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Evaluating intergenerational risks*

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Abstract

Climate policies have stochastic consequences that involve a great number of generations. This calls for evaluating social risk (what kind of societies will future people be born into) rather than individual risk (what will happen to people during their own lifetimes). We respond to this call by proposing and axiomatizing probability adjusted rank-discounted critical-level generalized utilitarianism (PARDCLU) through a key axiom ensuring that the social welfare order both is ethical and satisfies first-order stochastic dominance. PARDCLU yields a new useful perspective on intergenerational risks, is ethical in contrast to discounted utilitarianism, and avoids objections that have been raised against other ethical criteria. We show that PARDCLU handles situations with positive probability of human extinction and is linked to decision theory by yielding rank-dependent expected utilitarianism—but with additional structure—in a special case.

Keywords: Social evaluation, population ethics, decision-making under risk, critical-level utilitarianism, social discounting.

JEL Classification numbers: D63, D71, D81, H43, Q54, Q56.

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1 Introduction

This paper proposes a new normative criterion that can potentially be used for ranking climate policies. Climate policies seeking to abate anthropogenic greenhouse gas emissions have extremely long-term stochastic consequences, as greenhouse gas emissions cause environmental risks that extend into the far future. Therefore, to evaluate such policies one must assess risks that involve a great number of generations.

In this time frame, where people's lives are short compared to the time period for which the policies will have an effect, the objective social risk concerning

what kind of societies will future people be born into

might be more important than the subjective individual risk concerning

what will happen to people during their own lifetimes.

That is, it might be reasonable to be more concerned about reducing the probability that future people will live miserable lives, rather than avoiding volatility in the living conditions that people experience within their own lifetimes.

This motivates an approach that abstracts from lifetime fluctuations by assuming that people live for one period only. Moreover, the lives of the 'same' individual in two different future realizations might be considered as the lives of two different people, each living with the probability assigned to the realizations in question. Hence, if a future individual has equal probability of living a good or bad life, then this might be modeled as two different people, one living a good life and one living a bad life, where each has probability 0.5 of coming into existence.

Different normative considerations arise in a setting where people do not experience fluctuations and risk within their own lifetime. In particular, we are not concerned about individual risk attitudes and the risk generations may face from an abstract *ex ante* point of view. We are only concerned with the final distribution of well-being. The important question for the evaluation of policies with long-term intergenerational effects is how to handle inequality. Clearly, if, for each chosen policy, all people – now and in all future realizations – have the same level of lifetime well-being, then this uniform well-being level can be used to rank policies. Thus, in our context, only social aversion to inequality matters, while subjective aversions to individual fluctuations and risk play no role.

By focusing on social risks, our approach differs from the vast literature on the aggregation of preferences under risk and uncertainty stemming from Harsanyi's (1955) seminal contribution. This literature has focused on respecting people preferences, in a context where the society and the individuals face the 'same' uncertainty, in the sense that uncertainty does not concern the mere existence of people. The contributions have wavered between an ex ante approach that relaxes rationality (Diamond, 1967; Epstein and Segal, 1992) to allow for *ex ante* fairness, and an ex post approach that fails the ex ante Pareto principle (Broome, 1991; Fleurbaey, 2010) to allow for *ex post* fairness. In the present paper, these issues do not arise because we interpret individuals in different events as different individuals: individuals are born only after the realization of events relevant for their lives. This interpretation is consistent with other papers focusing on social risk rather than individual risks (for instance Asheim and Brekke, 2002; Piacquadio, 2015).

In the framework of Harsanyi's (1953) impartial observer theorem, Grant et al. (2010) have highlighted the distinction in social evaluation between lotteries over identities and lotteries over outcomes. We focus on lotteries over identities but add the complication that people may exist with different probabilities. We also depart from expected utility to address population ethics and equity concerns.

Our analysis will be confined to the case where there are objective assessments of the probabilities of different realizations. Hence, formally we will be concerned with risk rather than uncertainty. Moreover, we will assume that there is an indicator of lifetime well-being which is at least ordinally measurable and level comparable across people. Following the usual convention in population ethics, we will normalize the well-being scale so that lifetime well-being equal to 0 represents *neutrality*. Hence, a life with lifetime well-being above 0 is worth living; below 0, it is not.

We are concerned with normative evaluation where people are treated equally. This differs from the common use of discounted utilitarianism in integrated assessment models of climate change, where transformed well-being (*utility*) is discounted by a constant and positive per-period time-discount rate. As a matter of principle, utilitarianism with time-discounting means that people across time are not treated equally. As a matter of practical policy evaluation, this criterion is virtually insensitive to the long-term effects of climate change, beyond year 2100 when the most serious consequences will occur, in particular for poor groups who are expected to bear the highest costs (see for instance World Bank, 2013).

Equal treatment of people in axiomatic analysis is captured by the *Anonymity* axiom, whereby social evaluation is invariant to permuting two individuals' well-being. Combined with sensitivity for the interests of all people, as captured by the *Strong Pareto* principle, this leads to the *Suppes-Sen* principle (Suppes, 1966; Sen, 1970). This principle requires that one allocation be better than another if the former dominates the latter when being rank-ordered according to the levels of well-being. Conversely, the Suppes-Sen principle combined with the *Continuity* axiom implies both Anonymity and the Strong Pareto principle. A criterion that satisfies the Suppes-Sen principle is called *ethical* by Svensson (1980). In this paper, we characterize an ethical criterion that avoids objections raised against other ethical criteria, e.g. utilitarian and egalitarian criteria.

Undiscounted utilitarianism, where utility is summed without discounting, is one criterion which satisfies the Suppes-Sen principle. However, when modeling the many potential future people by assuming that there are infinitely many generations, this criterion assigns zero relative weight to the present generation's interests. It leads to the unappealing prescription that the present generation should endure heavy sacrifices even if it contributes to only a tiny gain for all future generations. Moreover, in a variable population setting with an unbounded number of potential people, it is subject to the *Repugnant conclusion*[1] or the *Very sadistic conclusion*.[2]

[1]The *Repugnant conclusion* (Parfit, 1976, 1982, 1984) states that, for any population in which people have high levels of well-being, there is a larger population in which people have lives barely worth living that is deemed socially better.

[2]The *Very sadistic conclusion* (Arrhenius, 2000, forthcoming) states that, for any population in

The egalitarian criterion of maximizing the well-being of the worst-off generation (*maximin*) also satisfies the Suppes-Sen principle, but assigns zero relative weight to all generations but the worst-off. It leads to the unappealing prescription that the present generation should not do an even negligible sacrifice for the benefit of better off future generations. Maximin has also problematic implications when applied in a variable population setting (Arrhenius, forthcoming; Asheim and Zuber, 2014).

This dilemma – that ethical criteria may to lead to extreme prescriptions in terms of sacrifice for future generations – motivates *rank-discounted generalized utilitarianism* (RDU), proposed and analyzed by Zuber and Asheim (2012). RDU discounts future utility as long as the future is better off than the present, thereby trading-off current sacrifice and future gain. However, if the present generation is better off than all future generations, then priority shifts to the future. In this case, zero relative weight is assigned to present utility. RDU is compatible with equal treatment of generations as discounting is made according to rank, not according to time. Asheim and Zuber (2014) extend RDU to a variable population setting by proposing and axiomatizing *rank-discounted critical-level generalized utilitarianism* (RDCLU). RDCLU avoids both the Repugnant and Very sadistic conclusions, thereby evading serious objections raised against other variable population criteria.

In the present paper we extend RDCLU to risky situations, including the case with positive probability of human extinction, by proposing the *probability adjusted rank-discounted critical-level generalized utilitarian* (PARDCLU) social welfare order (Definition 1). We start out in Section 2 by developing a framework where each (potential) individual is characterized by a level of lifetime well-being and a probability of existence. We show in Appendix A how this set-up is equivalent to a formulation where the individuals are distributed through time and over risky states. In this alternative dynamic framework individuals live for one period only and are not subjected to risk during their lifetime, reflecting our intergenerational

which people have terrible lives not worth living, there is a larger population in which everyone has a life worth living that is deemed socially worse.

perspective.

We then, in Section 3, present an axiomatic foundation for PARDCLU through Theorem 1. A key axiom, called *Probability adjusted Suppes-Sen*, generalizes the Suppes-Sen principle to a setting where people need not exist with probability one. In conjunction with the Continuity axiom, it implies invariance to permutations of individuals with the same well-being and the same probability of existence. It also entails invariance to the replacement of one individual with given well-being and probability with two individuals having the same well-being and whose probabilities of existence sum up the probability of the original individual. In the special case where the individual probabilities of existence sum up to one, Probability adjusted Suppes-Sen corresponds to first-order stochastic dominance. Hence, this axiom can be also considered as a generalization of first-order stochastic dominance to a normative multi-person setting.

Our axiomatic system is related to the one found in Asheim and Zuber (2014, Section 3). However, the modeling of individuals that exist with probabilities less than 1 allows us to use a much weaker critical-level axiom, requiring only that populations of different sizes must be comparable in a non-trivial way, in the sense that a larger population is not always better (and not always worse) than a smaller one. Like in Asheim and Zuber (2014) we obtain a critical level parameter c such that it is socially indifferent to add an individual at this level provided that people in the existing population has well-being that does not exceed c. But the existence of the level c is not imposed by the axiom; it is a result of the axiomatic system. Also, the proof of Theorem 1 (contained in Appendix B) shows that our main result is not a trivial extension, because probabilities are real numbers and only a weak continuity axiom is imposed.

In Section 4 we illustrate the usefulness of PARDCLU by showing how PARDCLU handles human extinction. Moreover, when individual probabilities of existence sum up to one, PARDCLU yields rank-dependent expected utilitarianism, but with additional structure. This additional structure derives from the axiom *Existence* *independence of the worst-off*, which plays the same role as Koopmans' (1960) stationarity postulate. In Section 5 we demonstrate additional properties (proven in Appendix C) of PARDCLU in terms of distributional equity and population ethics. In the final Section 6 we discuss some issues faced by the PARDCLU approach and provide concluding remarks.

2 The framework

In this section we introduce an abstract framework with a set of atemporal allocations, in which we develop our axiomatization for the sake of simplicity. Appendix A establishes a perfect correspondence between this setting and a more descriptive dynamic framework.

Individuals are described by two numbers: their lifetime well-being and their probability of existence. An allocation $\mathbf{x} \in (\mathbb{R} \times (0,1])^n$ determines the finite population size, $n(\mathbf{x}) = n$, and the distribution of pairs of well-being and probability,

$$\mathbf{x} = \left(x_1, \dots, x_{n(\mathbf{x})}\right) = \Big((x_1^w, x_1^p), \dots, (x_{n(\mathbf{x})}^w, x_{n(\mathbf{x})}^p)\Big) ,$$

among the $n(\mathbf{x})$ individuals who make up the population. For each $i \in \{1, \dots, n(\mathbf{x})\}$, x_i^w is the individual's well-being and x_i^p is his probability of existence. We denote by $\nu(\mathbf{x}) = \sum_{i=1}^{n(\mathbf{x})} x_i^p$ the probability adjusted population size of $\mathbf{x}$ and by

$$\mathbf{X} = \bigcup_{n \in \mathbb{N}} (\mathbb{R} \times (0,1])^n$$

the set of possible finite *allocations*.

As mentioned in the introduction, we follow the usual convention in population ethics, by letting lifetime well-being equal to 0 represents neutrality, above which a life, as a whole, is worth living, and below which, it is not.

A social welfare relation (SWR) on the set $\mathbf{X}$ is a binary relation $\succsim$, where for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ implies that the allocation $\mathbf{x}$ is deemed socially at least as good as $\mathbf{y}$. Let $\sim$ and $\succ$ denote the symmetric and asymmetric parts of $\succsim$.

For each $\mathbf{x} \in \mathbf{X}$, let $\pi : \{1, \ldots, n(\mathbf{x})\} \to \{1, \ldots, n(\mathbf{x})\}$ be a bijection that reorders individuals in non-decreasing well-being order:

$$x^w_{\pi(r)} \leq x^w_{\pi(r+1)} \quad \text{for all ranks } r \in \{1, \ldots, n(\mathbf{x}) - 1\}\,.$$

Let $\rho_0 = 0$ and define the probability adjusted rank ρ_r inductively as follows:

$$\rho_r = x^p_{\pi(r)} + \rho_{r-1}$$

for $r \in \{1, \ldots, n(\mathbf{x})\}$. Define the *rank-ordered allocation* $\mathbf{x}_{[\,]} : (0, \nu(\mathbf{x})] \to \mathbb{R}$ by

$$\mathbf{x}_{[\rho]} = x^w_{\pi(r)} \quad \text{for } \; \rho_{r-1} < \rho \leq \rho_r \; \text{ and } \; 1 < r \leq n(\mathbf{x})$$

and write $\mathbf{x}_{[0]} := \lim_{\rho \downarrow 0} \mathbf{x}_{[\rho]}$. Note that the permutation π need not be unique (if, for instance, $x^w_i = x^w_{i'}$ for some $i \neq i'$), but the resulting rank-ordered allocation $\mathbf{x}_{[\,]}$ is unique. Note also that the definitions imply that $\rho_{n(\mathbf{x})} = \nu(\mathbf{x})$.

For every $\nu \in \mathbb{R}$, write $\mathbf{X}_\nu = \{\mathbf{x} \in \mathbf{X} : \nu(\mathbf{x}) = \nu\}$ for the set of finite allocations with probability adjusted population size equal to ν. For $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\nu$, write $\mathbf{x}_{[\,]} > \mathbf{y}_{[\,]}$ if $\mathbf{x}_{[\rho]} \geq \mathbf{y}_{[\rho]}$ for all $\rho \in (0, \nu]$ and $\mathbf{x}_{[\rho']} > \mathbf{y}_{[\rho']}$ for some $\rho' \in (0, \nu]$; note that, by the definitions of the step functions $\mathbf{x}_{[\,]}$ and $\mathbf{y}_{[\,]}$, $\mathbf{x}_{[\rho']} > \mathbf{y}_{[\rho']}$ implies that $\mathbf{x}_{[\rho]} > \mathbf{y}_{[\rho]}$ for all ρ in a subset of $(0, \nu]$ that includes a non-empty proper interval.

For $z \in \mathbb{R}$, $p \in (0, 1]$ and $n \in \mathbb{N}$, let $\mathbf{x} \in (\mathbb{R} \times (0, 1])^n$ with $(x^w_i, x^p_i) = (z, p)$ for all $i \in \{1, \ldots, n\}$ be denoted by $(z)_\nu$, where $\nu = np$. For $\mathbf{x} \in \mathbf{X}$, $z \in \mathbb{R}$, $p \in (0, 1]$ and $n \in \mathbb{N}$, let $\mathbf{y} \in (\mathbb{R} \times (0, 1])^{n(\mathbf{x})+n}$ such that $(y^w_i, y^p_i) = (x^w_i, x^p_i)$ for all $i \in \{0, \ldots, n(\mathbf{x})\}$ and $(y^w_i, y^p_i) = (z, p)$ for all $i \in \{n(\mathbf{x}) + 1, \ldots, n(\mathbf{x}) + n\}$ be denoted by $\big(\mathbf{x}, (z)_{np}\big)$.

3 Axioms and representation result

Probability adjusted rank-discounted critical-level utilitarianism can be characterized by the following seven axioms.

The first three axioms are sufficient to ensure numerical representation of the SWR for any fixed probability adjusted population size. They also entail that individuals are treated anonymously and with sensitivity to their well-being.

Axiom 1 (Order) *The relation $\succsim$ is complete, reflexive and transitive on $\mathbf{X}$.*

An SWR satisfying Axiom 1 is called a social welfare order (SWO).

Axiom 2 (Continuity) *For all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x} \in \mathbf{X}_\nu$, the sets $\{\mathbf{y} \in \mathbf{X}_\nu : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbf{X} : \mathbf{x} \succsim \mathbf{y}\}$ are closed for the topology induced by the supnorm applied to rank-ordered allocations.*[3]

Axiom 3 (Probability adjusted Suppes-Sen) *For all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\nu$, if $\mathbf{x}_{[\,]} > \mathbf{y}_{[\,]}$, then $\mathbf{x} \succ \mathbf{y}$.*

Jointly with Axiom 2, Axiom 3 implies anonymity wrt. different individuals with the same probability of existence. Hence, permuting the well-being levels of individuals with the same probability of existence leads to an equally good allocation.

In line with Asheim and Zuber's (2014) axiomatization of rank-discounted critical-level utilitarianism we impose independence to adding an individual only if the added individual is best-off (relative to two allocations with the same probability adjusted population size) or worst-off.

Axiom 4 (Existence independence of the best-off) *For all $\nu \in \mathbb{R}_{++}$, $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\nu$, $p \in (0, 1]$, and $z \in \mathbb{R}$ satisfying $z \geq \max\{\mathbf{x}_{[\nu]}, \mathbf{y}_{[\nu]}\}$, $(\mathbf{x}, (z)_p) \succsim (\mathbf{y}, (z)_p)$ if and only if $\mathbf{x} \succsim \mathbf{y}$.*

Axiom 5 (Existence independence of the worst-off) *For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $p \in (0, 1]$, and $z \in \mathbb{R}$ satisfying $z \leq \min\{\mathbf{x}_{[0]}, \mathbf{y}_{[0]}\}$, $(\mathbf{x}, (z)_p) \succsim (\mathbf{y}, (z)_p)$ if and only if $\mathbf{x} \succsim \mathbf{y}$.*

The two axioms above are much weaker than unrestricted existence independence. Still, together with Axioms 1–3 they are sufficient for obtaining an additively separable representation for rank-ordered allocations (see Ebert, 1988, Theorem 1).

[3] This means that we use the metric $d(x, y) = sup_{r \in [0,\nu]} |x_{[r]} - y_{[r]}|$. In functional spaces, the topology induced by the sup metric is strong, so that the associated notion of continuity is weak. This is an advantage of our definition.

Axiom 5 is related to a condition named 'independent future' by Fleurbaey and Michel (2003), although it is here applied in a setting of rank-ordered allocations, not allocations ordered according to time. Their condition combines Koopmans' (1960) stationarity postulate (Postulate 4) with his postulate 3b (that the level of well-being of an unconcerned present generation does not affect decisions). In our application it implies that there is a constant rate of rank-discounting.

We next introduce an axiom stating that, for any allocation of a given size ν, adding an individual with non-negative well-being is not always better (or not always worse) than the original allocation in social evaluation.

Axiom 6 (Weak existence of a critical level) *There exists $\nu \in \mathbb{R}_{++}$ such that for all $\mathbf{x} \in \mathbf{X}_\nu$ and $p \in (0,1]$, there exist z', $z'' \in \mathbb{R}_+$ such that $(\mathbf{x}, (z')_p) \succsim \mathbf{x}$ and $(\mathbf{x}, (z'')_p) \precsim \mathbf{x}$.*

In the axiom, the numbers z' and z'' are allowed to be context-dependent in the sense that they may depend on the existing allocation $\mathbf{x}$ and on the probability of existence of the additional person. So this is a very weak notion. In particular, it is much weaker than the axiom of 'Existence of a critical level' used by Asheim and Zuber (2014): they impose that there is a uniform critical level c for all allocations such that if all existing individuals have well-being that does not exceed this critical level, then adding an individual with well-being c is indifferent in social evaluation.

In the case with no risk (i.e., for the subset of allocations with $x_i^p = 1$ for all $i \in \{1, \ldots, n(\mathbf{x})\}$), all axioms above are satisfied also by ordinary critical-level utilitarianism with critical level $c \geq 0$. However, as discussed by Arrhenius (forthcoming, Sect. 5.1), critical-level utilitarianism has the properties that adding sufficiently many individuals with well-being just above c makes the allocation better than any fixed alternative (thus leading to the Repugnant conclusion if $c = 0$) and adding sufficiently many individuals with well-being just below c makes the allocation worse than any fixed alternative (thus leading to the Very sadistic conclusion if $c > 0$). In conjunction with the other axioms the following axiom ensures that the Repugnant and Very sadistic conclusions are avoided, while not by itself contradicting these conclusions (as shown by Asheim and Zuber, 2014, p. 635, it is weaker than directly requiring avoidance of the two conclusions).

Axiom 7 (Existence of egalitarian equivalence) *For all* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *and* $p \in (0, 1]$, *if* $\mathbf{x} \succ \mathbf{y}$, *then there exists* $z \in \mathbb{R}$ *such that, for all* $N \in \mathbb{N}$, $\mathbf{x} \succ (z)_{np} \succ \mathbf{y}$ *for some* $n \geq N$.

We will now state our main result, namely that these seven axioms characterize the probability adjusted rank-discounted critical-level generalized utilitarian SWOs.

Definition 1 An SWR $\succsim$ on $\mathbf{X}$ is a *probability adjusted rank-discounted critical-level generalized utilitarian* (PARDCLU) SWO if there exist $c \in \mathbb{R}_+$, $\delta \in \mathbb{R}_{++}$, and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \int_0^{\nu(\mathbf{x})} e^{-\delta\rho} \left(u(\mathbf{x}_{[\rho]}) - u(c)\right) d\rho \;\geq\; \int_0^{\nu(\mathbf{y})} e^{-\delta\rho} \left(u(\mathbf{y}_{[\rho]}) - u(c)\right) d\rho\,.$$

Parameter δ is the rank utility discount rate. Note that, while the criterion is expressed as an integral over the continuum space of cumulative probability, our setting involves only a finite number of distinct individuals. As before, let $\pi : \{1, \ldots, n(\mathbf{x})\} \to \{1, \ldots, n(\mathbf{x})\}$ be a bijection that orders individuals in non-decreasing well-being order (i.e. $x^w_{\pi(r)} \leq x^w_{\pi(r+1)}$ for all ranks $r = 1, \ldots, n(\mathbf{x}) - 1$). Then it follows from Definition 1 that a PARDCLU SWO is represented by:

$$W(\mathbf{x}) = \frac{1}{\delta} \cdot \sum\nolimits_{r=1}^{n(\mathbf{x})} \left(e^{-\delta\rho_{r-1}} - e^{-\delta\rho_r}\right) \left(u(x^w_{\pi(r)}) - u(c)\right) \,,$$

for some $c \in \mathbb{R}_+$, $\delta \in \mathbb{R}_{++}$, and continuous and increasing function u, where ρ_r is the probability adjusted rank introduced earlier.[4]

Theorem 1 *The following two statements are equivalent.*

(1) The SWR $\succsim$ *satisfies Axioms 1–7.*

[4]This follows from Definition 1 by integrating the utility weights $e^{-\delta\rho}$, leading to the following cumulative utility weights: $\int_0^{\rho} e^{-\delta\rho'} d\rho' = -\left(e^{-\delta\rho} - 1\right)/\delta$.

(2) The SWR $\succsim$ is a PARDCLU SWO.

In the PARDCLU SWO, c is the well-being level which, if experienced by an added individual without changing the utilities of the existing population, leads to an alternative which is as good as the original *only if* $\mathbf{x}_{[\nu(\mathbf{x})]} \leq c$. If $\mathbf{x}_{[\nu(\mathbf{x})]} > c$, then there is a context-dependent critical level in the open interval $(c, \mathbf{x}_{[\nu(\mathbf{x})]})$ which depends on the well-being levels that exceed c (as well as the probability p with which the added individual exists). This follows from Definition 1, since adding an individual at well-being level $\mathbf{x}_{[\nu(\mathbf{x})]}$ increases welfare, while adding an individual at well-being level c lowers the weights assigned to individuals at well-being levels that exceed c and thereby reduces welfare.

4 Special cases

Cases with no risk correspond to situations where only allocations $\mathbf{x} = (x_1, \ldots, x_{n(\mathbf{x})}) = \Big((x_1^w, x_1^p), \ldots, (x_{n(\mathbf{x})}^w, x_{n(\mathbf{x})}^p)\Big)$ with $x_i^p = 1$ for all $i = 1, \ldots n(\mathbf{x})$ are considered. The implications of rank-discounted utilitarianism in such settings are discussed in Zuber and Asheim (2012) and Asheim and Zuber (2014). With no risk the modeling here translates exactly to the variable population framework of Asheim and Zuber (2014), while it specializes the fixed population framework of Zuber and Asheim (2012) to a situation with an unbounded but finite number of generations.

In this section, we highlight special cases *with* risk. First, we show how PARDCLU reduces to rank-dependent expected utilitarianism in the special case where the probability adjusted population size is equal to 1. Second, we discuss to what extent PARDCLU provides a foundation for discounting according to the probability of human extinction, as applied in, e.g., the Stern Review (2007, Ch. 2).

Rank-dependent expected utilitarianism. In the special fixed population case where only allocations $\mathbf{x}$ with probability adjusted population size $\nu(\mathbf{x}) = \sum_{i=1}^{n(\mathbf{x})} x_i^p$ equal to 1 is considered, the result of Theorem 1 leads to *rank-dependent expected utility maximization* – where the decision maker substitutes 'decision weights' for probability – but with additional structure, as explained below. Quiggin (1982) was the first to axiomatize such a theory for decisions under risk, even though the substitution of 'decision weights' for probability had been argued by earlier writers to explain behavior inconsistent with the vNM theory.

Consider any $\mathbf{x} = (x_1, \ldots, x_{n(\mathbf{x})}) = \Big((x_1^w, x_1^p), \ldots, (x_{n(\mathbf{x})}^w, x_{n(\mathbf{x})}^p)\Big)$ with $\sum_{i=1}^{n(\mathbf{x})} x_i^p = 1$. We may choose to interpret this as one person being subject to a lottery where the prizes $(x_1^w, \ldots, x_{n(\mathbf{x})}^w)$ are won with probabilities $(x_1^p, \ldots, x_{n(\mathbf{x})}^p)$, even though we thereby depart from our basic setting without individual risk.

Let $\pi : \{1, \ldots, n(\mathbf{x})\} \to \{1, \ldots, n(\mathbf{x})\}$ be a bijection that orders the prizes $(x_1^w, \ldots, x_{n(\mathbf{x})}^w)$ in non-decreasing order:

$$x_{\pi(r)}^w \leq x_{\pi(r+1)}^w \quad \text{for all ranks } r = 1, \ldots, n(\mathbf{x}) - 1\,.$$

Write $\mathbf{p} := (x_{\pi(1)}^p, \ldots, x_{\pi(n(\mathbf{x}))}^p)$. Then PARDCLU implies preferences for lotteries that are represented by:

$$\sum\nolimits_{r=1}^{n(\mathbf{x})} h_r(\mathbf{p}) u\big(x_{\pi(r)}^w\big)\,,$$

where the probability weighting functions $h_r : [0,1]^S \to [0,1]$ are defined by

$$h_r(\mathbf{p}) = f\left(\sum\nolimits_{r'=1}^{r} x_{\pi(r')}^p\right) - f\left(\sum\nolimits_{r'=1}^{r-1} x_{\pi(r')}^p\right)\,,$$

with $f : [0,1] \to [0,1]$ given by $f(\rho) = (1 - e^{-\delta\rho})/(1 - e^{-\delta})$ and using the convention $\sum_{r'=1}^{0} x_{\pi(r')}^p = 0$.[5] Note that the function f is concave; the plausibility of this property is discussed by Quiggin (1987). Our axioms (in particular, Axiom 5) lead to the special exponential structure displayed by function f. As can be easily checked by applying l'Hôpital's rule, f approaches the identify function as $\delta \downarrow 0$. Thus, if the probability adjusted population size equals 1, then PARDCLU approaches ordinary expected utility maximization as rank-discounting vanishes.

[5]This follows from Definition 1 by integrating the utility weights $e^{-\delta\rho}$, leading to the following cumulative utility weights: $\int_0^\rho e^{-\delta\rho'} d\rho' = -\left(e^{-\delta\rho} - 1\right)/\delta$. The function f is determined by multiplying these cumulative weights by $\delta/\left(1 - e^{-\delta}\right)$ so that $f(0) = 0$ and $f(1) = 1$.

Human extinction. By appealing to Harsanyi's (1953) original position and using Harsanyi's (1955) theorem, Dasgupta and Heal (1979, pp. 269–275) justified the use of discounted utilitarianism where the utility discount rate is the probability of human extinction. Also the Stern Review (2007, Ch. 2) argued that this probability is the primary justification for utility discounting (other contributions include Bommier and Zuber, 2008, and Roemer, 2011). Blackorby, Bossert and Donaldson (2007) supported this justification within a variable population framework. To what extent is PARDCLU consistent with this position?

The variable population case where population remains constant up to the time of human extinction can be captured by letting $\mathbf{x} = (x_1, \ldots, x_{n(\mathbf{x})}) = \big((x_1^w, x_1^p), \ldots, (x_{n(\mathbf{x})}^w, x_{n(\mathbf{x})}^p)\big)$ satisfy $x_t^p = \pi^t$. The interpretation is that population is constant and its size is normalized to 1 up to the time of extinction, x_t^w is the per capita well-being of generation t, and π^t is the probability of existence of generation t, with extinction occurring for sure after generation $n(\mathbf{x})$. If well-being is correlated with time so that $x_t^w \leq x_{t+1}^w$ for all periods $t \in \{1, \ldots, n(\mathbf{x})\}$, then PARDCLU implies preferences over streams that are represented by:

$$\sum\nolimits_{t=1}^{T} \left[f\left(\tfrac{\pi(1-\pi^t)}{1-\pi}\right) - f\left(\tfrac{\pi(1-\pi^{t-1})}{1-\pi}\right) \right] u(x_t^w)\,,$$

where, as above, $f : \mathbb{R}_+ \to \mathbb{R}_+$ is given by $f(\rho) = (1 - e^{-\delta\rho})/(1 - e^{-\delta})$, but with an extended domain. This follows from Definition 1 and the argument of footnote 5 by noting that $\pi + \cdots + \pi^t = \pi(1 - \pi^t)/(1 - \pi)$.

Note that as $\delta \downarrow 0$, f approaches the identity function:

$$f\left(\tfrac{\pi(1-\pi^t)}{1-\pi}\right) - f\left(\tfrac{\pi(1-\pi^{t-1})}{1-\pi}\right) \to \tfrac{\pi}{1-\pi}\left(\pi^{t-1} - \pi^t\right) = \pi^t\,.$$

Therefore, as rank-discounting vanishes, PARDCLU approaches the principle of discounting utility according to the probability of human extinction, as applied by the Stern Review (2007, Ch. 2). However, for $\delta > 0$, PARDCLU implies that *utility* is discounted according to both rank and the probability of human extinction. If well-being is correlated with time—which is the case considered above—discounting according to rank and the probability of human extinction reinforce each other, while they might pull in opposite directions otherwise. In all cases, *well-being* is also discounted according to the absolute well-being level if the function u is strictly concave, so that well-being is transformed into utility at a decreasing rate.

5 Equity and population ethics

We introduce equity concerns as inequality aversion with respect to the distribution of well-being. We thus follow the practice of expressing distributional equity ideals though a transfer axioms by considering a variation of the Pigou-Dalton transfer principle, but take into account the fact that people may have different probabilities of existing.

Axiom 8 (Probability adjusted Pigou-Dalton) *For all* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, *if* $n(\mathbf{x}) = n(\mathbf{y}) = n$ *and there exist* i, $j \in \{1, \cdots, n\}$ *and* $\varepsilon \in \mathbb{R}_{++}$ *such that* $y_i^w + \varepsilon = x_i^w \leq x_j^w = y_j^w - \varepsilon$, $y_i^p = x_i^p = x_j^p = y_j^p$ *and* $(x_k^w, x_k^p) = (y_k^w, y_k^p)$ *for all* $k \in \{1, \cdots, n\} \setminus \{i, j\}$, *then* $\mathbf{x} \succ \mathbf{y}$.

The condition under which Axiom 8 can be satisfied by PARDCLU criteria boils down to the concavity of the function u.

Proposition 1 *A PARDCLU SWO* $\succsim$ *on* $\mathbf{X}$ *satisfies Axiom 8 if and only if* u *is concave in the representation given in Definition 1.*

The concavity of u is a standard condition for generalized utilitarian criteria respecting the Pigou-Dalton transfer principle. It is however more surprising to find this condition for rank-dependent generalized utilitarian criteria. Indeed, in the similar case of risk aversion of rank-dependent expected utility criteria, Chateauneuf, Cohen and Meilijson (2005) proved that risk aversion implies that the probability transformation function must be more convex than the function u (function u must not be 'too convex'): the concavity of u is sufficient but not necessary for risk aversion. Zuber and Asheim (2012) showed that in the case of RDU criteria, the corresponding necessary and sufficient condition for inequality aversion was that an index of non-concavity of the function u was larger than $\beta = e^{-\delta}$. The difference in the presence setting is that we may have to compare individuals with arbitrarily small probabilities of existence so that the role of rank-discounting for ensuring inequality aversion becomes negligible: we are then close to the generalized-utilitarian case where the concavity of u is necessary.

Interestingly, the Probability adjusted Pigou-Dalton principle can also be used to characterize PARDCLU together with critical-level generalized utilitarian criteria without assuming Existence of egalitarian equivalence (Axiom 7). Let us first define probability adjusted critical-level generalized utilitarian SWOs.

Definition 2 An SWR $\succsim$ on $\mathbf{X}$ is a *probability adjusted critical-level generalized utilitarian* (PACLU) SWO if there exist $c \in \mathbb{R}_+$ and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \int_0^{\nu(\mathbf{x})} \big(u(\mathbf{x}_{[\rho]}) - u(c)\big)\, d\rho \;\geq\; \int_0^{\nu(\mathbf{y})} \big(u(\mathbf{y}_{[\rho]}) - u(c)\big)\, d\rho\,.$$

Proposition 2 *If The SWR $\succsim$ on $\mathbf{X}$ satisfies Axioms 1–6 and Axiom 8, then it is either a PARDCLU SWO with u being concave or a PACLU SWO with u being strictly concave.*

Asheim and Zuber (2014) discuss the population ethics of RDCLU when comparing populations of different sizes. In particular, they show that RDCLU criteria can avoid both the Repugnant conclusion and the Very sadistic conclusion. These results hold also for PARDCLU in terms of probability adjusted population size.

Through the following two propositions we provide additional results on the population ethics of PARDCLU. We first study what is the appropriate level of the critical-level parameter c in the representation of PARDCLU given in Definition 1. To do so, we consider the following principles discussed by Blackorby, Bossert and Donaldson (2005).

Axiom 9 (Priority to lives worth living) *For all x, $y \in \mathbb{R}$ and ν, $\nu' \in \mathbb{R}_{++}$, if $x > 0 \geq y$, then $(x)_\nu \succ (y)_{\nu'}$.*

Proposition 3 *A PARDCLU SWO $\succsim$ on $\mathbf{X}$ satisfies Axiom 9 if and only if $c = 0$ in the representation given in Definition 1.*

Proposition 3 suggests that setting $c = 0$ is a natural choice if we consider that lives above the neutrality level are worth living.

Another well-known principle of population ethics is the Mere addition principle, stating that it is always worth adding people with positive well-being. In our framework, it is formalized in the following way.

Axiom 10 (Mere addition) *For all $\mathbf{x} \in \mathbf{X}$, $z \in \mathbb{R}_{++}$, and $p \in (0, 1]$, $(\mathbf{x}, (z)_p) \succ \mathbf{x}$.*

PARDCLU criteria do not satisfy the Mere addition principle. This is clear when $c > 0$ because adding people with well-being below c decreases social welfare. This is also true when $c = 0$ because adding an individual with low positive well-being will decrease the weights on individuals with higher well-being and might thereby worsen the allocation. However, this drawback of PARDCLU must be considered in light of the following impossibility result.

Proposition 4 *There is no SWO $\succsim$ on $\mathbf{X}$ satisfying Axioms 1, 7, 8, 9 and 10.*

Proposition 4 is related to the finding of Carlson (1998) that the Mere addition principle and a Non-anti egalitarianism principle imply a conclusion similar to the Repugnant conclusion. In our framework, avoiding the Repugnant conclusion is represented by our Axiom 7 and we use a Pigou-Dalton transfer principle (Axiom 8) to represent egalitarian concerns. The form of PARDCLU criteria also indicates why we may not want to satisfy the Mere addition principle: adding people with positive but very low well-being may increase relative poverty by adding people at low ranks in the distribution. This is an objection which is often made against the Mere addition principle (Arrhenius, forthcoming, chap. 7).

6 Discussion and concluding remarks

The present paper contributes to the fields of population ethics and social evaluation in risky situations by proposing and axiomatizing the probability adjusted rank-discounted critical-level generalized utilitarian (PARDCLU) SWO. We have shown how the PARDCLU approach can be used to handle the situation where there is a positive probability of human extinction. We have established how the PARDCLU SWO reduces to rank-dependent expected utility maximization with additional structure in the special case where the probability adjusted population size equals 1, thereby linking our criterion to the theory of decisions under risk. We have also highlighted some of the properties of the PARDCLU approach in terms of distributive equity and population ethics. On the latter topic, we have showed that PARDCLU can avoid drawbacks of other equitable approaches (such as utilitarian and egalitarian approaches).

When evaluating consequences that stretch centuries into the future, it seems less important to consider the fluctuations in well-being and individual risk that people face during their own lifetimes. Rather, the important issues are interpersonal inequality and the social risk associated with what level of well-being future people will experience in the world they will be born into. Consequently, we have presented a framework where individuals live for one period only and are not subject to individual risk. In this framework one cannot differentiate between inequality aversion, fluctuation aversion, and risk aversion – a distinction that is sometimes highlighted in literature on climate change evaluation. Only inequality aversion matters in the present context.

Notwithstanding its advantages, PARDCLU also faces difficulties. First PARDCLU SWOs are not expected utilities. Hammond (1983) suggested that if social decision-making is consequentialist (that is, if social situations in each state of the world are assessed only on basis of their consequences in this state of the world), non-expected utility criteria induce time inconsistent choices. This issue also arises for the PARDCLU approach when there is no risk, as discussed in Zuber and Asheim

(2012), if decision making is time invariant. The problem of time consistency when there is risk is however more severe because not only the past, but also unrealized states of the world matter for social evaluation. This dependence on unrealized states of the world is not specific to PARDCLU: it also arises for criteria such as those suggested by Diamond (1967), Epstein and Segal (1992) and Grant et al. (2010).

Given this feature of PARDCLU criteria, there are two possible routes, which we want to explore in future work. One direction would be to reject consequentialism and assume that choices may depend on what could have happened. The issue then is whether the relevant information can be summarized in a practical way in specific economic environments, so that the dependence is manageable for public decision making. Another direction would be to accept that the social criterion is not time-consistent, and to device techniques such as sophisticated planning (see Pollak, 1968; Blackorby et al., 1973, for early references) to ensure the time consistency of choices (at the cost of optimality from the point of view of the initial criterion). We could compare in specific economic models such sophisticated planning to naive planning, in particular when fertility choices are endogenous.

A second important feature of evaluation based on PARDCLU is how it handles the risk on the planning horizon. According to PARDCLU, it is the total population, rather than the planning horizon, that matters. In particular, social evaluation based on PARDCLU is completely indifferent between having 10 billion people alive for 100 years and 1 billion people alive for 1000 years if all have the same well-being and live for sure, as total population is the same in both alternatives. One may object to this conclusion on the basis that people might prefer to live in a society with more people (so as to have richer scope for social interactions), or on the contrary to have more descendants.

Note that the issue extends to the case where population size is risky. If well-being is perfectly equal, then only expected total population size matters, so that society is completely risk neutral with respect to the risk on population size. Evaluation based on PARDCLU is indifferent whether n people exist for sure, or n_1

people exist with probability p and n_2 people with probability $1-p$, provided that $pn_1 + (1-p)n_2 = n$. This is in stark contrast with criteria exhibiting catastrophe avoidance in the sense of Bommier and Zuber (2008).

A last issue is sustainability. Zuber and Asheim (2012, Section 6) show how RDU leads to sustainable outcomes in models of economic growth within a setting where there are infinitely many time periods. This basic support for sustainability does not extend to the present criterion with endogenous population size and probability of existence, where the main concern is to avoid lives with low well-being. A stark conclusion is that it might be socially preferable to increase the per-period probability of extinction if per capita well-being is decreasing over time, as this increases the utility weight on the better off earlier generations. This points towards re-evaluating the concept of sustainability in a context where the number of future generations is bounded and their existence is uncertain, and where there might be a trade-off between the number of future people and their well-being.

A possible way to deal with both the indifference to risk on total population size and the willingness to ensure the existence of future people would be to include sentiments, and in particular the altruistic feelings parents have towards their descendants. There is a growing literature on social and altruistic preferences, both from a theoretical and from an experimental perspective (classical references include Fehr and Schmidt, 1999; Bolton and Ockenfels, 2000; Charness and Rabin, 2002; Andreoni and Miller, 2002). In general, there might be an argument in favor of distinguishing the conception of justice from the forces (like altruism) that are instrumental in attaining it, e.g., if impartiality follows from considering an original position where individuals do not have extensive times of natural sentiments (Rawls, 1999, p. 129). However, considering the social context in which people live seems essential when applying PARDCLU in a setting where population size and probability of existence are endogenous. In particular, a more pro-natal implication would follow if we assume that the well-being of individuals depends also on their reproductive choices, so that well-being of one generation increases with the size and living conditions of the next generation.

The implications of including sentiments for the PARDCLU approach (and for population ethics in general) is left for future research.

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Appendices

A Descriptive dynamic framework

In this appendix, we establish a perfect correspondence between the abstract framework with a set of atemporal allocations presented in Section 2 and a more descriptive dynamic framework. This exercise has both a theoretical and practical motivation:

(i) It provides a theoretical underpinning for our abstract framework in a dynamic model of risk where information is revealed through time and where the learning dynamics might depend on choices made by the decision maker.

(i1) It lies the groundwork for potential application in numerical models (e.g. of climate change) where individuals are distributed through time.

Let Ω be the space of *states* of the world. Endow Ω with the σ–algebra $\mathcal{F}$, being the collection of all Lebesgue measurable subsets of Ω, and a probability measure $p : \mathcal{F} \to [0, 1]$, so that $(\Omega, \mathcal{F}, p)$ is a probability space. The states are exogenously given and their probabilities cannot be influenced. One can think of the probabilities as a priori expert opinions used by the social decision maker in a Bayesian setting.

We assume that time is discrete. An *information structure*, $(I^t)_{t\in\mathbb{N}}$, determines the process through which the true state is learned. Formally, $I^1, I^2, \ldots, I^t, \ldots$ is a countable sequence of finite partitions of Ω where, for each $t \in \mathbb{N}$, (i) $p(i^t) > 0$ for all $i^t \in I^t$ and (ii) I^{t+1} is a weak refinement of I^t. The information structure $(I^t)_{t\in\mathbb{N}}$ determines a *filtration* $(\mathcal{F}_t)_{t\in\mathbb{N}}$ of $\mathcal{F}$ where, for each $t \in \mathbb{N}$, $\mathcal{F}_t$ is the collection of all unions of sets in I^t (including the empty set).

For given information structure $(I^t)_{t\in\mathbb{N}}$, an *allocation process*, $\mathbf{w} = (w^t)_{t\in\mathbb{N}}$, is a process from $\mathbb{N} \times \Omega$ to $\bigcup_{n\in\mathbb{N}_0} \mathbb{R}^n$ which is adapted to the filtration $(\mathcal{F}_t)_{t\in\mathbb{N}}$ so that, for each $t \in \mathbb{N}$, $w^t : \Omega \to \bigcup_{n\in\mathbb{N}_0} \mathbb{R}^n$ is an $\mathcal{F}_t$–measurable function. Here $\mathbb{R}^0$ represents the set containing only the situation where no one exists. The allocation process $\mathbf{w}$ maps each period-state pair $(t, \omega) \in \mathbb{N} \times \Omega$ into $w^t(\omega) \in \mathbb{R}^n$, which determines the population size, $n^t_{\mathbf{w}}(\omega) = n$ and the distribution of well-being,

$$w^t(\omega) = (w_1^t(\omega), \ldots, w_{n_{\mathbf{w}}^t(\omega)}^t(\omega)),$$

if $n_{\mathbf{w}}^t(\omega) > 0$ and the situation where no one exists if $n_{\mathbf{w}}^t(\omega) = 0$. Thus, $\mathbf{w}$ determines a *population process*, $\mathbf{n_w} = (n_{\mathbf{w}}^t)_{t\in\mathbb{N}}$, where, for each $t \in \mathbb{N}$, $n_{\mathbf{w}}^t : \Omega \to \mathbb{N}_0$ is an $\mathcal{F}_t$–measurable function. We require that, for any allocation process $\mathbf{w}$, there exists $t(\mathbf{w}) \in \mathbb{N}$ such that $n_{\mathbf{w}}^t(\omega) = 0$ for all $\omega \in \Omega$ if and only if $t > t(\mathbf{w})$. Hence, the total probability adjusted population size, $\sum_{t\in\mathbb{N}} \int_\Omega n_{\mathbf{w}}^t(\omega)dp(\omega)$, is positive and finite.

The social decision maker evaluates $((I^t)_{t\in\mathbb{N}}, \mathbf{w})$. For any such pair, we obtain, for each period $t \in \mathbb{N}$ and each smallest non-empty $\mathcal{F}_t$–measurable event $i^t \in I^t$ with $n_{\mathbf{w}}^t(i^t) > 0$, a distribution of pairs of well-being and probability:

$$\Big((w_1^t(i^t), p(i^t)), \ldots, (w_{n_{\mathbf{w}}^t(i^t)}^t(i^t), p(i^t))\Big).$$

Concatenating such vectors over all periods $t \in \mathbb{N}$ and all events $i^t \in I^t$ with $n^t(i^t) > 0$ yields a vector $\mathbf{x}$ in the set of possible finite allocations, $\mathbf{X}$, since there are only a finite number of period-event pairs (t, i^t) with positive population. In particular, the probability adjusted population size of $\mathbf{x}$, $\nu(\mathbf{x})$, equals $\sum_{t\in\mathbb{N}} \int_\Omega n_{\mathbf{w}}^t(\omega)dp(\omega)$. This establishes that any pair $((I^t)_{t\in\mathbb{N}}, \mathbf{w})$ can be mapped to an allocation in $\mathbf{X}$.

Conversely, we can map any allocation $\mathbf{x} = \Big((x_1^w, x_1^p), \ldots, (x_{n(\mathbf{x})}^w, x_{n(\mathbf{x})}^p)\Big) \in \mathbf{X}$ to a pair $((I^t)_{t\in\mathbb{N}}, \mathbf{w})$. To see that, first re-order the components of $\mathbf{x}$ to obtain a new allocation $\tilde{\mathbf{x}}$ such that $\tilde{x}_k^p \geq \tilde{x}_{k+1}^p$ for all $k = 1, \cdots, n(\mathbf{x}) - 1$ (noting that a permutation π such that $(\tilde{x}_k^w, \tilde{x}_k^p) = (x_{\pi(k)}^w, x_{\pi(k)}^p)$ for all $k = 1, \cdots, n(\mathbf{x})$ exists). Construct the sequence $(I^t)_{n\in\mathbb{N}}$ of partitions inductively in the following way:

1. If $\tilde{x}_1^p = 1$, then $I^1 = \{\Omega\}$ with E_1 denoting $\emptyset$, and if $\tilde{x}_1^p < 1$, then $I^1 = \{i_1, \Omega \setminus E_1\}$, where $i_1 \subsetneq \Omega$ is chosen so that $p(\Omega \setminus E_1) = \tilde{x}_1^p$ with E_1 denoting i_1;

2. For $t = 2, \cdots, n(\mathbf{x})$, if $\tilde{x}_{k+1}^p = \tilde{x}_k^p$, then $I^t = I^{t-1}$ with E_t denoting E_{t-1}, and if $\tilde{x}_{k+1}^p < \tilde{x}_k^p$, then $I^t = \big(I^{t-1} \cup \{i_t, \Omega \setminus E_t\}\big) \setminus \{\Omega \setminus E_{t-1}\}$, where $i_t \subsetneq \Omega \setminus E_{t-1}$ is chosen so that $p(\Omega \setminus E_t) = \tilde{x}_t^p$ with E_t denoting $E_{t-1} \cup i_t$.

3. For $t > n(\mathbf{x})$, $I^t = I^{n(\mathbf{x})}$ with E_t denoting Ω.

Construct the adapted allocation process $\mathbf{w} = (w^t)_{t\in\mathbb{N}}$ by, for all $t \in \mathbb{N}$, $w^t(\omega) = \tilde{x}_t^w$ if $\omega \in \Omega \backslash E_t$ and $w^t(\omega)$ maps to the situation where no one exists if $\omega \in E_t$. Hence, in each period there exists at most one individual and only in periods $t \leq n(\mathbf{x})$ and in states whose total probability is $\tilde{x}_t^p$. The well-being of the individual in period t when he exists is $\tilde{x}_t^w$. Clearly, the pair $((I^t)_{t\in\mathbb{N}}, \mathbf{w})$ permits to produce the allocation $\tilde{\mathbf{x}}$ of well-being and probabilities of existence, which is the same as $\mathbf{x}$ up to a permutation. There are of course more realistic ways of doing so, where e.g. individuals with the same probability of existence belong to the same generation.

The pair $((I^t)_{t\in\mathbb{N}}, \mathbf{w})$ is endogenously given by the policies chosen in the economy. The information structure $(I^t)_{t\in\mathbb{N}}$ can change depending on the decision maker's investment to learn about the true state of the world. For instance, in the case of climate change, more resources can be allocated to better understand how climate systems work and what are the effects of temperature change. This makes it possible to accelerate the refinement of the state space partition.

In this formulation, individuals' identities are implicitly defined by the information structure. An individual at time t exists only in one event i^t of the partition I^t that reflects the information available at period t. This also implies that individuals live for one period only and are not subjected to risk during their lifetime.[6] Each potential individual is born after the realization of the event relevant for his identity, and all risk in the economy is borne by society. Our focus on intergenerational issues motivates this abstraction from lifetime fluctuations and individual risk.

The choice of the pair $((I^t)_{t\in\mathbb{N}}, \mathbf{w})$ will be limited by the feasibility constraints concerning the possibility for learning and the development of well-being and population. As we are concerned with modeling the social decision maker's preferences over such pairs, such feasibility constraints will not be discussed here.

[6] Alternatively, one could assume that there is another layer of risk related to individual lifetime fluctuations and that the well-being measures w_i^t already incorporate it (they may be expected utilities or certainty equivalent measures).

B Proof of Theorem 1

To prove the Theorem 1, we need to introduce subsets of $\mathbf{X}$. For any $k \in \mathbb{N}$, denote by $\mathbf{X}_{1/k} = \{\mathbf{x} \in \mathbf{X} : x_i^p = 1/k, \ \forall i \in \{1, \ldots, n(\mathbf{x})\}\}$ the set of allocations where all individuals have the same rational probability $1/k$ of existing. Denote by $\mathbb{Q}_{++}$ the positive rational numbers and by $\mathbf{X}_{\mathbb{Q}_{++}} = \{\mathbf{x} \in \mathbf{X} : x_i^p \in \mathbb{Q}_{++}, \ \forall i \in \{1, \ldots, n(\mathbf{x})\}\}$ the set of allocations where all individuals have probabilities of existing which are positive rational numbers.

It is straightforward to show that (2) implies (1) in Theorem 1. We show that (1) implies (2) by proving the three following lemmas.

We start with Lemma 1, which establishes how the representation result of Asheim and Zuber (2014) can be extended to the present case for allocations with the same probability-adjusted population size.

Lemma 1 *If Axioms 1–5 hold, then there exist a number $\delta \in \mathbb{R}$ and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that for all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_\nu$,*

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \int_0^\nu e^{-\delta\rho} \big(u(\mathbf{x}_{[\rho]}) - u(c)\big)\, d\rho \geq \int_0^\nu e^{-\delta\rho} \big(u(\mathbf{y}_{[\rho]}) - u(c)\big)\, d\rho \qquad \text{(B.1)}$$

Proof. ***Case 1:* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_{1/k}$.** For any $k \in \mathbb{N}$, Axioms 1–5 above restricted to $\mathbf{X}_{1/k}$ collapse to Axioms 1–5 of Asheim and Zuber (2014). Hence, by Lemma 1 of Asheim and Zuber (2014) there exist $\beta_{1/k} \in \mathbb{R}_{++}$ and a continuous increasing function $u_{1/k} : \mathbb{R} \to \mathbb{R}$ such that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_{1/k}$, if $n(\mathbf{x}) = n(\mathbf{y}) = n$ then $\mathbf{x} \succsim \mathbf{y}$ if and only if

$$\textstyle\sum_{r=1}^{n} \beta_{1/k}^{r-1} u_{1/k}\big(x_{\pi(r)}^w\big) \geq \sum_{r=1}^{n} \beta_{1/k}^{r-1} u_{1/k}\big(y_{\pi(r)}^w\big). \qquad \text{(B.2)}$$

Consider any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_1$ such that $n(\mathbf{x}) = n(\mathbf{y}) = n$. For any $k \in \mathbb{N}$, construct $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_{1/k}$ such that $n(\hat{\mathbf{x}}) = n(\hat{\mathbf{y}}) = nk$ and, for any $i \in \{1, \ldots, n\}$, $\hat{x}_{ki-j}^w = x_i^w$ and $\hat{y}_{ki-j}^w = y_i^w$ for all $j \in \{0, \ldots, k-1\}$. By construction, $\nu(\mathbf{x}) = \nu(\mathbf{y}) = \nu(\hat{\mathbf{x}}) = \nu(\hat{\mathbf{y}}) = n$, $\mathbf{x}_{[\,]} = \hat{\mathbf{x}}_{[\,]}$ and $\mathbf{y}_{[\,]} = \hat{\mathbf{y}}_{[\,]}$. By Axioms 1, 2 and 3, we have $\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \hat{\mathbf{x}} \succsim \hat{\mathbf{y}}$, and therefore, using the above representation:

$$
\begin{aligned}
& \sum\nolimits_{r=1}^{n} \beta_1^{r-1} u_1\big(\mathbf{x}(\pi(r)\big) \geq \sum\nolimits_{r=1}^{n} \beta_1^{r-1} u_1\big(\mathbf{y}(\pi(r))\big) \\
\iff \; & \sum\nolimits_{r'=1}^{nk} \beta_{1/k}^{r'-1} u_{1/k}\big(\hat{\mathbf{x}}(\pi(r')\big) \geq \sum\nolimits_{r'=1}^{nk} \beta_{1/k}^{r'-1} u_{1/k}\big(\hat{\mathbf{y}}(\pi(r'))\big) \\
\iff \; & \sum\nolimits_{r=1}^{n} (\beta_{1/k}^{k})^{r-1} u_{1/k}\big(\mathbf{x}(\pi(r)\big) \geq \sum\nolimits_{r'=1}^{nk} (\beta_{1/k}^{k})^{r-1} u_{1/k}\big(\mathbf{y}(\pi(r))\big)
\end{aligned}
$$

since $(1-\beta_{1/k}) \cdot \sum\nolimits_{r'=1}^{k} \beta_{1/k}^{r'-1} = 1 - \beta_{1/k}^{k}$. Because additive representations are unique up to an affine transformation, the above equivalence implies $\beta_{1/k} = (\beta_1)^{1/k}$ and that we can set $u_{1/k} = u_1$, using the normalization $u_{1/k}(0) = u_1(0) = 0$.

Denoting $\delta = -\ln \beta_1$, this implies that $\beta_{1/k} = (\beta_1)^{1/k} = e^{-\delta/k}$. Moreover, since

$$
(\beta_{1/k})^{r-1} = \frac{1-e^{-\delta/k}}{1-e^{-\delta/k}} e^{-\delta(r-1)/k} = \frac{e^{-\delta(r-1)/k} - e^{-r\delta/k}}{1-e^{-\delta/k}} = \frac{\delta}{1-e^{-\delta/k}} \int_{(r-1)/k}^{r/k} e^{-\delta\rho} d\rho \,,
$$

and by denoting $u = u_1$ we can rewrite inequality (B.2) as:

$$
\int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho \geq \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho \,,
$$

where $\nu = n/k$.

***Case 2:* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_{\mathbb{Q}_{++}}$.** For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{\mathbb{Q}_{++}}$ such that $\nu(\mathbf{x}) = \nu(\mathbf{y}) = \nu$, let k be the least common denominator of all the probabilities in the two allocations. This means that for all $i \in \{1, \ldots, n(\mathbf{x})\}$ there exists a positive integer $\ell_i^{\mathbf{x}}$ such that $x_i^p = \ell_i^{\mathbf{x}}/k$. Similarly, for all $i \in \{1, \ldots, n(\mathbf{y})\}$, there exists a positive integer $\ell_i^{\mathbf{y}}$ such that $y_i^p = \ell_j^{\mathbf{y}}/k$.

We can construct $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_{1/k}$ in the following way:[7]

(a) For any $i \in \{1, \ldots, n(\mathbf{x})\}$, $\hat{x}^w_{\ell + \sum_{j=1}^{i-1} \ell_j^{\mathbf{x}}} = x_i^w$ for all $\ell \in \{1, \ldots, \ell_i^{\mathbf{x}}\}$;

(b) For any $i \in \{1, \ldots, n(\mathbf{y})\}$, $\hat{y}^w_{\ell + \sum_{j=1}^{i-1} \ell_j^{\mathbf{y}}} = y_i^w$ for all $\ell \in \{1, \ldots, \ell_i^{\mathbf{y}}\}$.

[7] Using the convention $\sum_{j=1}^{0} \ell_j^{\mathbf{x}} = \sum_{j=1}^{0} \ell_j^{\mathbf{y}} = 0$.

By construction, $\nu(\mathbf{x}) = \nu(\mathbf{y}) = \nu(\hat{\mathbf{x}}) = \nu(\hat{\mathbf{y}}) = n$, $\mathbf{x}_{[\,]} = \hat{\mathbf{x}}_{[\,]}$ and $\mathbf{y}_{[\,]} = \hat{\mathbf{y}}_{[\,]}$, and

$$\begin{aligned} \mathbf{x} \succsim \mathbf{y} \quad &\Longleftrightarrow \quad \hat{\mathbf{x}} \succsim \hat{\mathbf{y}} \\ &\Longleftrightarrow \quad \int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho \geq \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho \end{aligned}$$

by the result in Case 1.

Case 3: Extension to any* $\mathbf{x}$*,* $\mathbf{y} \in \mathbf{X}$*. Consider any $\nu \in \mathbb{R}_{++}$, and any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_{\nu}$. If $\mathbf{x}$, $\mathbf{y} \in \mathbf{Q}_{++}$ (so that $\nu \in \mathbb{Q}_{++}$), then we are back to Case 2, and the result holds. Assume therefore that $\mathbf{x}$, $\mathbf{y} \notin \mathbf{Q}_{++}$ and more specifically that $x_i^p \in \mathbf{Q}_{++}$ for all $i \in \{1, \ldots, n(\mathbf{x})-1\}$, $y_i^p \in \mathbf{Q}_{++}$ for all $j \in \{1, \ldots, n(\mathbf{y})-1\}$, and $x_{n(\mathbf{x})}$, $y_{n(\mathbf{y})} \notin \mathbf{Q}_{++}$. Assuming that the last individual is the one with an irrational probability of existing is made without loss of generality because of Axiom 3. Extension of the proof to more than one individual with an irrational probability of existing is similar to the one developed below. Because of Axiom 1, equivalence (B.1) holds if and only if the following equivalence holds:

$$\mathbf{x} \succ \mathbf{y} \Longleftrightarrow \int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho > \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho \,.$$

Step 1: $\mathbf{x} \succ \mathbf{y} \Longrightarrow \int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho > \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho$. Assume that $\mathbf{x} \succ \mathbf{y}$. By Axiom 2, there exists $\tilde{\mathbf{x}} \in \mathbf{X}_{\nu}$ such that $\tilde{\mathbf{x}}_{[\,]} < \mathbf{x}_{[\,]}$ and $\tilde{\mathbf{x}} \succ \mathbf{y}$. It is sufficient to show $\int_0^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho]}) d\rho \geq \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho$ since then it follows by the definitions of the step functions $\mathbf{x}_{[\,]}$ and $\tilde{\mathbf{x}}_{[\,]}$ that $\mathbf{x}_{[\,]} > \tilde{\mathbf{x}}_{[\,]}$ implies $\int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho > \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho$.

Let $\hat{\nu} \in \mathbf{Q}_{++}$ such that $0 < \hat{\nu} - \nu < 1$, and denote $\hat{p} = \hat{\nu} - \nu$. Let $p_{\tilde{x}} \in \mathbf{Q}_{++}$ be such that $0 < p_{\tilde{x}} < \tilde{x}^p_{n(\tilde{\mathbf{x}})}$ and denote $\varepsilon_{\tilde{x}} = \tilde{x}^p_{n(\tilde{\mathbf{x}})} - p_{\tilde{x}}$. Likewise, let $p_y \in \mathbf{Q}_{++}$ be such that $x^p_{n(\mathbf{y})} < p_y < \hat{p}$ and denote $\varepsilon_y = p_y - x^p_{n(\mathbf{y})}$.

Let $z = \max\{\tilde{\mathbf{x}}_{[\nu]}, \mathbf{y}_{[\nu]}\}$. By Axiom 4,

$$\tilde{\mathbf{x}} \succ \mathbf{y} \Longrightarrow (\tilde{\mathbf{x}}, z_{\hat{p}}) \succsim (\mathbf{y}, z_{\hat{p}}) \,.$$

Construct $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_{\hat{\nu}}$ such that

(a) $\hat{x}_i = \tilde{x}_i$ for all $i \in \{1, \dots, n(\tilde{\mathbf{x}}) - 1\}$; $\hat{x}_{n(\tilde{\mathbf{x}})} = (\tilde{x}^w_{n(\tilde{\mathbf{x}})}, p_{\tilde{x}})$; $\hat{x}_{n(\tilde{\mathbf{x}})+1} = (z, \hat{p} + \varepsilon_{\tilde{x}})$;

(b) $\hat{y}_i = y_i$ for all $i \in \{1, \dots, n(\mathbf{y}) - 1\}$; $\hat{y}_{n(\mathbf{y})} = (y^w_{n(\mathbf{y})}, p_y)$; $\hat{y}_{n(\mathbf{y})+1} = (z, \hat{p} - \varepsilon_y)$.

By construction, $\nu(\hat{\mathbf{x}}) = \nu(\hat{\mathbf{y}}) = \nu(\tilde{\mathbf{x}}, z_{\hat{p}}) = \nu(\mathbf{y}, z_{\hat{p}}) = \hat{\nu}$, $\hat{\mathbf{x}}_{[\,]} > (\tilde{\mathbf{x}}, z_{\hat{p}})_{[\,]}$ and $\hat{\mathbf{y}}_{[\,]} < (\mathbf{y}, z_{\hat{p}})_{[\,]}$. By Axioms 1 and 3,

$$(\tilde{\mathbf{x}}, z_{\hat{p}}) \succsim (\mathbf{y}, z_{\hat{p}}) \Longrightarrow \hat{\mathbf{x}} \succ \hat{\mathbf{y}}\,.$$

Also, by construction, $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_{\mathbb{Q}_{++}}$.[8] Hence, by the result in Case 2, we know that

$$\hat{\mathbf{x}} \succ \hat{\mathbf{y}} \Longleftrightarrow \int_0^{\nu'} e^{-\delta\rho} u(\hat{\mathbf{x}}_{[\rho]}) d\rho > \int_0^{\nu'} e^{-\delta\rho} u(\hat{\mathbf{y}}_{[\rho]}) d\rho$$

Let $\tilde{r}$ be the rank of $x_{n(\tilde{\mathbf{x}})}$ in $\tilde{\mathbf{x}}$ and r be the rank of $y_{n(\mathbf{y})}$ in $\mathbf{y}$. By definition of $\hat{\mathbf{x}}_{[\,]}$ and $\hat{\mathbf{y}}_{[\,]}$, we have:

$$\begin{aligned}
& \int_0^{\hat{\nu}} e^{-\delta\rho} u(\hat{\mathbf{x}}_{[\rho]}) d\rho \\
= & \int_0^{\rho_{\tilde{r}} - \varepsilon_{\tilde{x}}} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho]}) d\rho + e^{\delta\varepsilon_{\tilde{x}}} \int_{\rho_{\tilde{r}}}^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho_{\tilde{r}}]}) d\rho \\
& \qquad + \int_{\nu - \varepsilon_{\tilde{x}}}^{\nu} e^{-\delta\rho} u(z) d\rho + \int_{\nu}^{\hat{\nu}} e^{-\delta\rho} u(z) d\rho \\
= & \int_0^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho]}) d\rho + (e^{\delta\varepsilon_{\tilde{x}}} - 1) \int_{\rho_{\tilde{r}}}^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho_{\tilde{r}}]}) d\rho - \int_{\rho_{\tilde{r}} - \varepsilon_{\tilde{x}}}^{\rho_{\tilde{r}}} e^{-\delta\rho} u(x^w_{n(\tilde{\mathbf{x}})}) d\rho \\
& \qquad + \int_{\nu - \varepsilon_{\tilde{x}}}^{\nu} e^{-\delta\rho} u(z) d\rho + \int_{\nu}^{\hat{\nu}} e^{-\delta\rho} u(z) d\rho \\
= & \int_0^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho]}) d\rho + \int_{\nu}^{\hat{\nu}} e^{-\delta\rho} u(z) d\rho \\
& \qquad + (e^{\delta\varepsilon_{\tilde{x}}} - 1) \left(\int_{\rho_{\tilde{r}}}^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho_{\tilde{r}}]}) d\rho + \frac{e^{-\delta\nu} u(z) - e^{-\delta\rho_{\tilde{r}}} u(\tilde{x}^w_{n(\tilde{\mathbf{x}})})}{\delta} \right)
\end{aligned}$$

and likewise:

$$\int_0^{\hat{\nu}} e^{-\delta\rho} u(\hat{\mathbf{y}}_{[\rho]}) d\rho$$

[8] Indeed, $\nu - \tilde{x}^p_{n(\tilde{\mathbf{x}})}$ and $\nu - y^p_{n(\mathbf{y})}$ are rational number because all individuals but the last one have rational probabilities of existing.

$$
\begin{aligned}
&= \int_0^{\rho_r+\varepsilon_y} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho + e^{-\delta\varepsilon_y} \int_{\rho_r}^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho_r]}) d\rho \\
&\qquad - \int_{\nu}^{\nu+\varepsilon_y} e^{-\delta\rho} u(z) d\rho + \int_{\nu}^{\hat{\nu}} e^{-\delta\rho} u(z) d\rho \\
&= \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho + \int_{\nu}^{\hat{\nu}} e^{-\delta\rho} u(z) d\rho \\
&\qquad - (1 - e^{-\delta\varepsilon_y}) \left(\int_{\rho_r}^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho_r]}) d\rho + \frac{e^{-\delta\nu} u(z) - e^{-\delta\rho_r} u(x^w_{n(\mathbf{y})})}{\delta} \right)
\end{aligned}
$$

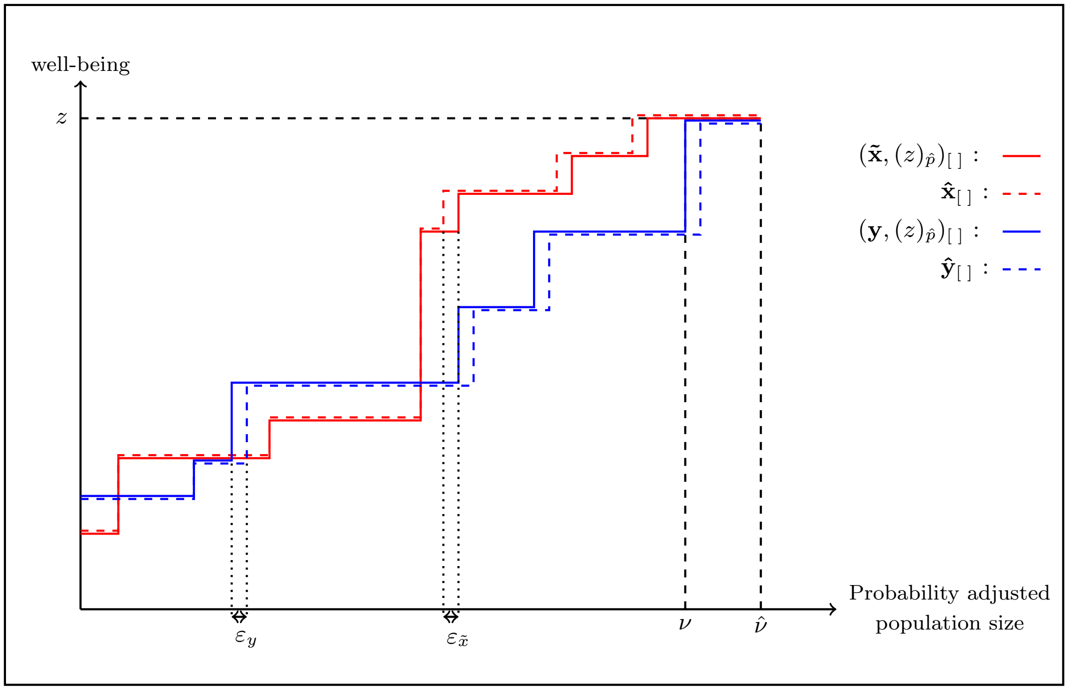

Figure 1: Allocations involved in Step 1 of the proof in Case 3

To sum up: $\mathbf{x} \succ \mathbf{y} \Longrightarrow \tilde{\mathbf{x}} \succ \mathbf{y} \Longrightarrow (\tilde{\mathbf{x}}, (z)_{\hat{p}}) \succsim (\mathbf{y}, (z)_{\hat{p}}) \Longrightarrow \hat{\mathbf{x}} \succ \hat{\mathbf{y}} \Longrightarrow$

$$
\begin{aligned}
&\int_0^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho]}) d\rho + (e^{\delta\varepsilon_{\tilde{x}}} - 1) \left(\int_{\rho_{\tilde{r}}}^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho_{\tilde{r}}]}) d\rho + \frac{e^{-\delta\nu} u(z) - e^{-\delta\rho_{\tilde{r}}} u(\tilde{x}^w_{n(\tilde{\mathbf{x}})})}{\delta} \right) > \\
&\int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho - (1 - e^{-\delta\varepsilon_y}) \left(\int_{\rho_r}^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho_r]}) d\rho + \frac{e^{-\delta\nu} u(z) - e^{-\delta\rho_r} u(x^w_{n(\mathbf{y})})}{\delta} \right).
\end{aligned}
$$

This implication is true for any $(\varepsilon_{\tilde{x}}, \varepsilon_y) \in \mathbb{R}^2_{++}$ as defined above. Since rational number are dense in the real number, it is possible to find a sequence of $((\varepsilon_{\tilde{x}}, \varepsilon_y)) \in$

$(\mathbb{R}^2)^{\mathbb{N}}$ such that each of $\varepsilon_{\tilde{x}}$ and ε_y tends to zero. Hence:

$$\mathbf{x} \succ \mathbf{y} \implies \int_0^{\nu} e^{-\delta\rho} u(\tilde{\mathbf{x}}_{[\rho]}) d\rho \geq \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho\,.$$

Figure 1 illustrates the construction of the different allocations involved in Step 1.

Step 2: $\int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho > \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho \Longrightarrow \mathbf{x} \succ \mathbf{y}$. Assume that

$$\int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho > \int_0^{\nu} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho\,.$$

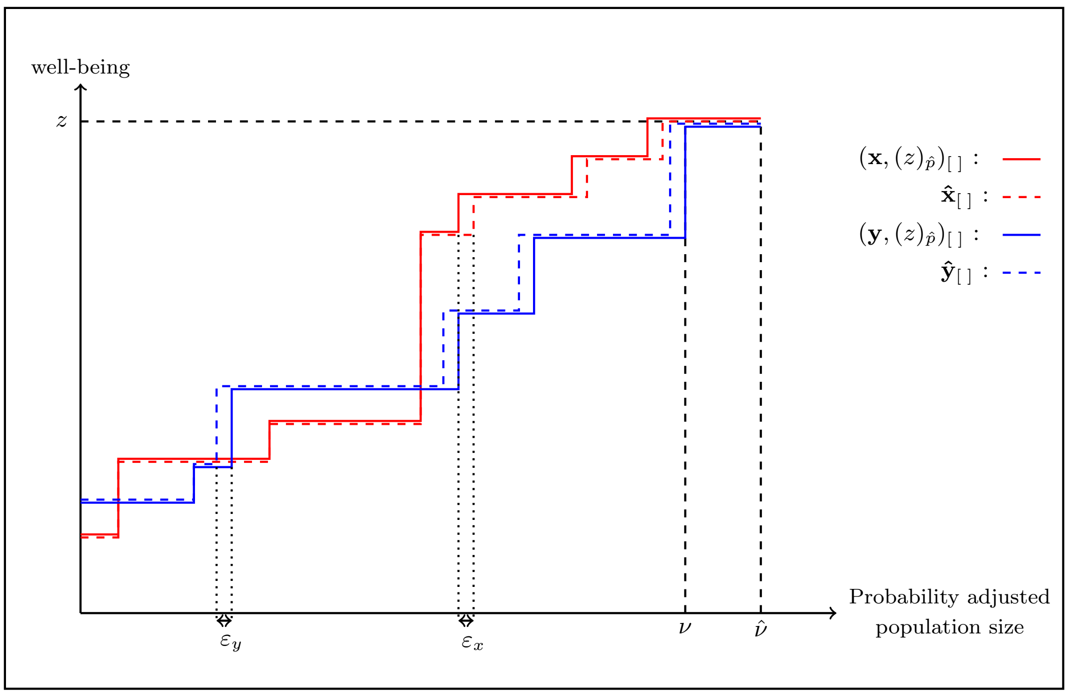

Figure 2: Allocations involved in Step 2 of the proof in Case 3

Since rational number are dense in real numbers, it is possible to find $(\varepsilon_x, \varepsilon_y) \in (0,1)^2$ such that $p_x = x^p_{n(\mathbf{x})} + \varepsilon_x$ and $p_y = y^p_{n(\mathbf{y})} - \varepsilon_y$ satisfy $(p_x, p_y) \in \mathbb{Q}^2_{++}$, and:

$$\int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho - (1 - e^{-\delta\varepsilon_x}) \left(\int_{\rho_{\tilde{r}}}^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho_{\tilde{r}}]}) d\rho + \frac{e^{-\delta\nu} u(z) - e^{-\delta\rho_{\tilde{r}}} u(x^w_{n(\mathbf{x})})}{\delta} \right) >$$

$$\textstyle\int_0^\nu e^{-\delta\rho}u(\mathbf{y}_{[\rho]})d\rho + (e^{\delta\varepsilon_y}-1)\Big(\int_{\rho_r}^\nu e^{-\delta\rho}u(\mathbf{y}_{[\rho_r]})d\rho + \frac{e^{-\delta\nu}u(z)-e^{-\delta\rho_r}u(x^w_{n(\mathbf{y})})}{\delta}\Big),$$

where $\tilde{r}$ is the rank of $x_{n(\mathbf{x})}$ in $\mathbf{x}$, r is the rank of $y_{n(\mathbf{y})}$ in $\mathbf{y}$, and $z = \max\{\tilde{\mathbf{x}}_{[\nu]}, \mathbf{y}_{[\nu]}\}$.

Let $\varepsilon_x < \hat{p} < 1$ be such that $\hat{\nu} = \nu + \hat{p}$ satisfies $\hat{\nu} \in \mathbb{Q}_{++}$. We can construct $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_{\hat{\nu}}$ in the following way:

(a) $\hat{x}_i = x_i$ for all $i \in \{1, \ldots, n(\mathbf{x}) - 1\}$; $\hat{x}_{n(\mathbf{x})} = (x^w_{n(\mathbf{x})}, p_x)$; $\hat{x}_{n(\mathbf{x})+1} = (z, \hat{p} - \varepsilon_x)$;

(b) $\hat{y}_i = y_i$ for all $i \in \{1, \ldots, n(\mathbf{y}) - 1\}$; $\hat{y}_{n(\mathbf{y})} = (y^w_{n(\mathbf{y})}, p_y)$; $\hat{y}_{n(\mathbf{y})+1} = (z, \hat{p} + \varepsilon_y)$;

so that $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_{\mathbb{Q}_{++}}$, $\nu(\hat{\mathbf{x}}) = \nu(\hat{\mathbf{y}}) = \nu(\mathbf{x}, (z)_{\hat{p}}) = \nu(\mathbf{y}, (z)_{\hat{p}}) = \hat{\nu}$, $\hat{\mathbf{x}}_{[\,]} < (\mathbf{x}, (z)_{\hat{p}})_{[\,]}$ and $\hat{\mathbf{y}}_{[\,]} > (\mathbf{y}, (z)_{\hat{p}})_{[\,]}$ (see Figure 2).

By the result in Case 2 and by construction of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$,

$$\begin{aligned}&\int_0^\nu e^{-\delta\rho}u(\mathbf{x}_{[\rho]})d\rho - (1-e^{-\delta\varepsilon_x})\Big(\int_{\rho_r}^\nu e^{-\delta\rho}u(\mathbf{x}_{[\rho_r]})d\rho + \frac{e^{-\delta\nu}u(z)-e^{-\delta\rho_r}u(x^w_{n(\mathbf{x})})}{\delta}\Big) > \\ &\int_0^\nu e^{-\delta\rho}u(\mathbf{y}_{[\rho]})d\rho + (e^{\delta\varepsilon_y}-1)\Big(\int_{\rho_{r'}}^\nu e^{-\delta\rho}u(\mathbf{y}_{[\rho_{r'}]})d\rho + \frac{e^{-\delta\nu}u(z)-e^{-\delta\rho_{r'}}u(x^w_{n(\mathbf{y})})}{\delta}\Big)\end{aligned}$$

$\implies \int_0^{\hat{\nu}} e^{-\delta\rho}u(\hat{\mathbf{x}}_{[\rho]})d\rho > \int_0^{\hat{\nu}} e^{-\delta\rho}u(\hat{\mathbf{y}}_{[\rho]})d\rho \implies \hat{\mathbf{x}} \succ \hat{\mathbf{y}}$. And by Axioms 1, 3 and 4, $\hat{\mathbf{x}} \succ \hat{\mathbf{y}} \implies (\mathbf{x}, (z)_{\hat{p}}) \succ (\mathbf{y}, (z)_{\hat{p}}) \implies \mathbf{x} \succ \mathbf{y}$. ∎

The next step is to show that our axioms imply the following stronger version of Axiom 6.

Axiom 11 (Existence of a critical level) *There exists $c \in \mathbb{R}_+$ such that for all $p \in (0,1]$, $\nu \in \mathbb{R}_{++}$, and $\mathbf{x} \in \mathbf{X}_\nu$ satisfying $x_{[\nu]} \leq c$, $(\mathbf{x}, (c)_p) \sim \mathbf{x}$.*

Lemma 2 *If the SWR $\succsim$ satisfies Axioms 1-6, then it satisfies Axiom 11.*

Proof. If the SWR $\succsim$ satisfies Axioms 1-5, then by Lemma 1 there exist a number $\delta \in \mathbb{R}$ and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that for any $\nu \in \mathbb{R}_{++}$ $\succsim$ is represented on X_ν by the function V_ν defined for any $\mathbf{x} \in \mathbf{X}_\nu$ by

$$V_\nu(\mathbf{x}) = \int_0^\nu e^{-\delta\rho}u(\mathbf{x}_{[\rho]})d\rho\,.$$

Step 1: Axiom 6 implies that there exists $\nu \in \mathbb{R}_{++}$ such that for all $\mathbf{x} \in \mathbf{X}_\nu$ and $p \in (0,1]$, there exists $z \in \mathbb{R}_+$ such that $(\mathbf{x},(z)_p) \sim \mathbf{x}$. By Axiom 6, there exists $\nu \in \mathbb{R}_{++}$ such that for all $\mathbf{x} \in \mathbf{X}_\nu$ and $p \in (0,1]$ there exists $z', z'' \in \mathbb{R}_+$ such that $V_{\nu+p}((\mathbf{x},(z')_p)) \geq V_\nu(\mathbf{x}) \geq V_{\nu+p}((\mathbf{x},(z'')_p))$. Since $V_{\nu+p}((\mathbf{x},(\tilde{z})_p))$ is continuous in $\tilde{z}$, there exists $z \in \mathbb{R}_+$ such that $V_{\nu+p}((\mathbf{x},(z)_p)) = V_\nu(\mathbf{x})$.

Step 2: If $x \in \mathbb{R}$ and $z \in \mathbb{R}$ are such that $x \leq z$, then for all $\nu \in \mathbb{R}_+$, $p \in (0,1]$, and $\mathbf{y} \in \mathbf{X}_\nu$ satisfying $y_{[\nu]} \leq x$, if $\big((x)_\nu, z_p\big) \sim (x)_\nu$, then $\big(\mathbf{y}, z_p\big) \sim \mathbf{y}$. For $0 < \varepsilon < \nu$, construct $\mathbf{y}^\varepsilon \in \mathbf{X}_\nu$ such that $\mathbf{y}^\varepsilon_{[r]} = \mathbf{y}_{[r]}$ for all $r \in [0, \nu-\varepsilon)$ and $\mathbf{y}^\varepsilon_{[r]} = x$ for all $r \in [\nu-\varepsilon, \nu]$. By existence independence of the worst-off (Axiom 5), if $\big((x)_\nu, z_p\big) \sim (x)_\nu$ then, by contracting, $\big((x)_\varepsilon, z_p\big) \sim (x)_\varepsilon$ and, by expanding, $\big(\mathbf{y}^\varepsilon, z_p\big) \sim \mathbf{y}^\varepsilon$. But $\lim_{\varepsilon\to 0} V_\nu\big(\mathbf{y}^\varepsilon\big) = V_\nu\big(\mathbf{y}\big)$ and $\lim_{\varepsilon \to 0} V_{\nu+p}\big(\mathbf{y}^\varepsilon, z_p\big) = V_{\nu+p}\big(\mathbf{y}, z_p\big)$. Thus we need $\big(\mathbf{y}, z_p\big) \sim \mathbf{y}$.[9]

Step 3: There exists $c \in \mathbb{R}_+$ such that for all $\hat{\nu} \in \mathbb{R}_+$, $(c)_{\hat{\nu}+1} \sim (c)_{\hat{\nu}}$. Let $c \in \mathbb{R}_+$ be such that $(0_\nu, c_1) \sim 0_\nu$, where ν is the number in the statement of Axiom 6 (so that we know that c exists by Step 1). By Step 2, we know that for any $x < 0$, $(x_\nu, c_1) \sim x_\nu$. Let x be such that $0 < x \leq c$: by Step 1, there exist $z \in \mathbb{R}_+$ such that $(x_\nu, z_1) \sim x_\nu$. If $z > c$, Step 2 would imply that we should have $(0_\nu, z_1) \sim 0_\nu$, which contradicts Axioms 1 and 3 given that $(0_\nu, c_1) \sim 0_\nu$. Hence $z \leq c$. If $z < c$ we need $z < x$ for all $x \leq c$, otherwise Step 2 would imply again that we should have $(0_\nu, z_1) \sim 0_\nu$, which contradicts Axioms 1 and 3 given that $(0_\nu, c_1) \sim 0_\nu$. In the case $z < c$ and $z < x$, we have $(x_\nu, z_1) \sim x_\nu \succ (x_{\nu-1}, z_1)$. By a reasoning similar to Step 2, this implies $(0_\nu, x_1) \succ 0_\nu$. This would contradict $(0_\nu, c_1) \sim 0_\nu$ by Axiom 3 because $x \leq c$. Thus we need $z = c$ for all $x \leq c$. In particular, $(c)_{\nu+1} \sim (c)_\nu$. And by existence independence of the worst-off (Axiom 5), this is true for all $\hat{\nu} \in \mathbb{R}_+$.

Step 4: Extension to all $p \in (0,1]$ and $\mathbf{x} \in \mathbf{X}_\nu$ satisfying $x_{[\nu]} \leq c$. By Step 3 and Axiom 1, there exists $c \in \mathbb{R}_+$ such that for all $\hat{\nu} \in \mathbb{R}_+$ and $k \in \mathbb{N}$ $(c)_{\hat{\nu}+k} \sim (c)_{\hat{\nu}}$.

[9] Indeed, if $\big(\mathbf{y}, z_p\big) \succ \mathbf{y}$, then it must be the case that for ε small enough $\big(\mathbf{y}, z_p\big) \succsim \mathbf{y}^\varepsilon$ by transitivity (Axiom 1) and the fact that $\lim_{\varepsilon\to 0} V_\nu\big(\mathbf{y}^\varepsilon\big) = V_\nu\big(\mathbf{y}\big)$. But, given that $\big(\mathbf{y}^\varepsilon, z_p\big) \succ \big(\mathbf{y}, z_p\big)$, this would imply by transitivity that $\big(\mathbf{y}^\varepsilon, z_p\big) \succ \mathbf{y}^\varepsilon$, which is a contradiction. A similar reasoning can be made if $\big(\mathbf{y}, z_p\big) \prec \mathbf{y}$.

Consider any $p \in (0,1]$ such that p is a rational number. Thus there exists $\ell \in \mathbb{N}$ such that $(\ell p) \in \mathbb{N}$. Assume that there exists $\hat{\nu} \in \mathbb{R}_+$ such that $\big(c_{\hat{\nu}+p}\big) \succ (c)_{\hat{\nu}}$. By Axioms 1 and 5, by adding ℓ people with probability of existence p at the welfare level c, we must have $\big(c_{\hat{\nu}+\ell p}\big) \succ (c)_{\hat{\nu}}$. This would contradict that for all $\hat{\nu} \in \mathbb{R}_+$ and $k \in \mathbb{N}$ $(c)_{\hat{\nu}+k} \sim (c)_{\hat{\nu}}$. The same reasoning can be made if $\big(c_{\hat{\nu}+p}\big) \prec (c)_{\hat{\nu}}$ to obtain a contradiction.

Now assume that $p \in (0,1]$ is irrational. Let $(q_k)_{k\in\mathbb{N}}$ be a sequence of rational numbers converging to zero. For any $\hat{\nu} \in \mathbb{R}_+$ and $k \in \mathbb{N}$ $(c)_{\hat{\nu}+q_k} \sim (c)_{\hat{\nu}}$. By Axioms 5 and 6, for any $p_k = p - q_k$ there exists $z_k \in \mathbb{R}_+$ such that $\big((c)_{\hat{\nu}+q_k}, (z_k)_{p_k}\big) \sim (c)_{\hat{\nu}+q_k}$. But $\lim_{k\to\infty} V_{\hat{\nu}+p}\big((c)_{\hat{\nu}+q_k}, (z_k)_{p_k}\big) = V_{\hat{\nu}+p}\big((c)_{\hat{\nu}+p}\big)$. By an argument similar to the one in Step 2, we obtain $(c)_{\hat{\nu}+p} \sim (c)_{\hat{\nu}}$.

Hence, we have proven that for any $\nu \in \mathbb{R}_+$ and any $p \in (0,1]$, $(c)_{\nu+p} \sim (c)_{\nu}$. By Step 2, this implies that, for any $\mathbf{x} \in \mathbf{X}_\nu$ such that $x_{[\nu]} \leq c$, $(\mathbf{x}, (c)_p) \sim \mathbf{x}$. ■

Finally, we extend the representation to the entire domain $\mathbf{X}$ of all finite allocations (thereby also considering allocations with different probability adjusted population sizes) by showing that any finite allocation $\mathbf{x}$ can be made as bad as an allocation where all individuals are at the critical level c by adding sufficiently many people at a low well-being level z, and thus indifferent to an egalitarian allocation where each individual's well-being equals $x \leq c$. This allows us to apply Axiom 11, thereby completing the demonstration of the result that statement (1) of Theorem 1 implies statement (2).

Lemma 3 *If Axioms 1-7 hold, then there exists $c \in \mathbb{R}_+$, $\delta \in \mathbb{R}_{++}$, and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,*

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \int_0^{\nu(\mathbf{x})} e^{-\delta\rho}\left(u(\mathbf{x}_{[\rho]}) - u(c)\right) d\rho \geq \int_0^{\nu(\mathbf{y})} e^{-\delta\rho}\left(u(\mathbf{y}_{[\rho]}) - u(c)\right) d\rho\,. \quad \text{(B.3)}$$

Proof. *Step 1: Representation when well-being does not exceed c.* Given that Axioms 1-6 hold, Axiom 11 also holds by Lemma 2. Let $c \in \mathbb{R}_+$ be the critical level parameter defined in Axiom 11.

Assume that $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ are such that $\mathbf{x}_{[\nu(\mathbf{x})]} \leq c$ and $\mathbf{y}_{[\nu(\mathbf{y})]} \leq c$. If $\nu(\mathbf{x}) = \nu(\mathbf{y})$, then equivalence (B.3) follows from Lemma 1. Therefore, assume that $\nu(\mathbf{x}) < \nu(\mathbf{y})$ (the case $\nu(\mathbf{x}) > \nu(\mathbf{y})$ can be treated similarly). Let $k := \min_{l \in \mathbb{N}}\{\ell : (\nu(\mathbf{y}) - \nu(\mathbf{x}))/\ell \leq 1\}$ and $p = (\nu(\mathbf{y}) - \nu(\mathbf{x}))/k$. Then, by k applications of Axiom 11, $\mathbf{x} \sim (\mathbf{x}, (c)_{kp})$. By Axiom 1 and Lemma 1:

$$\begin{aligned}
\mathbf{x} \succsim \mathbf{y} \iff & (\mathbf{x}, (c)_{kp}) \succsim \mathbf{y} \\
\iff & \int_0^{\nu(\mathbf{x})} e^{-\delta\rho} u\big(\mathbf{x}_{[\rho]}\big) d\rho + \int_{\nu(\mathbf{x})}^{\nu(\mathbf{y})} e^{-\delta\rho} u(c) d\rho \geq \int_0^{\nu(\mathbf{y})} e^{-\delta\rho} u\big(\mathbf{y}_{[\rho]}\big) d\rho \\
\iff & \int_0^{\nu(\mathbf{x})} e^{-\delta\rho} \Big(u\big(\mathbf{x}_{[\rho]}\big) - u(c)\Big) d\rho \geq \int_0^{\nu(\mathbf{y})} e^{-\delta\rho} \Big(u\big(\mathbf{y}_{[\rho]}\big) - u(c)\Big) d\rho\,.
\end{aligned} \tag{B.4}$$

We now show that $\delta \in \mathbb{R}_{++}$. Let $x, y \in \mathbb{R}$ satisfy that $c \geq x > y$ and let $\nu \in \mathbb{R}_{++}$. Assume that there exist $z \in \mathbb{R}$, $\ell \in \mathbb{N}$ and $p \in (0, 1]$ such that $(x)_\nu \succ (z)_{\ell p} \succ (y)_\nu$ (note that $z < c$, because otherwise, by Axioms 1, 3 and 11, $(z)_{\ell p} \succsim (c)_{\ell p} \sim (c)_\nu \succsim (x)_\nu$, a contradiction). By equivalence (B.4), this means that (for $\delta \neq 0$; the case $\delta = 1$ can be treated similarly):

$$\tfrac{1-e^{-\delta\nu}}{\delta}\big(u(x) - u(c)\big) > \tfrac{1-e^{-\delta\ell p}}{\delta}\big(u(z) - u(c)\big) > \tfrac{1-e^{-\delta\nu}}{\delta}\big(u(y) - u(c)\big)\,.$$

When $\delta < 0$, $\lim_{\ell\to\infty} \frac{1-e^{-\delta\ell p}}{\delta} = \infty$ so that $\lim_{\ell\to\infty} \frac{1-e^{-\delta\ell p}}{\delta}\big(u(z) - u(c)\big) = -\infty$. Hence there exists $N \in \mathbb{N}$ such that $\frac{1-e^{-\delta\ell p}}{\delta}\big(u(z) - u(c)\big) < \frac{1-e^{-\delta\nu}}{\delta}\big(u(y) - u(c)\big)$ for all $n \geq N$, a contradiction of Axiom 7.

Step 2: Equally distributed equivalent. For any $\nu \in \mathbb{R}_{++}$ and $\mathbf{x} \in \mathbf{X}_\nu$, let the ν–equally distributed equivalent of $\mathbf{x}$, denoted $e_\nu(\mathbf{x})$, be $x \in \mathbb{R}$ such that $(x)_\nu \sim \mathbf{x}$. Axioms 1–3 imply that $e_\nu : \mathbf{X}_\nu \to \mathbb{R}$ is well-defined. By Lemma 1, and since Axioms 1–5 hold, it is defined as follows:

$$e_\nu(\mathbf{x}) = u^{-1}\Big(\tfrac{\delta}{1-e^{-\delta\nu}} \int_0^{\nu} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho\Big)\,.$$

Let $\mathbf{x} \in \mathbf{X}_\nu$ and $z < \min\{\mathbf{x}_{[0]}, c\}$, for $k \in \mathbb{N}$ we obtain:

$$\begin{aligned} e_{\nu+k}\big(\mathbf{x}, (z)_k\big) &= u^{-1}\left(\tfrac{\delta}{1-e^{-\delta(\nu+k)}}\left(\int_0^k e^{-\delta\rho} u(z) d\rho + \int_k^{\nu+k} e^{-\delta\rho} u(\mathbf{x}_{[\rho-k]}) d\rho\right)\right) \\ &= u^{-1}\Big(\tfrac{1-e^{-\delta k}}{1-e^{-\delta(\nu+k)}} u(z) + \tfrac{e^{-\delta k}-e^{-\delta(\nu+k)}}{1-e^{-\delta(\nu+k)}} \cdot \tfrac{\delta}{1-e^{-\delta\nu}} \int_0^\nu e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho\Big)\,. \end{aligned}$$

Write $a(k) := \big(1 - e^{-\delta k}\big)/\big(1 - e^{-\delta(\nu+k)}\big)$; note that $a : \mathbb{N} \to \mathbb{R}$ is an increasing function of k converging to 1. Since $z < \mathbf{x}_{[0]} \le e_\nu(\mathbf{x})$ and

$$e_{\nu+k}\big(\mathbf{x}, (z)_k\big) = u^{-1}\big(a(k)u(z) + (1 - a(k))u\big(e_\nu(\mathbf{x})\big)\big)\,,$$

it follows that $e_{\nu+k}\big(\mathbf{x}, (z)_k\big)$ is a decreasing function of k converging to z as k approaches infinity. As $z < c$, we deduce that, for any $\mathbf{x} \in \mathbf{X}$, there exists $K(\mathbf{x}) \in \mathbb{N}$ such that, for all $k \ge K(\mathbf{x})$, $e_{\nu(\mathbf{x})+k}\big(\mathbf{x}, (z)_k\big) \le c$.

Step 3: Conclusion. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, choose z such that $z < \min\{\mathbf{x}_{[0]}, \mathbf{y}_{[0]}, z\}$. Let $\ell = \max\{K(\mathbf{x}), K(\mathbf{y})\}$, $x = e_{\nu(\mathbf{x})+\ell}\big(\mathbf{x}, (z)_\ell\big)$ and $y = e_{\nu(\mathbf{y})+\ell}\big(\mathbf{y}, (z)_\ell\big)$. By definition, $(\mathbf{x}, (z)_\ell) \sim (x)_{\nu(\mathbf{x})+\ell}$, $(\mathbf{y}, (z)_\ell) \sim (y)_{\nu(\mathbf{y})+\ell}$, $x \le c$ and $y \le c$. Hence, by repeated applications of Axioms 1 and 5, and by Step 1:

$$\begin{aligned} \mathbf{x} \succsim \mathbf{y} &\iff (\mathbf{x}, (z)_\ell) \succsim (\mathbf{y}, (z)_\ell) \\ &\iff (x)_{\nu(\mathbf{x})+\ell} \succsim (y)_{\nu(\mathbf{y})+\ell} \\ &\iff \int_0^{\nu(\mathbf{x})+\ell} e^{-\delta\rho}\,(u(x) - u(c))\,d\rho \ge \int_0^{\nu(\mathbf{y})+\ell} e^{-\delta\rho}\,(u(y) - u(c))\,d\rho\,. \end{aligned}$$

However, by the definition of equally distributed equivalents,

$$\begin{aligned} \int_0^{\nu(\mathbf{x})+\ell} e^{-\delta\rho} u(x) d\rho &= \int_0^\ell e^{-\delta\rho} u(z) d\rho + e^{-\delta\ell} \int_0^{\nu(\mathbf{x})} e^{-\delta\rho} u(\mathbf{x}_{[\rho]}) d\rho\,, \\ \int_0^{\nu(\mathbf{y})+\ell} e^{-\delta\rho} u(x) d\rho &= \int_0^\ell e^{-\delta\rho} u(z) d\rho + e^{-\delta\ell} \int_0^{\nu(\mathbf{y})} e^{-\delta\rho} u(\mathbf{y}_{[\rho]}) d\rho\,, \end{aligned}$$

Thereby we obtain equivalence (B.3). ■

C Proofs of results in Section 5

Proof of Proposition 1. *If a PARDCLU SWO satisfies Axiom 8, then u is concave.* Assume that a PARDCLU SWO satisfies Axiom 8. Consider $\mathbf{x} \in \mathbf{X}$ such that $n(\mathbf{x}) = 2$, $x_1^w = z \leq z' = x_2^w$ and $x_1^p = x_2^p = \pi$. Let $\mathbf{y} \in \mathbf{X}$ such that $n(\mathbf{y}) = 2$, $y_1^w = z - \varepsilon < z' + \varepsilon = y_2^w$ and $y_1^p = y_2^p = \pi$, with $\varepsilon > 0$. By Axiom 8, it must be the case that $\mathbf{x} \succ \mathbf{y}$. By the representation of PARDCLU SWOs in Definition 1, this implies that:

$$\left(1-e^{-\delta\pi}\right)u(z)+\left(e^{-\delta\pi}-e^{-\delta 2\pi}\right)u(z') > \left(1-e^{-\delta\pi}\right)u(z-\varepsilon)+\left(e^{-\delta\pi}-e^{-\delta 2\pi}\right)u(z'+\varepsilon)\,,$$

which can be rewritten:

$$1 > e^{-\delta\pi}\frac{u\left(x_{\tau'}+\varepsilon\right)-u\left(x_{\tau'}\right)}{u\left(x_{\tau}\right)-u\left(x_{\tau}-\varepsilon\right)}\,. \tag{C.1}$$

Equation (C.1) must be true for any arbitrarily small $\pi > 0$. Hence, we must have that

$$u(z)+u(z') \geq u(z-\varepsilon)+u(z'+\varepsilon)\,. \tag{C.2}$$

Equation (C.2) must be true for any $z, z' \in \mathbb{R}$ such that $z < z'$ and any $\varepsilon > 0$. This implies that u must be concave.

If a PARDCLU SWO is such that u is concave in its representation, then it satisfies Axiom 8. Consider the representation of a PARDCLU in Definition 1, and assume that u is concave.

Consider any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ such that $n(\mathbf{x}) = n(\mathbf{y}) = \nu$ and there exist $i, j \in \{1, \cdots, n\}$ and $\varepsilon \in \mathbb{R}_{++}$ such that $y_i^w+\varepsilon = x_i^w \leq x_j^w = y_j^w-\varepsilon$, $y_i^p = x_i^p = x_j^p = y_j^p = \pi$ and $(x_k^w, x_k^p) = (y_k^w, y_k^p)$ for all $k \in \Big\{1, \cdots, n\} \setminus \{i, j\}\Big)$. Let $\pi : \{1, \ldots, n(\mathbf{x})\} \rightarrow \{1, \ldots, n(\mathbf{x})\}$ be a bijection that reorders individuals in increasing well-being order in $\mathbf{x}$:

$$x_{\pi(r)}^w \leq x_{\pi(r+1)}^w \quad \text{for all } r \in \{1, \ldots, n(\mathbf{x})-1\}\,,$$

and define $\underline{\rho} = \sum_{r:\pi(r)<\pi(i)} x_{\pi(r)}^p$ and $\bar{\rho} = \sum_{r:\pi(r)<\pi(j)} x_{\pi(r)}^p$ (i.e. the cumulative probability weights of people whose well-being is below i and j respectively). Similarly, let $\tilde{\pi}:\{1,\ldots,n(\mathbf{x})\}\to\{1,\ldots,n(\mathbf{x})\}$ be a bijection that reorders individuals in increasing well-being order in $\mathbf{y}$:

$$y^w_{\tilde{\pi}(r)} \leq y^w_{\tilde{\pi}(r+1)} \quad \text{for all } r \in \{1,\ldots,n(\mathbf{x})-1\},$$

and define $\underline{\rho}' = \sum_{r:\tilde{\pi}(r)<\tilde{\pi}(i)} y^p_{\tilde{\pi}(r)}$ and $\bar{\rho}' = \sum_{r:\tilde{\pi}(r)<\tilde{\pi}(j)} x^p_{\tilde{\pi}(r)}$.

By Axiom 8, it must be the case that $\mathbf{x} \succ \mathbf{y}$. Given the representation of PARDCLU SWO in Definition 1, for this to be true we need to have (with $\nu = \nu(\mathbf{x}) = \nu(\mathbf{y})$):

$$\begin{aligned}
&\int_0^{\underline{\rho}} e^{-\delta\rho}\big(u(\mathbf{x}_{[\rho]})-u(c)\big)d\rho \\
&\qquad + \frac{e^{-\delta\underline{\rho}}(1-e^{-\delta\pi})}{\delta}\big(u(x^w_i)-u(c)\big) + \int_{\underline{\rho}+\pi}^{\bar{\rho}} e^{-\delta\rho}\,\big(u(\mathbf{x}_{[\rho]})-u(c)\big)\,d\rho \\
&\qquad + \frac{e^{-\delta\bar{\rho}}(1-e^{-\delta\pi})}{\delta}\big(u(x^w_j)-u(c)\big) + \int_{\bar{\rho}+\pi}^{\nu} e^{-\delta\rho}\,\big(u(\mathbf{x}_{[\rho]})-u(c)\big)\,d\rho \\
> &\int_0^{\underline{\rho}'} e^{-\delta\rho}\big(u(\mathbf{y}_{[\rho]})-u(c)\big)d\rho \\
&\qquad + \frac{e^{-\delta\underline{\rho}'}(1-e^{-\delta\pi})}{\delta}\big(u(x^w_i-\varepsilon)-u(c)\big) + \int_{\underline{\rho}'+\pi}^{\bar{\rho}'} e^{-\delta\rho}\,\big(u(\mathbf{y}_{[\rho]})-u(c)\big)\,d\rho \\
&\qquad + \frac{e^{-\delta\bar{\rho}'}(1-e^{-\delta\pi})}{\delta}\big(u(x^w_j+\varepsilon)-u(c)\big) + \int_{\bar{\rho}'+\pi}^{\nu} e^{-\delta\rho}\,\big(u(\mathbf{y}_{[\rho]})-u(c)\big)\,d\rho
\end{aligned}$$

By definition of $\mathbf{x}$ and $\mathbf{y}$ and using $\underline{\rho}' \leq \underline{\rho} \leq \bar{\rho} \leq \bar{\rho}'$, this can be simplified to:[10]

$$\begin{aligned}
&\int_{\underline{\rho}'}^{\underline{\rho}} e^{-\delta\rho}u(\mathbf{x}_{[\rho]})d\rho + \frac{e^{-\delta\underline{\rho}}(1-e^{-\delta\pi})}{\delta}u(x^w_i) \\
&\qquad + \frac{e^{-\delta\bar{\rho}}(1-e^{-\delta\pi})}{\delta}u(x^w_j) + e^{-\delta\pi}\int_{\bar{\rho}}^{\bar{\rho}'} e^{-\delta\rho}u(\mathbf{x}_{[\rho+\pi]})d\rho \\
> &\ \frac{e^{-\delta\underline{\rho}'}(1-e^{-\delta\pi})}{\delta}u(x^w_i-\varepsilon) + e^{-\delta\pi}\int_{\underline{\rho}'}^{\underline{\rho}} e^{-\delta\rho}u(\mathbf{y}_{[\rho+\pi]})d\rho
\end{aligned}$$

[10] In particular, $\int_0^{\underline{\rho}'} e^{-\delta\rho}u(\mathbf{x}_{[\rho]})d\rho = \int_0^{\underline{\rho}'} e^{-\delta\rho}u(\mathbf{y}_{[\rho]})d\rho$, $\int_{\underline{\rho}+\pi}^{\bar{\rho}} e^{-\delta\rho}u(\mathbf{x}_{[\rho]})d\rho = \int_{\underline{\rho}+\pi}^{\bar{\rho}} e^{-\delta\rho}u(\mathbf{y}_{[\rho]})d\rho$ and $\int_{\bar{\rho}'+\pi}^{\nu} e^{-\delta\rho}u(\mathbf{x}_{[\rho]})d\rho = \int_{\bar{\rho}'+\pi}^{\nu} e^{-\delta\rho}u(\mathbf{y}_{[\rho]})d\rho$.

$$+\int_{\bar{\rho}}^{\bar{\rho}'} e^{-\delta\rho}u(\mathbf{y}_{[\rho]})d\rho + \frac{e^{-\delta\bar{\rho}'}(1-e^{-\delta\pi})}{\delta}u(x_j^w+\varepsilon)$$

or rewritten:[11]

$$\begin{aligned}
&e^{-\delta\underline{\rho}}\Big(u(x_i^w) - u(x_i^w-\varepsilon)\Big) - e^{-\delta\bar{\rho}}\Big(u(x_j^w+\varepsilon) - u(x_j^w)\Big) \\
&> -\,\delta\left(\int_{\underline{\rho}'}^{\underline{\rho}} e^{-\delta\rho}\Big(u(\mathbf{x}_{[\rho]}) - u(x_i^w-\varepsilon)\Big)d\rho + \int_{\bar{\rho}}^{\bar{\rho}'} e^{-\delta\rho}\Big(u(x_j^w+\varepsilon) - u(\mathbf{y}_{[\rho]})\Big)d\rho\right).
\end{aligned}$$

The second term of the inequality is always negative, by definition of $\mathbf{x}$ and $\mathbf{y}$. The first term can be written

$$e^{-\delta\underline{\rho}}\Big(u(x_i^w) - u(x_i^w-\varepsilon)\Big)\left(1 - e^{-\delta(\bar{\rho}-\underline{\rho})}\frac{u(x_j^w+\varepsilon)-u(x_j^w)}{u(x_i^w)-u(x_i^w-\varepsilon)}\right),$$

which is strictly positive when u is concave, since then $e^{-\delta(\bar{\rho}-\underline{\rho})} < 1$ and

$$\frac{u(x_j^w+\varepsilon)-u(x_j^w)}{u(x_i^w)-u(x_i^w-\varepsilon)} \leq 1\,.$$

Hence the concavity of u is sufficient to guarantee that $\mathbf{x} \succ \mathbf{y}$ as required by Axiom 8. ■

The following lemma is used to prove Proposition 2.

Lemma 4 *If the SWR $\succsim$ on $\mathbf{X}$ satisfies Axioms 1–5 and 8, then one of the following must be true:*

1. *There exist a positive number $\delta \in \mathbb{R}_{++}$ and a continuous, increasing and concave function $u : \mathbb{R} \to \mathbb{R}$ such that for all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_\nu$,*

$$\mathbf{x} \succsim \mathbf{y} \Longleftrightarrow \int_0^\nu e^{-\delta\rho}\left(u(\mathbf{x}_{[\rho]}) - u(c)\right)d\rho \geq \int_0^\nu e^{-\delta\rho}\left(u(\mathbf{y}_{[\rho]}) - u(c)\right)d\rho \quad \text{(C.3)}$$

2. *There exist a continuous, increasing and strictly concave function $u : \mathbb{R} \to \mathbb{R}$*

[11] We use the fact that $\mathbf{x}_{[\rho]} = \mathbf{y}_{[\rho+\pi]}$ on the interval $[\underline{\rho}', \underline{\rho}]$ and $\mathbf{y}_{[\rho]} = \mathbf{x}_{[\rho+\pi]}$ on the interval $[\bar{\rho}, \bar{\rho}']$.

such that for all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_\nu$,

$$\mathbf{x} \succsim \mathbf{y} \iff \int_0^\nu \big(u(\mathbf{x}_{[\rho]}) - u(c)\big)\, d\rho \geq \int_0^\nu \big(u(\mathbf{y}_{[\rho]}) - u(c)\big)\, d\rho \tag{C.3$'$}$$

Proof. By Lemma 1, Axioms 1–5 imply that there exist a number $\delta \in \mathbb{R}$ and a continuous and increasing function $u : \mathbb{R} \to \mathbb{R}$ such that for any $\nu \in \mathbb{R}_{++}$ and for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\nu$,

$$\mathbf{x} \succsim \mathbf{y} \iff \int_0^\nu e^{-\delta\rho} \big(u(\mathbf{x}_{[\rho]}) - u(c)\big)\, d\rho \geq \int_0^\nu e^{-\delta\rho} \big(u(\mathbf{y}_{[\rho]}) - u(c)\big)\, d\rho.$$

Consider $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_{1/k}$ such that $n(\mathbf{x}) = n(\mathbf{x}) = 2$, $x_1^w = z \leq z' = x_2^w$, $y_1^w = z - \varepsilon < z' + \varepsilon = y_2^w$ and $x_1^p = x_2^p = y_1^p = y_2^p = 1/k$. By Axiom 8, it must be the case that $\mathbf{x} \succ \mathbf{y}$. This implies that:

$$1 > e^{-\delta/k} \frac{u(z' + \varepsilon) - u(z')}{u(z) - u(z + \varepsilon)}.$$

This must be true for any $z \leq z'$ and $\varepsilon > 0$. But Chateauneuf, Cohen and Meilijson (2005) proved that

$$\sup_{z \leq z', \varepsilon > 0} \frac{u(z' + \varepsilon) - u(z')}{u(z) - u(z + \varepsilon)} \geq 1\,.$$

Hence, if $\delta < 0$, there would exist $z \leq z'$ and $\varepsilon > 0$ such that

$$e^{-\delta/k} \frac{u_{1/k}(z' + \varepsilon) - u_{1/k}(z')}{u(z) - u(z + \varepsilon)} > 1\,,$$

a violation of Axiom 8. Thus, we need $\delta \geq 0$.

In the case $\delta > 0$, we can proceed like in the first part of the proof of Proposition 1 to show that u must be concave.

In the generalized utilitarian case ($\delta = 0$), we need[12]

$$2u(z) > u(z+\varepsilon) + u(z-\varepsilon)$$

for any $z \in \mathbb{R}$ and $\varepsilon > 0$. Thus, u must be strictly concave. ■

Proof of Proposition 2. By Lemma 4, there are two cases to consider.

Case 1: There exist a positive number $\delta \in \mathbb{R}_{++}$ and a continuous, increasing and concave function $u : \mathbb{R} \to \mathbb{R}$ such that for all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\nu$, $\mathbf{x} \succsim \mathbf{y}$ if and only if (C.3) holds.

The SWR $\succsim$ satisfies Axioms 1–5, so that Lemma 2 in Appendix B still holds. Given that the SWR $\succsim$ on $\mathbf{X}$ also satisfies Axiom 8, Lemma 3 in Appendix B can also be adapted: we just need to omit the reasoning at the end of step 1 of the proof of Lemma 3 using Axiom 7, given that it is only used to prove that $\delta > 0$, which is already proven.

Case 2: There exist a continuous, increasing and strictly concave function $u : \mathbb{R} \to \mathbb{R}$ such that for all $\nu \in \mathbb{R}_{++}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{X}_\nu$, $\mathbf{x} \succsim \mathbf{y}$ if and only if (C.3′) holds.

The SWR $\succsim$ satisfies Axioms 1–5, so that Lemma 2 in Appendix B still holds. Given that the SWR $\succsim$ on $\mathbf{X}$ also satisfies Axiom 8, Lemma 3 in Appendix B can be adapted: like before, we omit the reasoning at the end of step 1 of the proof of Lemma 3, and we change the definition of the equally-distributed equivalent (step 2) to obtain a similar conclusion (step 3). ■

Proof of Proposition 3. If $c = 0$ in the definition of the PARDCLU SWO, then for all $x, y \in \mathbb{R}$ and $\nu, \nu' \in \mathbb{R}_{++}$, $(x)_\nu \succ (y)_{\nu'} \iff (1 - e^{-\delta\nu})\big(u(x) - u(0)\big) > (1 - e^{-\delta\nu'})\big(u(y) - u(0)\big)$. But if $x > 0 \geq y$, it is also the case that $u(x) - u(0) > 0 \geq u(y) - u(0)$ because u is increasing. Hence we necessarily have $(x)_\nu \succ (y)_{\nu'}$ in that case.

[12] By considering $\mathbf{x}, \mathbf{y} \in \mathbf{X}_{1/k}$ such that $n(\mathbf{x}) = n(\mathbf{x}) = 2$, $x_1^w = x_2^w = z$, $y_1^w = z - \varepsilon < z' + \varepsilon = y_2^w$ and $x_1^p = x_2^p = y_1^p = y_2^p = 1/k$, and noting that Axiom 8 implies that $\mathbf{x} \succ \mathbf{y}$.

Conversely, assume that $c > 0$. Consider $c > x > 0 \geq y$ such that

$$u(x) - u(y) = \varepsilon < \frac{e^{-\delta}(1 - e^{-\delta})}{1 - e^{-2\delta}}\big(u(c) - u(0)\big)\,.$$

By continuity of u, it is always possible to find x and y close enough to 0 that satisfy this condition. Then $(1 - e^{-2\delta})\varepsilon < e^{-\delta}(1 - e^{-\delta})\big(u(c) - u(y)\big) = \big((1 - e^{-\delta}) - (1 - e^{-2\delta})\big)\big(u(y) - u(c)\big)$, so that $(1 - e^{-2\delta})\big(u(x) - u(c)\big) = (1 - e^{-2\delta})\big(u(y) + \varepsilon - u(c)\big) < (1 - e^{-\delta})\big(u(y) - u(c)\big)$. By definition of the PARDCLU SWO, this implies that $(y)_1 \succ (x)_2$, which is a violation of Axiom 9. ■

Proof of Proposition 4. Assume that the SWR $\succsim$ satisfies Axioms 1, 7, 8, 9 and 10, and consider any x, $y \in \mathbb{R}_{++}$ and n, $m \in \mathbb{N}$ such that $(x)_n \succ (y)_m$.[13] By Axiom 7, there exists $z \in \mathbb{R}$ such that, for all $N \in \mathbb{N}$, $(x)_n \succ (z)_k \succ (y)_m$ for some $k \geq N$. By Axiom 9, we must have $z > 0$, otherwise there exists k such that $(z)_k \succ (y)_m$ while $y > 0 \geq z$.

Let $N = \max\Big(n + 1, n\frac{x}{z} + 1\Big)$. There must exist $k \geq N$ such that $(x)_n \succ (z)_k$. Define $\varepsilon = \frac{k}{k-n}z - \frac{n}{k-n}x$; $\varepsilon > 0$ because $k \geq N$.[14] By repeated applications of Axioms 10 and 1, we know that $((x)_n, (\varepsilon)_{k-n}) \succsim (x)_n$. But $(z)_k$ can be obtained from $((x)_n, (\varepsilon)_{k-n})$ through a finite sequence of transfers from people with welfare x to people with welfare ε when $\varepsilon < x$, or through a finite sequence of transfers from people with welfare ε to people with welfare x when $\varepsilon > x$.[15] Given that all individuals have a probability 1 of existence, by repeated applications of Axioms 8 and 1, we obtain that $(z)_k \succ ((x)_n, (\varepsilon)_{k-n})$. By transitivity (Axiom 1), we obtain $(z)_k \succ (x)_n$, which contradicts $(x)_n \succ (z)_k$. ■

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[13] By Axioms 1, 8 and 10, there must exist such numbers.

[14] If $z \geq x$, $k \geq N = n + 1$ so that $\frac{k}{k-n} > \frac{n}{k-n}$, which (together with $z \geq x$) implies that $\varepsilon > 0$. If $0 < z < x$, $k \geq n\frac{x}{z} + 1$ so that $kz - nx \geq z > 0$ and $\frac{kz-nx}{k-n} > 0$.

[15] In the case $\varepsilon = x$, we already have $((x)_n, (\varepsilon)_{k-n}) = (z)_k \succsim (x)_n$, which contradicts that $(x)_n \succ (z)_k$.