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Abstract. The conditions for obtaining thermal convection in dry snow is examined. It is shown that thermal convection will occur in dry snow layers with strong vertical temperature gradients and large air permeabilities (i.e. old snow). Thermal convection will lower the insulating power of the snow and increase the flux of water vapour through the snow layer. The magnitude of the heat and mass flux in convection is computed for several values of the Rayleigh number.

INTRODUCTION

It is well known that the occurrence of a vertical temperature gradient in a snow layer will lead to a mass flux of water vapour in the snow. Snow layers with strong vertical temperature gradients occur frequently in polar regions and high mountain areas. The air temperature may here be down to -30°C or more whereas the snow temperature at the ground is about 0°C . Since the water vapour in the snow is approximately saturated, the vapour pressure is largest at the bottom and decreases upwards. Hence a diffusive flux of vapour must take place.

The temperature gradient and the attached mass flux is very important for the metamorphose of the snow. It is a well established empirical fact that a strong vertical temperature gradient favours the growth of the ice particles in the snow. The diameter of these may become as large as 1-2 cm, which is one order of magnitude larger than the characteristic diameter of the ice particles in the non-gradient case. Whilst the particles increase in size, the number of them decrease so strongly that the density of the snow diminishes compared to the density without a temperature gradient. Moreover, a change in the shape of the ice crystals takes place. In the more developed state, hollow cuplike crystals may be formed. The snow is then called "depth hoar". The properties of the snow is markedly changed. Thus the strength against shear stresses is lowered which in sloping terrain may give rise to slab avalanches. It is therefore considerable practical interest attached to the study of mass and heat transport in snow layers with temperature gradient.

Several writers have pointed out that the observed mass flux in gradient snow layers can be considerable larger than the computed transport due to molecular diffusion (see for example the excellent review article by M.R. de Quervain [1972] and the article by Trabandt and Benson [1972]). It has therefore been suggested that convection may, at least in certain cases, be an important part of the transport process. The convective motion may be set up by pressure fluctuations in the air or by unstable density stratification due to a temperature gradient (thermal convection). The intention of the present paper is to examine more closely the possibility for thermal convection to occur in real situations. We shall also study the intensity of the convective motion and its influence on the transport processes.

We assume that the temperature is so low that the water phase does not enter into the problem. The snow is considered as a porous medium with the ice particles constituting the fixed matrix, and the hollow spaces occupied by air and water vapour. The detailed study of the metamorphose of the snow is outside the scope of the paper.

MASS AND HEAT FLUX BY DIFFUSION

The mean mass flux m ($\text{g cm}^{-2}\text{s}^{-1}$) of water vapour in the vertical direction is given by

$$m = -nD \frac{\partial \bar{\rho}_v}{\partial z} \quad (1)$$

with n denoting a structural parameter characterizing the available pores for diffusion. n will here be chosen equal to the porosity. D denotes the diffusion coefficient of water vapour in air ($0.22 \text{ cm}^2\text{s}^{-1}$), ρ_v the vapour density and $\bar{\rho}_v$ the average vapour

density obtained by integrating in the horizontal plane, z is the vertical coordinate chosen positive upwards. Applying the equation of state for water vapour and neglecting a small term proportional to $\partial \bar{T} / \partial z$ (T the temperature and $\bar{T}$ the average temperature)

(1) may be written

$$m = - \frac{nD}{R_v \bar{T}} \frac{\partial \bar{p}_v}{\partial z} \quad (2)$$

where R_v is the specific gas constant for water vapour ($4.62 \cdot 10^3$ mbar g^{-1} $^{\circ}K^{-1}$), p_v the vapour pressure (mbar), and $\bar{p}_v$ the average vapour pressure.

It is found that the vapour pressure is approximately at saturation pressure with respect to the mean temperature (see the discussion later). The saturation pressure p_s is a function of T only and is approximately given by

$$p_s = A \exp[B(T - T_0)] \quad (3)$$

with $T_0 = 273$ $^{\circ}K$, $A = 6.42$ mbar and $B = 0.0857$ $^{\circ}K^{-1}$. Introducing (3) in (2) we obtain

$$m = - \frac{nD}{R_v \bar{T}} A B \exp[B(\bar{T} - T_0)] \cdot \partial \bar{T} / \partial z \quad (4)$$

In what follows we may to a good approximation consider $\bar{T}$ in the denominator in (4) as a constant. The accumulation of water vapour per unit time and unit volume, Q , is then given by

$$Q = - \frac{\partial m}{\partial z} = \frac{nD}{R_v \bar{T}} A B \left[\frac{\partial^2 \bar{T}}{\partial z^2} + B \left(\frac{\partial \bar{T}}{\partial z} \right)^2 \right] \exp[B(\bar{T} - T_0)] \quad (5)$$

This is the formula derived by Yosida [1955], Gidding and La Chapelle [1962] and de Quervain [1963]. It is noted that for constant temperature gradient, Q is positive (i.e. the density of the snow increases with time) everywhere, having its maximum value at the warmer end (the ground).

For later references, m and Q as function of z are displayed in Figures 3 and 4.

The flux predicted by (4) is, at least in some cases, too small to explain the flux obtained from observations. Furthermore, the solution (5) suffers from the inconsistency that for $\partial^2 T / \partial z^2$ zero (or positive), the only source for water vapour is the underlying ground. This defect is obviously due to the assumption that the vapour is saturated in the entire snow layer. It may therefore be of interest before proceeding further to examine this assumption somewhat closer.

In the pores very close to the ice particles the water vapour is saturated such that here $\rho_v = \rho_s$ (ρ_s the saturation density). Furthermore, ρ_v satisfies the Laplacian equation

$$\nabla^2 \rho_v = 0 \quad (6)$$

This equation together with the boundary condition $\rho_v = \rho_s$ at the ice particles and the proper conditions at the **upper** and lower boundaries determines ρ_v uniquely when the ice matrix is given. To discuss equation (6) we write

$$\rho_v = \bar{\rho}_v + \rho'_v \quad (7)$$

Introducing (7) in (6) we obtain

$$\frac{d^2 \bar{\rho}_v}{dz^2} + \nabla^2 \rho'_v = 0 \quad (8)$$

Since ρ_v is equal to ρ_s close to the ice particles, ρ'_v is of the order $\rho_s - \bar{\rho}_v$ and ∇^2 is of the order r^{-2} where r is the characteristic diameter of the ice particles (and therefore also the characteristic length scale for the distance between the ice particles). Equation (8) may therefore be written

$$\frac{d^2 \bar{\rho}_v}{dz^2} + \lambda \frac{\rho_s - \bar{\rho}_v}{r^2} = 0 \quad (9)$$

where λ is a dimensionless constant of order unity. Since the length scale for the variation of $\bar{\rho}_v$ is much larger than r , (9) leads to $\bar{\rho}_v \approx \rho_s$ which shows that the assumption made in the derivation of (4) and (5) is good. The assumption fails, however, close to the upper (snow-air) and lower (snow-ground) boundaries where we usually will have boundary layers (the two terms in equation (9) are then of the same order of magnitude). Such a boundary layer will occur, for example, if we have no flux of water vapour from the ground whereas $\partial T / \partial z \neq 0$. In this case the lower boundary layer, which has a thickness of order r , is the source for the water vapour. When this source is dried out, the layer above will constitute the source and so on.

It may perhaps be worth mentioning that a strong temperature gradient may give a noticeable contribution to the diffusion effect (the Sorét effect). We have examined the order of this effect in the actual case, but find that the Sorét effect is responsible for less than 1% of the vapour diffusion.

TRANSPORT OF HEAT BY CONDUCTION

Disregarding the pressure variation, the energy equation may be written

$$\frac{\partial(\rho h)}{\partial t} = - \nabla \cdot (\vec{J}_h + \vec{J}_q) \quad (10)$$

where h is the enthalpy per unit mass, $\vec{J}_h$ the flux of enthalpy, $\vec{J}_q$ the flux of heat and ρ the density of air and water vapour. Assuming for the moment no convection, the quantities entering (10) are functions of t and z only and the terms on the right hand side are due to molecular activity. Equation (10) may then be

written

$$\frac{\partial(\bar{\rho}h)}{\partial t} = - \frac{\partial}{\partial z} \left[nD \frac{\partial \bar{\rho}_v}{\partial z} h_v + k \frac{\partial \bar{T}}{\partial z} \right] \quad (11)$$

where h_v is the enthalpy for water vapour and k the thermal conductivity for the porous media. The term on the left hand side will be neglected since we are considering steady processes (strictly speaking, the term is not exactly zero due to the sublimation into ice. The term is, however, small since the enthalpy of ice is small compared to the enthalpy of water vapour). h_v may be set equal to L , the heat of sublimation, approximately. Integrating (11) with respect to z and applying the equation of state, we end up with

$$k_e \frac{\partial \bar{T}}{\partial z} = \text{constant} \quad (12)$$

where

$$k_e = k + \frac{nLD}{R_v \bar{T}} \frac{\partial p_s}{\partial T} \quad (13)$$

is called the effective thermal conductivity. Equation (12) determines the steady temperature profile. If k_e is assumed to be constant, the temperature is a linear function of z . If the variation of k_e with height is taken into account, the temperature profile is slightly curved. However, the effect of this curvature is small and will be cancelled.

THE CONDITIONS FOR THERMAL CONVECTION TO OCCUR

It is not clear ab initio whether thermal convection (Rayleigh convection) may occur or not in a snow layer. Strong vertical temperature gradients and great depth of snow layer obviously favour thermal convection. On the other side, the viscosity and thermal conductivity try to hinder the start of convection. To find an estimate for the necessary conditions for Rayleigh convection to

occur, we have to derive the critical Rayleigh number.

To determine the motion of the mixture of air and water vapour in the snow layer, we apply Darcy's law (for a derivation and discussion of this law, see Palm and Weber, 1971). We then have

$$-\frac{\mu}{k} \vec{v} - \rho \vec{g} - \nabla p = 0 \quad (14)$$

where μ is the viscosity, k the permeability, $\vec{v}$ the velocity, ρ the density, $\vec{g}$ the acceleration of gravity and p the pressure. It is customary, however, to define the permeability in snow by K where

$$K = \frac{k g \rho_w}{\mu} \quad (15)$$

Here ρ_w is the density of water. Applying the Boussinesq approximation, we write the equation of state in the simple form

$$\rho = \rho_1 (1 - \alpha(T - T_1)) \quad (16)$$

where ρ_1 and T_1 are reference values for density and temperature, respectively, and α is the coefficient of expansion for the mixture (which is put equal to α for air). The energy equation is given by (10) where now

$$\vec{j}_h = n(\rho_v h_v + \rho_a h_a) \vec{v}, \quad \vec{j}_q = k_e \nabla T \quad (17)$$

The diffusive transport of water vapour is incorporated into the $\vec{j}_q$ -term (k_e instead of k). ρ_a and h_a are density and enthalpy of the air, respectively. Introducing $h_v = L$ and $h_a = c_p T$ where c_p is the specific heat at constant pressure for air, equation (10) takes the form

$$n \vec{v} \cdot \nabla (\rho_v L + \rho_a c_p T) - k_e \nabla^2 T = 0 \quad (18)$$

Applying the equation of state for ρ_v and ρ_a and replacing the partial pressure of air by the total pressure, we obtain

$$n \rho_a \left(c_p + \frac{R_a}{R_v} \frac{L}{p} \frac{dp_s}{dT} \right) \vec{v} \cdot \nabla T = k_e \nabla^2 T \quad (19)$$

where R_a is the gas constant for air. This is essentially the equation derived by Yen [1962] in his work on the effect of ventilation. Equation (19) may be written

$$\vec{v} \cdot \nabla T = \kappa \nabla^2 T \quad (20)$$

where

$$\kappa = k_e / \rho_a \left(c_p + L \frac{R_a}{R_v} \frac{1}{p} \frac{dp_s}{dT} \right) n \quad (21)$$

For the present purpose we assume κ to be constant. Equation (20) is then formally the heat equation in its usual form.

The relevant equations may be thrown into a non-dimensional form by introducing $h, \Delta T$ (the difference in temperature between the upper and lower boundary), $\mu\kappa/k$, κ/h as units for length, temperature, pressure and velocity, respectively. We then end up with

$$-\nabla p + RT \vec{k} - \vec{v} = 0 \quad (22)$$

$$\vec{v} \cdot \nabla T = \nabla^2 T \quad (23)$$

where R is the Rayleigh number, defined by

$$R = \frac{\alpha \Delta T h K}{\kappa} \frac{\rho_a}{\rho_w} \quad (24)$$

and $\vec{k}$ is the vertical unit vector. In addition to (22) and (23) we also need the equation of continuity which in the Boussinesq approximation takes the form

$$\nabla \cdot \vec{v} = 0 \quad (25)$$

Introducing in (21)

$$\begin{aligned} T &= -15^{\circ}\text{C}, \quad p = 1000 \text{ mbar}, \quad L = 685 \text{ cal g}^{-1}, \\ \rho_a &= 1.36 \cdot 10^{-3} \text{ g cm}^3, \quad c_p = 0.24 \text{ cal g}^{-1}^{\circ}\text{C}^{-1}, \\ D &= 0.22 \text{ cm}^2\text{s}^{-1}, \quad \alpha = 5.39 \cdot 10^{-3}^{\circ}\text{C}^{-1}, \quad n = 0.75 \\ k_e &= 3.65 \cdot 10^{-4} \text{ cal cm}^{-1}\text{s}^{-1}^{\circ}\text{C} \quad \text{and} \quad R_a/R_v = 18/28.9 \end{aligned}$$

we find $\kappa = 1.10 \text{ cm}^2\text{s}^{-1}$. Moreover, introducing $\rho_a/\rho_w = 1.36 \cdot 10^{-3}$, (24) may be written

$$R = 6.7 \cdot 10^{-6} (\text{s}/(\text{cm}^2^{\circ}\text{C})) \cdot \Delta T \text{ h K} \quad (26)$$

The critical value of R necessary for the onset of convection, R_c , is dependent on the boundary conditions at the upper and lower boundaries. If we, for example, assume that the boundaries are perfect heat conductors and the vertical velocity is zero at the boundaries, it is easily shown from (22)-(25) that $R_c = 4\pi^2$ [Lapwood, 1948]. In the following table is shown the critical Rayleigh number for some other boundary conditions, obtained by a straightforward numerical procedure. The non-dimensional temperature T is written as

$$T = -z + \theta \quad (27)$$

whereby θ is the temperature due to the motion. The boundary conditions applied is

$$\text{Upper boundary:} \quad w + K_1 \partial w / \partial z = 0$$

$$\theta + S_1 \partial \theta / \partial z = 0$$

$$\text{Lower boundary:} \quad w + K_0 \partial w / \partial z = 0$$

$$\theta + S_0 \partial \theta / \partial z = 0$$

S_0 (or S_1) equal to zero corresponds to infinite conductivity whereas S_0 (or S_1) equal to infinity corresponds to insulating boundaries. a_c is the critical wave number (2π divided by the critical wave length).

K_0	S_0	K_1	S_1	R_c	a_c
0	0	0	0	$4\pi^2$	π
0	0	∞	0	27	2.3
0	0	0	∞	27	2.3
0	0	0.5	0.5	21	2.3
0	0	1.0	1.0	16	2.0
0	0	∞	∞	π^2	1.6
0	∞	∞	0	12	0.0
0	∞	∞	1.0	6	0.0
0	∞	∞	2.0	4	0.0
0	∞	∞	∞	3	0.0

Table 1.

It is not obvious which boundary conditions should be applied in the actual case. At the lower boundary we may assume that $w = 0$ ($K_0=0$). At this boundary we will often have an ice layer which has a thermal conductivity considerable larger than the conductivity of the snow. This suggest that we may approximate the boundary condition by $\theta = 0$ ($S_0 = 0$). At the upper boundary the conditions are more complicated. The boundary is partly "open", i.e. $w \neq 0$. If we assume that the pressure is constant ("open"boundary), we see from (22) and (25) that $w_z = 0$ ($K_1=\infty$). It seems reasonable to assume that K_1 is rather large. The thermal condition may be approximated by $\theta + S_1 \partial\theta/\partial z = 0$. The value of S_1 is, however, unknown. The air above the snow layer may act as a heat reservoir which would lead to a small value of S_1 . We see from the table that with the boundary conditions chosen at the lower layer, the critical Rayleigh number R_c will vary from $\pi^2(K_1=\infty, S_1=\infty)$ to about $27(K_1=\infty, S_1=0)$. Returning to formula (26) this implies that

K must be large for thermal convection to occur. If we, for example, put $h = 100$ cm and $\Delta T = 30^{\circ}\text{C}$, we find that for $R_c = \pi^2$ K must be larger than 500 and for $R_c = 27$ K must be larger than 1350.

We therefore conclude that with the values adopted for ΔT and h , K must be of the order 10^3 for obtaining thermal convection. This estimate is based on the assumption $w = 0$ and $\theta = 0$ at the lower boundary. The last condition may not always be the correct one. It is seen from the table that if S_0 is not chosen equal to zero, the critical Rayleigh number is lowered. Thereby also the critical value of K is lowered.

THE HEAT AND MASS TRANSFER IN THERMAL CONVECTION

When thermal convection occurs, the transport of heat and mass is strongly intensified. It is also expected that the space variations of the mass flux and accumulation are changed. To examine more closely the effect of thermal convection, we have to solve the system of non-linear equations (22)-(25). Since the actual Rayleigh numbers are only slightly supercritical, we may apply an amplitude expansion. In fact these equations have been discussed earlier [Palm, Weber and Kvernfold, 1971] for the actual regime of Rayleigh numbers. The boundary conditions applied in this paper are $\theta = w = 0$ at both the lower and upper boundary. The computed temperature distribution in convection is displayed in Figure 1. In Figures 2, 3 and 4 are shown the computed heat flux, mass flux and accumulation for various Rayleigh numbers. It is noted that the spatial distribution of accumulation has quite another form than in the case when only diffusive processes are present, even for small supercritical Rayleigh numbers. Close to the ground we find a thin layer of

positive accumulation. It should be noted, however, that the assumption of the water vapour being saturated is not fulfilled here so that the solution is not correct in this layer. Above this thin layer we find a rather thick and marked layer (depending on the Rayleigh number) of negative accumulation which is to be considered as the main source for water vapour. The upper half layer has a positive accumulation, the strength of which depends on the Rayleigh number.

The mass flux and the accumulation in the upper half layer obviously depend rather sensibly on the upper boundary conditions. It would therefore be of interest to study the effect of thermal convection for other upper boundary conditions. An approximation to the solution when $\theta_z = w_z = 0$ at the upper boundary is obtained from the solution by Palm et al. [1971], due to the symmetry of their problem, by considering the motion at the lower half part of the layer only. The corresponding mass flux and accumulation are shown in Figures 5 and 6. We see that in this case the upper layer of positive accumulation is drastically reduced. The displayed mass flux and accumulation in the Figures 3, 4 and 5, 6 are extremes. Therefore, in a real snow layer the positive accumulation in the upper layer most likely will be somewhat smaller than shown in Figure 4 and somewhat larger than shown in Figure 6.

Our results may be compared with the observations by Trabant and Benson [1972] from the winter season in Alaska over a period of several months with very low temperatures. They study the development of the density stratification in snow with and without thermal gradient. They find a marked relative decrease in density (combined with the formation of depth hoar) in the lower part of the snow layer and a relative increase in density in the upper part of the layer, in good agreement with the present theory. Trabant and Benson also estimate the mass flux and find it an order of

magnitude larger than obtained from the diffusion theory, indicating that thermal convection is present.

CONCLUSION

Our main conclusion is that thermal convection may occur in a snow layer, provided a strong temperature gradient is present and the snow has a high permeability (old snow). The occurrence of thermal convection will effectively increase the flux of water vapour. It will also lead to a pronounced decrease in snow density in the lower part of the snow layer and an increase in snow density in the upper part of the snow layer. Thermal convection may be important for the rate of formation of depth hoar. It is seen from Figure 5 that the occurrence of thermal convection may lead to a marked increase in loss of water vapour from the snow. Also the insulating power of the snow is changed.

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FIGURE LEGENDS

Figure 1.

Mean temperature profiles in the snow layer. The straight profile is for the case of no convective motion ($R \leq R_c$), and the curved profile when convection takes place.

Figure 2.

Mean heat flux through the layer (given by the Nusselt number) as a function of the Rayleigh number. The break in the curve for $R = R_c$ is due to the convection which starts at this Rayleigh number.

Figure 3.

Mean mass flux of water vapour (made dimensionless by $AD/(R_v T_o h)$) as a function of z for different values of the Rayleigh number. The boundary conditions applied are given by $K_o = S_o = K_1 = S_1 = 0$ (see Table 1).

Figure 4.

Mean accumulation of water vapour (made dimensionless by $AD/(R_v T_o h^2)$) as a function of z for different values of the Rayleigh number. The boundary conditions applied are given by $K_o = S_o = K_1 = S_1 = 0$ (see Table 1).

Figure 5.

Same as Figure 3 except $K_1 \neq 0$ and $S_1 \neq 0$.

Figure 6.

Same as Figure 4 except $K_1 \neq 0$ and $S_1 \neq 0$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6