NUMERICAL ANALYSIS OF CONSERVATION LAWS INVOLVING NON-LOCAL TERMS

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CHAPTER 1

INTRODUCTION

1.1 Outline

This thesis consists of a collection of five papers, which are summarised in this introduction. For the convenience of the readers, before presenting the article collection, a brief description of the necessary mathematical background and context is provided in the following way. In section 1.2 we are going to present the necessary theoretical concepts, notion of solutions *etc.* of hyperbolic scalar conservation law, under which the results have been concluded in the later part of the thesis and the tools (namely *compensated compactness* and *the finite volume method*) to be used later in our work, respectively. Then in section 1.3 and in section 1.4 we are going to briefly outline the derivation and the available results for the **Kuramoto–Sakaguchi equation** and the **Ostrovsky–Hunter equation** respectively, which have played a central role for the rest of the thesis. After setting up the context, in section 1.5 a summary of the contributions in Paper 1 to Paper 5 is given. Finally, in section 1.6 future research directions to be pursued is outlined. Some inevitable overlap of content in this introduction can be expected, while each paper is self-contained.

1.2 Hyperbolic Conservation Laws

Hyperbolic PDEs are ubiquitous throughout a broad spectrum of models of real life problems such as meteorology, gas dynamics, traffic dynamics, acoustics, elastodynamics, solar physics, oceanography and so on. Rigorous mathematical investigation of the qualitative properties of solutions to nonlinear PDEs appearing in many of the branches above, has witnessed a massive boost after the early 1950's. In practice, even if the initial data are smooth enough, the solutions of the considered problems have been found to be *non-smooth* and hence, solving the mathematical equation *classically* does not make sense. Eventually, this necessitates the introduction of much weaker notions of *distributional solution*, *entropy solution* etc., and the development of corresponding numerical methods to construct these solutions.

1.2.1 Scalar Conservation Laws

In all these applications of hyperbolic PDEs from different disciplines, our starting point is the initial value problem for hyperbolic conservation laws, which has the following form:

$$\begin{aligned}\partial_t u + \nabla_{\mathbf{x}} \cdot f(u) &= 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t \in \mathbb{R}_+ := (0, \infty), \\ u(x, 0) &= u_0(x),\end{aligned}\tag{1.1}$$

with the variable $\mathbf{x} \equiv (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ represents the location in d -dimensional space ($d \in \mathbb{N}$) and the variable t usually denotes the time. The unknown variable $u(\mathbf{x}, t)$ denotes the quantity under consideration, for which $\int_{\mathbb{R}^d} u(\mathbf{x}, t) d\mathbf{x}$ is conserved over time. Though it can be either a scalar or a vector-valued quantity, in this thesis we are going to consider u as a scalar quantity. Typical examples of such u are mass, momentum, energy, density, amplitude and so on. The function f is known as the flux function. The term *hyperbolic* corresponds to the finite speed of propagation, which is mathematically interpreted as diagonalizability of the jacobian matrix for f . For the ease of our further discussion we are going to consider the one dimensional case of (1.1) (*i.e.* for $d = 1$ we write $\mathbf{x} = x$) where $u(x, t)$ is a scalar valued function. Thus we are going to consider the scalar conservation law in one dimension

$$\begin{aligned}\partial_t u + \partial_x f(u) &= 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \\ u(x, 0) &= u_0(x),\end{aligned}\tag{1.2}$$

To derive (1.2), let $(x, t) \mapsto u(x, t)$ denote the *density* of the quantity under consideration in $D := [x_1, x_2]$, an interval in $\mathbb{R}$ and let the associated flux density through the endpoints of the endpoints of the interval is given by the flux function $(x, t) \mapsto f(u(x, t))$. Then that quantity is said to be *conserved* if the total quantity $\int_{\mathbb{R}} u(x, t) dx$ is constant for all time t . In mathematical terms, *balance law* states that assuming that there is no *production* or *destruction* of the quantity inside D , the rate of change of that quantity in D is given by the flux $f(u)$ through the boundary of D , *i.e.*

$$\frac{d}{dt} \int_D u(x, t) dx + \left(f(u(x_2, t)) - f(u(x_1, t)) \right) = 0.\tag{1.3}$$

Equation (1.3) is known as a *conservation law* which is now written as

$$\frac{d}{dt} \int_D u(x, t) dx = f(u(x_1, t)) - f(u(x_2, t)) \quad \left(= (\text{inflow at } x_1) - (\text{outflow at } x_2) \right).\tag{1.4}$$

In other words, the rate of change of $\int_D u(x, t) dx$ is equal to the inflow of u at x_1 , minus the outflow of u at x_2 , both determined by the flux density f . Since the equation (1.4) holds for any arbitrary $D \subset \mathbb{R}$, it is equivalent to the scalar one-dimensional conservation law (1.2) if we assume u is smooth enough in both the spatial and the temporal variable.

However, even if the initial data u_0 is smooth enough, the solution can turn out to be discontinuous in finite time. To see this, assume that the solution $u(x, t)$ of the following **Burgers' equation**

$$\begin{aligned}\partial_t u + \partial_x \left(\frac{u^2}{2} \right) &= 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}_+, \\ u(x, 0) &= u_0(x),\end{aligned}\tag{1.5}$$

is smooth. Then following the classical method of characteristics, for small enough time t , it can be implicitly defined as $u(x(t), t) = u_0(x_0)$ and the value of u_0 at x_0 follows the characteristic equation $x(t) = x_0 + tu_0(x_0)$. Now if $u_0(x)$ is differentiable with at least one point x such that $u'_0(x) < 0$, we can immediately check from the implicit definition that the smoothness of $u(x, t)$ at $x = x(t)$ will indeed break down at the finite time $t = -\frac{1}{\min_{x \in \mathbb{R}} u'_0(x)}$. In such cases, the classical notion of a solution of PDE breaks down. To allow such discontinuity the notion of solutions has to be extended (rather *weakened*). Consequently we need to consider the equation (1.2) in its integral form.

To that end, let $C_c^\infty(\mathbb{R} \times [0, \infty))$ be the space of smooth functions, compactly supported in $\mathbb{R} \times [0, \infty)$.

Definition 1.2.1 (Weak Solution). $u \in L^1(\mathbb{R} \times [0, \infty))$ is a weak solution of the equation (1.2) if for any test function $\varphi \in C_c^\infty(\mathbb{R} \times [0, \infty))$, we have

$$\int_0^\infty \int_{\mathbb{R}} (u \partial_t \varphi + f(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) dx = 0. \quad (1.6)$$

Equivalently, we say that $u \in L^1(\mathbb{R} \times [0, \infty))$ is a weak solution if it satisfies (1.2) **in the sense of distributions**.

Here we should note the fact that a smooth solution is indeed a weak solution too. The advantage of this notion of solution is weak solution allows certain discontinuities, namely those discontinuities which travel with a speed σ satisfying the **Rankine-Hugoniot** condition. If a discontinuity in u , connecting two states u_l and u_r travels with a speed σ which satisfy the Rankine-Hugoniot condition, then we have

$$\sigma = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \quad (1.7)$$

Due to neglected dissipative mechanisms, this notion of weak solution has its own downside, namely the loss of uniqueness. To single out a unique physically relevant solution among different weak solutions satisfying the same weak formulation (1.6) we need to impose **entropy conditions**. To obtain an entropy condition we consider a family of functions $\{u^\varepsilon\}_{\varepsilon > 0}$ parametrized by $\varepsilon > 0$, the so called viscous regularization of the scalar conservation law (1.2)

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_{xx} u^\varepsilon, \quad (1.8)$$

which is known to have a unique smooth solution. The core idea behind the approach is that the physical problem has some diffusion, and that the conservation law (1.2) represents a limit model when the diffusion is small. Consequently we are looking for the limits of the regularized equation, as $\varepsilon \rightarrow 0$. Choosing a $C^2(\mathbb{R})$ convex function $\eta := \eta(u)$ and multiplying (1.8) by η' we get

$$\begin{aligned} 0 &= \left(\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) - \varepsilon \partial_{xx}^2 u^\varepsilon \right) \eta'(u^\varepsilon) \\ &= \eta'(u^\varepsilon) \partial_t u^\varepsilon + f'(u^\varepsilon) \partial_x u^\varepsilon \eta'(u^\varepsilon) - \varepsilon \eta'(u^\varepsilon) \partial_{xx}^2 u^\varepsilon \\ &= \partial_t \eta(u^\varepsilon) + q'(u^\varepsilon) \partial_x u^\varepsilon - \varepsilon \partial_{xx}^2 \eta(u^\varepsilon) + \underbrace{\varepsilon \eta''(u^\varepsilon)}_{\geq 0} \underbrace{(\partial_x u^\varepsilon)^2}_{\geq 0} \\ &\geq \partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) - \varepsilon \partial_{xx}^2 \eta(u^\varepsilon), \end{aligned}$$

where the third equality is due to the following definition of the **entropy flux** $q(u)$, given by

$$q'(u) = f'(u)\eta'(u). \quad (1.9)$$

Definition 1.2.2 (Entropy/Entropy flux pair). *A pair of functions (η, q) is called an entropy/entropy flux pair for the equation (1.2) if $\eta \in C^2(\mathbb{R})$ is convex and $q \in C^2(\mathbb{R})$ satisfies (1.9).*

Thus we may rewrite the above inequality in the sense of distributions as

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) \leq \varepsilon \partial_{xx}^2 \eta(u^\varepsilon).$$

Assume for the moment $\{u^\varepsilon\}_{0 < \varepsilon < 1}$ to be uniformly bounded in L^∞ and also $u^\varepsilon \rightarrow u$ a.e. as $\varepsilon \rightarrow 0$ for some limit function u , and then passing to the limit as $\varepsilon \rightarrow 0$, we have in the sense of distributions

$$\partial_t \eta(u) + \partial_x q(u) \leq 0. \quad (1.10)$$

This motivates to define the notion of **entropy solution** in the following way

Definition 1.2.3 (Entropy Solution). *A weak solution u of the equation (1.2) is an entropy solution of (1.2) if for any convex function η with $q' = \eta' f'$, the following integral inequality*

$$\int_{\mathbb{R}} \int_0^\infty \left(\eta(u) \partial_t \varphi + q(u) \partial_x \varphi \right) dt dx \geq 0, \quad (1.11)$$

holds true for all non-negative test functions $\varphi \in C_c^\infty(\mathbb{R} \times (0, \infty))$, i.e., the inequality (1.10) holds in the sense of distributions.

Remark. Formally, integrating (1.10) in space, time and assuming $|q(u)| \rightarrow 0$ as $x \rightarrow \pm\infty$ we see that the total amount of entropy of an entropy solution decreases in time, i.e.

$$\int_{\mathbb{R}} \eta(u(x, t)) dx \leq \int_{\mathbb{R}} \eta(u_0(x)) dx, \text{ for any } t > 0. \quad (1.12)$$

For instance, taking $\eta(u) = |u|^p$ for some $p \in \mathbb{N}$ as an entropy for (1.2), then (1.12) provides an a-priori L^p -bound on $u(\cdot, t)$ for all $t > 0$.

Remark. By a standard approximation argument, if the inequality (1.11) holds for the *Kružkov function* $\eta(u, k) = |u - k|$ for all $k \in \mathbb{R}$, then this inequality holds for all the convex entropies η . The converse implication follows by choosing for any $k \in \mathbb{R}$, $\eta(u, k) = \eta_\delta(u, k) := \sqrt{(u - k)^2 + \delta^2}$ and letting $\delta \rightarrow 0$.

Remark. For all $k \in \mathbb{R}$,

$$\eta(u, k) = |u - k| \text{ and } q(u, k) = \text{sign}(u - k)(f(u) - f(k)),$$

constitute an entropy/entropy flux pair, known as the **Kružkov entropy pair**.

Before moving on let us mention one of the key concepts in the theory of conservation laws, namely the notion of **total variation**, denoted by $TV(u)$, of a function $u : \mathbb{R} \rightarrow \mathbb{R}$. This is defined by

$$TV(u) = \sup_{\mathcal{P}} \sum_i |u(x_i) - u(x_{i-1})|,$$

where the supremum is taken over the set of all finite partitions $\mathcal{P} = \{\{x_0, x_1, \dots, x_p\} | x_i < x_{i+1}, i = 0, 1, \dots, p-1\}$. In fact, this is a seminorm, denoted by $|u|_{BV} = T.V.(u)$. The set of all functions with finite total variation on $D \subset \mathbb{R}$, for any subset in $\mathbb{R}$ is denoted by $BV(D)$. For the scalar conservation laws, in 1970 Kruřkov showed well-posedness of the entropy solution in the class of bounded variation in the following theorem

Theorem 1.2.1 (Kruřkov [67]). *For each initial data $u_0 \in L^\infty(\mathbb{R})$ there exists a unique entropy solution u of the corresponding Cauchy problem for the scalar conservation law (1.2). Moreover, this function satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|u_0(\cdot)\|_{L^\infty(\mathbb{R})}, \quad \forall t > 0,$$

and it attains the initial data in the following sense: $\exists$ a set $\tau \subset [0, \infty)$ of zero measure such that for all $x_0 \in \mathbb{R}$ and $r > 0$,

$$\lim_{\substack{t \rightarrow 0 \\ t \notin \tau}} \int_{B_r(x_0)} |u(x, t) - u_0(x)| dx = 0.$$

Also if $u, \tilde{u} : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ are two entropy solutions of (1.2) with initial data $u_0, \tilde{u}_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, respectively, then

$$\int_{\mathbb{R}} |u(x, t) - \tilde{u}(x, t)| dx \leq \int_{\mathbb{R}} |u_0(x) - \tilde{u}_0(x)| dx, \quad \forall t > 0.$$

In particular, this implies

$$TV(u(\cdot, t)) \leq TV(u_0), \quad \text{for all } t > 0, \text{ if } u_0 \in BV(\mathbb{R}).$$

1.2.2 Method of Compensated Compactness

To establish the *existence* part of Theorem 1.2.1 Kruřkov considered a parabolic regularization (1.8) of the conservation law (1.2). Then exploiting *a priori* stability bounds and a local TV bound on the solution for this viscous problem, it was shown to converge strongly, upto a subsequence, to an entropy solution as the underlying diffusion coefficient goes to zero. Using Helly's theorem, to show the existence of a strongly convergent subsequence we need the sequence of solutions $\{u^\varepsilon\}_{\varepsilon>0}$ of the viscous regularization (1.8) to have uniformly bounded total variation. But this is in general, quite a strong requirement which cannot be expected for high-order accurate numerical methods.

To circumvent this issue Murat [77] and Tartar [93] devised a tool named *compensated compactness* which needs a much *weaker* bound on the gradients of the approximating sequence. To be precise, we need the entropy residual $\{\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon)\}_{\varepsilon>0}$ to lie in a compact subset of $H_{\text{loc}}^{-1}(\mathbb{R} \times \mathbb{R}^+)$. Before proceeding we need the following notion of *weak convergence*.

Definition 1.2.4 (Weak convergence in $L^p(\Omega)$ for $1 \leq p < \infty$). A sequence $\{u^\varepsilon\}_{\varepsilon>0} \subset L^p(\Omega)$ converges **weakly** to $u \in L^p(\Omega)$, denoted by

$$u^\varepsilon \rightharpoonup u \text{ in } L^p(\Omega),$$

if

$$\int_{\Omega} u^\varepsilon v \, dx \rightarrow \int_{\Omega} uv \, dx, \text{ for all } v \in L^{p'}(\Omega),$$

where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

The main result in the theory of compensated compactness, due to Murat [77] can be stated in the following way

Theorem 1.2.2 (Div-curl lemma). Let $\{D^\varepsilon\}_{\varepsilon>0}, \{E^\varepsilon\}_{\varepsilon>0} \in L^2(\Omega; \mathbb{R}^2)$ be two sequences such that $D^\varepsilon \rightharpoonup D$, $E^\varepsilon \rightharpoonup E$ weakly in $L^2(\Omega; \mathbb{R}^2)$ as $\varepsilon \rightarrow 0$ for some D, E . If the two sequences $\{\operatorname{div}(D^\varepsilon)\}_{\varepsilon>0}$ and $\{\operatorname{curl}(E^\varepsilon)\}_{\varepsilon>0}$ are precompact in $H^{-1}(\Omega)$, then $D^\varepsilon \cdot E^\varepsilon \rightarrow D \cdot E$ in $\mathcal{D}'(\Omega)$.

To invoke the div-curl lemma to conclude that the weak convergence $u^\varepsilon \rightharpoonup u$ indeed implies the strong convergence $u^\varepsilon \rightarrow u$, we need to have sufficient control on *oscillations* of u^ε . So we need the following quantities

$$\operatorname{div}(D^\varepsilon) = \partial_t u^\varepsilon + \partial_x f(u^\varepsilon), \quad \operatorname{curl}(E^\varepsilon) = \partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon),$$

to be in some *controlled* set. This condition translates to the precompactness condition of the div-curl lemma for the functions

$$D^\varepsilon := \begin{bmatrix} f(u^\varepsilon) & u^\varepsilon \end{bmatrix}^T, \quad E^\varepsilon := \begin{bmatrix} -\eta(u^\varepsilon) & q(u^\varepsilon) \end{bmatrix}^T,$$

for some entropy pair (η, q) . To invoke the method of compensated compactness, it is essential to show that $\{\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon)\}_{\varepsilon>0}$ is precompact in $H^{-1}(\Omega)$. To achieve this goal the following result due to Murat [77] is often used:

Lemma 1.2.3 (Murat's Lemma). Let for some $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\{a_j\}_{j \in \mathbb{N}}$ be a bounded sequence in the Sobolev space $W^{-1,p}(\Omega)$ for some $2 < p \leq \infty$. If we can express $a_j = b_j + c_j$, where $\{b_j\}_{j \in \mathbb{N}}$ is precompact in $H^{-1}(\Omega)$ and $\{c_j\}_{j \in \mathbb{N}}$ is bounded in $\mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ is the set of Radon measures on Ω . Then $\{a_j\}_{j \in \mathbb{N}}$ is precompact in $H^{-1}(\Omega)$.

1.2.3 Finite Volume Methods

To apply the theoretically obtained results in real-life problems, we first need to be able to solve the equation in computer. Thus we need efficient numerical schemes, which will generate approximate solutions of the actual conservation law, which can be computed directly using a computer.

To begin with, consider the Cauchy problem (1.2). We discretize the spatial domain using intervals $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ of uniform length $\Delta x := x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. We use the

notation $x_j := \frac{x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}}{2}$, the center of the interval I_j . Consider the time interval $I^n := [t^n, t^{n+1})$ with the time-step $\Delta t := t^{n+1} - t^n$. For notational convenience we will denote $I_j^n := I_j \times I^n$. A finite volume scheme approximates the *cell average* values at each time level t^n

$$u_j^n \approx \frac{1}{\Delta x} \int_{I_j} u(x, t^n) dx \quad (1.13)$$

in each cell I_j of the exact solution u to (1.2), and this enables us to treat possible discontinuities in the solution. Time integral of the flux along the interface $x_{j+\frac{1}{2}}$ is approximated by $\Delta t F_{j+\frac{1}{2}}^n$ which gives the following fully discrete finite volume scheme approximating (1.2)

$$\begin{aligned} u_j^{n+1} &= u_j^n - \frac{\Delta t}{\Delta x} \left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right), \\ u_j^0 &= \frac{1}{\Delta x} \int_{I_j} u_0(x) dx, \end{aligned} \quad (1.14)$$

where $F_{j+\frac{1}{2}}^n$ is called a numerical flux, which can be chosen in many different ways. Among those, the simplest choice of numerical flux is a two-point flux of the form $F_{j+\frac{1}{2}}^n := F(u_j^n, u_{j+1}^n)$. The time-step Δt has to be chosen in a way such that it satisfies something called “CFL condition”, which can be explained in the following manner. If the *update function* $G(u_{j-1}^n, u_j^n, u_{j+1}^n) := u_j^n - \frac{\Delta t}{\Delta x} \left(F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n \right)$ is shown to be nondecreasing in all three arguments, then the scheme (1.14) is called **monotone**. Equivalently, the scheme is **monotone** if the mapping $u \mapsto F(u, v)$ is nondecreasing and the mapping $v \mapsto F(u, v)$ is nonincreasing, where $\lambda := \frac{\Delta t}{\Delta x}$ satisfies the following inequality, named after three mathematicians Courant–Friedrichs–Lewy, the **CFL condition**

$$\max_j |f'(u_j^n)| \lambda \leq c. \quad (1.15)$$

Here the number c is known as the CFL number of the numerical method. Summing over (1.14) over the spatial mesh, we get $\sum_j u_j^{n+1} = \sum_j u_j^n$, for any time levels t^n and t^{n+1} ; we call such schemes (1.14) **conservative**. If the numerical flux satisfies $F(u, u) = f(u)$, then the scheme (1.14) is called **consistent**. Using the u_j^n given by (1.14) we define for the characteristic function $\chi_{I_j^n}$ of the cell I_j^n

$$u_{\Delta x}(x, t) := \sum_{j,n} \chi_{I_j^n} u_j^n. \quad (1.16)$$

The main aim of this method is to exploit the discrete analogues of the stability properties of solutions of (1.2). We are often interested in numerical methods which satisfy a discrete version of (1.10).

Definition 1.2.5 (Entropy stable scheme). *Let (η, q) be an entropy/entropy flux pair. The fully-discrete scheme (1.14) is said to be entropy stable if it satisfies a discrete cell entropy estimate of the form*

$$\eta(u_j^{n+1}) \leq \eta(u_j^n) - \frac{\Delta t}{\Delta x} \left(Q_{j+\frac{1}{2}}^n - Q_{j-\frac{1}{2}}^n \right), \quad (1.17)$$

where $Q_{j+\frac{1}{2}}^n := Q(u_j^n, u_{j+1}^n)$ is a numerical entropy flux, consistent with the entropy flux $q(u)$, i.e. $\bar{Q}(u, u) = q(u)$.

Moreover, we are going to use numerical schemes, which will preserve the following stability properties of the underlying continuous problem (1.2)

(C1) **L^∞ bounds** : $\|u^n\|_\infty \leq C$ for all $n \in \mathbb{N}$, for some generic constant C which is independent of the mesh parameters $\Delta x, \Delta t$. This estimate boils down to the property that approximate solution will not *blow up*.

(C2) **TV bounds** : $\text{TV}(u^n) \leq C$ for all $n \in \mathbb{N}$, with C , as described above. This estimate translates to the physical property that the approximate solution will not have many large oscillations.

(C3) **Time continuity bounds** :

$$\sum_j |u_j^n - u_j^m| \Delta x \leq C_F |u_0|_{\text{B.V.}} |t^n - t^m|, \quad (1.18)$$

with C_F being the Lipschitz constant of F . Note that we need the numerical flux of our choice to be Lipschitz continuous and the initial data to be in BV-space. This estimates translates to the fact that approximate solution is continuous in time.

Finally invoking the estimates (C1), (C2), (C3) we can invoke Helly's theorem and Arzela-Ascoli's theorem to conclude the compactness of the sequence $\{u_{\Delta x}\}_{\Delta x > 0}$ and hence we can extract a strongly convergent subsequence from this sequence and a limit function to which it converges. Then, using following **Lax-Wendroff theorem** we show that this limit function is indeed a weak solution of that conservation law. The Lax-Wendroff theorem is stated as follows

Theorem 1.2.4. (Lax-Wendroff) *Let u_j^n be numerical solutions of (1.2), computed by a conservative and consistent finite volume scheme (1.14) with a Lipschitz continuous numerical flux function F . Assuming that $u_0 \in L^\infty(\mathbb{R})$ and the approximating functions $u_{\Delta x}$*

(D1) are uniformly bounded, i.e., for some constant $C > 0$

$$\|u_{\Delta x}\|_{L^\infty(\mathbb{R} \times \mathbb{R}_+)} \leq C, \text{ for all } \Delta x > 0, \quad (1.19)$$

(D2) converge in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}_+)$, as $\Delta x, \Delta t \rightarrow 0$, to some function u ,

then u is a weak solution of (1.2).

Order of convergence

To calculate the *efficiency* of the numerical scheme we need to see *how fast this convergence $u_{\Delta x} \rightarrow u$ is happening*. To quantify this convergence, we first compute the approximate solution $u_{\Delta x}$ for very *fine* (i.e. Δx is very small) mesh and rename this $u_{\Delta x}$

as the **reference solution** u_{ref} for the scheme. Then we compute **percentage relative error**, which is defined by

$$\mathcal{E}_{\Delta x} := 100 \times \frac{\|u_{\Delta x} - u_{\text{ref}}\|_{L^1}}{\|u_{\text{ref}}\|_{L^1}}. \quad (1.20)$$

If there is any *exact* solution, namely u_{exact} is already known, then we substitute u_{ref} by u_{exact} in (1.20). Then we compute the **experimental order of convergence** (E.O.C.) using the following formula

$$\text{EOC}_{\Delta x, \Delta y} := \frac{\log(\mathcal{E}_{\Delta x}) - \log(\mathcal{E}_{\Delta y})}{\log(\Delta x) - \log(\Delta y)}. \quad (1.21)$$

For the ease of computation we use $\Delta y = 2\Delta x$.

1.3 Kuramoto Sakaguchi Equation

One of the numerous physical phenomena observed in nature, which has recently gained sufficient popularity among the mathematical community is *synchronization* or *consensus behaviour*. Ants forming colonies, birds flying in flocks, mobile networks coordinating a rendezvous, human opinions evolving into parties, flashing of fireflies, chorusing of crickets, synchronous firing of cardiac pacemakers and so on are one of many important examples of self-organized behavior that can be observed in natural and social regime. Winfree [101] and Kuramoto [68] pioneered the mathematical treatment of these synchronized phenomena by introducing phase models for large weakly coupled oscillator systems. In their work, later known as **Kuramoto model** they proved that the synchronized behavior of complex biological systems can emerge from the competing mechanisms of *intrinsic randomness* and *sinusoidal couplings*.

1.3.1 Brief derivation

In this subsection, we are going to derive and explain the basic structure of the Kuramoto phase model to motivate the reader for the work described in later chapter. To start, first consider the following complex-valued ODE describing the temporal dynamics of the Stuart-Landau oscillator

$$\frac{dz}{dt} = (1 - |z|^2 + i\Omega)z, \quad (1.22)$$

where $z(t) \in \mathbb{C}$ is the *state* of the oscillator at time t and $\Omega \in \mathbb{R}$ is the so-called *natural frequency*. The natural frequency is assumed to be a random variable with some given density function $g = g(\Omega)$, satisfying

$$\int_{\mathbb{R}} g(\Omega) d\Omega = 1, \quad \int_{\mathbb{R}} \Omega g(\Omega) d\Omega = 0, \quad \text{supp}(g) \subset \subset \mathbb{R}, \quad g \geq 0, \quad \text{and } g \in L^1(\mathbb{R}).$$

Substituting the polar form for the complex number $z(t)$, i.e. $z(t) = r(t)e^{i\theta(t)}$, where *amplitude* $r(t)$ and *phase* $\theta(t)$ we get the following system of ODEs

$$\frac{dr}{dt} = r(1 - r^2), \quad \frac{d\theta}{dt} = \Omega, \quad (1.23)$$

which has a stable limit cycle $r = 1$, on which it moves at its natural frequency Ω . At this point we should note that for $z \neq 0$, the phase θ is well defined (modulo 2π). Next considering a coupled system of N many Stuart–Landau oscillators with a linear all-to-all coupling, we have the following system of ODEs describing its temporal dynamics (see [15, 2])

$$\frac{dz_m}{dt} = (1 - |z_m|^2 + i\Omega_m)z_m + \frac{K}{N} \sum_{n=1}^N (z_n - z_m), \quad (1.24)$$

where the subscripts m, n denote the corresponding quantities for m -th, n -th oscillators and K being the positive *coupling strength*. To focus on the dynamics of phases for weakly coupled limit-cycle oscillators, invoking the polar form with unit amplitude as before, we get

$$i \frac{d\theta_m}{dt} = i\Omega_m + \frac{K}{N} \sum_{n=1}^N \left(e^{i(\theta_n - \theta_m)} - 1 \right), \text{ for } m = 1, 2, \dots, N. \quad (1.25)$$

Comparing imaginary sides of both sides of (1.25), in his seminal paper [68] Kuramoto introduced the following model (later known as **Kuramoto model**) to study synchronization for large populations of coupled limit cycle oscillators

$$\frac{d\theta_m}{dt} = \Omega_m + \frac{K}{N} \sum_{n=1}^N \sin(\theta_n - \theta_m), \text{ for } m = 1, 2, \dots, N. \quad (1.26)$$

Here the term Ω_m denotes the m -th oscillator's natural frequency which expresses *intrinsic randomness* via the probability density function and the sinusoidal term expresses the *nonlinear attractive coupling* between n -th and m -th oscillators. This model became a central object of attention from the beginning of this millenium. We complement the equation (1.26) with the following ODE

$$\frac{d\Omega_m}{dt} = 0, \quad (1.27)$$

to obtain the corresponding mean-field equation in Wasserstein framework on the extended phase space $(\theta, \Omega) \in \mathbb{T} \times \mathbb{R} := \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, to study the mesoscopic approximation of the infinite system consisting of infinitely many oscillators. Since each oscillator is coupled in the same way to all others, this represents a mean-field model for the set of oscillators, and following the works of Dobrushin [38] and Neunzert [79, 80], Lancelotti [71] investigated if the equations (1.26) and (1.27) does have a meaningful mean-field limit. To deal with this mean-field limit situation, following the standard Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY, in short) [55, 65] hierarchy letting $N \rightarrow \infty$ (i.e. passing to the mean field limit [38, 79]) he replaced individual oscillators by a continuum of oscillators. In this continuum model, he introduced a one-oscillator distribution function $f(\theta, \Omega, t)$ to describe the dynamical system $\{(\theta_i, \Omega_i)\}_{i \in \mathbb{N}}$. This function f obeys the following PDE

$$\partial_t f + \partial_\theta (\omega[f]f) = 0 \text{ for } (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, t > 0, \quad (1.28)$$

where $\omega[f](\theta, \Omega, t) := \Omega - KL[\rho](\theta, t)$ and $L[\rho](\theta, t) := \int_{\mathbb{T}} \sin(\theta - \theta_*) \rho(\theta_*, t) dt$. This equation is known as the **Kuramoto–Sakaguchi equation**. Here $\rho = \rho(\theta, t)$ is the density function in θ -variable, namely: $\rho(\theta, t) := \int_{\mathbb{R}} f(\theta, \Omega, t) d\Omega$. In particular, while

considering an *an ensemble of Kuramoto oscillators, at identical frequency* the natural frequency measure $g(\Omega) := \int_{\mathbb{T}} f(\theta, \Omega, t) d\theta$ becomes **monokinetic**, i.e. $g(\Omega) = \delta(\Omega)$. In that scenario, Balmforth [4], Benedetto *et al.* [6], Carrillo *et al.* [13], Amadori *et al.* [1, 2] have considered the PDE

$$\partial_t \rho - K \partial_\theta (L[\rho] \rho) = 0 \text{ for } \theta \in \mathbb{T} \text{ and } t > 0, \quad (1.29)$$

known as the Kuramoto–Sakaguchi equation *for identical oscillators*.

1.3.2 Available results

Rigorous mathematical treatment of the Kuramoto model initiated in the seminal works of Kuramoto [68, 69] and Winfree [101] half a century back. Since then, several works have been performed towards the direction of analyzing different nonlocal conservation law type models arising from *synchronization, traffic flow, aggregation-diffusion* and so on. Among them, for our purpose the work done by Carrillo *et al.* [13] which analyzed the time-asymptotic complete synchronization of an ensemble of Kuramoto oscillators in the setting of 1-Wasserstein norm, is of significant importance. Also, we have to keep in mind two papers by Amadori *et al.* [1, 2] where they have designed and analyzed a *front-tracking method* for the equation (1.29). Noting that the front-tracking method works only in one dimension, we have proposed in Paper 1 a finite volume scheme which is applicable to the multiple dimension equation. Moreover, it should be kept in mind that instead of the Kuramoto–Sakaguchi equation (1.28) or (1.29), we have considered more general non-local conservation law. An important example falling into this class of equation is the scalar conservation law where the integral dependence of the flux function is expressed through a convolution product, which has been explored in [18], where the authors have analyzed *stability estimates* and designed a high-order numerical scheme. Also, the authors have studied the well-posedness and regularity properties of equation from similar class in their work [32]. In the context of non-local conservation law in bounded domain, the work [33] is definitely worthy of mention due to its novelty of singling out proper boundary conditions by carefully using non-local operators in the presence of boundary.

1.4 Ostrovsky–Hunter Equation

Wave phenomena have been a central object of study for both mathematicians and physicists in view of its real-world applications and of the novelty of the problems. One such equation which exhibits wave phenomena is **Ostrovsky–Hunter equation**, which takes account of both of the nonlinearity of the model and *no-long-wave-dispersion* effects. In the following section we are going to briefly recall the origin and available literature for this equation.

1.4.1 Brief derivation

The main aim of the Ostrovsky–Hunter equation is to model nonlinear phenomena of dispersive type appropriately. The term **dispersive** has its root from the fact that the

solutions of this equation are waves that spread out spatially as long as no boundary conditions are imposed. Most natural objects to be modelled by this equation are waves on the free surface of a liquid (*i.e.* “water waves”), which are an important example of nonlinear dispersive waves. Keeping this in mind, we begin by considering a body of water of finite depth under the influence of gravity. For small slope of the free surface, *i.e.* in the *shallow water regime* the exact water wave equations can be approximated by the **Boussinesq equations** [58] which also includes rotational effects due to Coriolis force

$$\begin{aligned}\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0, \\ \partial_t u + u\partial_x u + v\partial_y u + g\partial_x h + \frac{1}{3}c_0^2 h_0 \frac{\partial}{\partial x}(\partial_{xx}^2 h + \partial_{yy}^2 h) - fv &= 0, \\ \partial_t v + u\partial_x v + v\partial_y v + g\partial_y h + \frac{1}{3}c_0^2 h_0 \frac{\partial}{\partial y}(\partial_{xx}^2 h + \partial_{yy}^2 h) + fu &= 0.\end{aligned}\tag{1.30}$$

In the above system of equations $h(x, y, t)$ is the *elevation* of the free surface, and u, v are the x, y -components of the fluid velocity, respectively. Also h_0, g , and f are three independent parameters involved in the equations (1.30) which represent the mean depth of fluid, the gravitational acceleration and the Coriolis frequency respectively. The quantity c_0 denotes the shallow water speed given by $c_0 := \sqrt{gh_0}$. For long internal waves, Ostrovsky [82] and Grimshaw [50] independently considered a wave propagating in the x -direction with wavelength λ . They considered the following scaling, for a small parameter $0 < \varepsilon \ll 1$

$$\frac{h - h_0}{h_0} = O(\varepsilon^2), \text{ i.e. small amplitude regime,} \tag{1.31}$$

$$\frac{\lambda}{h_0} = O\left(\frac{1}{\varepsilon}\right), \text{ i.e. shallow water regime,} \tag{1.32}$$

$$\frac{h_0 f}{c_0} = O(\varepsilon), \text{ i.e. slow rotation regime,} \tag{1.33}$$

which reduces the effects of *nonlinearity*, *shallow water dispersion*, and *rotational dispersion* of same order of magnitude respectively. Finally by the method of *singular perturbation*, letting $\varepsilon \rightarrow 0$ the quantity $\eta(x, t) = h(x, t) - h_0$ (*i.e.* the disturbance on the free surface) satisfies

$$\frac{\partial}{\partial x} \left(\partial_t \eta + c_0 \partial_x \eta + \frac{3c_0}{2h_0} \eta \partial_x \eta + \frac{c_0 h_0^2}{6} \partial_{xxx}^3 \eta \right) - \frac{f^2}{2c_0} \eta = 0. \tag{1.34}$$

After a Gallilean transformation and appropriate rescaling, Hunter [58, 59] reduces this equation to

$$\frac{\partial}{\partial x} \left(\partial_t u + u \partial_x u + \partial_{xxx}^3 u \right) \pm \sigma u = 0. \tag{1.35}$$

The case $\sigma = +1$ is significant only for rotating fluids with large surface tension in *low* gravitational fields. Thus it is not of high importance for geophysical problems in real world. Consequently, in most of the literatures, the case $\sigma = -1$ have been considered because of its importance to real life geophysical problems. The equation (1.35) is going

to be one of the cornerstones of the thesis as this leads to several interesting equations, including the **Ostrovsky–Hunter equation**. In the **no dispersion** (i.e. $\partial_{xxx}^3 u = 0$ and $\int^x u(y, t) dy = 0$) regime this equation (1.35) reduces to the well known *inviscid Burgers equation*

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0,$$

which is highly well-studied and serves as a fundamental example to study nonlinear conservation laws. Similarly, in the **no rotation** regime the equation (1.35) reduces to another well known nonlinear dispersive equation, namely the *Korteweg-de Vries equation* or KdV equation in short

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_{xxx}^3 u = 0, \quad (1.36)$$

One of the principal objects for our thesis has its root from the third regime, namely **no long wave dispersion** (i.e. $\partial_{xxx}^3 u = 0$) regime which leads to

$$\frac{\partial}{\partial x} \left(\partial_t u + \partial_x \left(\frac{u^2}{2} \right) \right) - u = 0. \quad (1.37)$$

The equation (1.37) is known via different terminologies by different investigators. For instance Hunter [58, 59] calls (1.37) the *short wave equation*, whereas following the work of Ostrovsky [82] and Hunter the equation came to be known as the *Ostrovsky–Hunter equation*. Also it is worth mentioning at this point in the works of Liu [75, 99], Brunelli and Sakovich [11], the equation (1.35) is derived in the no long wave dispersion regime with $\sigma = +1$. In their work, they have referred this equation as *Ostrovsky–Vakhnenko equation* after deriving it from the following Whitham equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \int_{\mathbb{R}} K(x - y) \partial_y u dy = 0, \quad (1.38)$$

where $K(x - y) = \frac{1}{2}|x - y|$. This equation intended to model a wave equation containing both breaking and peaking. The Ostrovsky–Hunter equation, which can be seen as a scalar conservation law with a *non-local* source term is going to play a central role in the later chapters (see Papers 3,4).

1.4.2 Glance at the available results

Since it's introduction in the work of Ostrovsky [82] in 1978 and Hunter [58, 59] in early 1980's the equation (1.37) came to the attention of the mathematical community from the early 2000. In his work [10], Boyd showed that the Fourier pseudospectral method gives first order convergence for (possibly with discontinuous slope) the limiting waves, namely *parabolic waves* solution to the equation (1.37). Moreover, near this limiting solution they showed the travelling waves, namely *paraboloidal waves* can be approximated [9] further investigated the phenomena of *wave breaking* (i.e. developing an infinite slope in finite time) and proved the famous Rosale's conjecture, which says one can find initial conditions for the equation (1.37) of arbitrarily small amplitude that will break, to be true! Later in 2006, following the work of Kenig *et al.* [64] Linares and Milanés [74] studied the initial value problem corresponding to the equation (1.35) and showed the

local well-posedness in $X_s := \{f \in H^s(\mathbb{R}) \mid \partial_x^{-1} f \in L^2(\mathbb{R})\}$, for $s > \frac{3}{4}$. Around the same time Gui and Liu [52] studied the Cauchy problem for the Ostrovsky equation with positive dispersion in the Sobolev space H^s of lower order s (precisely speaking $s \geq -\frac{7}{12}$) and established the local and global well-posedness for this Cauchy problem. Towards the end of last decade, Liu, Pelinovsky and Sakovich [75] examined the wave breaking phenomena for (1.37) to find sufficient conditions for wave breaking on both infinite and periodic domain. Moreover using method of characteristics they provided explicit blow-up rate at which the waves break. In spite of numerous papers by Coclite *et al.* [23, 24, 26] exploring well-posedness theory for Ostrovsky–Hunter equation, there were not many results from the numerical analysis perspective, approximating this equation, till the last decade. Ridder *et al.* [30, 86] in their papers have considered the Ostrovsky–Hunter equation with different boundary conditions and considered *finite difference method* to prove the existence, uniqueness for periodic entropy solutions. Also, in the case of *periodic boundary condition* they have proved a convergence rate of $O(\sqrt{\Delta x})$.

1.5 Summary of the Papers

The thesis consists of five papers. The first paper (cf. Chapter 2), being motivated by the Kuramoto–Sakaguchi equation discussed in Section 1.3 considers a class of conservation laws, namely continuity equations with nonlocal flux and contains convergence analysis of the numerical schemes for such equations. In the next three (cf. Chapters 3, 4, 5) papers, we have studied Ostrovsky–Hunter equation introduced in Section 1.4 but with *spatially dependent flux*. In the second paper of the thesis we have established the well-posedness of the initial value problem for the Ostrovsky–Hunter equation with spatially dependent flux. In the third and fourth papers of the thesis we have performed numerical analysis of the Ostrovsky–Hunter equation with the spatially dependent flux, continuously and discontinuously, respectively. The last paper (cf. Chapter 5) presents a proof of convergence of the second order TECNO scheme in multiple dimensions to the corresponding weak solution and it satisfies *one* of the entropy conditions. The contents of each paper is briefly described below.

1.5.1 Paper 1

(Written in collaboration with Ulrik Skre Fjordholm, accepted and to appear in *IMA Journal of Numerical Analysis*, doi : <https://doi.org/10.1093/imanum/dry074>)

In this paper we have derived and analyzed a Lax–Friedrichs type finite volume scheme approximating a particular subclass of conservation laws, namely continuity equations with non-local flux. Motivated by the Kuramoto–Sakaguchi equation (1.28) discussed in Section 1.3 we will consider a continuity equation, where the velocity term inside the flux generalizes the nonlocal term involved in the flux function of (1.28).

Our numerical analysis encompasses more generalized version of the equation (1.28) in multiple spatial dimensions. In this paper we have considered two particular regimes with respect to the regularity of the initial data. If the initial data is of bounded variation, then we have showed strong convergence while in the complementing scenario we have

proved its weak convergence to the measure valued solution. In this paper we have used the notion of measure valued solution following [13].

We have illustrated our theoretical results with several numerical examples, considering different initial data with different degree of singularity. For each numerical experiment we have computed the corresponding Experimental Order of Convergence (1.21) for L^1 - and 1-Wasserstein distance.

1.5.2 Paper 2

(Written in collaboration with Nils Henrik Risebro and Giuseppe Maria Coclite, accepted and to appear in Milan Journal of Mathematics.)

In this paper we have considered the Ostrovsky–Hunter equation (cf. Section 1.4), but with a spatial dependency of a *certain smoothness* in the flux function $f(x, u)$. Precisely speaking we are assuming for the flux function $f_x(x, u)$ to be locally uniformly Lipschitz and $f_u(x, u)$ to be uniformly bounded.

The paper contains a well-posedness theory for this equation, extending the results proved by Coclite and di Ruvo [26]. After proving an *a priori* energy estimate and *local* L^∞ bound on the solution of a viscous Ostrovsky–Hunter equation we have invoked the *method of compensated compactness* (cf. Subsection 1.2.2 of Section 1.2) to extract a subsequence convergent in some suitable sense, proving the existence of entropy solution. Then using a doubling of variable argument we have concluded the desired uniqueness.

1.5.3 Paper 3

(Written in collaboration with Nils Henrik Risebro, submitted for publication.)

Ridder *et al.* [30, 86] in their papers considered the Ostrovsky–Hunter equation and performed the analysis using the finite *difference* method. Extending their work, in this paper we have considered the Ostrovsky–Hunter equation with spatially dependent flux function, where the spatial dependency is continuous. Moreover we have translated the work done in the Paper 2 (see Chapter 3) into the discrete set up.

This paper contains the construction of the unique entropy solution to the *spatially dependent* periodic Ostrovsky–Hunter equation, as a limit of approximate solutions generated by a finite volume scheme. As mentioned in the subsection 1.2.3 of Section 1.2, we have proved the estimates of type (C1), (C2), and (C3) to prove the convergence to an entropy solution using Lax-Wendroff type theorem. Afterwards, by a Kuznetsov-type lemma we have shown that this convergence is of $O(\sqrt{\Delta x})$.

Similar to Paper 2 we are going to verify our theoretical results, by numerical simulation using initial data studied in [75, 99] and [30]. For each of these experiments we have computed the Experimental Order of Convergence for the L^1 -norm which is better than the theoretically proven order $\frac{1}{2}$.

1.5.4 Paper 4

Extending the work initiated in Paper 3 we have considered the Ostrovsky–Hunter equation (cf. Section 1.4) with spatial dependency in the flux function, but in particular the spatial dependency is multiplicative and *discontinuous*. Due to this discontinuity, usual *strong* TV bounds are not expected to hold [95, 96].

In this paper, we have designed a finite volume numerical scheme which generates a sequence of approximate solutions which strongly converge to a *weak* solution of the equation under consideration. For this purpose we have established few *a-priori* estimates which ensures the compactness of this sequence of approximate solutions. To compensate the lack of *strong* TV bound, we have followed the method of using a *Temple*-type singular functional [94] to derive the *weak* TV bounds. To show that the limit of this sequence is indeed a weak solution, we have used a Lax-Wendroff type argument.

Moreover we have exhibited several numerical simulations supporting our theoretical results. In each of these examples we have computed the Experimental Order of Convergence for L^1 -norm.

1.5.5 Paper 5

(Written in collaboration with Ulrik Skre Fjordholm, submitted for publication.)

In the work of LeFloch *et al.* [72], *higher* order accurate fully discrete (in both space and time) entropy conservative schemes were introduced, discretizing systems of conservation in one spatial dimension. Whereas in their papers Fjordholm *et al.* [39, 42] devised an entropy stable, arbitrarily higher order, non-*oscillatory* scheme known as the TECNO scheme approximating the scalar conservation law in one dimension and proved its convergence towards the unique entropy solution.

In this paper we have generalized these results to the spatially multidimensional case. The key to perform the convergence analysis of TECNO scheme, approximating scalar conservation law in multiple spatial dimension is a result on compensated compactness method in higher dimension, due to Panov [83].

As of now we have managed to show the second order version of the TECNO scheme, in the above mentioned setting converges to a weak solution, which satisfies at least *one* of the entropy conditions.

1.6 Future perspectives

In chapter 2, we have performed our numerical analysis and measured the error in L^1 and W_1 metric. Following the works of Carrillo *et al.* [13], a matter of future research could be investigation of the numerical scheme in W_p metric for $p > 1$. Also, we have not proved any theoretical rate of convergence, which need to be examined in future. In the recent years several efforts have been made to investigate similar questions for aggregation equation (see [7, 41, 60, 61]), which bears similarities to the equations I have considered so far. To the best of my knowledge there are no available results, in the investigation of this equation in several space dimension and with respect to W_p -norm for any $p \in \mathbb{N}$. So that avenue remains to be an interesting one to explore.

In chapters 3-5, we have considered the Ostrovsky–Hunter equation and performed the numerical analysis using the finite volume method. In 5 to single out the physically relevant solution we need to show an entropy condition and proving this condition is an obvious immediate future endeavour. The Ostrovsky–Hunter equation can be viewed as a conservation law, with a nonlocal source term. Hence, it is natural to perform the convergence analysis using **operator-splitting method**. Moreover, we have considered *only* spatially dependent flux function in the equation. Following the work of Coclite *et al.* [20] we can hope to extend the analysis for Ostrovsky–Hunter equation with *spatio-temporal* flux. One more possible direction of research in future sharing similar mathematical idea is to examine the **Hunter–Saxton equation** (see [14, 56]) with spatially, and then with spatio-temporally dependent flux. Since the Ostrovsky–Hunter equation and the Hunter–Saxton equation come up from the regime of fluids governed by shallow–water type equation and from the physics of nematic crystal, it would be an interesting problem to extend this numerical analysis for several spatial dimension.

In chapter 6, we have considered the second-order TECNO scheme in several space dimensions. One of the key tools to obtain our result is to have the so-called ENO conjecture (see chapter 6) to be true. But for *even* higher (more than two) order TECNO scheme our analysis does not work. Since ENO-conjecture has not been proved for even higher-order ENO reconstruction. This can be a challenging research problem in future.

CHAPTER 2

A CONVERGENT FINITE VOLUME METHOD FOR THE KURAMOTO EQUATION AND RELATED NON-LOCAL CONSERVATION LAWS

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Abstract

We derive and study a Lax–Friedrichs type finite volume method for a large class of nonlocal continuity equations in multiple dimensions. We prove that the method converges weakly to the measure-valued solution, and converges strongly if the initial data is of bounded variation. Several numerical examples for the kinetic Kuramoto equation are provided, demonstrating that the method works well both for regular and singular data.

Keywords: Nonlocal conservation laws; finite volume methods; Kuramoto equation

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2.1 Introduction

The purpose of the present paper is to derive a non-oscillatory and convergent numerical method for a large class of nonlocal continuity equations of the form

$$\begin{aligned}\partial_t \mu + \nabla \cdot (V[\mu]\mu) &= 0 & x \in \mathcal{K}, \ t > 0 \\ \mu(x, 0) &= \mu_0(x) & x \in \mathcal{K}\end{aligned}\tag{2.1}$$

on some domain $\mathcal{K} \subset \mathbb{R}^d$. Here, $\mu(x, t)$ is interpreted as a density of particles at time t , and $V[\mu] : \mathcal{K} \rightarrow \mathbb{R}^d$ depends nonlocally on μ , typically the convolution of μ with some kernel. The initial data is assumed to be some probability measure $\mu_0 \in \mathcal{P}(\mathcal{K})$. Conservation laws of this form arise in a large variety of applications, such as the simulation of crowd dynamics, traffic flow [32], microbiology [12], in the analytical study of chemotaxis [60], consensus problem [7, 61] and more [12, 68, 101]. One particular instance of (2.1) which has been studied extensively is the so-called *Kuramoto–Sakaguchi equation*, also called the *kinetic Kuramoto equation*,

$$\begin{aligned}\partial_t f + \partial_\theta(\omega[f]f) &= 0 & (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}, \ t > 0 \\ f(\theta, \Omega, 0) &= f_0(\theta, \Omega) & (\theta, \Omega) \in \mathbb{T} \times \mathbb{R}\end{aligned}\tag{2.2}$$

where $\mathbb{T} = \mathbb{R}/2\pi$ is the one-dimensional torus and

$$\omega[f](\theta, \Omega, t) := \Omega - K \int_{\mathbb{T}} \sin(\theta - \theta^*) \rho(\theta^*, t) d\theta^*, \quad \rho(\theta, t) := \int_{\mathbb{R}} f(\theta, \Omega, t) d\Omega$$

This equation arises as the mean-field limit of the *Kuramoto equation*, a system of ordinary differential equations for coupled oscillators which was first studied by Winfree and Kuramoto [69, 68, 101]. The unknowns in the Kuramoto equation are the phase $\theta \in \mathbb{T}$ and the natural frequency $\Omega \in \mathbb{R}$ of each oscillator, and the interaction between the oscillators depends on the coupling strength $K > 0$ and the relative difference in phase $\theta - \theta^*$ between pairs of oscillators. The mean-field limit as the number of oscillators goes to infinity is a probability distribution $f = f(\theta, \Omega, t)$ obeying the above nonlocal continuity equation; see e.g. [38, 79, 19]. See also the recent paper [13] for some qualitative properties of (2.2). We will also let $g(\Omega, t) := \int_{\mathbb{T}} f(\theta, \Omega, t) d\theta$ denote the distribution function for natural frequencies. From (2.2) it is easily seen that g is constant in time, $g(\Omega, t) \equiv g(\Omega)$. Here and elsewhere we will assume that f_0 (and hence also g) is compactly supported.

The Kuramoto equation (2.2) is valid for oscillators with non-identical natural frequencies. The situation is somewhat simpler in the case of identical oscillators, i.e. oscillators whose natural frequencies coincide. This corresponds to g being a Dirac measure, and without loss of generality we can assume that $g = \delta$, the Dirac measure located at $\Omega = 0$. Consequently,

$$f(\theta, \Omega, t) = \rho(\theta, t) \delta(\Omega), \quad \omega[f]f = L[\rho]\rho, \quad L[\rho](\theta, t) := -K \int_{\mathbb{T}} \sin(\theta - \theta^*) \rho(\theta^*, t) d\theta^*.$$

Therefore, (2.2) reduces to the following equation for ρ :

$$\begin{aligned}\partial_t \rho + \partial_\theta(L[\rho]\rho) &= 0 \\ \rho(\theta, 0) &= \rho_0(\theta).\end{aligned}\tag{2.3}$$

Both equation (2.2) and (2.3) are instances of general nonlocal conservation laws of the form (2.1).

The purpose of the present paper is to derive and analyze a finite volume numerical method for a large class of equations of the form (2.1) (including the kinetic Kuramoto equations (2.2), (2.3)). In particular, we prove that the scheme converges strongly to the unique weak solution μ whenever $\mu_0 \in L^1 \cap BV$, and in all other cases converges in the sense of measures to the so-called *measure-valued* solution μ . We emphasize that the stability and convergence properties of the scheme are valid even when μ_0 (and hence also the exact solution) has point-mass singularities.

In the recent works [2, 1], Amadori et al. designed and analyzed a front-tracking numerical method for the Kuramoto–Sakaguchi equation for identical oscillators (2.3). In contrast to their method, our finite volume method works in any number of dimensions, does not require any regularity on the initial data, and does not impose “entropy conditions” on solutions. We also mention the work by Crippa and Lécureux-Mercier [34] where well-posedness of a *system* of nonlocal continuity equations of the form (2.1) is established. The extension of our finite volume scheme to such systems should be straightforward.

2.1.1 Nonlocal conservation laws

Nonlocal conservation laws of the form (2.1) were first studied by Neunzert [79] and Dobrushin [38]. Using techniques which by now are standard, they showed existence and uniqueness of solutions of (2.1). We will consider solutions which are weakly continuous maps $\mu : \mathbb{R}_+ \rightarrow \mathcal{M}(\mathcal{K})$, mapping time t to probability measures $\mu_t \in \mathcal{P}(\mathcal{K})$. Here, $\mathcal{M}(\mathcal{K}) = (C_0(\mathcal{K}))^*$ is the space of bounded Radon measures on $\mathcal{K}$, $\mathcal{P}(\mathcal{K})$ is the subset of probability measures, and by weakly continuous we mean that $t \mapsto \langle \mu_t, \varphi \rangle := \int_{\mathcal{K}} \varphi(x) d\mu_t(x)$ is continuous for every $\varphi \in C_0(\mathcal{K})$. We define the *1-Wasserstein* metric

$$d_W(\nu_1, \nu_2) := \sup \left\{ \int_{\mathcal{K}} \psi(x) d(\nu_2 - \nu_1)(x) : \psi \in C_b(\mathcal{K}), \|\psi\|_{\text{Lip}(\mathcal{K})} \leq 1 \right\}, \quad \nu_1, \nu_2 \in \mathcal{P}(\mathcal{K}).$$

It can be shown that d_W metrizes the topology of weak (or *narrow*) convergence in $\mathcal{P}_1(\mathcal{K})$, the set of probability measures ν with finite first moment, $\int_{\mathcal{K}} |x| d\nu(x) < \infty$ (see e.g. [100]).

Definition 2.1.1. $\mu \in C_w(\mathbb{R}_+; \mathcal{M}(\mathcal{K}))$ is a *measure-valued solution* of (2.1) if it satisfies (2.1) in the sense of distributions, i.e.

$$\int_0^\infty \int_{\mathcal{K}} \partial_t \varphi(x, t) + V[\mu_t](x) \cdot \nabla \varphi(x, t) d\mu_t(x) dt + \int_{\mathcal{K}} \varphi(x, 0) d\mu_0(x) = 0 \quad (2.4)$$

for every $\varphi \in C_c^\infty(\mathcal{K} \times \mathbb{R}_+)$.

Since μ is assumed to be weakly continuous in time, one can show that μ is a measure valued solution *if and only if*

$$\langle \mu_t, \psi \rangle - \langle \mu_s, \psi \rangle = \int_s^t \int_{\mathcal{K}} V[\mu_\tau](x) \cdot \nabla \psi(x) d\mu_\tau(x) d\tau \quad (2.5)$$

for every $0 \leq s < t$ and every $\psi \in C_c^\infty(\mathcal{K})$. We say that μ is a *weak solution* of (2.1) if it is a measure-valued solution which is absolutely continuous with respect to Lebesgue measure.

We will henceforth assume that $\mathcal{K}$ is of the form $\mathcal{K} = \mathbb{K}_1 \times \cdots \times \mathbb{K}_d$, where each $\mathbb{K}_i$ is either the torus $\mathbb{T}$ or the whole real line $\mathbb{R}$. For the well-posedness of (2.1) and the convergence of our numerical scheme, we also need some of the following assumptions on V :

- (A1) $\exists C_1 > 0$ such that $\|V[\nu]\|_{L^\infty(\mathcal{K})} \leq C_1$ for all $\nu \in \mathcal{P}(\mathcal{K})$
- (A2) $\exists C_2 > 0$ such that $|V[\nu](x) - V[\nu](y)| \leq C_2|x - y|$ for all $x, y \in \mathcal{K}$ and $\nu \in \mathcal{P}(\mathcal{K})$
- (A3) $\exists C_3 > 0$ such that $\|V[\nu_1] - V[\nu_2]\|_{L^\infty(\mathcal{K})} \leq C_3 d_W(\nu_1, \nu_2)$ for all $\nu_1, \nu_2 \in \mathcal{P}(\mathcal{K})$
- (A4) $\exists C_4 > 0$ such that $\|\nabla V[\nu]\|_{\text{Lip}(\mathcal{K})} \leq C_4$ for all $\nu \in \mathcal{P}(\mathcal{K})$.

The main well-posedness result for the nonlocal conservation law is the following.

Theorem 2.1.1 (Neunzert [79], Dobrushin [38]). *Let $\mu_0 \in \mathcal{P}_1(\mathcal{K})$ and let V satisfy assumptions (A1), (A2), (A3). Then there exists a unique measure-valued solution of (2.1). This solution is Lipschitz in time, $d_W(\mu_t, \mu_s) \leq C_1|t - s|$. If $\mu_0 \in L^1(\mathcal{K})$ then $\mu_t \in L^1(\mathcal{K})$ for all $t > 0$.*

Remark. It is easy to see that both kinetic Kuramoto equations (2.2) and (2.3) satisfy assumptions (A1)–(A4), and are therefore well-posed by Theorem 2.1.1. Indeed, in the case of (2.2) we let $d = 2$, $\mathcal{K} = \mathbb{T} \times \mathbb{R}$ and $V[\nu] = (V^1[\nu], V^2[\nu])$, where $V^2[\nu] \equiv 0$ and

$$V^1[\nu](\theta, \Omega) = \begin{cases} \Omega - K \int_{\mathcal{K}} \sin(\theta - \theta^*) d\nu(\theta^*, \Omega^*) & \text{for } \Omega \in \text{supp } g \\ 0 & \text{otherwise.} \end{cases}$$

We can then choose $C_1 = K + \max\{|\Omega| : \Omega \in \text{supp } g\}$, $C_2 = 1 + K$ and $C_3 = C_4 = K$. In the case of (2.3) we let $d = 1$, $\mathcal{K} = \mathbb{T}$, $V = L$ and $C_1 = C_2 = C_3 = C_4 = K$.

“Kinetic” PDEs (2.1) under our assumptions (A1)–(A3) are rather well-behaved under approximations. For instance, in a *particle approximation* one approximates the solution as a convex combination of Dirac measures, $\mu_t^M = \sum_{i=1}^M \delta_{x_i(t)} \mu_i$, where $x_i(t)$ is the position of the i th particle and $\mu_i \in (0, \infty)$ is its mass. It is straightforward to see that μ^M is in fact a measure-valued solution of (2.1) provided $x_i(t)$ satisfy the system of ODEs

$$\frac{dx_i}{dt}(t) = V[\mu^M](x_i(t)). \quad (2.6)$$

Assumptions (A1), (A2) guarantee that this system has a unique solution. Taking an approximating sequence μ_0^M converging weakly to μ_0 , assumption (A3) guarantees that the limit is a measure-valued solution. Thus, if one can solve the system of ODEs (2.6) then the question of convergence boils down to the approximation of the initial data by Dirac measures (see [38, 71, 79, 80, 71]).

Although the dynamical system (2.6) can be studied qualitatively in certain simple cases, it is computationally infeasible to solve (2.6) in more realistic settings. In the next section we proceed to construct a simple, computationally efficient numerical approximation to the nonlocal PDE (2.1).

2.2 The Lax–Friedrichs scheme

2.2.1 Derivation of the method

For the sake of simplicity we derive the method in one space dimension with either $\mathcal{K} = \mathbb{R}$ or $\mathcal{K} = \mathbb{T}$, and then simply state the method in multiple space dimensions (Section 2.2.5). We start by deriving a *staggered* version of the method but—again for the sake of simplicity—we will only analyze the unstaggered version of this method.

Consider a mesh $\dots < x_{i-\frac{1}{2}} < x_{i+\frac{1}{2}} < \dots$, where i run over $\mathcal{I} := \mathbb{Z}$ in the unbounded case $\mathcal{K} = \mathbb{R}$, or over some finite set $\mathcal{I} := \{1, \dots, N_x\}$ in the periodic case $\mathcal{K} = \mathbb{T}$, and such that $\bigcup_i [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) = \mathcal{K}$. For simplicity we assume a uniform mesh, $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \equiv \Delta x$.

We will denote $x_i := \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$. Let $0 = t^0 < t^1 < \dots < t^N = T$ be a partition of the time interval $[0, T]$ with uniform step size $t^{n+1} - t^n \equiv \Delta t$. Assuming that we are given an approximate solution $\mu^n = \sum_i \delta_{x_i} \mu_i^n$ at time t^n , we compute an approximation μ^{n+1} at $t = t^{n+1}$ as follows. Let μ_t be the exact solution of

$$\begin{aligned} \partial_t \mu + \partial_x (V[\mu] \mu) &= 0 & t^n < t < t^{n+1} \\ \mu_{t^n} &= \mu^n. \end{aligned} \quad (2.7)$$

Define $\mu_{i+\frac{1}{2}}^{n+1} := \langle \mu_{t^{n+1}}, \psi_{i+\frac{1}{2}} \rangle$, where ψ are the usual “witch’s hat” finite element basis functions,

$$\psi_{i+\frac{1}{2}}(x) := \begin{cases} \frac{x - x_{i-\frac{1}{2}}}{\Delta x} & x_{i-\frac{1}{2}} \leq x < x_{i+\frac{1}{2}} \\ \frac{x_{i+\frac{3}{2}} - x}{\Delta x} & x_{i+\frac{1}{2}} \leq x < x_{i+\frac{3}{2}} \\ 0 & \text{else,} \end{cases}$$

and the index i is taken over all $i \in \mathcal{I}$ if n is even and over $i \in \mathcal{I} - \frac{1}{2}$ if n is odd. The approximation μ^{n+1} at time t^{n+1} is then defined to be the projected solution $\mu^{n+1} := \sum_i \mu_{i+\frac{1}{2}}^{n+1} \delta_{x_{i+\frac{1}{2}}}$.

We can derive a simplified expression of $\mu_{i+\frac{1}{2}}^{n+1}$ as follows. Since the initial data μ^n in (2.7) is a convex combination of Dirac measures, the solution of (2.7) can be written as $\mu_t = \sum_i \delta_{x_i(t)} \mu_i^n$, where x_i solve the system of ODEs (2.6). If we assume the *CFL condition*

$$\frac{\Delta t}{\Delta x} \|V[\mu]\|_{L^\infty(\mathcal{K})} \leq \frac{1}{2} \quad (2.8)$$

then the particle with position $x_i(t)$ stays in the interval $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ for all $t \in [t^n, t^{n+1}]$. In particular, the atoms $\delta_{x_i(t)}$ in μ_t stay away from the kinks in $\psi_{i+\frac{1}{2}}$, and hence we may use the (non-differentiable) function $\psi_{i+\frac{1}{2}}$ as a test function in the weak formulation (2.5).

Using the fact that $\langle \mu^n, \psi_{i+\frac{1}{2}} \rangle = \frac{\mu_i^n + \mu_{i+1}^n}{2}$, we obtain

$$\begin{aligned} \mu_{i+\frac{1}{2}}^{n+1} &= \frac{\mu_i^n + \mu_{i+1}^n}{2} + \int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} V[\mu_t](x) \partial_x \psi(x) d\mu_t(x) dt \\ &= \frac{\mu_i^n + \mu_{i+1}^n}{2} - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} V[\mu_t](x_{i+1}(t)) \mu_{i+1}^n - V[\mu_t](x_i(t)) \mu_i^n dt. \end{aligned}$$

Approximating $V[\mu_t](x_i(t)) \approx V[\mu^n](x_i)$ yields our final scheme,

$$\mu_{i+\frac{1}{2}}^{n+1} = \frac{\mu_i^n + \mu_{i+1}^n}{2} - \Delta t \frac{V[\mu^n]_{i+1}\mu_{i+1}^n - V[\mu^n]_i\mu_i^n}{\Delta x} \quad (2.9)$$

(where we denote $V[\mu^n]_i := V[\mu^n](x_i)$). The initial data is set as $\mu_i^0 = \langle \mu_0, \psi_i \rangle$. We refer to (2.9) as the *staggered Lax–Friedrichs method*, after [78].

For the sake of simplicity, we will only consider an unstaggered version of (2.9), obtained by inserting an extra mesh point between all pairs of neighboring mesh points. The *unstaggered Lax–Friedrichs method* is then

$$\begin{aligned} \mu_i^0 &= \langle \mu_0, \psi_i \rangle \\ \mu_i^{n+1} &= \frac{\mu_{i-1}^n + \mu_{i+1}^n}{2} - \Delta t \frac{V[\mu^n]_{i+1}\mu_{i+1}^n - V[\mu^n]_{i-1}\mu_{i-1}^n}{2\Delta x} \end{aligned} \quad (2.10)$$

where $\mu^n = \sum_i \mu_i^n \delta_{x_i}$.

2.2.2 Properties of the method

In this section we prove several stability properties of the unstaggered Lax–Friedrichs method. Since the multi-dimensional method shares the same properties as the one-dimensional method, we will prove these properties for the one-dimensional method (2.10) and merely state the properties for the two-dimensional method (2.15) in Section 2.2.5.

Proposition 1. *Let $\mu_0 \in \mathcal{P}(\mathcal{K})$ and Consider the unstaggered, one-dimensional Lax–Friedrichs method (2.10) and define the piecewise linear measure-valued map*

$$\mu_t^{\Delta x} := \frac{t^{n+1} - t}{\Delta t} \mu^n + \frac{t - t^n}{\Delta t} \mu^{n+1} \quad \text{for } t \in [t^n, t^{n+1}) \quad (2.11)$$

where $\mu^n := \sum_i \mu_i^n \delta_{x_i}$. Assume that V satisfies condition (A1), and that the CFL condition

$$\lambda_0 \leq \frac{\Delta t}{\Delta x} \leq C_1^{-1} \quad (2.12)$$

is satisfied for some $\lambda_0 > 0$ (where C_1 is the constant in (A1)). Then for all $t \geq 0$

- (i) $\mu_t^{\Delta x} \geq 0$
- (ii) $\|\mu_t^{\Delta x}\|_{\mathcal{M}(\mathbb{K})} = 1$
- (iii) if $\text{supp } \mu_0 \subset B_M(0)$ then $\text{supp } \mu_t^{\Delta x} \subset B_{M+\lambda_0^{-1}t}(0)$
- (iv) $d_W(\mu_t^{\Delta x}, \mu_s^{\Delta x}) \leq \lambda_0^{-1}|t - s|$ for all $s \geq 0$.

Proof. Writing

$$\mu_i^{n+1} = \mu_{i-1}^n \frac{1 + \frac{\Delta t}{\Delta x} V[\mu^n]_{i-1}}{2} + \mu_{i+1}^n \frac{1 - \frac{\Delta t}{\Delta x} V[\mu^n]_{i+1}}{2},$$

it is clear that (2.12) and (A1) ensure that μ_i^{n+1} is a linear combination of nonnegative terms, whence (i) follows. Property (ii) follows from the fact that the method (2.10) is

conservative. From (2.10) we see that if $\mu_{i-1}^n = \mu_i^n = \mu_{i+1}^n = 0$ then $\mu_i^{n+1} = 0$. Hence, after n time steps the support of $\mu_t^{\Delta x}$ can have grown at most a distance $n\Delta x \leq n \frac{\Delta t}{\lambda_0} = \lambda_0^{-1}t^n$, and (iii) follows. For (iv), it is clear that by the definition (2.11) of $\mu^{\Delta x}$, it is enough to prove the claim for $s = t^n$, $t = t^{n+1}$. Let $\psi \in C_b(\mathbb{K})$ be Lipschitz continuous and write $\psi_i = \psi(x_i)$. Then

$$\langle \mu^{n+1} - \mu^n, \psi \rangle = \sum_i \psi_i \left(\frac{\mu_{i-1}^n - 2\mu_i^n + \mu_{i+1}^n}{2} - \frac{\Delta t}{2\Delta x} (V[\mu^n]_{i+1}\mu_{i+1}^n - V[\mu^n]_{i-1}\mu_{i-1}^n) \right)$$

(summation by parts)

$$\begin{aligned} &= -\frac{1}{2} \sum_i (\psi_{i+1} - \psi_i) \left(\mu_{i+1}^n - \mu_i^n - \frac{\Delta t}{\Delta x} (V[\mu^n]_i \mu_i^n + V[\mu^n]_{i+1} \mu_{i+1}^n) \right) \\ &= \frac{1}{2} \sum_i (\psi_{i+1} - \psi_i) \left(\mu_i^n \left(1 + \frac{\Delta t}{\Delta x} V[\mu^n]_i \right) + \mu_{i+1}^n \left(\frac{\Delta t}{\Delta x} V[\mu^n]_{i+1} - 1 \right) \right) \\ &\leq \frac{1}{2} \Delta x \|\psi\|_{\text{Lip}} \sum_i \mu_i^n \left(1 + \frac{\Delta t}{\Delta x} V[\mu^n]_i \right) + \mu_{i+1}^n \left(1 - \frac{\Delta t}{\Delta x} V[\mu^n]_{i+1} \right) \\ &= \Delta x \|\psi\|_{\text{Lip}}, \end{aligned}$$

the last two steps following from the CFL condition and (ii). Taking the supremum over ψ with $\|\psi\|_{\text{Lip}} \leq 1$ yields (iv). $\square$

2.2.3 Weak convergence of the method

We split the proof of convergence of the numerical method to the (unique) measure-valued solution into two parts. First we show that our method is consistent (Lemma 2.2.1), in the sense that *if* the method converges, then the limit is the measure-valued solution. Next, we show that the method indeed converges either weakly (Theorem 2.2.3) or strongly (Theorem 2.2.4), depending on the assumptions on the velocity field V and the initial data μ_0 .

Lemma 2.2.1. *Assume that V satisfies conditions (A2), (A3) and define $\mu^{\Delta x}$ by (2.11). Assume that $d_W(\mu_t^{\Delta x}, \mu_t) \rightarrow 0$ as $\Delta x \rightarrow 0$ uniformly on bounded intervals $t \in [0, T]$, for some $\mu \in C_w([0, \infty); \mathcal{M}(\mathbb{K}))$. Then μ is the measure-valued solution of (2.1).*

Proof. The convergence $d_W(\mu_t^{\Delta x}, \mu_t) \rightarrow 0$ implies that $\mu_t \in \mathcal{P}(\mathcal{K})$ for all t . Let $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$; we want to show that μ satisfies (2.4) with $\mathcal{K} = \mathbb{K}$. Define the measure $\rho^{\Delta x}$ by

$$\rho_t^{\Delta x} = \mu^n \quad \text{for } t \in [t^n, t^{n+1}).$$

Due to the Lipschitz continuity in time of $\mu^{\Delta x}$ (Proposition 1 (iv)), we have that also $d_W(\rho_t^{\Delta x}, \mu_t) \rightarrow 0$ as $\Delta x \rightarrow 0$ uniformly on bounded intervals $t \in [0, T]$.

Let $\varphi \in C_c^\infty(\mathbb{K} \times \mathbb{R})$ and define $\varphi_i^n := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \varphi(x_i, t) dt$. Then

$$\int_0^\infty \int_{\mathbb{K}} \varphi(x, t) d\rho_t^{\Delta x}(x) dt = \sum_i \sum_{n=0}^\infty \int_{t^n}^{t^{n+1}} \varphi(x_i, t) \mu_i^n dt = \sum_i \sum_{n=0}^\infty \varphi_i^n \mu_i^n \Delta t. \quad (2.13)$$

Multiplying (2.10) by φ_i^n and summing over i, n we get:

$$\begin{aligned}
 0 &= \sum_{n=0}^{\infty} \sum_i \left(\varphi_i^n \mu_i^{n+1} - \varphi_i^n \frac{\mu_{i-1}^n + \mu_{i+1}^n}{2} + \Delta t \varphi_i^n \frac{V[\mu^n]_{i+1} \mu_{i+1}^n - V[\mu^n]_{i-1} \mu_{i-1}^n}{2\Delta x} \right) \\
 &\quad (\text{summation by parts}) \\
 &= -\Delta t \sum_{n=0}^{\infty} \sum_i \frac{\frac{\varphi_{i-1}^n + \varphi_{i+1}^n}{2} - \varphi_i^{n-1}}{\Delta t} \mu_i^n - \Delta t \sum_{n=0}^{\infty} \sum_i \frac{\varphi_{i+1}^n - \varphi_{i-1}^n}{2\Delta x} V[\mu^n]_i \mu_i^n - \sum_i \varphi_i^{-1} \mu_i^0 \\
 &= -\int_0^\infty \int_{\mathbb{K}} \frac{\frac{\varphi(x-\Delta x, t) + \varphi(x+\Delta x, t)}{2} - \varphi(x, t - \Delta t)}{\Delta t} \\
 &\quad + \frac{\varphi(x + \Delta x, t) - \varphi(x - \Delta x, t)}{2\Delta x} V[\mu_t^{\Delta x}](x) d\rho_t^{\Delta x}(x) dt \\
 &\quad - \int_{\mathbb{K}} \varphi(x, -\Delta t) d\mu_0^{\Delta x}(x),
 \end{aligned}$$

where we in the last step have used (2.13). By condition (A3) we know that $V[\mu_t^{\Delta x}] \rightarrow V[\mu_t]$ uniformly on $\mathbb{R} \times [0, T]$, and by condition (A2), the functions $V[\mu_t^{\Delta x}]$ are uniformly Lipschitz. Moreover, the divided differences in the integrand converge to $\partial_t \varphi$ and $\partial_x \varphi$, respectively, by the mean value theorem. Finally, a straightforward projection error estimate in d_W shows that $\mu_0^{\Delta x}$ converges weakly to μ_0 as $\Delta x \rightarrow 0$. It follows that the above integral converges to

$$-\int_0^\infty \int_{\mathbb{K}} \partial_t \varphi(x, t) + \partial_x \varphi(x, t) V[\mu_t](x) d\mu_t(x) dt - \int_{\mathbb{K}} \varphi(x, 0) d\mu_0(x)$$

and the proof is complete. $\square$

For the convergence proof we use the following compactness lemma, whose proof is postponed to the appendix.

Lemma 2.2.2. *Let $K \subset \mathcal{P}_1(\mathbb{R}^d)$ be a bounded set, i.e. $\sup_{\mu \in K} d_W(\mu, \bar{\mu}) < \infty$ for some $\bar{\mu} \in \mathcal{P}(\mathbb{R}^d)$. Then K is relatively compact in the metric space $(\mathcal{P}_1(\mathbb{R}^d), d_W)$.*

Theorem 2.2.3. *Assume that V satisfies conditions (A1), (A2), (A3) and let $\mu_0 \in \mathcal{P}(\mathbb{K})$ have compact support. Assume that the CFL condition (2.12) is satisfied. Then the Lax–Friedrichs method (2.10) converges weakly to the measure-valued solution of (2.1). More precisely, if $\mu^{\Delta x}$ is given by (2.11) then*

$$\sup_{t \in [0, T]} d_W(\mu_t^{\Delta x}, \mu_t) \rightarrow 0 \quad \text{as } \Delta x \rightarrow 0$$

for any $T > 0$.

Proof. By Proposition 1 (i), (ii) and (iii), the measures $\mu_t^{\Delta x}$ stay in $\mathcal{P}(\mathbb{R})$ and have uniformly bounded support for all $\Delta x > 0, t \in [0, T]$, and by (iv), the maps $\mu^{\Delta x} : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^d)$ are Lipschitz. Noting that the projection map $\mu_0 \mapsto \mu_0^{\Delta x}$ moves mass a distance

of at most Δx leads to the estimate $d_W(\mu_0, \mu_0^{\Delta x}) \leq \Delta x$, we have that $d_W(\mu_0, \mu_t^{\Delta x}) \leq d_W(\mu_0, \mu_0^{\Delta x}) + d_W(\mu_0^{\Delta x}, \mu_t^{\Delta x}) \leq \Delta x + \frac{1}{\lambda_0}t$, so by Lemma 2.2.2 the set

$$K := \{\mu_t^{\Delta x} : 0 < \Delta x < 1, t \in [0, T]\}$$

is relatively compact with respect to d_W -convergence. Hence, by Ascoli's theorem, there exists a subsequence $\Delta x_k \rightarrow 0$ as $k \rightarrow \infty$ and some Lipschitz map $\mu : [0, T] \rightarrow \mathcal{P}(\mathbb{R}^d)$ such that $d_W(\mu_t^{\Delta x_k}, \mu_t) \rightarrow 0$ as $k \rightarrow \infty$, uniformly in $t \in [0, T]$. By Lemma 2.2.1, the limit μ is the measure-valued solution of (2.1). But this solution is unique, which implies that the whole sequence $\mu^{\Delta x}$ converges to μ . $\square$

2.2.4 Strong convergence of the method

If the initial data and the velocity field are sufficiently smooth then we can show that the numerical method in fact converges *strongly*. We assume that the initial data is absolutely continuous with respect to Lebesgue measure, and hence has a density function $\mu_0(x)$. This data is sampled by its cell averages,

$$\hat{\mu}_i^0 := \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mu_0(x) dx$$

and the numerical solution is realized as the linear-in-time, piecewise constant L^1 function

$$\hat{\mu}_t^{\Delta x}(x) = \frac{t^{n+1} - t}{\Delta t} \hat{\mu}^n + \frac{t - t^n}{\Delta t} \hat{\mu}^{n+1} \quad \text{for } t \in [t^n, t^{n+1}), \quad (2.14)$$

where

$$\hat{\mu}^n(x) = \sum_i \mu_i^n \mathbb{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})}(x).$$

and μ_i^n is the solution of the Lax–Friedrichs scheme (2.10) with initial data $\hat{\mu}_i^0$.

Proposition 2. *For the unstaggered, one-dimensional Lax–Friedrichs method (2.10) assume that V satisfies conditions (A1), (A2), (A4), and that the CFL condition (2.12) is satisfied. Then for all $t \in [0, T]$*

$$(i) \quad \hat{\mu}_t^{\Delta x} \geq 0$$

$$(ii) \quad \|\hat{\mu}_t^{\Delta x}\|_{L^1(\mathbb{K})} = 1$$

$$(iii) \quad TV(\hat{\mu}_t^{\Delta x}) \leq TV(\mu_0)e^{C_2 t} + (e^{C_2 t} - 1)\frac{C_4}{2C_2}$$

$$(iv) \quad \|\hat{\mu}_t^{\Delta x} - \hat{\mu}_s^{\Delta x}\|_{L^1(\mathbb{K})} \leq C_T |t - s| \text{ for all } s \geq 0.$$

(In (iv), $C_T = C_2 + \lambda_0^{-1}c_T$, where c_T is the upper bound from (iii), $c_T := TV(\mu_0)e^{C_2 T} + (e^{C_2 T} - 1)\frac{C_4}{2C_2}$.)

Proof. Properties (i) and (ii) obviously follow from Proposition 1 (i) and (ii). To show (iii) we split $\text{TV}(\hat{\mu}^{n+1})$ as follows:

$$\begin{aligned} \text{TV}(\hat{\mu}^{n+1}) &\leq \frac{1}{2} \sum_i \left| (\mu_{i+2}^n - \mu_{i+1}^n) \left(1 - \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_{i+2} \right) + (\mu_i^n - \mu_{i-1}^n) \left(1 + \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_i \right) \right. \\ &\quad \left. - \frac{\Delta t}{\Delta x} \mu_{i+1}^n (V[\hat{\mu}^n]_{i+2} - V[\hat{\mu}^n]_{i+1}) + \frac{\Delta t}{\Delta x} \mu_{i-1}^n (V[\hat{\mu}^n]_i - V[\hat{\mu}^n]_{i-1}) \right| \\ &\leq A^n + B^n \end{aligned}$$

where

$$A^n = \frac{1}{2} \sum_i |\mu_{i+2}^n - \mu_{i+1}^n| \left(1 - \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_{i+2} \right) + |\mu_i^n - \mu_{i-1}^n| \left(1 + \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_i \right) \leq \text{TV}(\hat{\mu}^n)$$

and

$$\begin{aligned} B^n &= \frac{\Delta t}{2\Delta x} \sum_i \left| \mu_{i+1}^n (V[\hat{\mu}^n]_{i+2} - V[\hat{\mu}^n]_{i+1}) - \mu_{i-1}^n (V[\hat{\mu}^n]_i - V[\hat{\mu}^n]_{i-1}) \right| \\ &\leq \frac{\Delta t}{2} \sum_i \left[|\mu_{i+1}^n - \mu_i^n| + |\mu_i^n - \mu_{i-1}^n| \right] \frac{|V[\hat{\mu}^n]_{i+2} - V[\hat{\mu}^n]_{i+1}|}{\Delta x} \\ &\quad + \frac{\Delta t}{2} \sum_i \mu_{i-1}^n \frac{|V[\hat{\mu}^n]_{i+2} - V[\hat{\mu}^n]_{i+1} - V[\hat{\mu}^n]_i + V[\hat{\mu}^n]_{i-1}|}{\Delta x^2} \Delta x \\ &\leq C_2 \Delta t \text{TV}(\hat{\mu}^n) + \frac{C_4}{2} \Delta t. \end{aligned}$$

Iterating over all n gives

$$\text{TV}(\hat{\mu}^{n+1}) \leq (1 + C_2 \Delta t) \text{TV}(\hat{\mu}^n) + \frac{C_4}{2} \Delta t \leq \dots \leq (1 + C_2 \Delta t)^{n+1} \left(\text{TV}(\mu_0) + \frac{C_4}{2C_2} \right) - \frac{C_4}{2C_2}$$

and (iii) follows, after applying the inequality $1 + \varepsilon \leq e^\varepsilon$.

Finally, we show the time continuity (iv):

$$\begin{aligned} \|\hat{\mu}^{n+1} - \hat{\mu}^n\|_{L^1(\mathbb{K})} &= \sum_i |\mu_i^{n+1} - \mu_i^n| \Delta x \\ &= \frac{\Delta x}{2} \sum_i \left| (\mu_{i+1}^n - \mu_i^n) \left(1 - \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_{i+1} \right) \right. \\ &\quad \left. - (\mu_i^n - \mu_{i-1}^n) \left(1 + \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_{i-1} \right) \right. \\ &\quad \left. - \frac{\Delta t}{\Delta x} \mu_i^n (V[\hat{\mu}^n]_{i+1} - V[\hat{\mu}^n]_{i-1}) \right| \\ &\leq \frac{\Delta x}{2} \sum_i |\mu_{i+1}^n - \mu_i^n| \left(1 - \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_{i+1} \right) \\ &\quad + |\mu_i^n - \mu_{i-1}^n| \left(1 + \frac{\Delta t}{\Delta x} V[\hat{\mu}^n]_{i-1} \right) \\ &\quad + \frac{\Delta t}{\Delta x} \mu_i^n |V[\hat{\mu}^n]_{i+1} - V[\hat{\mu}^n]_{i-1}| \\ &\leq \Delta x \text{TV}(\hat{\mu}^n) + C_2 \Delta t. \end{aligned}$$

Iterating over n yields (iv). $\square$

Theorem 2.2.4. *Assume that V satisfies conditions (A1), (A2), (A3), (A4) and let $\mu_0 \in \mathcal{P}(\mathbb{K})$ be compactly supported and absolutely continuous with respect to Lebesgue measure with density $\mu_0 \in BV(\mathbb{K})$. Assume that the CFL condition (2.12) holds. Then the measure-valued solution μ of (2.1) is absolutely continuous, and the one-dimensional Lax–Friedrichs method (2.10) converges strongly to μ . More precisely, if $\hat{\mu}^{\Delta x}$ is given by (2.14) then*

$$\sup_{t \in [0, T]} \|\hat{\mu}_t^{\Delta x} - \mu_t\|_{L^1(\mathbb{K})} \rightarrow 0 \quad \text{as } \Delta x \rightarrow 0$$

for any $T > 0$.

Proof. By Proposition 2 (ii) and (iii), the set

$$K := \{\hat{\mu}_t^{\Delta x} : 0 < \Delta x < 1, t \in [0, T]\}$$

is uniformly bounded in $BV(\mathbb{K})$, so by Helly’s theorem, K is relatively compact in $L^1(\mathbb{K})$. Hence, by Ascoli’s theorem there is a subsequence $\Delta x_k \rightarrow 0$ as $k \rightarrow \infty$ and some Lipschitz map $\tilde{\mu} : [0, T] \rightarrow L^1(\mathbb{K})$ such that $\hat{\mu}_t^{\Delta x_k} \rightarrow \tilde{\mu}_t$ in $L^1(\mathbb{K})$ as $k \rightarrow \infty$ uniformly in $t \in [0, T]$. Since convergence in L^1 implies convergence in d_W , also $d_W(\hat{\mu}_t^{\Delta x_k}, \tilde{\mu}_t) \rightarrow 0$. On the other hand, Theorem 2.2.3 implies that $d_W(\mu_t^{\Delta x_k}, \mu_t) \rightarrow 0$, where μ is the measure-valued solution of (2.1). Noting that $d_W(\hat{\mu}_t^{\Delta x_k}, \mu_t^{\Delta x_k}) \leq \frac{\Delta x}{2}$, we can conclude $\mu = \tilde{\mu}$. The conclusion follows. $\square$

2.2.5 Multiple dimensions

The extension to multiple space dimensions is done in a tensorial fashion; we limit ourselves to the two-dimensional variant for the sake of notational simplicity. As before, let $\mathbb{K}_1, \mathbb{K}_2$ be either the torus $\mathbb{T}$ or the real line $\mathbb{R}$. Let $(x_{i+\frac{1}{2}})_i$ and $(y_{j+\frac{1}{2}})_j$ be discretizations of $\mathbb{K}_1$ and $\mathbb{K}_2$ with mesh lengths $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} \equiv \Delta x$ and $y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} \equiv \Delta y$, respectively. With obvious notation we get the unstaggered, two-dimensional Lax–Friedrichs method:

$$\begin{aligned} \mu_{i,j}^0 &= \langle \mu_0, \varphi_{i,j} \rangle \\ \mu_{i,j}^{n+1} &= \frac{\mu_{i-1,j}^n + \mu_{i+1,j}^n + \mu_{i,j-1}^n + \mu_{i,j+1}^n}{4} - \Delta t \frac{V^1[\mu^n]_{i+1,j} \mu_{i+1,j}^n - V^1[\mu^n]_{i-1,j} \mu_{i-1,j}^n}{2\Delta x} \\ &\quad - \Delta t \frac{V^2[\mu^n]_{i,j+1} \mu_{i,j+1}^n - V^2[\mu^n]_{i,j-1} \mu_{i,j-1}^n}{2\Delta y} \end{aligned} \quad (2.15)$$

where we denote $\mu^n = \sum_i \sum_j \mu_{i,j}^n \delta_{(x_i, y_j)}$. As before, μ is interpolated linearly between time steps,

$$\mu_t^{\Delta x, \Delta y} := \frac{t^{n+1} - t}{\Delta t} \mu^n + \frac{t - t^n}{\Delta t} \mu^{n+1} \quad \text{for } t \in [t^n, t^{n+1}). \quad (2.16)$$

We omit the proofs of the following stability and convergence results, as they are straightforward generalizations of their one-dimensional counterparts.

Proposition 3. *For the unstaggered, two-dimensional Lax–Friedrichs method (2.15) assume that V satisfies condition (A1), and that the CFL condition*

$$\lambda_0 \leq \frac{\Delta t}{\min(\Delta x, \Delta y)} C_1 \leq \frac{1}{2} \quad (2.17)$$

is satisfied for some $\lambda_0 > 0$. Then for all $t \geq 0$

- (i) $\mu_t^{\Delta x, \Delta y} \geq 0$
- (ii) $\|\mu_t^{\Delta x, \Delta y}\|_{\mathcal{M}(\mathcal{K})} = 1$
- (iii) $d_W(\mu_t^{\Delta x, \Delta y}, \mu_s^{\Delta x, \Delta y}) \leq \frac{\Delta x + \Delta y}{2\Delta t} |t - s|$ for all $s \geq 0$.

Theorem 2.2.5. *Consider the two-dimensional equation (2.1). Assume that V satisfies conditions (A1), (A2), (A3) and let $\mu_0 \in \mathcal{P}(\mathbb{K}_1 \times \mathbb{K}_2)$ have compact support. Assume that the CFL condition (2.17) is satisfied. Then the Lax–Friedrichs method (2.15) converges weakly to the measure-valued solution of (2.1). More precisely, if $\mu^{\Delta x, \Delta y}$ is given by (2.16) then*

$$\sup_{t \in [0, T]} d_W(\mu_t^{\Delta x, \Delta y}, \mu_t) \rightarrow 0 \quad \text{as } \Delta x, \Delta y \rightarrow 0$$

for any $T > 0$.

As in the 1-dimensional case, sufficient smoothness for the initial data and the velocity field will yield *strong* convergence of the numerical method. Assume the initial datum is absolutely continuous with respect to the Lebesgue measure with density function $\mu_0(x, y)$. This initial data is sampled by its cell averages:

$$\hat{\mu}_{i,j}^0 := \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mu_0(x, y) dx dy$$

Similarly we define the numerical solution as piecewise linear L^1 function as

$$\hat{\mu}_t^{\Delta x, \Delta y} = \frac{t^{n+1} - t}{\Delta t} \hat{\mu}^n + \frac{t - t^n}{\Delta t} \hat{\mu}^{n+1} \quad \text{for } t \in [t^n, t^{n+1}) \quad (2.18)$$

where

$$\hat{\mu}^n(x, y) = \sum_{i,j} \hat{\mu}_{i,j}^n \mathbb{1}_{[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})}(x, y).$$

Proposition 4. *For the unstaggered, one-dimensional Lax–Friedrichs method (2.10). Assume that V satisfies the conditions (A1), (A2), (A4) and that the CFL condition (2.17) is satisfied. Then for all $t \in [0, T]$*

- (i) $\hat{\mu}_t^{\Delta x, \Delta y} \geq 0$
- (ii) $\|\hat{\mu}_t^{\Delta x, \Delta y}\|_{L^1(\mathbb{K}_1 \times \mathbb{K}_2)} = 1$
- (iii) $TV(\hat{\mu}_t^{\Delta x, \Delta y}) \leq c_T := TV(\hat{\mu}_0) e^{C_2 T} + (e^{C_2 T} - 1) \frac{C_4}{2C_2}$
- (iv) $\|\hat{\mu}_t^{\Delta x, \Delta y} - \hat{\mu}_s^{\Delta x, \Delta y}\|_{L^1(\mathbb{K}_1 \times \mathbb{K}_2)} \leq C_T |t - s|$ for all $s \geq 0$.

(In (iv), $C_T = C_2 + \lambda_0^{-1} c_T$, where c_T is the upper bound from (iii).)

So now we can finally state:

Theorem 2.2.6. *Assume that V satisfies conditions (A1), (A2), (A3), (A4) and let $\mu_0 \in \mathcal{P}(\mathbb{K}_1 \times \mathbb{K}_2)$ be compactly supported and absolutely continuous with respect to Lebesgue measure with density $\mu_0 \in BV(\mathbb{K}_1 \times \mathbb{K}_2)$. Assume that the CFL condition (2.17) holds. Then the measure-valued solution μ of (2.1) takes values in $L^1(\mathbb{K}_1 \times \mathbb{K}_2)$, and the two-dimensional Lax–Friedrichs method (2.15) converges strongly to μ . More precisely, if $\hat{\mu}^{\Delta x, \Delta y}$ is given by (2.18) then*

$$\sup_{t \in [0, T]} \|\hat{\mu}_t^{\Delta x, \Delta y} - \mu_t\|_{L^1(\mathbb{K}_1 \times \mathbb{K}_2)} \rightarrow 0 \quad \text{as } \Delta x, \Delta y \rightarrow 0$$

for any $T > 0$.

2.3 Numerical experiments

In this section we illustrate our analytical results by performing several numerical experiments for the Kuramoto equation with identical (2.3) and non-identical (2.2) natural frequencies. Recall that in these equations, θ takes values in the periodic domain $\mathbb{T}$, while Ω lies in $\mathbb{R}$.

2.3.1 One-dimensional simulations

We consider first the one-dimensional Kuramoto equation with identical oscillators (2.3). In all experiments we have used the unstaggered Lax–Friedrichs method (2.10) with CFL number 0.4 and with $T = 0.5$ as the final time. All simulations were computed on a sequence of meshes with $N = 32, 64, 128, 256, 512$ grid points, as well as a reference solution with $N = 4096$ points. The tables in each subsection shows the number of grid points as well as the error and the experimental order of convergence (EOC)

$$\text{EOC}_{\Delta x, \Delta x'} := \frac{\log(E_{\Delta x}/E_{\Delta x'})}{\log(\Delta x/\Delta x')}, \quad E_{\Delta x} = d(\mu_T^{\Delta x}, \mu_T)$$

computed both in the two metrics $d = d_W$ and $d(u, v) = \|u - v\|_{L^1(\mathcal{K})}$.

The experiments in this section solve the Kuramoto equation with progressively more singular initial data. As will be seen from the tables, the EOC in the 1-Wasserstein distance is close to 1 in all cases, while in L^1 it is close to 1 only for smooth data.

Polynomial initial data

The initial data is taken to be the continuous and piecewise parabolic function

$$\mu_0(\theta) = \begin{cases} \frac{6}{\pi^3}(\frac{3\pi}{2} - \theta)(\theta - \frac{\pi}{2}) & \text{if } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

As shown in Figure 2.1, the numerical solution is a non-oscillatory and reasonable approximation at all mesh resolutions. Table 2.1 shows that the numerical method seems to converge at a rate close to 1, both in the 1-Wasserstein and the L^1 distances.

Figure 2.1: Comparing solutions on different meshes at $t = 0.5$ for smooth initial data.

N	1-Wasserstein		L^1	
	Error	EOC	Error	EOC
32	0.1351		0.2811	
64	0.0634	1.09	0.1336	1.07
128	0.0303	1.07	0.0663	1.01
256	0.0153	0.99	0.0336	0.98
512	0.0073	1.07	0.0157	1.10

Table 2.1: Errors and EOC in the 1-Wasserstein and L^1 distances at $t = 0.5$ for smooth initial data.

Piecewise constant initial data

This experiment was taken from [2] and uses piecewise constant initial data,

$$\mu_0(\theta) = \begin{cases} \frac{2}{3\pi} & \text{if } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}) \\ \frac{1}{3\pi} & \text{otherwise.} \end{cases}$$

Figure 2.2 shows again that the approximation is reasonably accurate even on coarse meshes. Table 2.2 shows that the rate of convergence in 1-Wasserstein is again 1, while it is around 0.7 in L^1 .

Figure 2.2: Comparing solutions on different meshes for piecewise constant initial data.

N	1-Wasserstein		L^1	
	Error	EOC	Error	EOC
32	0.0517		0.0610	
64	0.0258	1.00	0.0315	0.95
128	0.0131	0.98	0.0196	0.69
256	0.0067	0.97	0.0123	0.67
512	0.0034	0.98	0.0071	0.79

Table 2.2: Errors and EOC in the 1-Wasserstein and L^1 distances at $t = 0.5$ for piecewise constant initial data.

Figure 2.3: Comparing solutions on different meshes at $t = 0.5$ for initial data with Dirac masses.

Singular initial data

The final experiment uses the singular initial data

$$\mu_0 = \frac{1}{4}(\delta_{\frac{3\pi}{4}} + \delta_{\frac{5\pi}{4}}) + \frac{1}{2}\chi_{[\frac{\pi}{2}, \frac{3\pi}{2}]}$$

where $\chi_{[a,b]}$ is the indicator function of the interval $[a,b]$. As shown in Figure 2.3, the numerical approximation is nonoscillatory and seems to converge, even in the presence of Dirac singularities. This is confirmed in Table 2.3: The method seems to converge at a rate of around $3/4$ in d_W , while it does not converge at all in L^1 , as expected.

N	1-Wasserstein		L^1	
	Error	EOC	Error	EOC
32	0.1737		0.8703	
64	0.1119	0.63	0.7945	0.13
128	0.0687	0.70	0.7131	0.16
256	0.0433	0.67	0.6153	0.21
512	0.0260	0.74	0.4861	0.34

Table 2.3: Errors and EOC in the 1-Wasserstein and L^1 distances at $t = 0.5$ for singular initial data.

2.3.2 Polynomial initial data in 2-D

The final numerical experiment approximates the two-dimensional Kuramoto equation (2.2). We consider the piecewise linear initial data

$$\mu_0(\theta, \Omega) = \begin{cases} \frac{64}{3\pi^2}\theta\Omega & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}) \text{ and } \Omega \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Figure 2.4 shows the solution at $t = 0.5$ and Table 2.4 shows the L^1 errors. (Due to the complexities of computing the Wasserstein distance for multi-dimensional measures, we only compute the L^1 errors in this experiment.) The convergence rate in L^1 is seen to be about the same as in the piecewise constant one-dimensional example in Section 2.3.1.

Figure 2.4: Comparing solutions on different meshes for two-dimensional polynomial initial data.

N	L^1	
	Error	EOC
32	1.2912	
64	1.0054	0.36
128	0.7346	0.45
256	0.4925	0.58
512	0.2999	0.72
1024	0.1638	0.87

Table 2.4: Table of L^1 -errors for two-dimensional initial data.

2.4 Appendix

Proof of Lemma 2.2.2. We claim that K is tight. Define

$$\chi(x) := \begin{cases} 1 & |x| \leq 1 \\ 2 - |x| & 1 < |x| < 2 \\ 0 & 2 \leq |x|, \end{cases} \quad \chi_M(x) := \chi(x/M).$$

For an $\varepsilon > 0$, let $M > 0$ be such that $\bar{\mu}(\mathbb{R} \setminus B_M(0)) \leq \varepsilon$ and let $C := \sup_{\mu \in K} d_W(\mu, \bar{\mu})$. If $\mu \in K$ then

$$\begin{aligned} \mu(\mathbb{R} \setminus B_{2M}(0)) &\leq \langle \mu, 1 - \chi_M \rangle = \langle \bar{\mu}, 1 - \chi_M \rangle + \langle \mu - \bar{\mu}, 1 - \chi_M \rangle \\ &\leq \bar{\mu}(\mathbb{R} \setminus B_M(0)) + \|1 - \chi_M\|_{\text{Lip}} d_W(\mu, \bar{\mu}) \\ &\leq \varepsilon + \frac{C}{M}, \end{aligned}$$

which can be made arbitrarily small, independently of μ . Hence, K is tight, so by Prokhorov's theorem, K is relatively compact in the topology of narrow (or “weak”) convergence. But narrow convergence is equivalent to d_W -convergence. This completes the proof. $\square$

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CHAPTER 3

WELLPOSEDNESS OF THE INITIAL VALUE PROBLEM FOR THE OSTROVSKY–HUNTER EQUATION WITH SPATIALLY DEPENDENT FLUX

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Abstract

In this paper we study the Ostrovsky–Hunter equation for the case where the flux function $f(x, u)$ may depend on the spatial variable with certain smoothness. Our main results are that if the flux function is smooth enough, then there exists a unique entropy solution. Our proof relies on proving some *a-priori* estimates, followed by methods of *compensated compactness* and *doubling of variables*.

3.1 Introduction

To model numerous physical phenomena such as the propagation of undular bores in shallow water, the flow of liquids containing gas bubbles, the propagation of waves in

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an elastic tube filled with a viscous fluid, weakly nonlinear plasma waves with certain dissipative effects *etc.* the following nonlinear evolution equation, known as *Korteweg-de Vries-Burgers equation*

$$u_t + \left(f(u)\right)_x - \alpha u_{xx} - \beta u_{xxx} = 0, \quad \alpha, \beta \in \mathbb{R}, \quad f(u) = \frac{u^2}{2}, \quad (3.1)$$

has been extensively investigated in the recent years (see [45, 66, 88] and references therein). Also considering the effects of background rotation through the Coriolis force (κ being the force parameter and C_0 is the linear shallow water speed) (3.1) takes the following form

$$\left(u_t + \left(f(u)\right)_x - \alpha u_{xx} - \beta u_{xxx}\right)_x = \gamma u, \quad \gamma = \frac{\kappa^2}{2C_0} > 0. \quad (3.2)$$

To model small-amplitude long waves in a rotating fluid of finite depth [82] and to study long internal waves in a rotating fluid [58] both the viscous dissipation term and the high-frequency dispersion term has to be dropped, *i.e.* $\alpha, \beta = 0$; which leads to

$$\left(u_t + \left(f(u)\right)_x\right)_x = \gamma u, \quad (3.3)$$

which is known as the *Ostrovsky–Hunter equation*, as Ostrovsky also independently derived them [82]. This equation is also used to model high frequency waves in a relaxing medium [97]. In the cases described above the flux is considered to be of Burger’s type, *i.e.* $f(u) = \frac{u^2}{2}$.

Also by including the effects of background rotation in the shallow water equation, and then using singular perturbation methods (3.3) has been derived previously (see [36], [59]). In the recent years enormous amount of research has been carried out investigating (3.3). Among those works in [82], [85], [89] the equation (3.3) is also known as the reduced Ostrovsky equation, in [58] it is called short wave equation, whereas in [8], [11] (3.3) is known as Ostrovsky–Vakhnenko equation and as Vakhnenko equation in [98]. Moreover, the equation (3.2) is used to model ultra short light pulses in silica optical fibres (see [3], [75]), in which case $f(u) = -\frac{1}{6}u^3$. In this case equation (3.2) is sometimes referred to as the *short-pulse-equation*.

In his seminal paper [58], Hunter showed the connection between the KdV equation (3.2) and the Ostrovsky–Hunter equation (3.3) as the no-rotation and no-long wave dispersion limits of the same equation. When the oceanic waves approach shore, the waves usually propagate through a background with varying properties. It is natural to expect the linear phase speed of the wave which encoded in the flux function, in such a variable medium, should have a *spatial dependency*. In the context of KdV equation, Johnson [62] for water waves and Grimshaw [49] for internal waves derived the variable coefficient equation (see also [51] for a detailed review). Motivated by this, it is immediate to pose the question of design and analysis of numerical scheme for the Ostrovsky–Hunter equation with a spatial dependency in the flux function. In [16], we investigated the *spatially dependent* Ostrovsky–Hunter equation in the fully-discretized setting to prove convergence of the corresponding numerical method to the unique entropy solution and we proved its *order of convergence*. Whereas in this paper, our aim is to establish well-posedness of the problem in continuous set up.

The results obtained in this paper are the following. If the function $f_x(x, u)$ is uniformly Lipschitz continuous locally in u , the function $f_u(x, u)$ to be uniformly locally bounded, and the initial data are square integrable and satisfy zero-mean condition, then there exists an entropy solution via method of compensated compactness. Furthermore, for two entropy solutions u and v , with initial data u_0 and v_0 respectively, we establish the following estimate

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1((0, R))} \leq e^{Ct} \|u_0(\cdot) - v_0(\cdot)\|_{L^1((0, R+Lt))},$$

for some constants C, L and R to be specified later. The rest of this paper is organized as follows. In Section 3.2 we give detailed descriptions of the notations used, the precise assumptions of the regularity of the flux function and the initial data. Also, apart from stating our main result as a theorem, we state the definition of entropy solution to be used. In Section 3.3 we prove few useful *a priori* estimates (namely energy estimate and L_{loc}^∞ bound) for the purpose of compensated compactness technique. In Section 3.4 we first state the two results due to Murat and Tartar in the form of two lemmas, using which we will employ a compensated compactness argument to show the existence of entropy solution of the equation under consideration. Moreover, we establish an L^1 contraction type estimate mentioned above using the technique of doubling of variables.

3.2 Preliminaries and Notation

Throughout this paper $u(x, t)$ is the conserved quantity and f is the flux which is dependent on the spatial variable x and $u(x, t)$, denoted by $f(x, u(x, t))$. For notational consistency, we mention the following chain rule keeping the notation of $f_x(x, u) \neq f(x, u)_x$

$$\begin{cases} \frac{\partial f}{\partial u}(x, u) = f_u(x, u), \\ \partial_x f(x, u) = f(x, u)_x = f_u(x, u) \frac{\partial u}{\partial x} + f_x(x, u), \\ \frac{\partial u}{\partial t}(x, t) = u_t(x, t). \end{cases}$$

We are interested in the initial boundary value problem for (3.3), but with *spatially dependent* flux, and hence we augment the equation with the initial datum

$$u(x, 0) = u_0(x), \text{ for } x > 0. \quad (3.4)$$

Keeping that in mind, following the works of [25], [22] and [99] integrating the equation (3.3) on the interval $(0, x)$ we get the integro-differential formulation of the problem under consideration and setting $\gamma = 1$,

$$\begin{cases} u_t + f(x, u)_x = \int_0^x u(y, t) dy, & t > 0, x > 0, \\ u(x, 0) = u_0(x), & x > 0, \\ u(0, t) = 0, & t > 0. \end{cases} \quad (3.5)$$

Denoting $P[t, x] := \int_0^x u(y, t) dy$, we get the following equivalent formulation

$$\begin{cases} u_t + f(x, u)_x = P[t, x], & t > 0, x > 0, \\ P_x = u(x, t), & t > 0, x > 0, \\ P(t, 0) = u(0, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x > 0. \end{cases} \quad (3.6)$$

3.2. PRELIMINARIES AND NOTATION

For the initial datum, we assume the following *zero-mean condition* and regularity assumption respectively

$$\int_0^\infty u_0(x) dx = 0, \quad (3.7)$$

and

$$u_0(x) \in L^2(\mathbb{R}_+) \cap L_{\text{loc}}^\infty(\mathbb{R}_+), \quad (3.8)$$

where $\mathbb{R}_+ := (0, \infty)$ will be denoting the unbounded positive half line throughout the paper. Similarly Π will be used to denote $(0, \infty)^2$. Also the flux f is assumed to satisfy the following hypothesis:

- (A1) $f(x, \cdot)$ is *genuinely nonlinear*, i.e. $f_{uu}(x, u) \neq 0$ for a.e. $(x, u) \in \mathbb{R}_+ \times \mathbb{R}$ and $\lim_{x \rightarrow \infty} \partial_x f(x, u) = \lim_{x \rightarrow \infty} f(x, u) = \lim_{x \rightarrow 0} f(x, u) = \lim_{x \rightarrow 0} \partial_x f(x, u) = 0$ for all u ,
- (A2) $\exists$ a constant $C > 0$ such that $|f_{xu}(x, u)| \leq C$ and $|f_x(x, u)| \leq C|u|$ for all u ,
- (A3) $\exists$ a constant $L_1 > 0$ such that $|f_x(x, u) - f_x(x, v)| \leq L_1|u - v|$, for all u, v ,
- (A4) $\exists$ a constant $L > 0$ such that $|f_u(x, u)| \leq L$, for all u .

Even if the initial data is smooth enough, solutions of (3.6) generically develop discontinuities. Hence solutions must be considered in the weak sense. A function u is a *weak solution* of (3.6) if

$$\iint_{\Pi} u \varphi_t + f(x, u) \varphi_x + P[t, x] \varphi \, dx dt + \int_{\mathbb{R}_+} u_0(x) \varphi(x, 0) \, dx = 0, \quad (3.9)$$

for all test functions $\varphi = \varphi(x, t) \in C_c^\infty(\Pi)$. Moreover, from (3.6) we have that

$$u \in L_{\text{loc}}^\infty(\Pi) \Rightarrow P \in L_{\text{loc}}^\infty(\mathbb{R}_+; W_{\text{loc}}^{1, \infty} \mathbb{R}_+). \quad (3.10)$$

Following [5] we define entropy solutions as

Definition 3.2.1 (Entropy Solution). *We say that $u \in L_{\text{loc}}^\infty(\Pi)$ is an entropy solution of the initial boundary value problem (3.6), if*

- u satisfies (3.9) ;
- for every smooth, non negative test function $\varphi \in C_c^2(\Pi)$ and $c \in \mathbb{R}$

$$\begin{aligned} & \iint_{\Pi} \left(|u - c| \partial_t \varphi + \text{sign}(u - c) \left(f(x, u) - f(x, c) \right) \partial_x \varphi - \text{sign}(u - c) f_x(x, c) \varphi \right) dt dx \\ & + \iint_{\Pi} \text{sign}(u - c) P \varphi \, dt dx - \int_{\mathbb{R}_+} \text{sign}(c) \left(f(0, u_0^\tau) - f(0, c) \right) \varphi(0, t) \, dt \\ & + \int_{\mathbb{R}_+} |u_0(x) - c| \varphi(x, 0) \, dx \geq 0. \end{aligned} \quad (3.11)$$

As an immediate consequence of (3.10) if a map $u \in L_{\text{loc}}^\infty(\Pi)$ satisfies the following equivalent entropy inequality for every convex entropy/entropy flux pair (i.e. for $\eta \in C^2(\mathbb{R})$, $q(x, u) := \int_0^u \eta'(v) f_v(x, v) dv$)

$$\partial_t \eta(u) + \partial_x q(x, u) + \eta'(u) f_x(x, u) - q_x(x, u) - \eta'(u) P \leq 0, \quad (3.12)$$

in the sense of distributions, then by Theorem 1.1 of [28] on the boundary $x = 0$ strong trace u_0^τ exists. By a standard approximation argument equivalently any *convex entropy/entropy flux pair* (η, q) in (3.12) can be replaced by Kruřkov entropy pair namely for $c \in \mathbb{R}$, $\eta(u) = |u - c|$ and $q(x, u) = \int_0^u \text{sign}(u - c) f_v(x, v) dv$. The main result of this paper is the following theorem.

Theorem 3.2.1. *Assuming (3.7) and (3.8), the Cauchy problem (3.5), or equivalently (3.6) possesses a unique entropy solution u in the sense of Definition 3.2.1. Moreover, if u and v are two entropy solutions of (3.5), or equivalently (3.6) in the sense of Definition 3.2.1, the following estimate holds for a given $0 < t \leq T$*

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1((0, R))} \leq e^{Ct} \|u_0(\cdot) - v_0(\cdot)\|_{L^1((0, R+Lt))} \quad (3.13)$$

for almost every $T > t > 0$, $R > 0$ and $L > 0$ being the bound $|f_u(x, u)| \leq L$, where the constant C depends on T , R , and L .

Before proceeding to prove this theorem, it is worth mentioning that Coclite *et al.* [25], [22] have showed the well-posedness of the initial-boundary value problem and the Cauchy problem for the Ostrovsky–Hunter Equation (3.5), but without any spatial dependency in the flux. Throughout the next section we will extend their results following the papers cited just above.

3.3 A-Priori Estimates

The existence argument is based on passing to the limit in the following vanishing viscosity approximation of (3.6) (see [26]). Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(x, t)$ be the unique classical solution of the following problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(x, u_\varepsilon) = P_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0, x > 0 \\ \partial_x P_\varepsilon = u_\varepsilon, & t > 0, x > 0 \\ P_\varepsilon(t, 0) = u_\varepsilon(0, t) = 0, & t > 0 \\ u_\varepsilon(x, 0) = u_{\varepsilon,0}(x), & x > 0, \end{cases} \quad (3.14)$$

where $u_{\varepsilon,0}$ is a $C^\infty(\mathbb{R}_+)$ approximation of u_0 such that

$$\|u_{\varepsilon,0}\|_{L^2(\mathbb{R}_+)} \leq \|u_0\|_{L^2(\mathbb{R}_+)}, \quad \int_{\mathbb{R}_+} u_{\varepsilon,0}(x) dx = 0, \quad (3.15)$$

and on the viscous source term for $x > 0$, $P_{\varepsilon,0}(x) := \int_0^x u_{\varepsilon,0}(y) dy$ we assume that

$$\begin{cases} \|P_{\varepsilon,0}\|_{L^2(\mathbb{R}_+)}^2 = \int_{\mathbb{R}_+} \left(\int_0^x u_{\varepsilon,0}(y) dy \right)^2 dx < \infty, \\ \int_{\mathbb{R}_+} P_{\varepsilon,0}(x) dx = \int_{\mathbb{R}_+} \left(\int_0^x u_{\varepsilon,0}(y) dy \right) dx = 0. \end{cases} \quad (3.16)$$

Clearly, (3.14) is equivalent to the integro-differential problem

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(x, u_\varepsilon) = \int_0^x u_\varepsilon(y, t) dy + \varepsilon \partial_{xx}^2 u_\varepsilon & t > 0, x > 0, \\ u_\varepsilon(x, 0) = u_{\varepsilon,0}(x) & x > 0. \end{cases} \quad (3.17)$$

The existence of such solutions can be obtained by fixing a small number $1 > \delta > 0$ and considering the further approximation of (3.17) (see for the whole real line [22, 25]; for the half line [23, 24, 26] and the references therein). We are going to use the following result from previous works of Coclite *et al.* (see [22, 27] and references therein).

Theorem 3.3.1. *Let $T > 0$. Assuming that conditions (3.15) and (3.16) hold, there exists a unique classical solution to the Cauchy problem of (3.17) such that*

$$\begin{cases} u_\varepsilon \in L_{loc}^\infty((0, T) \times \mathbb{R}_+) \cap C((0, T); H^l(\mathbb{R}_+)), \text{ for all } l \in \mathbb{N}, \\ P_\varepsilon \in L_{loc}^\infty((0, T) \times \mathbb{R}_+) \cap L^2((0, T) \times \mathbb{R}_+), \\ \int_0^\infty u_\varepsilon(x, t) dx = 0, \quad t \geq 0. \end{cases} \quad (3.18)$$

Now let us prove some *a priori* estimates on u_ε .

Lemma 3.3.2. *We have the equivalence of following two equalities*

$$\int_{\mathbb{R}_+} u_\varepsilon(x, t) dx = 0; \quad t \geq 0, \quad (3.19)$$

$$\begin{aligned} & \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \\ &= \|u_{\varepsilon,0}(\cdot)\|_{L^2(\mathbb{R}_+)}^2 + 2 \int_0^t \left[\int_{\mathbb{R}_+} \left[\int_0^{u_\varepsilon} v f_{xv}(x, v) dv - u_\varepsilon f_x(x, u_\varepsilon) \right] dx \right] ds; \quad t > 0. \end{aligned} \quad (3.20)$$

Proof. Let $t > 0$. First we will prove that (3.19) implies (3.20). Multiplying equation (3.17) by $u_\varepsilon(x, t)$ we get

$$u_\varepsilon \partial_t u_\varepsilon + u_\varepsilon f_x(x, u_\varepsilon) (\partial_x u_\varepsilon) + u_\varepsilon f_x(x, u_\varepsilon) = u_\varepsilon \int_0^x u_\varepsilon(y, t) dy + \varepsilon u_\varepsilon \partial_{xx}^2 u_\varepsilon. \quad (3.21)$$

In this equality, consider the term $u_\varepsilon \int_0^x u_\varepsilon(y, t) dy$. We are going to show that after integration this term vanishes. Clearly the equation (3.19) implies that

$$\int_{\mathbb{R}_+} u_\varepsilon(x, t) \left[\int_0^x u_\varepsilon(y, t) dy \right] dx = \int_{\mathbb{R}_+} P_\varepsilon (\partial_x P_\varepsilon) dx = \int_{\mathbb{R}_+} \partial_x \left[\frac{1}{2} P_\varepsilon^2 \right] dx = 0. \quad (3.22)$$

Using $\eta(u) := \frac{1}{2}u^2$ into (3.21) we get

$$[u_\varepsilon^2(x, t)]_t + 2[q(x, u_\varepsilon)]_x - 2\varepsilon u_\varepsilon \partial_{xx}^2 u_\varepsilon = 2 \left[\int_0^{u_\varepsilon} v f_{xv}(x, v) dv - u_\varepsilon f_x(x, u_\varepsilon) \right] + \partial_x \left[\frac{1}{2} P_\varepsilon^2 \right].$$

Integrating this expression over $\mathbb{R}_+$ and invoking (3.22) we get

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+)}^2 = 2 \int_{\mathbb{R}_+} \left[\int_0^{u_\varepsilon} v f_{xv}(x, v) dv - u_\varepsilon f_x(x, u_\varepsilon) \right] dx. \quad (3.23)$$

And finally integrating over the $(0, t)$ we obtain

$$\begin{aligned} & \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \\ &= \|u_{\varepsilon,0}(\cdot)\|_{L^2(\mathbb{R}_+)}^2 + 2 \int_0^t \left[\int_{\mathbb{R}_+} \left[\int_0^{u_\varepsilon} v f_{xv}(x, v) dv - u_\varepsilon f_x(x, u_\varepsilon) \right] dx \right] ds, \end{aligned}$$

which proves (3.20). Now we are going to prove the opposite implication. Assume that $\int_{\mathbb{R}_+} u_\varepsilon(x, t) dx \neq 0$ for some $t > 0$, which implies

$$P_\varepsilon^2(t, +\infty) = \left(\int_{\mathbb{R}_+} u_\varepsilon(x, t) dx \right)^2 \neq 0,$$

which results in

$$\frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+)}^2 \neq 2 \int_{\mathbb{R}_+} \left[\int_0^{u_\varepsilon} v f_{xv}(x, v) dv - u_\varepsilon f_x(x, u_\varepsilon) \right] dx,$$

ultimately contradicting our assumption (3.20). This concludes the proof. $\square$

Lemma 3.3.3. *For each $t \geq 0$, (3.19) holds. In particular we have that for a constant $C > 0$ coming from (A2), independent of ε*

$$\begin{aligned} & \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \leq \|u_0\|_{L^2(\mathbb{R}_+)}^2 \\ & \quad + \hat{C} \int_0^t \|u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \end{aligned} \quad (3.24)$$

where $\hat{C}$ is any constant greater than C .

Proof. From the equation (3.17) we have

$$\partial_x (\partial_t u_\varepsilon + \partial_x f(x, u_\varepsilon) - \varepsilon \partial_{xx}^2 u_\varepsilon) = u_\varepsilon.$$

Integrating both sides with respect to x we get

$$\partial_t u_\varepsilon + \partial_x f(x, u_\varepsilon) - \varepsilon \partial_{xx}^2 u_\varepsilon \Big|_0^\infty = \int_{\mathbb{R}_+} u_\varepsilon dx.$$

Observe that from $u_\varepsilon(0, t) = 0$ of (3.14) we have $\partial_t u_\varepsilon(0, t) = 0$ which, due to (A1)

$$\varepsilon \partial_{xx}^2 u_\varepsilon(0, t) = \partial_t u_\varepsilon(0, t) + \partial_x f(x, u_\varepsilon) \Big|_{x=0} - \int_0^0 u_\varepsilon(y, t) dy = 0. \quad (3.25)$$

Invoking the property (A1), (3.25) and the smoothness of $u_\varepsilon(x, t)$ from Theorem 3.3.1 we can conclude $\int_{\mathbb{R}} u_\varepsilon(x, t) dx = 0$, which proves (3.19). So by Lemma 3.3.3 the relation (3.20) holds. To estimate the last term of the relation (3.20) due to our assumption (A2) for any constant $\hat{C} \geq C$ we get

$$\left| \int_0^t \left[\int_{\mathbb{R}_+} \left[\int_0^{u_\varepsilon} v f_{xv}(x, v) dv - f_x u_\varepsilon(s, x) \right] dx \right] ds \right| \leq \hat{C} \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R}_+)}^2 ds. \quad (3.26)$$

Consequently in (3.20) inserting (3.15) and (3.26) we have:

$$\begin{aligned} & \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \\ & \leq \|u_{\varepsilon,0}\|_{L^2(\mathbb{R}_+)}^2 + \hat{C} \int_0^t \|u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \\ & \leq \|u_0\|_{L^2(\mathbb{R}_+)}^2 + \hat{C} \int_0^t \|u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds, \end{aligned}$$

which concludes the proof. $\square$

Remark. It follows from (3.24) that

$$\begin{aligned} \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)}^2 & \leq \|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds \\ & \leq \|u_0\|_{L^2(\mathbb{R}_+)}^2 + \hat{C} \int_0^t \|u_\varepsilon(\cdot, s)\|_{L^2(\mathbb{R}_+)}^2 ds. \end{aligned}$$

Thus by an application of Gronwall's inequality, we have

$$\|u_\varepsilon(\cdot, t)\|_{L^2(\mathbb{R}_+)} \leq e^{\hat{C}t} \|u_0\|_{L^2(\mathbb{R}_+)}. \quad (3.27)$$

Lemma 3.3.4. *The family*

$$\{u_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in} \quad L_{loc}^\infty(\Pi). \quad (3.28)$$

And consequently the family

$$\{P_\varepsilon\}_{\varepsilon>0} \quad \text{is bounded in} \quad L_{loc}^\infty(\Pi). \quad (3.29)$$

Proof. By Hölder inequality we have the following estimate

$$\begin{aligned}
 \partial_t u_\varepsilon + \partial_x f(x, u_\varepsilon) - \varepsilon \partial_{xx}^2 u_\varepsilon &= \int_0^x u_\varepsilon(t, y) dy \leq \left| \int_0^x u_\varepsilon(t, y) dy \right| \\
 &\leq \int_0^x |u_\varepsilon(t, y)| dy, \quad \text{by Hölder's inequality,} \\
 &\leq \sqrt{x} \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}_+)}, \quad \text{using (3.27)} \\
 &\leq \sqrt{x} e^{\hat{C}t} \|u_0\|_{L^2(\mathbb{R}_+)}.
 \end{aligned}$$

Now assume v_ε and w_ε be the solutions of the following equations respectively

$$\begin{cases} \partial_t v_\varepsilon + \partial_x f(x, v_\varepsilon) = \|u_0\|_{L^2(\mathbb{R}_+)} \sqrt{x} + \varepsilon \partial_{xx}^2 v_\varepsilon, & t > 0, x > 0, \\ v_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x > 0, \end{cases} \quad (3.30)$$

$$\begin{cases} \partial_t w_\varepsilon + \partial_x f(x, w_\varepsilon) = -\|u_0\|_{L^2(\mathbb{R}_+)} \sqrt{x} + \varepsilon \partial_{xx}^2 w_\varepsilon, & t > 0, x > 0, \\ w_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x > 0. \end{cases} \quad (3.31)$$

Then u_ε , v_ε , and w_ε are respectively a solution, a supersolution, and a subsolution of the parabolic problem (3.17). Following [44, Theorem 9, Chapter 2] we have that $w_\varepsilon \leq u_\varepsilon \leq v_\varepsilon$. Moreover from [2], $\{w_\varepsilon\}_{\varepsilon>0}$ and $\{v_\varepsilon\}_{\varepsilon>0}$ are uniformly bounded in $L_{\text{loc}}^\infty(\Pi)$. Define the following two functions:

$$W := \inf_{\varepsilon>0} w_\varepsilon \text{ and } V := \sup_{\varepsilon>0} v_\varepsilon.$$

Clearly therefore $W, V \in L_{\text{loc}}^\infty(\Pi)$ and they satisfy the inequality

$$W \leq w_\varepsilon \leq u_\varepsilon \leq v_\varepsilon \leq V.$$

This proves (3.28).

Now since $|P_\varepsilon(t, x)| = \left| \int_0^x u_\varepsilon(t, y) dy \right| \leq \int_0^x |u_\varepsilon(t, y)| dy$, (3.29) follows from (3.28). This completes the proof. $\square$

3.4 Proof of the Main Theorem

In this section we prove Theorem 3.2.1. Using the compensated compactness method, (see [93, 83]) we are going to construct a solution of (3.5) or equivalently of (3.6) by passing to the limit in sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of the viscosity approximations (3.14). The compensated compactness method due to Panov (see Theorem 5 of [83], or Lemma 2.2 of [29]) to be used here can be stated as the following lemma

Lemma 3.4.1. *Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of functions defined on Π . If $\{v_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L_{\text{loc}}^\infty(\Pi)$ and the family $\{\partial_t \eta(v_\varepsilon) + \partial_x q(x, v_\varepsilon)\}_{\varepsilon>0}$ is compact in $H_{\text{loc}}^{-1}(\Pi)$ for every convex $\eta \in C^2(\mathbb{R})$, where $q_u(x, u) = \eta'(u) f_u(x, u)$. Then there exist a sequence $\{\varepsilon_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_+$, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, and a map $v \in L_{\text{loc}}^\infty(\Pi)$ such that $v_{\varepsilon_k} \rightarrow v$ a.e. and in $L^p(\Pi)$ $1 \leq p < \infty$, as $k \rightarrow \infty$.*

3.4. PROOF OF THE MAIN THEOREM

The following compact embedding result of Murat [77] will be also used,

Lemma 3.4.2. *Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geq 2$. Suppose that the sequence $\{\mathcal{L}_\varepsilon\}_{\varepsilon \in \mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. In addition, suppose that $\mathcal{L}_\varepsilon = \mathcal{L}_{1,\varepsilon} + \mathcal{L}_{2,\varepsilon}$; where $\{\mathcal{L}_{1,\varepsilon}\}_{\varepsilon \in \mathbb{N}}$ lies in a compact subset of $H_{loc}^{-1}(\Omega)$ and $\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon \in \mathbb{N}}$ lies in a bounded subset of $L_{loc}^1(\Omega)$. Then $\{\mathcal{L}_\varepsilon\}_{\varepsilon \in \mathbb{N}}$ lies in a compact subset of $H_{loc}^{-1}(\Omega)$.*

First we are going to extract a limit function u from the collection u_ε and then we are going to show that this u satisfies (3.11).

Lemma 3.4.3. *The family $\{u_\varepsilon\}_{\varepsilon > 0}$ has a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ and a limit function $u \in L_{loc}^\infty(\Pi)$ such that*

$$u_{\varepsilon_k} \rightarrow u \quad \text{a.e. and in } L_{loc}^p(\Pi), 1 \leq p < \infty. \quad (3.32)$$

Moreover, we have

$$P_{\varepsilon_k} \rightarrow P \quad \text{a.e. and in } L_{loc}^p(\mathbb{R}_+; W_{loc}^{1,p}(\mathbb{R}_+)), 1 \leq p < \infty, \quad (3.33)$$

where

$$P(t, x) = \int_0^x u(t, y) dy, \quad t \geq 0, x \geq 0.$$

Moreover, (3.11) is satisfied.

Proof. Let . Multiplying the equation (3.17) by $\eta'(u_\varepsilon)$, we get

$$\partial_t u_\varepsilon \eta'(u_\varepsilon) + \partial_x f(x, u_\varepsilon) \eta'(u_\varepsilon) = P_\varepsilon \eta'(u_\varepsilon) + \varepsilon \partial_{xx}^2 u_\varepsilon \eta'(u_\varepsilon),$$

which can be rewritten as

$$\partial_t \eta(u_\varepsilon) + f_{u_\varepsilon}(x, u_\varepsilon) (u_\varepsilon)_x \eta'(u_\varepsilon) + f_x(x, u_\varepsilon) \eta'(u_\varepsilon) = P_\varepsilon \eta'(u_\varepsilon) + \varepsilon \partial_{xx}^2 u_\varepsilon \eta'(u_\varepsilon).$$

From the definition of $q(x, u_\varepsilon)$ we have $q_{u_\varepsilon}(x, u_\varepsilon) = \eta'(u_\varepsilon) f_{u_\varepsilon}(x, u_\varepsilon)$. Inserting this into the above expression we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(x, u_\varepsilon) + f_x(x, u_\varepsilon) \eta'(u_\varepsilon) - q_x(x, u_\varepsilon) = P_\varepsilon \eta'(u_\varepsilon) + \varepsilon \partial_{xx}^2 u_\varepsilon \eta'(u_\varepsilon).$$

This can be written as

$$\begin{aligned} \partial_t \eta(u_\varepsilon) + \partial_x q(x, u_\varepsilon) &= \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{\mathcal{L}_\varepsilon^1} - \underbrace{\varepsilon \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2}_{\mathcal{L}_\varepsilon^2} + \underbrace{\eta'(u_\varepsilon) P_\varepsilon}_{\mathcal{L}_\varepsilon^3} \\ &\quad + \underbrace{q_x(x, u_\varepsilon)}_{\mathcal{L}_\varepsilon^4} - \underbrace{f_x(x, u_\varepsilon) \eta'(u_\varepsilon)}_{\mathcal{L}_\varepsilon^5}. \end{aligned} \quad (3.34)$$

From Lemma 3.3.3 we have

$$\mathcal{L}_\varepsilon^1 \rightarrow 0, \text{ in } H_{loc}^{-1}(\Pi), \{\mathcal{L}_\varepsilon^2\}_{\varepsilon > 0} \text{ is uniformly bounded in } L_{loc}^1(\Pi).$$

To show $\{\mathcal{L}_\varepsilon^3\}_{\varepsilon>0}$ is uniformly bounded in $L_{\text{loc}}^1(\Pi)$, let K be any bounded subset of Π . Then, by Lemma 3.3.4,

$$\|\eta'(u_\varepsilon)P_\varepsilon\|_{L^1(K)} \leq \|\eta'(u_\varepsilon)\|_{L^\infty(K)}\|P_\varepsilon\|_{L^\infty(K)}|K|.$$

So it remains to show that $f_x(x, u_\varepsilon)\eta'(u_\varepsilon)$ and $q_x(x, u_\varepsilon)$ are uniformly bounded in $L_{\text{loc}}^1(\Pi)$. To that end observe that

$$\begin{aligned} \|f_x(x, u_\varepsilon)\eta'(u_\varepsilon)\|_{L^1(K)} &= \int_K |f_x(x, u_\varepsilon)\eta'(u_\varepsilon)| dx dt \text{ (by (A2) and } |\eta'(u_\varepsilon)| \leq C|u_\varepsilon|) \\ &\leq \tilde{C} \int_K |u_\varepsilon|^2 dx dt \text{ (for some constant } \tilde{C} > 0) \\ &< \infty \end{aligned}$$

So $\{f_x(x, u_\varepsilon)\eta'(u_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L_{\text{loc}}^1(\Pi)$. Similarly we have

$$\begin{aligned} \|q_x(x, u_\varepsilon)\|_{L^1(K)} &= \int_K \left| \int_0^{u_\varepsilon} \eta'(v) f_{xv}(v) dv \right| dx dt \text{ (by (A2))} \\ &\leq C \int_K \int_0^{u_\varepsilon} |\eta'(v)| dv dx dt \\ &< \infty. \end{aligned}$$

Consequently, $\{q_x(x, u_\varepsilon)\}_{\varepsilon>0}$ is uniformly bounded in $L_{\text{loc}}^1(\Pi)$.

Therefore, by Lemma 3.4.2 we can conclude that

$$\{\partial_t \eta(u_\varepsilon) + \partial_x q(x, u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H_{\text{loc}}^{-1}(\Pi). \quad (3.35)$$

Therefore using the L_{loc}^∞ bound obtained from Lemma 3.3.4, (3.35) and Lemma 3.4.1 we can conclude that there exists a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ and a limit function $u \in L_{\text{loc}}^\infty(\Pi)$ such that (3.32) holds. By the Hölder inequality and the definition of P_ε , (3.33) follows from (3.32).

We remark that the entropy inequality (3.12) can be obtained from (3.34) by the standard argument of letting $\varepsilon \rightarrow 0$ and using convexity of $\eta(\cdot)$. Thus by [28, Theorem 1.1], strong trace u_0^τ for u on $x = 0$ does exist. Now we are going to prove (3.11). From the Definition 3.2.1 for (3.17) and using (3.12) we get for Krüzkov entropy/entropy flux pair (η, q)

$$\begin{aligned} &\partial_t |u_{\varepsilon_k} - c| + \partial_x \left(\text{sign}(u - c)(f(x, u) - f(x, c)) \right) \\ &\quad - \text{sign}(u_{\varepsilon_k} - c)P_{\varepsilon_k} - \varepsilon_k \partial_{xx}^2 |u_{\varepsilon_k} - c| \leq 0. \end{aligned}$$

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Multiplying by a non-negative test function $\varphi \in C_c^2(\Pi)$ and integrating over Π , we get

$$\begin{aligned} & \iint_{\Pi} \left(|u_{\varepsilon_k} - c| \partial_t \varphi + \left(\text{sign}(u_{\varepsilon_k} - c)(f(x, u) - f(x, c)) \right) \partial_x \varphi \right. \\ & \quad \left. - \text{sign}(u_{\varepsilon_k} - c) f_x(x, c) \varphi + \text{sign}(u_{\varepsilon_k} - c) P_{\varepsilon_k} \varphi \right) dt dx \\ & - \varepsilon_k \iint_{\Pi} \partial_x |u_{\varepsilon_k} - c| \partial_x \varphi dt dx + \int_{\mathbb{R}_+} |u_0(x) - c| \varphi(x, 0) dx \\ & + \int_{\mathbb{R}_+} \text{sign}(c) f(0, c) \varphi(0, t) dt - \varepsilon_k \int_{\mathbb{R}_+} \partial_x |u_{\varepsilon_k}(0, t) - c| \varphi(0, t) dt \geq 0. \end{aligned}$$

Invoking Lemmas 3.3.3, 3.3.4, and 3.4.3, letting $k \rightarrow \infty$, we have

$$\begin{aligned} & \iint_{\Pi} \left(|u - c| \partial_t \varphi + \left(\text{sign}(u - c)(f(x, u) - f(x, c)) \right) \partial_x \varphi \right. \\ & \quad \left. - \text{sign}(u - c) f_x(x, c) \varphi + \text{sign}(u - c) P \varphi \right) dt dx \\ & + \int_{\mathbb{R}_+} |u_0(x) - c| \varphi(x, 0) dx + \int_{\mathbb{R}_+} \text{sign}(c) f(0, c) \varphi(0, t) dt \\ & - \lim_{k \rightarrow \infty} \varepsilon_k \int_{\mathbb{R}_+} \partial_x |u_{\varepsilon_k}(0, t) - c| \varphi(0, t) dt \geq 0. \end{aligned} \tag{3.36}$$

Consequently to show (3.11) it is enough to prove that

$$\lim_{k \rightarrow \infty} \varepsilon_k \int_{\mathbb{R}_+} \partial_x |u_{\varepsilon_k}(0, t) - c| \varphi(0, t) dt = \int_{\mathbb{R}_+} \text{sign}(c) f(0, u_0^\tau(t)) \varphi(0, t) dt. \tag{3.37}$$

In order to prove this we need to employ a particular choice of test function. Let $\{\Psi_m\}_{m \in \mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be a sequence of non-negative test functions satisfying

$$\begin{cases} \Psi_m(0) = 1, \text{ for all } m \in \mathbb{N}, \\ |\Psi'_m| \leq m, \text{ and} \\ \Psi_m(x) = 0, \text{ for all } x \geq \frac{1}{m}. \end{cases} \tag{3.38}$$

Multiplying equation (3.17) by the test function $\Psi_m(x) \varphi(x, t)$ we get after an integration by parts

$$\begin{aligned} & \iint_{\Pi} \left(u_{\varepsilon_k} \partial_t \varphi \Psi_m + f(x, u_{\varepsilon_k})(\Psi_m \partial_x \varphi + \Psi'_m \varphi) + P_{\varepsilon_k} \Psi_m \varphi \right) dt dx \\ & - \iint_{\Pi} \varepsilon_k \partial_x u_{\varepsilon_k} (\Psi_m \partial_x \varphi + \Psi'_m \varphi) dt dx + \int_{\mathbb{R}_+} u_0(x) \varphi(x, 0) \Psi_m(x) dx \\ & - \int_{\mathbb{R}_+} f(0, u_{\varepsilon_k}(0, t)) \varphi(0, t) dt - \varepsilon_k \int_{\mathbb{R}_+} \partial_x u_{\varepsilon_k}(0, t) \varphi(0, t) dt = 0. \end{aligned} \tag{3.39}$$

Employing the strong convergence $u_{\varepsilon_k} \rightarrow u$ from Lemma 3.4.3, passing to the limit $k \rightarrow \infty$, $m \rightarrow \infty$ respectively and using the properties of Ψ_m in the above relation (3.39) we get

$$\lim_{k \rightarrow \infty} \varepsilon_k \int_{\mathbb{R}_+} \partial_x u_{\varepsilon_k}(0, t) \varphi(0, t) dt = - \int_{\mathbb{R}_+} f(0, u_0^\tau(t)) \varphi(0, t) dt,$$

which in turn proves (3.37). Combining (3.36) and (3.37) we have obtained the desired inequality (3.11).

This completes the proof. $\square$

Consequently we have established the existence of an entropy solution (in the sense of Definition 3.2.1) $u(x, t)$ of the equation (3.5) or equivalently of (3.6). Now in order to prove the uniqueness of entropy solutions we are going to prove (3.13), *i.e.* we will prove Theorem 3.2.1.

Proof. (of Theorem 3.2.1 :) Let u and v be two entropy solutions of (3.5) or equivalently (3.6). We will use the doubling of variables. For $\Pi := (0, \infty)^2$ and $\Pi^2 := (0, \infty)^4$ let $\varphi(t, \tau, x, y) \in C_c^\infty(\Pi^2)$ be a non-negative test function. Since u and v are entropy solutions of (3.6), we have

$$\begin{aligned} & \iint_{\Pi} \left[|u(x, t) - v(y, \tau)| \partial_t \varphi(t, \tau, x, y) + [f(x, u(x, t)) - f(y, v(y, \tau))] \right. \\ & \quad \text{sign}(u(x, t) - v(y, \tau)) \partial_x \varphi(t, \tau, x, y) \\ & \quad \left. - \text{sign}(u(x, t) - v(y, \tau)) [f_x(x, v(y, \tau)) - P_u(x, t)] \varphi(t, \tau, x, y) \right] dt dx \geq 0, \end{aligned} \quad (3.40)$$

and

$$\begin{aligned} & \iint_{\Pi} \left[|v(y, \tau) - u(x, t)| \partial_\tau \varphi(t, \tau, x, y) + [f(y, v(y, \tau)) - f(x, u(x, t))] \right. \\ & \quad \text{sign}(v(y, \tau) - u(x, t)) \partial_y \varphi(t, \tau, x, y) \\ & \quad \left. - \text{sign}(v(y, \tau) - u(x, t)) [f_y(y, u(x, t)) - P_v(y, \tau)] \varphi(t, \tau, x, y) \right] d\tau dy \geq 0. \end{aligned} \quad (3.41)$$

Then integrating (3.40) with respect to τ, y ; (3.41) with respect to t, x ; and adding the two outcomes we obtain,

$$\begin{aligned} & \iiint_{\Pi^2} \left[|u(x, t) - v(y, \tau)| (\partial_t \varphi(t, \tau, x, y) + \partial_\tau \varphi(t, \tau, x, y)) + [f(x, u(x, t)) \right. \\ & \quad - f(y, v(y, \tau))] \text{sign}(u(x, t) - v(y, \tau)) (\partial_x \varphi(t, \tau, x, y) + \partial_y \varphi(t, \tau, x, y)) \\ & \quad + \text{sign}(u(x, t) - v(y, \tau)) (P_u(x, t) - P_v(y, \tau)) \varphi(t, \tau, x, y) \\ & \quad \left. - \text{sign}(u(x, t) - v(y, \tau)) (f_x(x, v(y, \tau)) - f_y(y, u(x, t))) \right. \\ & \quad \left. \varphi(t, \tau, x, y) \right] dt d\tau dx dy \geq 0. \end{aligned} \quad (3.42)$$

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For $\rho_\varepsilon \rightarrow \delta_0$ as $\varepsilon \rightarrow 0$, where δ_0 is the Dirac mass concentrated at 0, where

$$\rho_\varepsilon(z) := \varepsilon \rho(\varepsilon z), \quad \text{and} \quad \alpha_\varepsilon(z) := \int_{-\infty}^z \rho_\varepsilon(x) dx, \quad (3.43)$$

for some non-negative $\rho \in C_c^\infty([-1, 1])$ with total mass being 1. Now let us define the particular test function

$$\varphi_\varepsilon(t, \tau, x, y) = \psi\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \rho_\varepsilon\left(\frac{\tau-t}{2}\right) \rho_\varepsilon\left(\frac{y-x}{2}\right), \quad (3.44)$$

where $\psi \in C_c^\infty(\Pi)$ is a non-negative, test function. Inserting the function (3.44) into the last inequality (3.42), we get

$$\begin{aligned} & \iiint_{\Pi^2} \left[\rho_\varepsilon\left(\frac{\tau-t}{2}\right) \rho_\varepsilon\left(\frac{y-x}{2}\right) \left\{ |u(t, x) - v(\tau, y)| \partial_t \psi\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \right. \right. \\ & \quad + \left(f(x, u(t, x)) - f(y, v(\tau, y)) \right) \text{sign}(u(t, x) - v(\tau, y)) \partial_x \psi\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \Big\} \\ & \quad + \gamma \psi\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \rho_\varepsilon\left(\frac{\tau-t}{2}\right) \rho_\varepsilon\left(\frac{y-x}{2}\right) \text{sign}(u(t, x) - v(\tau, y)) \\ & \quad \quad (P_u(t, x) - P_v(\tau, y)) - \text{sign}(u(t, x) - v(\tau, y)) (f_x(x, v(\tau, y)) \\ & \quad \quad - f_y(y, u(t, x))) \Big] \psi\left(\frac{t+\tau}{2}, \frac{x+y}{2}\right) \rho_\varepsilon\left(\frac{\tau-t}{2}\right) \rho_\varepsilon\left(\frac{y-x}{2}\right) dt d\tau dx dy \geq 0. \end{aligned} \quad (3.45)$$

By standard limiting argument of doubling of variable technique, passing to the limit as $\varepsilon \rightarrow 0$ we obtain from the previous inequality (3.45) that for all test functions ψ as mentioned above,

$$\begin{aligned} & \iint_{\Pi} \left[|u(t, x) - v(t, x)| \partial_t \psi + \left(f(x, u) - f(x, v) \right) \text{sign}(u(t, x) - v(t, x)) \right] \partial_x \psi dt dx \\ & \quad + \iint_{\Pi} \text{sign}(u(t, x) - v(t, x)) ((P_u(t, x) - P_v(t, x)) \psi) dt dx \\ & \quad + \iint_{\Pi} \text{sign}(u(t, x) - v(t, x)) \left(f_x(x, u(t, x)) - f_x(x, v(t, x)) \right) \psi dt dx \geq 0. \end{aligned} \quad (3.46)$$

Following Kruřkov's argument [67] if we consider the sets for $T, R > 0$

$$\Omega_{R,T} := \{(t, x) \in [0, T] \times [0, R]; \quad 0 \leq s \leq t, \quad 0 \leq x \leq R + L(t - s)\}, \quad (3.47)$$

and define the following non-negative test function

$$\varphi_\varepsilon(t, x) := [\alpha_\varepsilon(s) - \alpha_\varepsilon(s - t)][1 - \alpha_\varepsilon(x - R - L(t - s))],$$

where α_ε is defined in (3.43) and L is defined in (A4). Clearly observe that φ_ε is an approximation of the characteristic function of $\Omega_{R,T}$. From definition $\alpha'_\varepsilon = \rho_\varepsilon \geq 0$. Using

φ_ε as the test function in (3.46) and similarly as before letting $\varepsilon \rightarrow 0$, we get

$$\begin{aligned} \|u(\cdot, t) - v(\cdot, t)\|_{L^1(0, R)} &\leq \|u_0 - v_0\|_{L^1(0, R+Lt)} \\ &\quad + \int_{\Omega_{R, T}} \text{sign}(u(x, t) - v(x, t))(P_u - P_v) dx ds \\ &\quad + \int_{\Omega_{R, T}} \text{sign}(u(x, t) - v(x, t)) \left(f_x(x, u) - f_x(x, v) \right) dx ds. \end{aligned} \quad (3.48)$$

With

$$I(s) := [0, R + L(t - s)], \quad (3.49)$$

note that

$$\begin{aligned} \int_{\Omega_{R, T}} \text{sign}(u - v) \left(f_x(x, u) - f_x(x, v) \right) dx ds &\leq \int_0^t \int_{I(s)} |f_x(x, u) - f_x(x, v)| dx ds \\ &\quad \text{(Using (A3))} \\ &\leq \int_0^t \int_{I(s)} L_1 |u - v| dx ds \\ &\leq \int_0^t L_1 \|u - v\|_{L^1(I(s))} ds. \end{aligned} \quad (3.50)$$

Since

$$\begin{aligned} \int_{\Omega_{R, T}} \text{sign}(u - v)(P_u - P_v) ds dx &\leq \int_0^t \int_{I(s)} |P_u - P_v| ds dx \\ &\leq \int_0^t \int_{I(s)} \left(\left| \int_0^x |u - v| dy \right| \right) ds dx \\ &\leq \int_0^t \int_{I(s)} \left(\left| \int_{I(s)} |u - v| dy \right| \right) ds dx \\ &= \int_0^t |I(s)| \|u(\cdot, s) - v(\cdot, s)\|_{L^1(I(s))} ds, \end{aligned} \quad (3.51)$$

and,

$$|I(s)| = R + L(t - s) \leq R + Lt \leq R + LT. \quad (3.52)$$

We consider the following continuous function:

$$G(t) := \|u(\cdot, t) - v(\cdot, t)\|_{L^1(I(t))}, \quad t \geq 0. \quad (3.53)$$

Then we can combine (3.48), (3.50), (3.51) and (3.52) to obtain

$$G(t) \leq G(0) + \int_0^t (|I(s)| + L_1)G(s)ds \quad \text{with} \quad |I(s)| = R + L(t - s). \quad (3.54)$$

Consequently, by Gronwall's inequality we can conclude:

$$G(t) \leq G(0)e^{\int_0^t (|I(s)| + L_1)ds}, \quad \text{for a.e. } 0 < t < T,$$

i.e.

$$G(t) \leq G(0)e^{(Rt + \frac{1}{2}Lt^2) + L_1t} \leq G(0)e^{(R + \frac{1}{2}LT)t + L_1T}, \quad \text{for a.e. } 0 < t < T.$$

Consequently we have the estimate (3.13), namely

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1((0, R))} \leq e^{Ct} \|u(\cdot, 0) - v(\cdot, 0)\|_{L^1(0, R+Lt)}, \quad \text{for a.e. } 0 < t < T, \quad (3.55)$$

where the constant C depends on T , R , L_1 and L .

This completes the proof. □

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