$$

is zero for $s < 2t$, e.g., for $s - t < 0$ (this holds if X is regular, hence over any regular base scheme S). The case $t = 0$ implies by [3, Proposition 4] that $f_0 \mathbf{KGL}$ is connective, and by [3, Lemma 10] that $f_0 \mathbf{KGL}$ is the very effective cover. $\square$

Remark 2.5: Lemma 2.4 holds more generally for motivic Landweber exact spectra over a field in the sense of [20].

Over a noetherian scheme S of finite Krull dimension d , the presheaf on $\mathbf{Sm}_S$

$$X \mapsto [\Sigma^{s,t} X_+, \mathbf{KGL}] = KH_{s-2t}(X)$$

is zero for $s - 2t < -d$ by [15], since $\mathbf{KGL}$ represents homotopy K -theory over S [4]. Thus for $t \geq q$, the presheaf

$$X \mapsto [\Sigma^{s,t} X_+, f_q \mathbf{KGL}] = K_{s-2t}(X)$$

is zero for $s - t + d < q$, and $f_q \mathbf{KGL}$ is q -connected in the sense of [25, Definition 3.16]. If the very effective slice filtration coincides with the combination of the homotopy t -structure and the effective slice filtration over S , then $f_0 \mathbf{KGL}$ is the very effective cover, i.e., the effective and very effective slices of $\mathbf{KGL}$ agree. We can argue differently for $\mathbf{KGL}/2$ when 2 is invertible as follows (this proof can also be adapted to motivic Landweber exact spectra).

Lemma 2.6: *Over a base scheme S as in (1.6) on which 2 is invertible, the effective and very effective covers of $\mathbf{KGL}/2$ coincide in $\mathbf{SH}$.*

Proof. We claim $\mathbf{KGL}/2$ affords the description as a homotopy quotient of $\mathbf{MGL}/2$ for the generators of the Lazard ring $x_i \in \pi_{2i,i}\mathbf{MGL}$. Since $\mathbf{MGL}$ is effective the orientation map for $\mathbf{KGL}$ factors through

$$\phi: \mathbf{MGL} \rightarrow f_0\mathbf{KGL}.$$

For $i \geq 2$ we have $\pi_{2i,i}\phi(x_i) = 0$, so that ϕ admits a factorization

$$\mathbf{MGL}/(x_2, x_3, \dots) \rightarrow f_0\mathbf{KGL}.$$

We claim there is a canonically induced motivic weak equivalence

$$\psi: \mathbf{MGL}/(2, x_2, x_3, \dots) \xrightarrow{\cong} f_0\mathbf{KGL}/2.$$

The map ψ yields an isomorphism on slices by [27, Theorem 10.3] and an appropriate adaption of [26, Proposition 5.4]. We show that $\text{holim}_{q \rightarrow \infty} f_q \psi$ is a map between contractible motivic spectra, i.e., ψ is a map between slice complete spectra. For $\mathbf{KGL}/2$ this follows by the argument prior to Lemma 2.6: By [15] we know $f_q\mathbf{KGL}$ is q -connected in the sense of [25, Definition 3.16]. Thus $\text{holim}_{q \rightarrow \infty} f_q\mathbf{KGL} \cong *$, and likewise for $f_0\mathbf{KGL}/2$. The contractibility of $\text{holim}_{q \rightarrow \infty} f_q\mathbf{MGL}/(2, x_2, x_3, \dots)$ follows from the description of the covers $f_q\mathbf{MGL}$ in the proof of [26, Theorem 4.6]. To conclude for ψ we use that slices detect motivic weak equivalences between slice complete motivic spectra, cf. [11, §8.3]. Recall that $\mathbf{MGL}$ is a very effective motivic spectrum [28, Theorem 5.7]. The lemma follows from the canonically induced motivic weak equivalences in the commutative diagram

$$\begin{array}{ccccc} \tilde{f}_0\mathbf{MGL}/(2, x_2, x_3, \dots) & \xrightarrow{\cong} & \tilde{f}_0f_0\mathbf{KGL}/2 & \xrightarrow{\cong} & \tilde{f}_0\mathbf{KGL}/2 \\ \cong \downarrow & & \downarrow & & \downarrow \\ f_0\mathbf{MGL}/(2, x_2, x_3, \dots) & \xrightarrow{\cong} & f_0f_0\mathbf{KGL}/2 & \xrightarrow{\cong} & f_0\mathbf{KGL}/2. \end{array}$$

□

The Bott element $\mathbf{P}^1 \rightarrow \mathbf{KGL}$ lifts canonically to a map $\beta: \mathbf{P}^1 \rightarrow \mathbf{kgl}$ because $\mathbf{P}^1$ is very effective. Let γ denote the canonical composite

$$\mathbf{kgl} \rightarrow f_0(\mathbf{KGL}) \rightarrow s_0\mathbf{KGL}.$$

Proposition 2.7: *Over F multiplication with the Bott element induces the cofiber sequence*

$$\Sigma^{2,1}\mathbf{kgl} \xrightarrow{\beta} \mathbf{kgl} \xrightarrow{\gamma} \mathbf{MZ} \xrightarrow{\delta} \Sigma^{3,1}\mathbf{kgl}.$$

Proof. By Lemma 2.4 we have $f_0(\mathbf{KGL}_{\geq 0}) \cong \mathbf{kgl}$ and by $(2,1)$ -periodicity $f_{-1}(\mathbf{KGL}_{\geq -1}) \cong \Sigma^{-2,-1}\mathbf{kgl}$. Let $\beta': f_1(\mathbf{KGL}_{\geq 1}) \rightarrow f_0(\mathbf{KGL}_{\geq 0})$ denote the natural map in the very effective slice filtration for $\mathbf{KGL}$. The commutative diagram

$$\begin{array}{ccccccc} \Sigma^{2,1}f_0(\mathbf{KGL}_{\geq 0}) & \xrightarrow{\cong} & f_1(\Sigma^{2,1}\mathbf{KGL}_{\geq 0}) & \xrightarrow{\cong} & f_1((\Sigma^{2,1}\mathbf{KGL})_{\geq 1}) & \xrightarrow{\cong} & f_1(\mathbf{KGL}_{\geq 1}) \\ \beta \downarrow & & & & & & \downarrow \beta' \\ f_0(\mathbf{KGL}_{\geq 0}) & \xrightarrow{\quad \text{id} \quad} & & & & & f_0(\mathbf{KGL}_{\geq 0}) \end{array}$$

shows that it corresponds to multiplication with the Bott element on $\mathbf{kgl}$. The cofiber sequence

$$f_1(\mathbf{KGL}_{\geq 1}) \xrightarrow{\beta'} f_0(\mathbf{KGL}_{\geq 0}) \rightarrow \mathbf{MZ},$$

for the very effective zero slice of $\mathbf{KGL}$ [3, Lemma 7], which coincides with the usual zero slice $s_0\mathbf{KGL} \cong \mathbf{MZ}$ computed in [16], [31], concludes the proof. $\square$

Proposition 2.8: *Over a base scheme S as in (1.6) on which 2 is invertible, multiplication with the Bott element induces the cofiber sequence*

$$\Sigma^{2,1}\mathbf{kgl}/2 \xrightarrow{\beta} \mathbf{kgl}/2 \xrightarrow{\gamma} \mathbf{MZ}/2 \xrightarrow{\delta} \Sigma^{3,1}\mathbf{kgl}/2.$$

Proof. This follows from Lemma 2.6. $\square$

Lemma 2.9: *If 2 is invertible on a base scheme S as in (1.6), then the composite*

$$\mathbf{MZ} \wedge \mathbf{MZ}/2 \xrightarrow{\delta \wedge \mathbf{MZ}/2} \Sigma^{3,1}\mathbf{kgl} \wedge \mathbf{MZ}/2 \xrightarrow{\gamma \wedge \mathbf{MZ}/2} \Sigma^{3,1}\mathbf{MZ} \wedge \mathbf{MZ}/2$$

is given by multiplication with the first Milnor operation

$$Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1 : \mathbf{MZ}/2 \rightarrow \Sigma^{3,1}\mathbf{MZ}/2.$$

Proof. The proof of Lemma 2.6 shows $\mathbf{KGL}$ and $f_0\mathbf{KGL}/2$ are invariant under base change, being homotopy quotients of $\mathbf{MGL}$. The same holds for $\mathbf{MZ}$ and the dual motivic Steenrod algebra $\mathbf{MZ}/2 \wedge \mathbf{MZ}/2$ by [27, Section 9, Theorem 10.26]. Hence it suffices to show the statement in the case $S = \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$. The inclusion $\mathbb{Z}[\frac{1}{2}] \hookrightarrow \mathbb{C}$ induces a bijection on the bidegree $(3,1)$ summand of the motivic Steenrod algebra (which in both cases is given as in [27, Theorem 10.26, Remark 10.27] over the mod two motivic cohomology of the base), because the Picard group of $\mathbb{Z}[\frac{1}{2}]$ vanishes. Hence it remains to prove the statement over $\mathbb{C}$, where it follows by complex realization from Lemma 2.3 and the corresponding topological result [2, p. 366]. $\square$

Remark 2.10: Following [13, Theorem 5.4], Lemma 2.9 shows the mod-2 motivic cohomology $\mathbf{MZ}/2^*\mathbf{kgl}$ is the quotient of the mod-2 motivic Steenrod algebra $\mathcal{A}^*$ by the augmentation ideal of the $\mathbf{MZ}/2^*$ -subalgebra generated by $Q_0 = Sq^1$ and Q_1 .

Proposition 2.11: *Over a field of characteristic $\neq 2$, multiplication with the Hopf map η induces a cofiber sequence*

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{f} \mathbf{kgl} \xrightarrow{h} \Sigma^{2,1}\mathbf{kq}. \quad (2.1)$$

Here f and h are functorially induced by the forgetful and hyperbolic maps between algebraic and hermitian K -theory, respectively.

Proof. Consider the fiber F of the naturally induced forgetful map $f_{\geq 0}: \mathbf{KQ}_{\geq 0} \rightarrow \mathbf{KGL}_{\geq 0}$. Since f_0 is a triangulated functor, $f_0(F)$ is the fiber of $f := f_0(f_{\geq 0})$. The composite map

$$\Sigma^{1,1}\mathbf{kq} \xrightarrow{\eta} \mathbf{kq} \xrightarrow{f} \mathbf{kgl}$$

is trivial because the first negative algebraic K -group $\pi_{1,1}\mathbf{kgl} = \pi_{1,1}\mathbf{KGL} = K_{-1}$ vanishes over regular schemes. We show there is an induced motivic weak equivalence $\Sigma^{1,1}\mathbf{kq} \rightarrow f_0(F)$ of effective motivic spectra by checking the map of homotopy sheaves $\pi_{s,t}$ is an isomorphism for $t \geq 0$. This follows if (2.1) induces a long exact sequence of sheaves for $t \geq 0$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{s,t}\Sigma^{1,1}\mathbf{kq} & \xrightarrow{\eta} & \pi_{s,t}\mathbf{kq} & \xrightarrow{f} & \pi_{s,t}\mathbf{kgl} \xrightarrow{h} \cdots \\ & & \cong \downarrow & & \downarrow = & & \downarrow = \\ \cdots & \longrightarrow & \pi_{s-1,t-1}\mathbf{kq} & \xrightarrow{\eta} & \pi_{s,t}\mathbf{kq} & \xrightarrow{f} & \pi_{s,t}\mathbf{kgl} \xrightarrow{h} \cdots \end{array} \quad (2.2)$$

By construction, (2.2) is exact for $t \geq 1$ and $s \geq t$ since in the said range it coincides with the long exact sequence

$$\cdots \rightarrow \pi_{s,t}\Sigma^{1,1}\mathbf{KQ} \xrightarrow{\eta} \pi_{s,t}\mathbf{KQ} \xrightarrow{f} \pi_{s,t}\mathbf{KGL} \xrightarrow{h} \cdots$$

induced by the Wood cofiber sequence for η , $\mathbf{KQ}$, and $\mathbf{KGL}$ [23, Theorem 3.4].

If $t \geq 1$ and $s = t$, $\pi_{t,t}(f): \pi_{t,t}\mathbf{kq} \rightarrow \pi_{t,t}\mathbf{kgl}$ is surjective since its target is trivial. Thus (2.2) is exact for $t \geq 1$ and all s ; recall that $\pi_{s,t}\mathbf{kq} = \pi_{s,t}\mathbf{kgl} = 0$ for all $s < t$.

It remains to consider the case $t = 0$. By [3, Theorem 16] the composite

$$f_0(\mathbf{KQ}_{\geq 0}) \rightarrow f_{-1}(\mathbf{KQ}_{\geq 0}) \rightarrow f_{-1}(\mathbf{KQ}_{\geq -1})$$

is an equivalence. The canonical map $\mathbf{KQ}_{\geq 0} \rightarrow \mathbf{KQ}_{\geq -1}$ is an isomorphism on homotopy sheaves $\pi_{s,t}$ for all $t \geq -1$ and all s . When $s < t - 1$ and $s \geq t$ this follows by construction. The case $s = t - 1$ holds since $\pi_{t-1,t}\mathbf{KQ} = 0$ for all $t \geq -1$. More precisely, the vanishing for $t \geq 0$ is implied by comparison with Witt theory because $\pi_{t-1,t}\mathbf{KW} = 0$ for all t . The case $t = -1$ follows from the long exact sequence

$$\cdots \rightarrow \pi_{0,0}\mathbf{KGL} \xrightarrow{0} \pi_{-2,-1}\mathbf{KQ} \xrightarrow{\eta} \pi_{-1,0}\mathbf{KQ} \xrightarrow{f} \pi_{-1,0}\mathbf{KGL} = 0,$$

and surjectivity of the rank map $f: \pi_{0,0}\mathbf{kq} \rightarrow \pi_{0,0}\mathbf{kgl}$. It follows that there is a canonical motivic weak equivalence

$$f_0(\mathbf{KQ}_{\geq 0}) \xrightarrow{\cong} f_{-1}(\mathbf{KQ}_{\geq 0}),$$

which implies exactness of (2.2) for $t = 0$. □

Lemma 2.12: *If 2 is invertible on a base scheme S as in (1.6), then the composite*

$$\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{h \wedge \mathbf{M}\mathbb{Z}/2} \Sigma^{2,1}\mathbf{kq} \wedge \mathbf{M}\mathbb{Z}/2 \xrightarrow{f \wedge \mathbf{M}\mathbb{Z}/2} \Sigma^{2,1}\mathbf{kgl} \wedge \mathbf{M}\mathbb{Z}/2$$

is given by multiplication with $\mathrm{Sq}^2: \mathbf{M}\mathbb{Z}/2 \rightarrow \Sigma^{2,1}\mathbf{M}\mathbb{Z}/2$.

Proof. As in the proof of Lemma 2.9 it suffices to work over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and hence over $\mathbb{C}$. The result follows from Lemma 2.13 and the corresponding topological statement. $\square$

Lemma 2.13: *If F admits a complex embedding, the Betti realization of $\mathbf{kq}$ coincides with the connective cover ko of the real topological K -theory spectrum KO in the topological stable homotopy category.*

Proof. This follows since the Betti realization sends $\mathbf{KQ}$ to KO , $\mathbf{kgl}$ to ku by Lemma 2.3, and preserves the Wood cofiber sequence. $\square$

Remark 2.14: As in [13, Theorem 5.11], Lemma 2.9 identifies $\mathbf{M}\mathbb{Z}/2^\star \mathbf{kq}$ with the quotient of the mod-2 motivic Steenrod algebra $\mathcal{A}^\star$ by the augmentation ideal of the $\mathbf{M}\mathbb{Z}/2^\star$ -subalgebra generated by Sq^1 and Sq^2 , and the homotopy of $\mathbf{kq} \wedge \mathbf{M}\mathbb{Z}/2$ as a comodule over the dual motivic Steenrod algebra recorded by the $\mathbf{M}\mathbb{Z}/2$ -based Adams spectral sequence for $\mathbf{kq}$ (1.3).

Remark 2.15: If F admits a real embedding, the Betti realization of $\mathbf{kq}$ acquires the structure of a genuine $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant spectrum [8]. Equivariant forms of connective real and complex topological K -theory exist. The survey [5], pointed out to us by John Rognes, emphasizes a particularly well-behaved form. A natural question following Lemma 2.13 is whether the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant Betti realization of $\mathbf{kq}$ is this $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant connective real topological K -theory spectrum. Dan Isaksen sketched an argument, based on the Adams spectral sequence, that the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -equivariant Betti realization of $\mathbf{kq}$ coincides with the form employed in [6], at least after completion at the prime two.

Next we observe that $\mathbf{kgl}$ differs from the cover of algebraic K -theory introduced in [13]. By the cofiber sequence

$$\mathbf{kgl} = f_0(\mathbf{KGL}_{\geq 0}) \rightarrow f_{-1}(\mathbf{KGL}_{\geq -1}) \rightarrow s_{-1}\mathbf{KGL} = \Sigma^{-2,-1}\mathbf{M}\mathbb{Z},$$

we obtain a long exact sequence and an isomorphism

$$\cdots \rightarrow \pi_{0,-1}s_{-1}\mathbf{KGL} \rightarrow \pi_{-1,-1}\mathbf{kgl} \xrightarrow{\cong} \pi_{-1,-1}f_{-1}(\mathbf{KGL}_{\geq -1}) \rightarrow \pi_{-1,-1}s_{-1}\mathbf{KGL} \rightarrow \cdots. \quad (2.3)$$

The outer terms in (2.3) are trivial. Since $\pi_{-1,-1}f_{-1}(\mathbf{KGL}_{\geq -1}) \cong \pi_{-1,-1}\mathbf{KGL}$ it follows that $\pi_{-1,-1}\mathbf{kgl} \cong K_1(F) \cong F^\times$. Over the complex numbers, this calculation distinguishes $\mathbf{kgl}$ from the (2-complete) positive cellular cover of $\mathbf{KGL}$ in [13] because $\pi_{-1,-1}$ of the latter is trivial by construction.

Finally, we remark that $\mathbf{kq}$ does not coincide with the effective cover $f_0\mathbf{KQ}$ featuring in the solution of the homotopy limit problem for the C_2 -action on $\mathbf{kgl}$ in [24].

3 Slice computations

We shall identify the slices of $\mathbf{kq}$ similarly to the slices of $\mathbf{KQ}$ in [23]. The crucial ingredients are the Wood cofiber sequence (2.1) and the slices of connective algebraic K -theory $\mathbf{kgl}$.

Theorem 3.1: *Over F the canonical map $\mathbf{kgl} \rightarrow \mathbf{KGL}$ induces an isomorphism on all nonnegative slices. The negative slices of $\mathbf{kgl}$ are zero.*

Proof. Since $\mathbf{kgl} = f_0 \mathbf{KGL}$ by Lemma 2.4, this follows by construction. $\square$

Identifying the slices of $\mathbf{kq}$ is more involved because $\mathbf{kq} \neq f_0 \mathbf{KQ}$.

Theorem 3.2: *When $\text{char}(F) \neq 2$ the nonnegative slices of $\mathbf{kq}$ are given as*

$$s_q \mathbf{kq} = \begin{cases} \Sigma^{2n, 2n} \mathbf{MZ}/2 \vee \Sigma^{2n+2, 2n} \mathbf{MZ}/2 \vee \dots \vee \Sigma^{4n-2, 2n} \mathbf{MZ}/2 \vee \Sigma^{4n, 2n} \mathbf{MZ} & q = 2n, \\ \Sigma^{2n+1, 2n+1} \mathbf{MZ}/2 \vee \Sigma^{2n+3, 2n+1} \mathbf{MZ}/2 \vee \dots \vee \Sigma^{4n+1, 2n+1} \mathbf{MZ}/2 & q = 2n+1. \end{cases}$$

The negative slices of $\mathbf{kq}$ are zero. Moreover, the canonical map $\mathbf{kq} \rightarrow \mathbf{KQ}$ induces a natural inclusion on slices, and respects the multiplicative structure.

Proof. Since $\mathbf{kq} = f_0(\mathbf{KQ}_{\geq 0})$ is (very) effective, its negative slices are zero. Applying the slice functor to (2.1) yields a cofiber sequence. The natural isomorphism $s_q \circ \Sigma^{1,1} \cong \Sigma^{1,1} \circ s_{q-1}$ of [23, Lemma 2.1] shows the forgetful map $f: \mathbf{kq} \rightarrow \mathbf{kgl}$ induces an isomorphism on zero slices

$$s_0 \mathbf{kq} \xrightarrow{\cong} s_0 \mathbf{kgl},$$

and likewise for the unit map $\mathbf{1} \rightarrow \mathbf{kq}$.

For the 1-slices there is a cofiber sequence

$$\Sigma^{1,1} s_0 \mathbf{kq} = \Sigma^{1,1} \mathbf{MZ} \xrightarrow{\eta} s_1 \mathbf{kq} \xrightarrow{s_1 f} s_1 \mathbf{kgl} = \Sigma^{2,1} \mathbf{MZ} \xrightarrow{s_1 h} \Sigma^{2,1} s_0 \mathbf{kq} = \Sigma^{2,1} \mathbf{MZ}.$$

Here $s_1 h$ can be identified with an integer $n \in \mathbb{Z}$. Comparison with the hyperbolic map $\mathbf{KGL} \rightarrow \mathbf{KQ}$ in [23, §4.3] shows that $n = 2$, so that $s_1 \mathbf{kq} = \Sigma^{1,1} \mathbf{MZ}/2$.

For the 2-slices there is a cofiber sequence

$$\Sigma^{1,1} s_1 \mathbf{kq} = \Sigma^{2,2} \mathbf{MZ}/2 \xrightarrow{\eta} s_2 \mathbf{kq} \xrightarrow{s_2 f} s_2 \mathbf{kgl} = \Sigma^{4,2} \mathbf{MZ} \xrightarrow{s_2 h} \Sigma^{2,1} s_1 \mathbf{kq} = \Sigma^{3,2} \mathbf{MZ}/2.$$

Hence $s_2 h = 0$, the cofiber sequence splits, and we get $s_2 \mathbf{kq} = \Sigma^{2,2} \mathbf{MZ}/2 \vee \Sigma^{4,2} \mathbf{MZ}$. Moreover, $s_2 f$ is the projection map onto $\Sigma^{4,2} \mathbf{MZ}$.

For the 3-slices there is a cofiber sequence

$$\Sigma^{1,1} s_2 \mathbf{kq} = \Sigma^{3,3} \mathbf{MZ}/2 \vee \Sigma^{5,3} \mathbf{MZ} \xrightarrow{\eta} s_3 \mathbf{kq} \xrightarrow{s_3 f} s_3 \mathbf{kgl} = \Sigma^{6,3} \mathbf{MZ} \xrightarrow{s_3 h} \Sigma^{2,1} s_2 \mathbf{kq}.$$

Here $s_3 h$ maps trivially to $\Sigma^{4,3} \mathbf{MZ}/2$, while the component of $s_3 h$ mapping to $\Sigma^{6,3} \mathbf{MZ}$ can be identified with an integer $n \in \mathbb{Z}$. We deduce $n = 2$ by comparison with the hyperbolic map $\mathbf{KGL} \rightarrow \mathbf{KQ}$ in [23, §4.3]. Hence we obtain $s_3 \mathbf{kq} \cong \Sigma^{3,3} \mathbf{MZ}/2 \vee \Sigma^{5,3} \mathbf{MZ}/2$.

Iterating these arguments produces the claimed calculation. $\square$

Remark 3.3: Contrary to the calculation of the slices of $\mathbf{KQ}$ in [23] there is no “mysterious summand” appearing in Theorem 3.2 thanks to the connectivity of $\mathbf{kq}$. Each slice of $\mathbf{kq}$ is a finite sum of motivic Eilenberg-MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. The odd slices of $\mathbf{kq}$ are cellular of finite type for every F [25, §3.3], and likewise for all the slices when $\text{char}(F) = 0$.

The multiplicative structure on the graded slices $\mathbf{s}_*\mathbf{kq}$ can be identified similarly to $\mathbf{s}_*\mathbf{KQ}$ as in [22, Theorem 3.3]. In more details, there is a motivic weak equivalence

$$\mathbf{s}_*\mathbf{kq} \cong \mathbf{MZ}[\eta, \sqrt{\alpha}]/(2\eta = 0, \eta^2 \xrightarrow{\delta} \sqrt{\alpha})$$

where η has bidegree $(1, 1)$ and $\sqrt{\alpha}$ is a class of bidegree $(4, 2)$ arising from the $(8, 4)$ -periodicity operator on $\mathbf{KQ}$ mentioned in the introduction. The multiplicative structure is not quite that of a polynomial ring; for example, the multiplication $\mathbf{s}_1\mathbf{kq} \wedge_{\mathbf{s}_0\mathbf{kq}} \mathbf{s}_1\mathbf{kq} \rightarrow \mathbf{s}_2\mathbf{kq}$ is given as the composition

$$\mathbf{s}_1\mathbf{kq} \wedge_{\mathbf{s}_0\mathbf{kq}} \mathbf{s}_1\mathbf{kq} \cong \Sigma^{2,2}\mathbf{MZ}/2 \vee \Sigma^{3,2}\mathbf{MZ}/2 \xrightarrow{\text{id} \vee \delta} \Sigma^{2,2}\mathbf{MZ}/2 \vee \Sigma^{4,2}\mathbf{MZ}$$

where $\delta: \mathbf{MZ}/2 \rightarrow \Sigma^{1,0}\mathbf{MZ}$ is the connecting map. Moreover, the action of the Hopf map η on the slices of $\mathbf{kq}$ can be read off from the proof of Theorem 3.2, giving us the next result.

Theorem 3.4: *When $\text{char}(F) \neq 2$ the slices of $\mathbf{kq}[\frac{1}{\eta}] = \mathbf{KW}_{\geq 0}$ are given by*

$$\mathbf{s}_q(\mathbf{KW}_{\geq 0}) = \Sigma^{q,q} \left(\mathbf{MZ}/2 \vee \Sigma^{2,0}\mathbf{MZ}/2 \vee \Sigma^{4,0}\mathbf{MZ}/2 \vee \dots \right),$$

and

$$\mathbf{s}_*(\mathbf{KW}_{\geq 0}) \cong \mathbf{MZ}[\eta^{\pm 1}, \sqrt{\alpha}]/(2\eta = 2\sqrt{\alpha} = 0, \eta^2 \xrightarrow{\text{Sq}^1} \sqrt{\alpha}).$$

The canonical map $\mathbf{KW}_{\geq 0} \rightarrow \mathbf{KW}$ induces the natural inclusion on slices, and respects the multiplicative structure.

As in the case of $\mathbf{s}_*\mathbf{kq}$, the multiplicative structure is not quite polynomial, and because of the occurrence of Sq^1 not $\mathbf{MZ}/2$ -linear. Let $\mathbf{d}_1^{\mathbf{kq}}(q): \mathbf{s}_q\mathbf{kq} \rightarrow \Sigma^{1,0}\mathbf{s}_{q+1}\mathbf{kq}$ denote the first slice differential as a map of motivic spectra, and similarly for $\mathbf{KW}_{\geq 0}$. By Theorem 3.2, $\mathbf{d}_1^{\mathbf{kq}}(q)$ is a map between finite sums of motivic Eilenberg-MacLane spectra for the groups $\mathbb{Z}$ and $\mathbb{Z}/2$. Thus $\mathbf{d}_1^{\mathbf{kq}}(q)$ can be described via its restriction $\mathbf{d}_1^{\mathbf{kq}}(q, i)$ to the summand corresponding to the unique suspension $\Sigma^{q+i, q}$. We note that $\mathbf{d}_1^{\mathbf{kq}}(q, i)$ splits into at most three nontrivial components.

Let $\tau \in h^{0,1} \cong \mu_2(F)$ and $\rho \in h^{1,1} \cong F^\times/2$ denote the classes represented by $-1 \in F$; $h^{p,q}$ is shorthand for the mod-2 motivic cohomology group of F in degree p and weight q . There are canonical maps $\text{pr}: \mathbf{MZ} \rightarrow \mathbf{MZ}/2$ and $\partial: \mathbf{MZ}/2 \rightarrow \Sigma^{1,0}\mathbf{MZ}$.

Theorem 3.5: *When $\text{char}(F) \neq 2$ the $\mathbf{d}_1$ -differential in the slice spectral sequence for $\mathbf{kq}$ is given by*

$$\begin{aligned} \mathbf{d}_1^{\mathbf{kq}}(q, i) &= \begin{cases} (\text{Sq}^3\text{Sq}^1, \text{Sq}^2, 0) & q-1 > i \equiv 0 \pmod{4} \\ (\text{Sq}^3\text{Sq}^1, \text{Sq}^2 + \rho\text{Sq}^1, \tau) & q-1 > i \equiv 2 \pmod{4} \end{cases} \\ \mathbf{d}_1^{\mathbf{kq}}(q, q) &= \begin{cases} (0, \text{Sq}^2 \circ \text{pr}, 0) & q \equiv 0 \pmod{4} \\ (0, \text{Sq}^2 \circ \text{pr}, \tau \circ \text{pr}) & q \equiv 2 \pmod{4} \end{cases} \\ \mathbf{d}_1^{\mathbf{kq}}(q, q-1) &= \begin{cases} (\partial\text{Sq}^2\text{Sq}^1, \text{Sq}^2, 0) & q \equiv 1 \pmod{4} \\ (\partial\text{Sq}^2\text{Sq}^1, \text{Sq}^2 + \rho\text{Sq}^1, \tau) & q \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Proof. Use Theorem 3.2 and the identification of $\mathbf{d}_1^{\mathbf{KQ}}$ for $\mathbf{KQ}$ in [23, Theorem 5.5]. $\square$

Theorem 3.6: *When $\text{char}(F) \neq 2$ the $\mathbf{d}_1$ -differential in the slice spectral sequence for $\mathbf{KW}_{\geq 0}$ is given by*

$$\mathbf{d}_1^{\mathbf{KW}_{\geq 0}}(q, i) = \begin{cases} (\text{Sq}^3 \text{Sq}^1, \text{Sq}^2, 0) & i \equiv 0 \pmod{4} \\ (\text{Sq}^3 \text{Sq}^1, \text{Sq}^2 + \rho \text{Sq}^1, \tau) & i \equiv 2 \pmod{4}. \end{cases}$$

Proof. This follows from Theorem 3.4 and the identification of $\mathbf{d}_1^{\mathbf{KW}}$ for $\mathbf{KW}$ recorded in [23, Theorem 5.3]. $\square$

Following [22, §4] we calculate the first slice differentials for $\mathbf{kq}$ and $\mathbf{KW}_{\geq 0}$ in terms of the multiplicative generators for their slices.

We note that $\mathbf{d}_1^{\mathbf{kq}}(\sqrt{\alpha}^m \eta^n)$ is given by

$$\begin{cases} \tau \sqrt{\alpha}^{m-1} \eta^{n+3} + (\text{Sq}^2 + \rho \text{Sq}^1) \sqrt{\alpha}^m \eta^{n+1} + \text{Sq}^3 \text{Sq}^1 \sqrt{\alpha}^{m+1} \eta^{n-1} & m \equiv 1(2), n > 1 \\ \text{Sq}^2 \sqrt{\alpha}^m \eta^{n+1} + \text{Sq}^3 \text{Sq}^1 \sqrt{\alpha}^{m+1} \eta^{n-1} & m \equiv 0(2), n > 1 \\ \tau \sqrt{\alpha}^{m-1} \eta^4 + (\text{Sq}^2 + \rho \text{Sq}^1) \sqrt{\alpha}^m \eta^2 + \delta \text{Sq}^2 \text{Sq}^1 \sqrt{\alpha}^{m+1} & m \equiv 1(2), n = 1 \\ \text{Sq}^2 \sqrt{\alpha}^m \eta^2 + \delta \text{Sq}^2 \text{Sq}^1 \sqrt{\alpha}^{m+1} & m \equiv 0(2), n = 1 \\ \tau \text{pr} \sqrt{\alpha}^{m-1} \eta^3 + \text{Sq}^2 \text{pr} \sqrt{\alpha}^m \eta & m \equiv 1(2), n = 0 \\ \text{Sq}^2 \sqrt{\alpha}^m \eta & m \equiv 0(2), n = 0, \end{cases} \quad (3.1)$$

while $\mathbf{d}_1^{\mathbf{KW}_{\geq 0}}(\sqrt{\alpha}^m \eta^n)$ is given by

$$\begin{cases} \tau \sqrt{\alpha}^{m-1} \eta^{n+3} + (\text{Sq}^2 + \rho \text{Sq}^1) \sqrt{\alpha}^m \eta^{n+1} + \text{Sq}^3 \text{Sq}^1 \sqrt{\alpha}^{m+1} \eta^{n-1} & m \equiv 1(2) \\ \text{Sq}^2 \sqrt{\alpha}^m \eta^{n+1} + \text{Sq}^3 \text{Sq}^1 \sqrt{\alpha}^{m+1} \eta^{n-1} & m \equiv 0(2). \end{cases} \quad (3.2)$$

Remark 3.7: The corresponding formula for $\mathbf{d}_1^{\mathbf{KQ}}(\sqrt{\alpha}^m \eta^n)$ in [22, §4] contains a typo when $m \equiv 1(2), n = 0$. We thank Bert Guillou for pointing this out to us.

Remark 3.8: Bachmann [3] determined the very effective slices of $\mathbf{KQ}$ and hence of $\mathbf{kq}$ up to extensions. Additional work is needed to identify the corresponding first very effective slice differentials. A first step is to calculate the endomorphisms of the very effective zero slice of $\mathbf{KQ}$. The very effective slices of $\mathbf{KW}_{\geq 0}$ were determined up to extensions in [3, Lemma 6].

4 Homotopy computations

First we identify the target of the slice spectral sequences for the sphere and very effective hermitian K -theory.

Theorem 4.1: *Over a field F of characteristic $\neq 2$ there are conditionally convergent slice spectral sequences*

$$\pi_* \mathbf{s}_* \mathbf{1} \implies \pi_* \mathbf{1}_\eta^\wedge, \quad (4.1)$$

and

$$\pi_* s_* \mathbf{kq} \implies \pi_* \mathbf{kq}_\eta^\wedge. \quad (4.2)$$

Proof. Here (4.1) is shown in [25, §3]. The only issue in (4.2) is to identify the quotient of $\mathbf{kq}$ by η with a slice complete spectrum [24, §4]. This follows directly from Lemma 2.4, Proposition 2.11 and [24, Lemma 3.11]. $\square$

To formulate our identification of the 0-line of $\mathbf{kq}$ we recall the definition of Milnor-Witt K -theory $K_*^{MW}(F)$ in [19]. It is the quotient of the free associative integrally graded ring on the set of symbols $[F^\times] := \{[u] \mid u \in F^\times\}$ in degree 1 and η in degree -1 by the homogeneous ideal enforcing the relations

- (1) $[uv] = [u] + [v] + \eta[u][v]$ (η -twisted logarithm),
- (2) $[u][v] = 0$ for $u + v = 1$ (Steinberg relation),
- (3) $[u]\eta = \eta[u]$ (commutativity), and
- (4) $(2 + [-1]\eta)\eta = 0$ (hyperbolic relation).

Milnor-Witt K -theory is ε -commutative for $\varepsilon = -(1 + [-1]\eta)$. By work of Morel [17] there is an isomorphism with the graded ring of endomorphisms of the sphere

$$K_*^{MW}(F) \cong \bigoplus_{n \in \mathbb{Z}} \pi_{n,n} \mathbf{1}.$$

Moreover, $K_0^{MW}(F) \cong GW(F)$, the Grothendieck-Witt ring of quadratic forms with its standard presentation, inverting η in $K_*^{MW}(F)$ yields the ring of Laurent polynomials $W(F)[\eta^{\pm 1}]$ over the Witt ring, and $K_*^{MW}(F)/\eta = K_*^M(F)$, the Milnor K -theory ring of F .

Theorem 4.2: *Over a field F of characteristic $\neq 2$ the unit map $\mathbf{1} \rightarrow \mathbf{kq}$ induces an isomorphism on 0-lines*

$$K_*^{MW}(F) = \bigoplus_{n \in \mathbb{Z}} K_n^{MW}(F) \xrightarrow{\cong} \bigoplus_{n \in \mathbb{Z}} \pi_{-n, -n} \mathbf{kq}. \quad (4.3)$$

Proof. Recall from [25, §5] the short exact sequence

$$0 \rightarrow \pi_{n,n} \mathbf{1} \rightarrow \pi_{n,n} \mathbf{1}_\eta^\wedge \oplus \pi_{n,n} \mathbf{1}[\tfrac{1}{\eta}] \rightarrow \pi_{n,n} \mathbf{1}_\eta^\wedge[\tfrac{1}{\eta}] \rightarrow 0. \quad (4.4)$$

Similarly, following [25, §3], the η -arithmetic square

$$\begin{array}{ccc} \mathbf{kq} & \longrightarrow & \mathbf{kq}[\tfrac{1}{\eta}] \\ \downarrow & & \downarrow \\ \mathbf{kq}_\eta^\wedge & \longrightarrow & \mathbf{kq}_\eta^\wedge[\tfrac{1}{\eta}] \end{array}$$

for very effective K -theory yields a short exact sequence

$$0 \rightarrow \pi_{n,n} \mathbf{kq} \rightarrow \pi_{n,n} \mathbf{kq}_\eta^\wedge \oplus \pi_{n,n} \mathbf{kq}[\tfrac{1}{\eta}] \rightarrow \pi_{n,n} \mathbf{kq}_\eta^\wedge[\tfrac{1}{\eta}] \rightarrow 0. \quad (4.5)$$

Here we use the vanishing of $\pi_{n+1,n} \mathbf{kq}_\eta^\wedge[\tfrac{1}{\eta}]$ and $\pi_{n-1,n} \mathbf{kq}$. On the terms contributing to the 0-line, the map from (4.1) to (4.2) is an isomorphism. Theorem 3.5 combined with the same computations as in [25, §4] show the said isomorphism persists to the E^∞ -page. By invoking Theorem 4.1 we conclude $\pi_{n,n} \mathbf{1}_\eta^\wedge \xrightarrow{\cong} \pi_{n,n} \mathbf{kq}_\eta^\wedge$ and $\pi_{n,n} \mathbf{1}_\eta^\wedge[\tfrac{1}{\eta}] \xrightarrow{\cong} \pi_{n,n} \mathbf{kq}_\eta^\wedge[\tfrac{1}{\eta}]$. As noted above, by [17] we have $\pi_{n,n} \mathbf{1}_\eta^\wedge[\tfrac{1}{\eta}] \xrightarrow{\cong} \pi_{n,n} \mathbf{kq}[\tfrac{1}{\eta}] \cong \pi_{n,n} \mathbf{KW}_{\geq 0} \cong W(F)$. A straightforward comparison between (4.4) and (4.5) allows us to deduce (4.3). $\square$

Remark 4.3: It was pointed out to us by Bachmann that the results of [3] yield an isomorphism of the zeroth generalized slices $\tilde{s}_0 \mathbf{1} \cong \tilde{s}_0 \mathbf{KQ}$. This gives another proof for Theorem 4.2.

We note the isomorphism $\pi_{n+1,n} \mathbf{kq} \xrightarrow{\cong} \pi_{n+1,n} \mathbf{kq}_\eta^\wedge$ follows as in [25, Proposition 5.3]. Thus for the purpose of identifying the 1-line of $\mathbf{kq}$ we may use Theorem 3.5 and computations as in [25, §4] to deduce:

Proposition 4.4: *The only nontrivial terms in (4.2) contributing to $\pi_{n+1,n} \mathbf{kq}$ are*

$$E_{n+1,q,n}^\infty(\mathbf{kq}) = \begin{cases} h^{-n+1,-n+2}/\mathbf{Sq}^2(h^{-n-1,-n+1}) & q = 2 \\ h^{-n,-n+1}/\mathbf{Sq}^2 \text{pr}(H^{-n-2,-n}) & q = 1 \\ H^{-n-1,-n} & q = 0. \end{cases}$$

Here $h^{i,j}$ and $H^{i,j}$ denote the mod-2 and integral motivic cohomology groups of F in degree i and weight j . This determines the 1-line of $\mathbf{kq}$ up to extensions; these are nontrivial in general, as already the classical computation of $K_3(\mathbb{Q})$ implies. When $n > 1$ we read off the vanishing $\pi_{n+1,n} \mathbf{kq} = 0$. The first nontrivial group on the 1-line is $\pi_{2,1} \mathbf{kq} \cong \mu_2(F) \cong \mathbb{Z}/2$. When $n = 0$ we obtain $\pi_{1,0} \mathbf{kq} \cong \pi_{1,0} \mathbf{KQ} \cong F^\times/2 \oplus \mu_2(F)$. Furthermore, there is a short exact sequence

$$0 \rightarrow h^{2,3}/\mathbf{Sq}^2(h^{0,2}) \rightarrow \pi_{0,-1} \mathbf{kq} \rightarrow h^{1,2} \rightarrow 0. \quad (4.6)$$

When $n \leq -2$ the group $\pi_{n+1,n} \mathbf{kq}$ surjects onto the integral motivic cohomology group $H^{-n-1,-n}$, with kernel described by Proposition 4.4.

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