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Finite difference representations
of nonlinear waves

by

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Abstract

The work concerns cnoidal and solitary wave solutions of the Boussinesq equations in the context of finite difference methods. Existence of permanent wave forms is discussed and approximate solutions are found for one particular discrete formulation by perturbation techniques. These techniques can be applied to most difference and regular element discretizations of equations describing weakly nonlinear and dispersive waves. No proof of convergence of the perturbation expansions is established, but the appropriateness of the solutions is confirmed through comparison with the results of computer simulations. From the perturbation series it is found that the phase speed of solitons is useless for evaluation of numerical methods, and that asymmetries in the discrete equations may ruin the soliton solution. The availability of a discrete closed form solution also enables discussion of existence and accuracy of permanent wave forms with oblique orientations relative to the grid.

Contents

1	Introduction	2
2	Formulation	3
2.1	The Boussinesq equation	4
2.2	The numerical procedure	4
3	Cnoidal waves	5
3.1	The linear harmonic solution	6
3.2	The superharmonics	7
4	The soliton solution	7
4.1	The perturbation expansion	9
4.2	The zeroth order solution	11
4.3	The first order solution	12
4.4	Higher order terms	13
4.5	Verification and comparison with simulation	13
4.6	Three dimensional solutions	15
4.7	Asymmetric difference representations	19
5	Conclusions	20

1 Introduction

One of the most instructive ways of evaluating a numerical method is to investigate its capability of reproducing simple, analytical solutions. Often, this involves a large number of test runs on a computer with uncertainties concerning code-errors, limitations on grid resolutions, introduction of unwanted boundary conditions etc. However, in many cases "manual" analysis of the methods may provide an attractive alternative. For wave problems the most important example is Fourier analysis which is applicable to linear equations with constant coefficients. Unfortunately, such analysis is seldom carried beyond the establishment of dispersion relations and simple stability criteria. Nevertheless, classical methods of analysis, traditionally applied to differential equations etc., often work equally well with discretized problems. Thus, discrete solutions of closed form may occasionally be derived for problems involving coupling between modes, nontrivial boundary conditions or nonlinearities. Such solutions are of great value both for evaluating numerical methods and for validation of computer programs. The solutions could be viewed as analytical solutions to discrete problems. However, we will avoid such terms and reserve the notation analytical for direct treatment of differential equations. In the present paper we will study numerical representations of some simple solutions of the nonlinear Boussinesq equation.

For surface gravity waves in shallow water there are two important nonlinear solutions of permanent form, namely the soliton and the periodic cnoidal wave. Both are important in the context of complex wave patterns as well as representatives of nonlinear waves in their own right. Particularly the soliton has received an appreciable attention in the literature. The remarkable collision properties of these nonlinear

waves has been studied in a series of papers through the last decades, some of which involve numerical integration. Also the propagation of solitons over shelves and their run up on vertical or inclined planes have been extensively studied ([7], [8], [10], [11], [17], [21] and many more). These works do not focus exclusively on the peculiar properties of the soliton solution itself, but also on features concerning nonlinear waves in general. An example is the question of nonlinear modifications to Greens' law for amplification in shoaling water. In the last years the topic of nonlinear ship wave patterns in shallow water has become popular. Particularly, attention has been paid to the upstream radiation of solitons that has been observed in the transcritical regime ([3], [4], [9], [14], [19], [29]). The nonlinear shallow water theory do also describe ocean waves in coastal regions and their impact on harbours etc. These are the motivations for works like [1], [2], [23] and [25].

Even though soliton and cnoidal wave solutions also are widely used for validation of numerical techniques, the problem of finding numerical counterparts to the analytical solutions has hardly been addressed. In the limit of very refined grid resolutions one would always expect to reproduce the solutions fairly well, at least for moderate time intervals. Still, many problems involving slow evolution, refraction, interaction etc. of solitons, will be very sensitive to even extremely slow numerical growth/damping or disintegration of the wave forms. Besides, such cases generally require discretization of large domains with corresponding limitations on the grid increments. As a consequence, the results of the computations may be confused or even dominated by spurious effects.

The succeeding sections will contain approximate close form solutions for solitary and cnoidal waves moving in a finite difference grid. The study of the latter wave type will be confined to the Stokes wave limit (small Ursell numbers). Cases of intermediate and large Ursell numbers can probably be treated analogously as for the soliton, but we believe that the addition of such an analysis will give but little new insight of the numerics. The work will be concentrated on a shallow water formulation and numerical techniques that have been used previously by the author. Nevertheless, the principles of the analysis will apply equally well to most finite difference and finite element discretization on a regular mesh.

2 Formulation

The equations will be formulated in a coordinate system with horizontal axes, ox^* and oy^* , in the undisturbed water level and the vertical axis, oz^* , pointing upwards. The asterisks indicate dimensional quantities. The fluid is confined to $-h^* < z^* < \eta^*$ and the depth averaged velocity and the velocity potential are denoted by $\mathbf{v}^*$ and ϕ^* respectively. We introduce a characteristic depth h_0 , wavelength ℓ , amplitude: αh_0 , and dimensionless variables, according to

$$\begin{aligned} z^* &= h_0 z & x^* &= \ell x & y^* &= \ell y \\ t^* &= \ell (gh_0)^{-\frac{1}{2}} t & h^* &= h_0 h(x, y, t) & \eta^* &= \alpha h_0 \eta \\ \phi^* &= \alpha \ell (gh_0)^{\frac{1}{2}} \phi & \mathbf{v}^* &= \alpha (gh_0)^{\frac{1}{2}} \mathbf{v} \end{aligned} \quad (1)$$

where g is the acceleration of gravity. The x - and y - components of $\mathbf{v}$ will be referred to as u and v respectively. In various sections we will use different horizontal length scales, ℓ . To make comparisons easier within the report, as well as with other works,

we introduce a reference scaling based on using the depth as both horizontal and vertical length scale. This leads to the following definitions of $\hat{x}$, $\hat{y}$, $\hat{t}$ and $\hat{\eta}$:

$$\hat{x} = x^*/h_0 \quad \hat{y} = y^*/h_0 \quad \hat{t} = (gh_0)^{1/2}t^*/h_0 \quad \hat{\eta} = \eta^*/h_0 \quad (2)$$

2.1 The Boussinesq equation

For long waves in shallow water, weakly dispersive and nonlinear theories can be established through expansions in the small parameters α and $\epsilon \equiv h_0^2/\ell^2$. The resulting equations may be cast into a variety of forms due to the choice of variables, extra assumptions etc. In the subsequent sections we will mainly be concerned with one particular version of the Boussinesq equations. Other important shallow water equations are the KdV (Korteweg-deVries) equation, which describes unidirectional waves, and the KP (Kadomtsev-Petviashvili) which is a weakly three-dimensional generalization of the KdV equation. Derivations and discussion of Boussinesq equations are given in many text books and articles. The references [22], [28], [19] and [20] are of particular relevance to the present work.

In the case of constant depth, $h \equiv 1$, the Boussinesq equations expressed in terms of the depth averaged velocity potential read:

$$\eta_t = -\nabla \cdot \{(1 + \alpha\eta)\nabla\phi\} + O(\alpha\epsilon) \quad (3)$$

$$\phi_t + \frac{1}{2}\alpha(\nabla\phi)^2 + \eta - \frac{1}{3}\epsilon\nabla^2\phi_t = O(\epsilon^2, \alpha\epsilon) \quad (4)$$

In three dimensions this set is solved numerically in [19] and [25]. Recognizing $\nabla\phi$ as $\mathbf{v}$, the average horizontal velocity, the equations may be rewritten to give:

$$\eta_t = -\nabla \cdot ((1 + \alpha\eta)\mathbf{v}) \quad (5)$$

$$\mathbf{v}_t + \frac{\alpha}{2}\nabla(\mathbf{v}^2) = -\nabla\eta + \frac{1}{3}\epsilon\nabla^2\mathbf{v}_t \quad (6)$$

which is another formulation that is frequently used.

2.2 The numerical procedure

We will concentrate the study on the numerical techniques discussed and used in [18], [19], [20] and [23]. In the present paper we will only recapitulate the methods briefly, including some useful nomenclature.

The approximation to a quantity f at a grid-point with coordinates $(\beta\Delta x, \gamma\Delta y, \kappa\Delta t)$ where Δx , Δy and Δt are the grid increments, is denoted by $f_{\beta,\gamma}^{(\kappa)}$. To improve the readability of the difference equations we introduce the symmetric difference operator, δ_x :

$$\delta_x f_{\beta,\gamma}^{(\kappa)} = \frac{1}{\Delta x} (f_{\beta+\frac{1}{2},\gamma}^{(\kappa)} - f_{\beta-\frac{1}{2},\gamma}^{(\kappa)}) \quad (7)$$

and the midpoint average operator $\bar{x}$ by:

$$(\bar{f}^x)_{\beta,\gamma}^{(\kappa)} = \frac{1}{2} (f_{\beta-\frac{1}{2},\gamma}^{(\kappa)} + f_{\beta+\frac{1}{2},\gamma}^{(\kappa)}) \quad (8)$$

Difference and average operators with respect to the other coordinates y and t are defined correspondingly. We note that all combinations of these operators are commutative. To abbreviate the expressions further we also group terms of identical

indices inside square brackets, leaving the super- and subscripts outside the bracket. These notations are adopted from [18].

In [19] the set (3) and (4) was discretized exclusively by centered differences and averaging. Hence, both the continuity and the Bernoulli equation gave difference equations that were implicit in space. To decouple the implicitness in the two equations, a grid staggered in time was used. This also improves the accuracy significantly for particular combinations of grid increments. The difference equations read:

$$[\delta_t \eta = -\delta_x \{(1 + \alpha \bar{\eta}^{xt}) \delta_x \phi\} - \delta_y \{(1 + \alpha \bar{\eta}^{yt}) \delta_y \phi\} + \nu C_1]_{i,j}^{(n+\frac{1}{2})} \quad (9)$$

$$[\delta_t \phi + T + \eta - \frac{\epsilon}{3}(\delta_x^2 + \delta_y^2) \delta_t \phi + \nu C_2 = 0]_{i,j}^{(n)} \quad (10)$$

where

$$[T^{(n)} = \frac{\alpha}{2} \{(\delta_x \bar{\phi}^x)^{(n-\frac{1}{2})} (\delta_x \bar{\phi}^x)^{(n+\frac{1}{2})} + (\delta_y \bar{\phi}^y)^{(n-\frac{1}{2})} (\delta_y \bar{\phi}^y)^{(n+\frac{1}{2})}\}]_{i,j} \quad (11)$$

and νC_1 , νC_2 are numerical correction terms that remove second order errors from the hydrostatic parts of the equations. Thus, the overall discretization error is $O(\Delta x^2, \Delta y^2, \Delta t^2)$ for $\nu = 0$ and $O(\epsilon \Delta x^2, \alpha \Delta x^2, \Delta x^4, \epsilon \Delta y^2, \alpha \Delta y^2, \dots)$ for $\nu = 1$. The correction terms read:

$$C_1 = -\frac{1}{12}(\Delta x^2 + \Delta y^2) \delta_x^2 \delta_y^2 \phi \quad (12)$$

$$C_2 = \frac{1}{12}(\Delta x^2 - \Delta t^2) \delta_x^2 \delta_t \phi + \frac{1}{12}(\Delta y^2 - \Delta t^2) \delta_y^2 \delta_t \phi \quad (13)$$

When ν is set to zero, it is shown in [20] that the difference equations (9) and (10) are equivalent to the following representation of (6) and (5):

$$[\delta_t \eta = -\delta_x \{(1 + \alpha \bar{\eta}^{xt}) u\} - \delta_y \{(1 + \alpha \bar{\eta}^{yt}) v\}]_{i,j}^{(n+\frac{1}{2})} \quad (14)$$

$$[\delta_t u + \delta_x T = -\delta_x \eta + \frac{\epsilon}{3}(\delta_x^2 + \delta_y^2) \delta_t u = 0]_{i+\frac{1}{2},j}^{(n)} \quad (15)$$

$$[\delta_t v + \delta_y T = -\delta_y \eta + \frac{\epsilon}{3}(\delta_x^2 + \delta_y^2) \delta_t v = 0]_{i,j+\frac{1}{2}}^{(n)} \quad (16)$$

$$[T^{(n)} = \frac{\alpha}{2} \{(\bar{u}^x)^{(n-\frac{1}{2})} (\bar{u}^x)^{(n+\frac{1}{2})} + (\bar{v}^y)^{(n-\frac{1}{2})} (\bar{v}^y)^{(n+\frac{1}{2})}\}]_{i,j} \quad (17)$$

These are the constant depth versions of the difference equations in [18] and [23].

3 Cnoidal waves

It is well known that the nonlinear KdV or Boussinesq equations possess a family of plane and periodic wave solutions of permanent form, generally referred to as cnoidal waves. The solutions are governed by two parameters, the amplitude and wave length, that determine the shape and frequency. A thorough discussion of cnoidal waves is found in [26]. For small ratios α to ϵ the cnoidal wave solution coincides with the shallow water Stokes' wave. The latter solution is found by expanding the field variables and the frequency simultaneously in powers of α . The series for the frequency is achieved by requiring the expansions for the field variables to be consistent and uniformly valid in space and time. Details on the Stokes' wave expansion are given in [12].

Assuming no y -dependence and denoting the wave number and frequency by k and ω respectively, we may define the Stokes' wave expansion according to:

$$\eta = Y_0(\theta) + \alpha Y_1(\theta) + \dots \quad (18)$$

$$\phi = Rt + Fx + \Phi_0(\theta) + \alpha \Phi_1(\theta) + \dots \quad (19)$$

$$\begin{aligned} \omega &= \omega_0 + \alpha \omega_1 + \alpha^2 \omega_2 + \dots \\ &= k(c_0 + \alpha c_1 + \alpha^2 c_2 + \dots) \end{aligned} \quad (20)$$

where $\theta = kx - \omega t$ and the functions Y_j, Φ_j are periodic. We will choose α as the amplitude of the first harmonic of η , which means that $Y_0(\theta) = \cos \theta$. A similar expansion can be applied to the difference equations (9) and (10). Letting $\Delta x, \Delta t \rightarrow 0$ we will then reproduce the analytical solution of (3) and (4). We shall not discuss all the details of the tedious and rather straightforward arithmetics, but we note that the calculations are substantially simplified by the following observations and definitions:

- The horizontal length scale is chosen according to $\ell = 1/k^*$, which means that $k = 1$. A new time variable is defined by $\tau = \omega t$. We note that $\delta_t F(\theta) = \omega \delta_\tau F(\theta)$ where $\Delta \tau = \omega \Delta t$
- We define $a_x = \frac{2}{\Delta x} \sin(\frac{\Delta x}{2})$, $m_x = \cos(\frac{\Delta x}{2})$ and a_τ, m_τ correspondingly to obtain the relations:
 $\delta_x \sin \theta = a_x \cos \theta \quad \overline{\sin \theta^x} = m_x \sin \theta$ etc.
- It is easily shown that
 $\delta_x \sin 2\theta = 2m_x a_x \cos 2\theta \quad \overline{\sin 2\theta^x} = (2m_x^2 - 1) \sin 2\theta$ etc.

Naturally, in the limit $\Delta x, \Delta \tau \rightarrow 0$ the expansion becomes identical to the analytical one.

3.1 The linear harmonic solution

Inserting the series (18) through (20) in (9) and (10) we obtain to leading order:

$$Y_0 = \cos \theta \quad \Phi_0 = B_0 \sin \theta \quad c_0^2 = \frac{a_x^2}{a_\tau^2(1 + \frac{\tilde{\epsilon}}{3} a_x^2)} \quad (21)$$

where $B_0 = c_0 a_\tau / a_x^2$ and $\tilde{\epsilon} = \epsilon + \frac{1}{4}(\Delta t^2 - \Delta x^2)$. The expression for c_0 is naturally the discrete, linear dispersion relation. By restoring the grid increments into the right hand side and adopting the scaling according to equation (2) we obtain the stability criterion:

$$\Delta \hat{t}^2 \leq \Delta \hat{x}^2 + \frac{4}{3} \quad (22)$$

and the leading numerical dispersion terms according to:

$$\hat{\omega} = \hat{k} \left\{ 1 - \frac{1}{6} \hat{k}^2 - \frac{1}{24} (\Delta \hat{x}^2 - \Delta \hat{t}^2) \hat{k}^2 + O(\hat{k}^4) \right\} \quad (23)$$

We observe that leading numerical error terms enter the expression to the same order in $\hat{k}$ as the dispersion terms. This has bearing for the choice of scaling indicated by equation (37) in section 4.

3.2 The superharmonics

To the next order we have to apply additional conditions to obtain unique solution. When the averaged surface elevation and horizontal mass flux are prescribed to be zero we find:

$$c_1 = 0 \quad Y_1 = A_1 \cos 2\theta \quad \Phi_1 = B_1 \sin 2\theta + R_1 \tau + F_1 x \quad (24)$$

where

$$A_1 = \frac{1}{4} \frac{m_x^4 + 2m_\tau^2 + \frac{8\bar{\epsilon}}{3} m_x^2 m_\tau^2 a_x^2}{m_\tau^2 - m_x^2 + \frac{\bar{\epsilon}}{3} (m_x^2 - 1) a_x^2 m_x^2} \quad (25)$$

and

$$B_1 = \frac{c_0 a_\tau m_\tau m_x^2}{4a_x^2} \frac{(\frac{3}{2} + \frac{\bar{\epsilon}}{3} a_x^2)}{m_\tau^2 - m_x^2 + \frac{\bar{\epsilon}}{3} (m_x^2 - 1) a_x^2 m_x^2} \quad (26)$$

$$R_1 = \frac{1}{4} a_x^2 m_x^2 (2m_\tau^2 - 1) \quad (27)$$

$$F_1 = \frac{1}{2} B_0 \quad (28)$$

It is emphasized that the appearance of the coefficient R_1 is related to the choice of the constant in the Bernoulli equation. Letting $\Delta\tau, \Delta x \rightarrow 0$ we find $\alpha A_1 \rightarrow O(\alpha\epsilon^{-1})$. Hence, the expansion is meaningful only if $\epsilon \gg \alpha \gg \epsilon^2$ which imply that the Ursell parameter, α/ϵ has to be small. Assuming $O(\Delta x^2 \Delta\tau^2) \leq O(\epsilon)$ the expression for A_1 can be expanded in orders of $\Delta x^2, \Delta\tau^2$ to give:

$$A_1 = \frac{3 + \hat{k}(\frac{8}{3} + \frac{1}{12}\{(8\nu - 6)\Delta\hat{t}^2 - (8\nu + 6)\Delta\hat{x}^2\} + O(\Delta\hat{x}^4, \Delta\hat{t}^4)}{\hat{k}^2\{\frac{4}{3} + (1 - \frac{\nu}{3})(\Delta\hat{x}^2 - \Delta\hat{t}^2) + O(\Delta\hat{x}^4, \Delta\hat{t}^4)} \quad (29)$$

Noting that $\Delta\hat{x} = \Delta\hat{t} = 0$ corresponds to the analytical (non-discrete) case, we observe that, regardless of the wavelength, we must require $\Delta\hat{x}, \Delta\hat{t} \ll 1$, or $\nu = 1$ and $\Delta\hat{x} = \Delta\hat{t}$, to reproduce A_1 properly. In dimensional coordinates the first requirement corresponds to $\Delta x^* \ll h^*$ etc. We note that the correction terms do not improve the value of A_1 markedly.

4 The soliton solution

The shallow water solitary wave is a single crested wave of permanent shape with exponentially decaying tails extending to infinity. The propagation speed is slightly larger than that for infinitely long, linear waves. Solitary wave solutions are inherent in the weakly dispersive and nonlinear theories corresponding to Boussinesq or KdV equations. Strongly nonlinear solutions have been obtained by perturbing the full, inviscid set, as in [13] and [27]. The solitary wave solution has a large amount of interesting features and mathematics attached to it. Most remarkable is maybe the particle-like collision properties which have motivated the soliton name. Within the weakly nonlinear and dispersive approximation, solitons regain their original shapes after interaction. A survey on the theory of shallow water solitons can be found in [26] and [16]. In the present paper, we will not attempt to discuss the performance of numerical methods in relation to interaction behavior etc. of solitons. Instead we

will confine ourselves to the more basic question of existence of single crested waves of permanent shape. Still, we will use the term soliton also for the discrete solutions.

As demonstrated in section 3, Stokes wave expansion for the difference equations was, at least principally, as simple as for the differential equations. This is due to the existence of Fourier transforms that are essentially identical for the discrete and the non-discrete case. However, the description of a general wave of permanent form moving in a grid is less straightforward, and the existence of such solutions is far from obvious.

We will start with a brief look at a special case for which the concept of a permanent wave form carries directly over to the discrete description. When the plane permanent wave form $\eta = Y(x - ct)$, $\phi = \Phi(x - ct)$ are substituted into the differential equations (3) and (4) we obtain a set of ordinary differential equations in the composite variable $\theta = x - ct$. But, even though we assume no y -dependence and prescribe $\eta_i^{(j)} = Y(i\Delta x - cj\Delta t)$ etc., the partial difference equations will degenerate to ordinary ones. To achieve an equation that involves single indices only, we must assume $c\Delta t = \Delta x$ which means that the wave profile moves exactly one space increment pr. time step. This is an awkward requirement, particularly since c generally has to be found as a part of the solution. Anyway, if we assume $c\Delta t = \Delta x$, define $\theta_j = (j - \gamma)\Delta x + c\gamma\Delta t$ and invoke

$$\eta_i^n = Y(\theta_{i+n}) \quad \phi_i^{(n+\frac{1}{2})} = \Phi(\theta_{i+n+\frac{1}{2}}) \quad (30)$$

in (3) and (4) we obtain:

$$[cY = (1 + \alpha \bar{Y}^{\theta\theta})U]_i \quad (31)$$

$$[cU - \frac{\alpha}{2}T - Y - \frac{c}{3}\{\alpha + \frac{\nu}{4}\Delta x^2(c^2 - 1)\}\delta_\theta^2 U = 0]_i \quad (32)$$

where we have exploited that U and Y vanish at infinity and where U and T are defined according to:

$$[\delta_\theta \Phi = U]_i \quad T_i = \bar{U}_{i-\frac{1}{2}} \bar{U}_{i+\frac{1}{2}} \quad (33)$$

The ordinary difference equations (31), (32) together with the conditions at infinity define a nonlinear eigenvalue problem for c . This problem is probably solvable by combining asymptotic approximations at infinity with a shooting technique, and a soliton solution is likely to exist. However, the existence of a solution for this particular choice of grid increments, is not a reliable indication for the existence of such solutions in general. We will not pursue this case any further but instead turn to another and more general approach.

Provided $c\Delta t/\Delta x$ is a non-rational number all grid-points are associated with different values $\theta_i^{(j)} = i\Delta x - cj\Delta t$ and the definition of a permanent wave form is not trivial. A simple definition is to link the existence of a solitary wave to the existence of an analytical function $Y(\theta)$ which satisfies the relation $Y(\theta_i^{(j)}) = \eta_i^{(j)}$ for all i and j . If Y exist it is uniquely determined because the set $\{\theta_i^{(j)}\}$ always has accumulation points¹. When an analytic function is substituted into (9) and (10), or (14) through (17), all differences can be replaced by series of derivatives that are found by Taylor series expansions. Retaining only the leading discretization error terms we will then

¹Accumulation point in the usual mathematical sense: a point for which we may find arbitrarily close $\theta_i^{(j)}$

find approximate solutions to the resulting differential equations. This correspond to the first few steps of a perturbation procedure that in principle can be carried out to any order, thereby indicating that discrete solitary wave solutions do exist. However, no proof of convergence of the perturbation series is established.

4.1 The perturbation expansion

When the dependent variables in the difference equations are regarded as grid point values of analytical functions, we may apply Taylor series expansion. Expanding the quantities in (9) and (10) about $(i\Delta x, j\Delta y, (n+\frac{1}{2})\Delta t)$ and $(i\Delta x, j\Delta y, n\Delta t)$ respectively, we deduce the differential equations:

$$\begin{aligned} \eta_t + \frac{\Delta t^2}{24}\eta_{ttt} + \frac{\Delta t^4}{1920}\eta_{ttttt} = & -\{(1+\alpha\eta)\phi_x\}_x + D_{(x)} - \{(1+\alpha\eta)\phi_y\}_y + D_{(y)} \\ & -\frac{\nu}{12}(\Delta x^2 + \Delta y^2)(\phi + \frac{\Delta x^2}{12}\phi_{xx} + \frac{\Delta y^2}{12}\phi_{yy})_{xxyy} \\ & + O(\alpha\Delta x^{2i}\Delta y^{2j}\Delta t^{2k}, \Delta x^{2I}\Delta y^{2J}\Delta t^{2K}) \end{aligned} \quad (34)$$

$$\begin{aligned} \phi_t + \frac{\Delta t^2}{24}\phi_{ttt} + \frac{\Delta t^4}{1920}\phi_{ttttt} + \frac{\alpha}{2}(\phi_x)^2 + \frac{\alpha}{2}(\phi_y)^2 + \eta \\ -\{\frac{\epsilon}{3} + \frac{\nu}{12}(\Delta t^2 - \Delta x^2)\}\phi_{xxt} + R_{(x)} - \{\frac{\epsilon}{3} + \frac{\nu}{12}(\Delta t^2 - \Delta y^2)\}\phi_{yyt} + R_{(y)} \\ + O(\alpha\Delta x^{2i}\Delta y^{2j}\Delta t^{2k}, \epsilon\Delta x^{2i}\Delta y^{2j}\Delta t^{2k}, \Delta x^{2I}\Delta y^{2J}\Delta t^{2K}) \end{aligned} \quad (35)$$

where $i + j + k = 2$, $I + J + K = 3$ and

$$\begin{aligned} D_{(x)} = & -\frac{\Delta x^2}{12}\phi_{xxxx} - \frac{\Delta x^4}{360}\phi_{xxxxxx} \\ & -\frac{\Delta x^2}{24}\alpha\{(\eta\phi_x)_{xxx} + (\eta\phi_{xxx})_x + 3(\eta_{xx}\phi_x)_x\} - \frac{\Delta t^2}{8}(\eta_{tt}\phi_x)_x \\ R_{(x)} = & +\frac{\Delta x^2}{6}\alpha\phi_{xxx}\phi_x + \frac{\Delta t^2}{4}\alpha(\phi_x\phi_{xtt} - (\phi_{xt})^2) \\ & -\{\frac{\epsilon}{3} + \frac{\nu}{12}(\Delta t^2 - \Delta x^2)\}\{\frac{\Delta x^2}{12}\phi_{xxxxt} + \frac{\Delta t^2}{24}\phi_{xtttt}\} \end{aligned}$$

Expressions for $E_{(y)}$ and $D_{(y)}$ are obtained simply by replacing x by y in the above definitions. We seek permanent wave solutions as functions of the composite variable

$$\theta = k(x \cos \psi + y \sin \psi - ct) \quad (36)$$

where ψ is the angle between the x -axis and the direction of wave advance. Prescribing:

$$\alpha = \epsilon \quad \Delta x^2 = O(\alpha) \quad \Delta y^2 = O(\alpha) \quad \Delta t^2 = O(\alpha) \quad (37)$$

we may assume the expansions:

$$\eta = Y(\theta) = Y_0 + \alpha Y_1 + \alpha^2 Y_2 + \dots \quad (38)$$

$$U \equiv \partial\phi/\partial x = U_0 + \alpha U_1 + \alpha^2 U_2 + \dots \quad (39)$$

$$k = 1 + \alpha k_1 + \alpha^2 k_2 + \dots \quad (40)$$

$$c = 1 + \alpha c_1 + \alpha^2 c_2 + \dots \quad (41)$$

Equation (37) corresponds to rather large grid increments, namely $\Delta x^* \sim h^*$ etc. However, in 3-D simulations the resolution often have to be very coarse. According to [19] and [20] sufficient accuracy may be still be achieved by introduction of simple correction terms, or sometimes by choosing an optimal time step. The simpler case of a more refined grid will be discussed later. The expansion of k is necessary because the shape and length of the soliton, as the speed of propagation, depend on the amplitude. The zeroth order term of k can be chosen arbitrarily without loss of generality. At $\theta = \pm\infty$ we require $Y = Y' = Y'' = \dots = U = U' = U'' = \dots = 0$. A definition of α can be given by $\max Y_0 = 1$. This definition is not precise unless we add the condition that solutions of the same mathematical form as Y_0 must be excluded from higher order terms. Substitution into (34),(35) and integration of the former give:

$$cY = (1 + \alpha Y)U + \alpha k^2(\gamma_1 U'' - \gamma_2 c^2 Y'') \quad (42)$$

$$+ \alpha^2(\kappa_1 U'''' - \kappa_2 Y'''' + \kappa_3(YU)'' + \kappa_4 Y''U + \kappa_5 YU'') + O(\alpha^3) \quad (43)$$

$$cU - \frac{\alpha}{2}U^2 - Y - \alpha\left(\frac{1}{3} + \mu - \gamma_3 c^2\right)ck^2 U''$$

$$- \alpha^2(\kappa_6 U'''' + \kappa_7 UU'' - \kappa_8 (U')^2) + O(\alpha^3)$$

The coefficients are given by:

$$\begin{aligned} \gamma_2 = \gamma_3 &= \frac{\Delta t^2}{24\alpha} & \gamma_1 &= \frac{e_2}{12\alpha} & \mu &= \frac{\nu}{12\alpha}(\Delta t^2 - \cos^2 \psi \Delta x^2 - \sin^2 \psi \Delta y^2) \\ \kappa_1 &= \frac{e_4}{360\alpha^2} + \frac{\nu}{144}(\Delta x^2 + \Delta y^2)(\Delta x^2 \cos^2 \psi + \Delta y^2 \sin^2 \psi) \cos^2 \psi \sin^2 \psi & (44) \\ \kappa_2 &= \frac{\Delta t^4}{1920\alpha^2} & \kappa_3 = \kappa_5 &= \frac{e_2}{24\alpha} & \kappa_4 &= \frac{e_2 + \Delta t^2}{8\alpha} & \kappa_8 &= \frac{\Delta t^2}{4\alpha} \\ \kappa_6 &= \frac{2e_2 + (3\mu + 1)\Delta t^2}{72\alpha} - \kappa_2 \\ &+ \frac{\nu}{144}(\Delta x^2 \cos^4 \psi (\Delta t^2 - \Delta x^2) + \Delta y^2 \sin^4 \psi (\Delta t^2 - \Delta y^2)) \\ \kappa_7 &= \frac{2e_2 + 3\Delta t^2}{12\alpha} \end{aligned}$$

where $e_2 = \Delta x^2 \cos^4 \psi + \Delta y^2 \sin^4 \psi$ and $e_4 = \Delta x^4 \cos^6 \psi + \Delta y^4 \sin^6 \psi$. The distinctions between coefficients with equal values are kept for generality. Invoking the series expansion and requiring that every order of α shall vanish, we obtain a hierarchy of equations of the form:

$$Y_i - U_i = R_i(Y_0 \dots Y_{i-1}, U_0 \dots U_{i-1}; \dots c_i, \dots k_{i-1}) \quad (45)$$

$$U_i - Y_i = H_i(Y_0 \dots Y_{i-1}, U_0 \dots U_{i-1}; \dots c_i, \dots k_{i-1}) \quad (46)$$

where $R_0 = H_0 = 0$. This means that, after having computed Y_{j-1}, U_{j-1} , we find Y_j, U_j by solving the set:

$$Y_j - U_j = R_j \quad R_{j+1} + H_{j+1} = 0 \quad (47)$$

The latter equation is a solubility condition for the next order.

4.2 The zeroth order solution

For the leading terms of the expansion we obtain $U_0 = Y_0$ and

$$2\sigma Y_0'' + \left(\frac{3}{2}Y_0 - 2c_1\right)Y_0 = 0 \quad (48)$$

where $\sigma = \frac{1}{2}(\frac{1}{3} + \mu + \gamma_1 - \gamma_2 - \gamma_3)$. Integration of Y_0 times (48) immediately gives:

$$\sigma(Y_0')^2 + \left(\frac{1}{2}Y_0 - c_1\right)Y_0^2 = 0 \quad (49)$$

The above equation, together with (48), will be used to remove derivatives of Y_0 from the right hand side of the higher order equations. This will work as long as the sum of the number of derivations applied to all factors of each term is even. To avoid negative solitons with negative c_1 we have to require $\sigma > 0$. Anyway, this turns out to be a much weaker condition than the linear stability criterion (22). From the above equation and the constraint $\max Y_0 = 1$ we find:

$$c_1 = \frac{1}{2} \quad (50)$$

$$Y_0 = \cosh^{-2}\left(\frac{\theta}{2\sqrt{2\sigma}}\right) \quad (51)$$

A few important observations should be made at this stage. We note that to order α the propagation speed, c , is independent of the discretization, whereas the shape of the soliton is not. In fact, even an erroneous factor before the dispersion term will not alter c to this order. Therefore, considerations on propagation speeds of solitons alone should never be used to validate numerical methods.

The quantity σ has a simple interpretation. If the nonlinear term of (48) is omitted the equation must describe linear solutions of permanent form, which means sinusoidal waves. In this case the perturbation parameter α have to be regarded as a measure of wavelength instead of amplitude. To get a unique definition of the parameter we demand that the dimensionless wave number equals unity and find $c_1 = -\sigma$ which means:

$$c = 1 - \sigma(k^*h_0)^2 + O((k^*h_0)^4) \quad (52)$$

The analysis in the present section has so far been concerned only with the difference formulation (9) and (10), alternatively (14) and (15). However, any consistent and regular discretization of the Boussinesq equations will lead to a zeroth order solution of the form (51) with σ having the above interpretation. A consequence is that the zeroth order solution for the discrete solitary wave can be deduced from the linear hydrostatic dispersion relation. (The "analytical part" of σ is always $\frac{1}{6}$.) This is different for KdV or KP(Kadomtsev-Petsiavili) equations which are expressed in terms of a composite variable $x - t$ in addition to a slow time variation.

We note that to the present order all numerical errors disappear when the correction terms are retained ($\nu = 1$), or if $\Delta t^2 = \Delta x^2 \cos^4 \psi + \Delta y^2 \sin^4 \psi$. In these cases the difference equations are accurate to the same order as the Boussinesq equations, provided that (37) holds.

4.3 The first order solution

Most of the important features of discrete solitons are revealed by the leading terms alone. In fact, the $O(\alpha\epsilon, \epsilon^2)$ error terms of the Boussinesq equations themselves would enter the perturbation scheme at the next stage. Still, we will advance the perturbation expansion one step further to indicate the existence of solitary wave solutions of the set (9) and (10). Besides, it is essential for verification of numerical methods and programs to have as precise descriptions as possible of both analytical and numerical solutions, even though the accuracy of the basic theory is limited. Rewriting R_1 , R_2 and H_2 from equation (45) and (46), repeatedly using (48) and (49) to remove derivatives of Y_0 and finally eliminating U_1 we obtain an equation of the form

$$L(Y_1) \equiv 2\sigma Y_1'' + (3Y_0 - 1)Y_1 = aY_0 + bY_0^2 + fY_0^3 \quad (53)$$

where

$$a = 2c_2 - 2k_1 - \frac{3}{4} + \frac{1}{2}\sigma^{-1}(\gamma_1 - \frac{1}{2}\gamma_2 + \gamma_3) + \frac{1}{4}\sigma^{-2}\gamma_2(\gamma_1 - \gamma_2) - \frac{1}{4}\sigma^{-2}(\kappa_1 - \kappa_2 + \kappa_6) \quad (54)$$

$$b = 3k_1 + 4 + \sigma^{-1}(-\frac{11}{4}\gamma_1 + \frac{35}{8}\gamma_2 - \frac{3}{4}\gamma_3) - \frac{1}{2}\sigma^{-1}(4\kappa_3 + \kappa_4 + \kappa_5 + \kappa_7 - \kappa_8) - \frac{15}{8}\sigma^{-2}\gamma_2(\gamma_1 - \gamma_2) + \frac{15}{8}\sigma^{-2}(\kappa_1 - \kappa_2 + \kappa_6) \quad (55)$$

$$f = -3 + \sigma^{-1}(\frac{9}{4}\gamma_1 - \frac{19}{4}\gamma_2) + \frac{3}{4}\sigma^{-1}(\frac{10}{3}\kappa_3 + \kappa_4 + \kappa_5 + \kappa_7 - \kappa_8) + \frac{15}{8}\sigma^{-2}\gamma_2(\gamma_1 - \gamma_2) - \frac{15}{8}\sigma^{-2}(\kappa_1 - \kappa_2 + \kappa_6) \quad (56)$$

Contrary to (48) equation (53) is linear, but inhomogeneous, and have variable coefficients. However, by differentiating (48) it is immediately realized that $V = Y_0'$ is a solution of the homogeneous equation $L(V) = 0$. The other homogeneous solution grows exponentially as $\theta \rightarrow \pm\infty$. If $H^{(j)}$ is defined as the symmetric and finite solution of $L(V) = Y_0^j$ it is easily shown that:

$$\begin{aligned} H^{(1)} &= \frac{1}{2}\theta Y_0' + Y_0 \\ H^{(2)} &= \frac{2}{3}Y_0 \\ H^{(j+1)} &= \frac{1}{3 - \frac{1}{2}j - j^2}(Y_0^j - (j^2 - 1)H^{(j)}) \quad j \geq 2 \end{aligned} \quad (57)$$

For Y_1 we thus find:

$$Y_1 = \beta Y_0' + a(\frac{1}{2}\theta Y_0' + Y_0) + (\frac{2}{3}b + f)Y_0 - \frac{1}{2}fY_0^2 \quad (58)$$

The first term on the right hand side corresponds to a constant phase shift of order α in Y_0 . Thus, requiring that the maximum shall be located at $\theta = 0$ to all orders yields $\beta = 0$. The second term violates the uniformity of the expansion (secular term) and we must have $a = 0$. Finally, the third term only modify the definition of α and must be prescribed to vanish to give unique values for c_2 and k_1 . This gives:

$$Y_1 = -\frac{1}{2}fY_0^2 \quad (59)$$

For a more refined grid we may assume that the increments are of order α instead of $\alpha^{\frac{1}{2}}$. In this case the discretization errors will first appear in the equations for Y_1 and U_1 and only cause a modification of k_1 . We shall not go into details but merely state that when the grid increments are restored into (51) the resulting expression is valid also in this case.

4.4 Higher order terms

Provided sufficient terms from the Taylor series expansion are retained in (34) and (35), the expansion can be carried out to any order, although the arithmetics becomes unbearable. It is easily realized that only terms of even derivatives will enter the counterparts of (42) and (43). For product terms the sum of the derivatives of each factor has to be even. We may then recursively obtain equations of the form:

$$L(Y_n) = \sum_{i=1}^{n+2} a_i^{(n)} Y_0^i \quad (60)$$

where $a_1^{(n)}$ and $a_2^{(n)}$ are functions of $c_{(n+1)}$ and k_n . This equation can be handled in the same manner as (53). Solutions Y_n , that vanish at infinity, can thus be found for all n . Formally, the total solution for Y will become a power series in Y_0 . We note that the a_1 -coefficients stem exclusively from the linear parts of the equations. Thus, requiring these coefficients to be zero must correspond to some dominant balance between linear terms. Since the solutions have form as power series in Y_0 , the behaviour at the outskirts of the soliton will be:

$$\eta \sim 4e^{-\frac{|\eta|}{\sqrt{2}\sigma}} \quad \phi \sim 4Be^{-\frac{|\phi|}{\sqrt{2}\sigma}} \quad (61)$$

which means that the linear terms have to dominate at infinity. Substituting the above expressions into the linearized versions of (9) and (10) we find

$$\Omega^2(1 - \frac{\alpha}{3}K^2) = K^2 \quad B = -\Omega K^{-2} \quad (62)$$

where

$$K = \frac{2}{\Delta x} \sinh(\frac{k\Delta x}{2}) \quad \Omega = \frac{2}{\Delta t} \sinh(\frac{kc\Delta t}{2}) \quad (63)$$

Exploiting this relation and invoking the power series directly in the equations corresponding to (34) and (35) will simplify the arithmetics to some extent. Still, we have no intention of calculating the higher Y_n . Neither will we attempt to derive bounds on the higher order coefficients, thereby leaving the question of convergence open.

4.5 Verification and comparison with simulation

In the present subsection it is more convenient to describe the solitary waves in terms of the total wave height, A , instead of the perturbation parameter α . We will also introduce the renormalized quantities $\hat{x}$, $\hat{t}$ and $\hat{\eta}$, which are defined at the beginning of section 2 and confine the discussions to two dimensional cases. The exact solution will be denoted by $\hat{\eta}_{(e)}$ for the analytical (non-discrete) as well as the discrete case. Correspondingly, $\hat{\eta}_{(0)}$ and $\hat{\eta}_{(1)}$ denote two levels of approximation. Naturally, the

perturbation expansion described in section 4.1–4.3 is applicable also to the analytical case, just by putting $\Delta x = \Delta t = 0$.

The zeroth order approximation to a soliton of height A is defined as:

$$\begin{aligned}\hat{\eta}_{(0)} &= AY_0(\theta) \\ \theta &= \sqrt{A}(\hat{x} - c\hat{t}) \\ c &= 1 + \frac{1}{2}A\end{aligned}\tag{64}$$

Correspondingly the first order approximation is given by:

$$\begin{aligned}\hat{\eta}_{(1)} &= \alpha Y_0(\theta) - \frac{1}{2}\alpha^2 f Y_0(\theta)^2 \\ \theta &= (1 + \alpha k_1)\sqrt{\alpha}(\hat{x} - c\hat{t}) \\ c &= 1 + \frac{1}{2}\alpha + c_2\alpha^2\end{aligned}\tag{65}$$

where $\alpha(A)$ is a solution of:

$$A = \alpha - \frac{1}{2}f\alpha^2\tag{66}$$

The above definitions imply that the height of the soliton becomes correct, whereas the length, shape and speed will display deviations.

Letting $\Delta x, \Delta t \rightarrow 0$ we find by some simple manipulations:

$$c = 1 + \frac{1}{2}A - \frac{5}{24}A^2 + O(A^3)\tag{67}$$

which is consistent with the result of [20]. In figure 1 we have compared the analytical $\hat{\eta}_{(0)}$ and $\hat{\eta}_{(1)}$ to the “exact” soliton solution, $\hat{\eta}_{(e)}$, of the set (3) and (4). This solution is obtained by accurate numerical integration as described in [19]. A similar procedure of solution is not available for the discrete case. For $A = 0.1$ the zeroth order solution follows $\hat{\eta}_{(e)}$ closely, whereas the first order solution hardly can be distinguished from the exact one. Even for $A = 0.5$, which is clearly outside the validity range of the Boussinesq equations, both approximations agree well with $\hat{\eta}_{(e)}$.

To confirm the validity of the perturbation expansion in the discrete case we have compared $\hat{\eta}_{(0)}$ and $\hat{\eta}_{(1)}$ to solutions of the difference equations (9) and (10) that are assumed to be to solitons. These solutions are generated by evolution from initial conditions. The corresponding analytical case is thoroughly analyzed by the inverse scattering technique [26]. If the leading pulse that evolves reach a permanent state, we accept this as a soliton solution. In the computer calculations the exponentially decaying head and tail of the soliton have to be truncated, and care must be taken to avoid that this will influence the solution. When a soliton solution of (3) and (4) is used as initial conditions we find that the amplitude, A , changes slowly in time. Meanwhile a trough and a dispersive tail evolve. In figure 2 we have displayed $A(\hat{t})$ for $A_0 \equiv A(0) = 0.1$ and $A_0 \equiv A(0) = 0.3$. Both functions clearly seems to approach a finite limit as $\hat{t} \rightarrow \infty$. The fluctuations, which are particularly noticeable for $A_0 = 0.3$, are due to the spline interpolation that is used to find $A(\hat{t})$. Such an interpolation is sensitive to the location of the maximum relative to the nearest grid point. In figure 3 we have compared the calculated wave profiles to the various soliton solutions. The agreement between the discrete solution and $\hat{\eta}_{(1)}$ is nearly perfect. These results strongly support the relevance of the perturbation solution.

Figure 1: Wave profiles. Dashed/dotted line : exact solution ($\hat{\eta}_{(e)}$), fully drawn line : first order solution ($\hat{\eta}_{(1)}$), dashed line : zeroth order solution ($\hat{\eta}_{(0)}$). Both cases correspond to $\Delta\hat{x} = \Delta\hat{t} = 0$.

When the correction terms are retained ($\nu = 1$), we still get a noticeable decrease in amplitude relative to A_0 . This is partly due to the initial conditions, which are not corrected. Anyway, the crucial point is that the wave profile which eventually evolves, is much closer to the exact one than for $\nu = 0$. This is illustrated in figure 4 which displays profiles for the same set of parameters as figure 3a. The improvement of the numerical solution (marks) is substantial. The deviation between the discrete solution and $\hat{\eta}_{(1)}$ is still small, but noticeably larger than in the previous cases. Other comparisons have revealed that the accuracy of the perturbation solution becomes poorer when the correction terms are nearly equal to the dispersion terms. A moderate reduction of the grid increments causes $\hat{\eta}_{(1)}$ and the discrete solution to once again coincide almost exactly, as shown in figure 5

4.6 Three dimensional solutions

In computer calculations, solitons of oblique orientation relative to the grid axes generally must be treated as fully three dimensional waves. It is possible to simulate only a finite length of the crest and some nontrivial conditions have to be applied at the lateral boundaries. The outcome of such calculations would be influenced as least as much by these conditions as by the difference discretization itself. Thus, the question of existence of oblique solitons can not easily be answered through studies of evolution from initial conditions. We may state that whereas the discrete perturbation solution made conclusions concerning the 2-D case firmer, more general and easier to reach, a proper discussion of the 3-D solitons might rely entirely on the closed form solution. An exception is the case $\tan\psi = \Delta x/\Delta y$, for which the discrete equations (3) and (4) reduce to their two dimensional form with $\cos\psi\Delta x$ as new space increment.

Figure 2: Wave height, A , as function of time when (9) and (10) are solved with an "exact" solitary wave solution as initial state. a: $A_0 = 0.1$, $\Delta\hat{x} = 1.5$ and $\Delta\hat{t} = 0.5$, b: $A_0 = 0.3$, $\Delta\hat{x} = 1.0$ and $\Delta\hat{t} = 0.3$.

Figure 3: Wave profiles. The marks correspond to values of the surface elevation at the grid points, the meaning of the different lines is as in figure 1. The location of the origin has no relevance.

a: $A_0 = 0.1$, $\Delta\hat{x} = 1.5$ and $\Delta\hat{t} = 0.5$, $\hat{t} = 1000$.
b: $A_0 = 0.3$, $\Delta\hat{x} = 1.0$ and $\Delta\hat{t} = 0.3$, $\hat{t} = 300$.

Figure 4: The wave profile for $\nu = 1$ and the same grid increments as in figure 3a.

Figure 5: Wave profiles at $\hat{t} = 1000$ for $A_0 = 0.1$, $\Delta \hat{x} = 1.0$ and $\Delta \hat{t} = 0.2$. (a): $\nu = 1$
(b): $\nu = 0$

(a)

(b)

Figure 6: Soliton profiles for $\psi \neq 0$. The units on the coordinate axis correspond to dimensionless length along the direction of wave advance. Fully drawn line : $\hat{\eta}_{(1)}$ for $\nu = 1$, short dashes : $\hat{\eta}_{(1)}$ for $\nu = 0$ and long dashes : exact solution of the differential equations.

(a): $\psi = 45^\circ$, $A = 0.1$, $\Delta\hat{x} = \Delta\hat{t} = \Delta\hat{y} = 1.0$.

(b): $\psi = 30^\circ$, $A = 0.2$, $\Delta\hat{x} = \Delta\hat{t} = 1.5$, $\Delta\hat{y} = 1.0$.

There are no traceable differences between the cases $\psi = 0$ and $\psi \neq 0$ as far as the perturbation expansion is concerned. Therefore, it is likely that also oblique soliton solutions exist and are closely approximated by $\hat{\eta}_{(1)}$. In figure 6 we have compared the approximate discrete solution $\hat{\eta}_{(1)}$ to the full analytical solution $\hat{\eta}_{(e)}$ for two sets of parameters giving different resolutions. The improvement due to the correction term ($\nu = 1$) is once again apparent, and yields good agreement even for the coarser grid.

4.7 Asymmetric difference representations

The use of up-wind differences for convective terms or any sort of explicitness in the time stepping will violate the symmetry of the difference equations. Generally, implicit equations like (9) and (10) must be solved by some iterative technique. This may, at least in principle, introduce asymmetry. The effect of up-winding or iteration will often be numerical dissipation which makes the existence of permanent wave forms unlikely. However, neither the Boussinesq equations (3), (4) nor the numerical counterparts are exactly energy conserving and solitary waves do still exist. In view of this, the question of existence of solitons for weakly asymmetric cases calls for a more careful discussion. Besides, it would be instructive to see how the perturbation scheme from the previous sections eventually fails.

In [19] the continuity equation (9) is solved by pointwise iteration. If only one iteration is used this method correspond to an explicit representation of the nonlinear term yielding an error of order $\alpha\Delta t$. Even though this error is comparatively small, it causes a noticeable growth in amplitude that rapidly makes the results worthless. Even the application of two iterations yield a very small, but observable, amplification which indicates that no permanent wave form do evolve. This is demonstrated in figure 7 which also shows that four iterations gives no apparent growth for the displayed time interval ($\hat{t} < 3000$). The use of n iterations will give second order accuracy provided $n > 1$, but will also introduce an error of order Δt^{2n+1} in (34) which corresponds to a term of order $\alpha^{n+\frac{1}{2}}$ in (42). One implication is that we have to expand in half powers of α , even though the half powers less than n will vanish. Another is that we at stage $n + \frac{1}{2}$ obtain an equation

$$L(Y_{n+\frac{1}{2}}) = q(Y_0, c_{n+\frac{1}{2}}, k_{n-\frac{1}{2}}) + F \quad (68)$$

where q is a quadratic function of Y_0 and F is an odd function in the sense that $F(\theta) = -F(-\theta)$. The odd term stems from the $O(\alpha^{n+\frac{1}{2}})$ term in the counterpart of (42) which involves an odd number of derivations. However, no solution of (68) can vanish at both $\theta = +\infty$ and $\theta = -\infty$ unless the right hand side is even in θ . As a consequence the perturbation expansion breaks down. To prove the statement concerning (68) we multiply both sides by Y'_0 , integrate from $\theta = -\infty$ to $\theta = +\infty$, apply integration by parts to the left hand side and finally invoke equation (48) to obtain:

$$\{2\sigma Y'_0 Y'_{n+\frac{1}{2}} + (\frac{3}{2}Y_0 - 1)Y_0 Y_{n+\frac{1}{2}}\}|_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty} F Y'_0 d\theta \geq 0 \quad (69)$$

For nonzero F this equation can be satisfied only if

$$Y_{n+\frac{1}{2}} \sim e^{\frac{|\theta|}{\sqrt{2\sigma}}} \quad (70)$$

at $\theta = \infty$ or $\theta = -\infty$. We may conclude that the soliton solution is violated for any finite number of iterations. Still, if n is large the higher order imperfection in the expansion will be noticeable only for extremely large times. For the present pointwise iteration scheme, the growth in amplitude is so weak that even $n = 2$ will suffice in most cases. However, for the calculations on steady ship-wave patterns in [19] the effect of using only two iterations was noticeable and n was increased to 5. The actual calculations involved slow energy exchange between (nearly) solitons and were carried out for large time intervals.

From the above discussion it also follows that up-wind differences will ruin the solitary wave solution. This is not the case for line by line iterations as described in [18], [19] and [20] which do not introduce odd error terms. Use of any kind of smoothing will, on the other hand, introduce damping which again may distort the soliton solution.

5 Conclusions

Applying perturbation expansions to the difference equations (9) and (10) we have established formal power series solutions corresponding to the cnoidal and solitary wave. For the solitary wave the validity of the approach have been demonstrated by comparison between a two term approximation and the full (or presumably exact) solution, although no rigorous proof of convergence is given. The full solution is obtained by accurate numerical integration for the differential equations, and by time evolution of a steady permanent wave form in the 2-D discrete case. We conclude that discrete solitary waves do exist.

Quantitatively, the perturbation solution shows that the phase speed becomes accurate regardless of the discretization and sometimes even of the consistency of the numerical method. As a consequence, propagation speed of solitary waves can not be used for validation of numerical methods. The discrete, closed form solution also reveals that the inclusion of simple correction terms improve the accuracy substantially for coarse grids. In two dimensions this is confirmed by computer calculations. In addition, the series solution gives information on the accuracy of representations of solitons that are oblique, in the sense that the direction of wave advance do not coincide with any of the grid axes. A proper investigation of existence and accuracy of such discrete waves would be extremely difficult without access to "hand made" solutions of the difference equations.

Accurate representation of the second harmonic of a cnoidal wave requires a refined grid mesh. This result is confined to small Ursell numbers, for moderate and high Ursell numbers the behavior will probably be as for solitary waves.

The methods used are immediately applicable to any symmetric and homogeneous difference or element discretization. By symmetric we mean that all error terms are of even order in the grid increments. On the other hand, if the symmetry is violated by iterative equation solvers, up-winding, predictor -corrector representations etc, the perturbation expansion collapses. This is demonstrated for a case involving pointwise iteration. It is also shown that the absence of a perfect solitary wave solution is of no consequence, unless the number of iterations is very small. Generally the acceptable level for damping of the "imperfect solitons" depends on the other time scales of the actual problem. Anyway, the question of existence or damping rates of near solitons

should always be considered when studying development of steady state solutions etc.

Figure 7: Time stories for the maximum wave height when pointwise iteration is used. The grid increments are $\Delta\hat{x} = \Delta\hat{t} = 1.0$ for all cases.

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