SYMMETRIC INSTABILITY OF STRATIFIED GEOSTROPHIC FLOW

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Abstract

The linear stability of a stably stratified geostrophic current between parallel horizontal planes has been investigated theoretically. The considered disturbances are two-dimensional with axes aligned along the flow direction. Diffusive processes enter the stability problem through the Ekman number E and the Prandtl number Pr . The analysis is valid for general vertical stable stratification, characterized by $S = N^2/f^2$ where N is the Brunt-Väisälä frequency and f the (constant) Coriolis parameter. When E is small, the horizontal scale of the marginally stable disturbance is $\propto H(1 + PrS)^{\frac{1}{2}} E^{\frac{1}{3}}$ where H is the distance between the bounding planes. Increasing values of E stabilizes the system. For given viscosity, and $S = 0$, the system is destabilized by increasing Pr , while for $S \gg 1$ destabilization occur if $Pr \neq 1$.

INTRODUCTION

Vortex rolls aligned along the mean flow are commonly observed in nature. When they occur in the ocean, they are often referred to as Langmuir circulations; see Faller (1971) for a comprehensive review of the subject. In the atmosphere the phenomenon usually manifests itself as cloud streets, see for example Kuettner (1971).

These circulations occur in a variety of situations and are probably not explained by one single physical mechanism. When the stratification is unstable, buoyancy-driven convection in the form of longitudinal rolls may account for the phenomenon (Kuo 1963), while in neutral or stable conditions instability of the Ekman boundary layer flow is a possible candidate for the explanation of such rolls (Faller 1963, 1965, Brown 1970).

Another type of roll-like structure may arise from the inertial instability of a rectilinear geostrophic current with respect to two-dimensional disturbances with axes aligned along the basic flow. This phenomenon is well-known in meteorology, and is only a special case of the more general theory of symmetric instability of the baroclinic vortex, see the review by Eliassen and Kleinschmidt (1957). When diffusion of momentum and heat are taken into account double-diffusive destabilization of the vortex may occur when the Prandtl number is different from one (McIntyre 1970), since then angular momentum and heat diffuse at different rates. With particular reference to the ocean, Calman (1977) has performed laboratory experiments in a rotating salt stratified system, where angular momentum diffuses much faster than salt. He observed the formation of layers which may

explain the occurrence of micro-structure (of order 1 m or less) in certain places of the ocean.

In contrast to McIntyre (1970), who works in an infinite domain, we assume that the basic flow is confined between horizontal parallel planes in order to model a shallow atmosphere or ocean. This is similar to Kuo (1954), Lilly (1966) and Gammelsrød (1975), for a vertically homogeneous fluid, and to Walton (1975), who considers a fluid with strong vertical stability in the limit of small Ekman numbers. The present paper investigates the linear stability problem for arbitrary vertical (stable) stratification. The Ekman number does not necessarily have to be small. However, in a physical model it has to be so small that the Ekman boundary layers only fill a small portion of the gap between the bounding planes.

MODEL AND BASIC FLOW

We consider the flow of a rotating viscous fluid bounded by two parallel horizontal planes at a distance H . A Cartesian coordinate system (x,y,z) with unit vectors $(\vec{i}, \vec{j}, \vec{k})$ is defined such that the z -axis is vertical and positive upwards, and the upper plane is situated at $z = H$. In the present paper emphasis will be put on basic principles, and accordingly we simplify the problem as much as possible without losing the fundamental physics involved. In the light of this, the Boussinesq approximation will be assumed. The governing equations may then be written

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + f \vec{k} \times \vec{v} = - \nabla \pi + \nu \nabla^2 \vec{v} - g \sigma \vec{k} \quad (1)$$

$$\frac{\partial \sigma}{\partial t} + \vec{v} \cdot \nabla \sigma = \kappa \nabla^2 \sigma \quad (2)$$

$$\nabla \cdot \vec{v} = 0 \quad (3)$$

where $\vec{v}(=u,v,w)$ is the velocity vector, f the constant Coriolis parameter, π the pressure per reference density ρ_0 , σ a dimensionless density ($=\rho/\rho_0$), g the acceleration of gravity, ν the kinematic viscosity and κ the thermal diffusivity.

The above formulation is probably most applicable to the ocean, although an alternative formulation for air (perfect gas) qualitatively leads to the same stability problem.

We assume that the basic density distribution Σ is linear in the y - and z -direction, and does not vary with x , i.d.

$$\Sigma = -\beta y - \gamma z \quad (4)$$

Since stable vertical stratification is most commonly found in the ocean and the atmosphere, we take $\gamma > 0$. We also choose $\beta > 0$, i.e. the y -axis pointing towards lighter fluid.

Assuming a stationary velocity distribution of the form

$$\vec{v} = U(z)\vec{i} \quad (5)$$

we obtain from (1) that

$$U = -g\beta z/f \quad (6)$$

which is often referred to as the thermal wind. In the derivation of the solution we have neglected the frictional influence of the horizontal boundaries, which means that (5) is valid outside the Ekman

boundary layers $\sim(2\nu/f)^{\frac{1}{2}}$. Alternatively, the upper boundary may move with the thermal wind velocity such that an Ekman layer only occurs at the bottom boundary.

PERTURBATION ANALYSIS

We investigate the stability of the basic state by introducing small perturbations into the solution. In particular we consider disturbances which are independent of the x-coordinate. These are often referred to as symmetric disturbances in meteorology, since the basic motion may be considered as a baroclinic vortex with infinite radius. When $\partial/\partial x = 0$, we may define a streamfunction ψ such that the perturbation velocity can be written

$$\vec{v} = (u, \psi_z, -\psi_y) \quad (7)$$

where subscripts denote partial differentiation. It proves convenient to introduce non-dimensional variables into the problem. This is achieved by taking

$$H, \quad 1/f, \quad Hf, \quad H^2f^2, \quad Hf^2/g \quad (8)$$

as units of length, time, velocity, pressure and density, respectively. Eliminating the pressure, the linearized perturbation equations may be written

$$u_t = -B\psi_y + \psi_z + E\nabla_1^2 u \quad (9)$$

$$\nabla_1^2 \psi_t = -u_z + \sigma_y + E\nabla_1^4 \psi \quad (10)$$

$$\sigma_t = B\psi_z - S\psi_y + Pr^{-1}E\nabla_1^2 \sigma \quad (11)$$

where $\nabla_1^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the two-dimensional Laplacian.

The dimensionless parameters governing the stability of the system are

$$\begin{aligned} B &= g\beta/f^2 \quad \text{the dimensionless thermal wind shear } (=|U_z|/f) \text{ which} \\ &\quad \text{is a measure of the baroclinicity} \\ S &= g\gamma/f^2 \quad \text{the stratification parameter } (=N^2/f^2) \text{ where } N \text{ is} \\ &\quad \text{the BruntVäisälä frequency} \end{aligned} \tag{12}$$

$$E = \nu/fH^2 \quad \text{the Ekman number}$$

$$Pr = \nu/\kappa \quad \text{the Prandtl number}$$

It often proves convenient to combine B and S to yield a Richardson number Ri . Since $B = |U_z|/f$ and $S = N^2/f^2$ we obtain

$$Ri = S/B^2 \tag{13}$$

We take the horizontal boundaries to be impermeable. Hence the normal velocity must vanish there, i.e.

$$\psi = 0, \quad z = 0, 1. \tag{14}$$

Concerning the stresses and the density variations some extreme cases usually are considered. At a rigid boundary a no-slip condition must be applied, i.e. $u = \psi_z = 0$, while for a stress-free boundary we have $u_z = \psi_{zz} = 0$. If the boundary is a perfect conductor of heat, then $\sigma = 0$ there, while at an insulating boundary $\sigma_z = 0$.

By assuming disturbances of the form $\exp(i\mathbf{l}y + \omega t)$ where l is a real wavenumber in the y -direction and ω the complex growth rate,

the equations are

$$\begin{aligned}
 \omega u &= -ilB\psi + D\psi + E(D^2 - l^2)u \\
 \omega(D^2 - l^2)\psi &= -Du + il\sigma + E(D^2 - l^2)^2\psi \\
 \omega\sigma &= BD\psi - ilS\psi + Pr^{-1}E(D^2 - l^2)\sigma
 \end{aligned} \tag{15}$$

where $D = d/dz$.

By specifying appropriate boundary conditions, the system of equations (15) can be solved numerically by Galerkin's method to yield the eigenvalues ω for any given set B, E, S, Pr and l . The procedure is straight forward, but rather lengthy, and requires large amounts of computer time, see for example Weber (1978) for a related problem.

To obtain some information about the qualitative behaviour of the problem, we shall introduce a simplification, namely that of replacing the vertical viscous and diffusive terms in the momentum and heat equations by a wave mode assumption, i.e.

$$\frac{\partial^2}{\partial z^2}(\vec{v}, \sigma) = -m^2(\vec{v}, \sigma) \tag{16}$$

This assumption reduces the order of the equations and we get no generation of vorticity at the boundaries. Accordingly this assumption makes no sense in a problem where this production is essential as for example for the stability of plane Poiseuille flow, while it is an acceptable qualitative approximation in a problem with stress-free and perfectly conducting boundaries where instability occurs as a result of conversion of potential energy, or as here, when energy is extracted from the shear of the basic flow.

With this simplification equations (15) reduce to

$$\Omega_2(\Omega_1^2+1)D^2\psi - i1B(\Omega_1+\Omega_2)D\psi - \Omega_1 l^2(\Omega_1\Omega_2+S)\psi = 0 \quad (17)$$

where

$$\Omega_1 = \omega + E(l^2+m^2)$$

$$\Omega_2 = \omega + \text{Pr}^{-1}E(l^2+m^2)$$

The boundary condition (14) at the lower plane is satisfied by a solution of the form $\psi \propto \exp(ir_1 z) - \exp(ir_2 z)$, where

$$\left. \begin{matrix} r_1 \\ r_2 \end{matrix} \right\} = \frac{1}{2a} \left[b \pm (b^2 - 4ac)^{\frac{1}{2}} \right] \quad (18)$$

Here

$$a = \Omega_2(\Omega_1^2+1)$$

$$b = B(\Omega_1+\Omega_2)$$

$$c = \Omega_1(\Omega_1\Omega_2+S)$$

The boundary condition at the upper plane requires that

$$r_1 - r_2 = 2n\pi, \quad n = 1, 2, 3, \dots \quad (19)$$

which may be rearranged to yield an expression for the growth rate. However, McIntyre (1970) found for an infinite domain that, although oscillatory instability did occur for $\text{Pr} \neq 1$ the most unstable disturbance was monotonic. This was also shown by Walton (1975), for large S and small E , to apply for a model of finite vertical extent. This suggests that we should concentrate on monotonic disturbances which seem to be those most likely encountered in a physical problem. Then the transition to instability occurs via $\omega = 0$. With $n = 1$ which can be shown to be the most "dangerous" mode, (19) reduces to

$$B = \frac{2}{1+Pr} \left[PrS + \frac{\pi^2}{l^2} + (PrS + 1 + \frac{2\pi^2}{l^2})E^2k^2 + (1 + \frac{\pi^2}{l^2})E^4k^4 \right]^{\frac{1}{2}} \quad (20)$$

where $k \equiv l^2 + m^2$ is a sort of squared overall wavenumber.

The evaluation of the coefficient m defined by (16) is somewhat arbitrary, but essentially m must correspond to an effective vertical wavenumber, i.e. it must be of the same order of magnitude as r_1, r_2 from (18). As Lilly (1966) we shall assume a root-mean-square value i.e.

$$m^2 = \frac{1}{2}(r_1^2 + r_2^2) \quad (21)$$

For the familiar Rayleigh-Bénard convection problem with stress-free and perfectly conducting boundaries this choice may be shown to yield the exact critical Rayleigh number. This may not be so in a more complicated problem, but we still expect (21) to be qualitatively correct for stress-free and perfectly conducting boundaries. *)

By insertion from (18) into (21) we obtain

$$E^2k^3 - 2E^2(l^2 + \pi^2)k^2 + k - 2\pi^2 - l^2(1 + PrS) = 0 \quad (22)$$

To find k from this equation is an elementary task and we leave out the details. Its value is then substituted back into (20). For fixed values of Pr, S and E , we compute the minimum value of B when l varies. This minimum is denoted by B_c , and the corresponding wavelength

*) During the final preparation of this paper the author became aware of the PhD-thesis by Emanuel (1978) who has done some numerical computations by a variational method. Comparisons with his results show that the agreement is very good, also quantitatively.

by l_c . Since k from (22) is a somewhat complicated function of l , this is most easily done by evaluating the left-hand side of (20) by a computer for sufficiently small steps in l and then determine the minimum by an interpolation procedure. However, for sufficiently small values of E , explicit expressions may be obtained for B_c and l_c as demonstrated in the following section.

RESULTS AND DISCUSSION

The non-diffusive case

When $E = 0$ and $Pr^{-1}E = 0$ in (17) we are back to the well-known non-diffusive case. The growth rate is then easily found to be

$$\omega = \pm \left[\frac{-(S+1+2n^2\pi^2/l^2) \pm [(S-1)^2 + 4(1+n^2\pi^2/l^2)B^2]^{\frac{1}{2}}}{2(1+n^2\pi^2/l^2)} \right]^{\frac{1}{2}} \quad (23)$$

It is seen that for the upper sign of the inequality

$$B \lesssim (S+n^2\pi^2/l^2)^{\frac{1}{2}} \quad (24)$$

we have two pairs of purely imaginary roots, corresponding to two set of progressive inertia/gravity waves. When the lower sign applies, the one pair of roots become real, and hence exponential growth first occur when

$$B > S^{\frac{1}{2}} \quad \text{or} \quad Ri < 1 \quad \text{for} \quad l = \infty \quad (25)$$

which is Stone's (1966) result. But as seen from (23), when $B = (S+n^2\pi^2/l^2)^{\frac{1}{2}}$, we have a double root $\omega^2 = 0$ which leads to a

linear growth in time. The meridional stream function may then be written (taken $n = 1$) as

$$\psi = (A_0 + A_1 t) \sin \pi z \cos l(y + Bz) \equiv (A_0 + A_1 t) \varphi(y, z) \quad (26)$$

where A_0 is an initial value, and we have let the real part represent the physical solution. Hence, for given wavelength and stratification, the phase speed of the slowest waves decreases towards zero, when B increases, before they can extract energy from the basic flow. Accordingly, the situation characterized by $Ri = 1$ must also be considered as unstable (although weaker than exponentially) when diffusive effects are not present. This point seems to have been overlooked in previous investigations.

From (9) and (11) we then obtain for u and σ on the critical curve

$$u = u_0 + (A_0 t + \frac{1}{2} A_1 t^2) (-B \varphi_y + \varphi_z) \quad (27)$$

$$\sigma = \sigma_0 + (A_0 t + \frac{1}{2} A_1 t^2) (B \varphi_z - S \varphi_y) \quad (28)$$

where u_0, σ_0 are initial values and φ is defined by (26).

Inclusion of diffusive processes

When E is small, but non-zero, several interesting results can be obtained explicitly from (20). Since an inviscid model turns unstable for zero wavelength, it is reasonable to assume that the effect of viscosity sets the length scale in the diffusive case, i.e. we assume

$$l = l_0 E^{-\alpha} \quad (29)$$

as a leading term, where $\alpha > 0$ and $E \ll 1$ (see also Walton 1975). Two different cases will be considered. First we assume $S = 0$, that is a vertically homogeneous fluid. Then, in the inviscid case instability will commence for any $B > 0$. This suggests that for $E \neq 0$, but small, we should take

$$B = B_0 E^\beta \quad (30)$$

as a leading term, where $\beta > 0$. By putting $S = 0$ in (21) and (22) and expressing all the variables in powers of E , balance to lowest order requires

$$\alpha = \beta = 1/3 \quad (31)$$

and

$$B_0^2 = \frac{4}{(1+Pr)^2} \left(\frac{\pi^2}{l_0^2} + l_0^4 \right) \quad (32)$$

Minimizing B with respect to l_0 , we obtain the critical value

$$B_c = \frac{2 \cdot 3^{\frac{1}{2}}}{1+Pr} \left(\frac{\pi^2}{2} \right)^{\frac{1}{3}} E^{\frac{1}{3}} + O(E) \quad (33)$$

for

$$l_c = \left(\frac{\pi^2}{2} \right)^{\frac{1}{6}} E^{-\frac{1}{3}} + O\left(E^{\frac{1}{3}}\right) \quad (34)$$

The assessment of the order of the next terms in the expansions comes from the fact that B and l should change sign when f , or E , changes sign. In particular we note that increasing values of Pr destabilizes the flow for given E . This is obvious, since when κ is small (Pr large) the density of an individual particle is nearly

conserved, and hence meridional vorticity is generated nearly as in the non-diffusive case where instability commences for any $B > 0$. On the other hand, when κ is large (Pr small) the perturbation density σ nearly vanishes and the production of meridional vorticity (proportional to $-u_z + \sigma_y$ in non-dimensional notation) is reduced and reaches a minimum when $Pr = 0$, that is when σ vanishes identically. We further note from (34) that in this case the horizontal scale of the most unstable disturbance is determined by viscosity alone.

When the vertical stratification is moderate or large, i.e. $S \gtrsim 0(1)$, we still expect (29) to represent the proper length scale, but l_0 will now depend on S . Introducing the Richardson number, $Ri = S/B^2$ and assuming that $R \sim 0(1)$, we expand Ri in powers of E :

$$Ri = R_0 + R_1 E^\beta + O(E^{2\beta}) \quad (35)$$

By inserting (29) and (35) into (20), we now find that balance requires

$$\begin{aligned} \alpha &= 1/3 \\ \beta &= 2/3 \end{aligned} \quad (36)$$

Equating equal powers in E yields

$$\begin{aligned} R_0 &= \frac{(1+Pr)^2}{4Pr} \\ R_1 &= -\frac{(1+Pr)^2}{4Pr^2 S} \left(\frac{\pi^2}{l_0^2} + (PrS+1)^3 l_0^4 \right) \end{aligned} \quad (37)$$

By maximizing Ri with respect to l_0 , we finally obtain

$$Ri^c = \frac{(1+Pr)^2}{4Pr} \left[1 - 3(1+Pr^{-1}S^{-1}) \left(\frac{\pi^2}{2} \right)^{\frac{2}{3}} E^{\frac{2}{3}} + O(E^{\frac{4}{3}}) \right] \quad (38)$$

$$l_c = (PrS+1)^{-\frac{1}{2}} \left(\frac{\pi^2}{2} \right)^{\frac{1}{6}} E^{-\frac{1}{3}} + O(E^{\frac{1}{3}}) \quad (39)$$

By letting $S \gg 1$ and rescaling the horizontal length such that $l' = Bl$, (38) and (39) are seen to be identical to the results obtained by Walton (1975). Again the stabilizing effect of viscosity is noted, while for ν fixed, the system is destabilized if $Pr \neq 1$, analogous to the findings of McIntyre (1970). As expected, we see from (39) that strong vertical stratification (S large) gives a tendency towards cells with large horizontal extent. This tendency, however, is opposed when the heat diffusion coefficient is large (Pr small), since then a particle more easily adjusts its temperature to that of the surroundings. Hence the constraining effect of vertical stratification is less felt.

For larger values of E , the series expansions do not converge. In principle we can find B_c and l_c from (20) and (22) for general values of the parameters by minimizing B with respect to l for fixed values of Pr, S and E . This is rather laborious, however, and it proves much quicker to obtain B_c (or Ri^c) by numerical interpolation.

When $S = 0$, it is easily seen from (20) and (22) that only two independent parameters occur in the problem. One is the Ekman number E and the other may be defined as

$$F = (1+Pr)B/2 \quad (40)$$

In figure 1 we have displayed the critical baroclinicity parameter B_c as a function of E for small and moderate values of S and $Pr = 1$. The curve for $S = 0$ represents in fact, as explained above, the parameter $F_c = (1+Pr)B_c/2$ and is accordingly valid for all Pr . The dashed curve represents formulae (33), which is seen to be a good approximation only for very small E . The open circles are values for $S = 0$ from Emanuel (1978) for stressfree boundaries, and the agreement is seen to be quite good. The analytical results of Kuo (1954) for neutral vertical stratification (not displayed in the figure) is very close to Emanuel's and ours. Unfortunately neither Kuo nor Emanuel report results for E smaller than 0.001. The figure clearly exhibits, for given Pr , the stabilizing effect of vertical stratification and viscous diffusion, while as discussed before for small E and $S = 0$, the effect of increasing the Prandtl number is to destabilize the system.

In figure 2 we have plotted the critical Richardson number as a function of E for $S \gtrsim 0(1)$ and again chosen $Pr = 1$. This value is not an unreasonable order of magnitude estimate for many problems in the earth's atmosphere and oceans involving turbulent heat transfer. For $Pr = 1$ it can also easily be shown that the instability is monotonic. The dashed curves represent the analytical result (38). The formulae is seen to be valid for larger E the larger S is, though it should never be extended beyond $E = 0.01$. Again the stabilizing effect of viscous diffusion is obvious.

When $S \gg 1$ one is justified in doing the hydrostatic approximation. By rescaling the previous horizontal wavenumber l as

$$l = l' \frac{B}{S} \left(\frac{1+Pr}{Pr} \right) \quad (41)$$

it is easily shown that the stationary problem again is governed by only two independent parameters; one is E as before, while the other may be defined as

$$G = \frac{4Pr}{(1+Pr)^2} Ri \quad (42)$$

(see also Emanuel (1978)). This is also obvious from the solutions (38) and (39), which in the case $S \gg 1$ reduce to

$$G_c = 1 - 3 \left(\frac{\pi^2}{2} \right)^{\frac{2}{3}} E^{\frac{2}{3}} \quad (43)$$

$$l'_c = \frac{1}{2} \left(\frac{\pi^2}{2} \right)^{\frac{1}{6}} E^{-\frac{1}{3}}$$

For $Pr = 1$ we have $G = Ri$ from (42), and hence the curve for $S \gg 1$ in figure 2 can be directly compared with Emanuel's results for the hydrostatic case. We have plotted some of his results for free boundaries (open circles) and rigid boundaries (crosses) within the Ekman number region he works in, and the agreement with the present analysis is seen to be quite satisfactorily.

In figure 3 we have plotted the critical wavenumber corresponding to the situation depicted in figure 2. We have chosen to plot l_c times $(S+1)^{\frac{1}{2}}$ as a function of E , and we see that the analytical results (39) (dashed curve) is valid for all S when E is small. In particular it is valid for larger E the larger S is, but not beyond $E \sim 0.01$. For comparison we have plotted some of Emanuel's

results for $S = 0$ and free boundaries (open circles).

In figure 4 we have compared our non-dimensional critical wavelength L (solid curve), defined by $2\pi/l'$ from (41) with Emanuel's result for $S \gg 1$ and free boundaries. Again the agreement is not too bad.

Unfortunately Calman (1977) in his experiments with large Prandtl, or more correctly Schmidt numbers, does not give sufficient information about the values of his parameters, which makes direct comparisons with his results impossible.

Even though a Prandtl number of order unity in some cases does provide a reasonable approximation for a turbulent ocean or atmosphere, there are cases where this is definitely not so. For example in other planetary atmospheres or in stellar material, where the effective conductivity may be greatly enhanced by radiation, the Prandtl number may become quite small.

Let us for simplicity consider the limit $Pr \rightarrow 0$. Then, as also discussed in connection with the solutions (38) and (39), the exchange of heat between a particle and its surroundings takes place instantaneously. Hence a particle never "feels" the stratification, and the buoyancy effect drops out of the problem, or from (11): E finite, $Pr \rightarrow 0$ implies that $\sigma \rightarrow 0$. This is in fact the situation considered by Gammelsrød (1975), but with a different choice for m in (16). The relevant stability boundary for this case, i.e. B_c , is obtained from figure 1 as twice the value of B_c for $S = 0$ and $Pr = 1$. It can also be shown that the most unstable disturbance in this case is monotonic.

To apply the present analysis to the earth's atmosphere or oceans, appropriate values of the non-dimensional parameters involved must be ascertained. Assuming for the atmosphere that $H = 1-10$ km, $\nu = 10^5$ cm²/s, $f = 10^{-4}$ s⁻¹, $N = 10^{-2}$ s⁻¹ we obtain

$$S = 10^4, \quad E = 10^{-3}-10^{-1} \quad (44)$$

while for the ocean we take $H = 100-1000$ m, $\nu = 10^2$ cm²/s, $f = 10^{-4}$ s⁻¹, $N = 10^{-3}-10^{-2}$ s⁻¹ which leads to

$$S = 10^2-10^4, \quad E = 10^{-4}-10^{-2} \quad (45)$$

It is thus seen that S and E roughly are comparable in the atmosphere and the ocean.

An interesting thing to note from the present results is that for many "normal" situations the critical wavelength becomes quite large. Thus for $E = 10^{-2}$, $S = 10^4$ and $Pr = 1$, which is not unreasonable values for the ocean and the atmosphere, we find a critical value

$$Ri^C = 0.6 \quad \text{for} \quad l_c = 0.062 \quad (46)$$

which corresponds to a critical wavelength $\lambda_c = 2\pi/l_c = 100$, or dimensionally $\lambda_* = 100 H$.

It can be shown that the cell structure in the meridional plane when $Pr = 1$ does not differ much from that of the inviscid case. The most unstable horizontal wavelength is of course finite (and rather large in this example), but the cells are tilted towards heavier fluid and the tilt lines nearly coincides with the isopycnals of the basic flow.

FINAL REMARKS

Since shear of the same order of magnitude as the Brunt-Väisälä frequency is not uncommonly found in the ocean or the atmosphere, there should be a theoretical possibility for the generation of roll vortices by the symmetric instability mechanism discussed in this paper. The pertinent question is: Have cell structures originating from this mechanism been observed in nature, and if not, where should one look to find them ?

Concerning the explanation of the Langmuir circulations observed in wind-driven shear flow in a shallow mixed layer of the upper ocean, symmetric instability may be of importance as demonstrated by Gammelsrød (1975). This type of instability may also be associated with larger current systems reaching down to a greater depth. Due to the large wavelengths, however, typically ten to hundred times the depth, the cells may be difficult to observe.

A frequently observed cloud street situation in the atmosphere occurs when cold arctic air flows over open warm sea. The cloud base is typically situated at a height of 1 km, just below the inversion layer. In such cases thermal convection has often been offered as the most obvious explanation of the phenomenon. However, in these cases the Rayleigh number may be of order $10^{12} - 10^{15}$, which is very much larger than the critical Rayleigh number corresponding to the onset of convection ($\sim 10^3$). Accordingly the convective motions are turbulent with a characteristic scale much less than the height H of the inversion layer, and the heat and momentum transports may be considered

as eddy processes. This may give rise to larger secondary convection cells, and so on, as demonstrated by Foster (1972) where he develops the idea of a hierarchy of convection leading to an increasingly larger scale. However, due to the large vertical eddy transfer associated with the primary convection, the mean vertical density stratification will be approximately neutral in the main part of the layer, i.e. $S = 0$ in our analysis. When we have a non-zero shear then, the symmetric instability mechanism discussed in the present paper may be important. Kuettner (1971) reports a vertical shear of the order 10^{-2} s^{-1} in typical cloud street situations. This implies a baroclinicity parameter B of order 10^2 . Taking $\kappa = \nu = 10^5 \text{ cm}^2/\text{s}$, $f = 10^{-4} \text{ s}^{-1}$ and $H = 1000\text{m}$, i.e. $Pr = 1$, $E = 0.1$, the present analysis yields $B_c = 15.1$ so that symmetric instability may well be possible. The dimensional critical wavelength λ_* is found to be $3.7 H$, which is in excellent agreement with the observed spacing of cloud streets (Kuettner 1971). For $B = 100$, the growth rate in this example is $4.4 \times 10^{-4} \text{ s}^{-1}$, corresponding to an e-folding time of about 38 min.

Cloud streets have also been observed at much higher levels (cirrus streets), and often connected with the jet stream (Doswell and Schaefer 1976). In such cases, where the generation process occurs above the Ekman layer in a stably stratified atmosphere, the instability mechanism discussed here may be of relevance.

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REFERENCES

- Brown, R.A. 1970
A secondary flow model for the planetary boundary layer, J.Atmos.Sci., 27, 742-757.
- Calman, J. 1977
Experiments on high Richardson number instability of a rotating stratified shear flow. Dyn.Atm.Oceans, 1, 277-297.
- Doswell, C.A. & Schaefer, J.T. 1976
On the relationship of cirrus clouds to the jet stream. Mon.Wea.Rev. 104, 105-106.
- Eliassen, A. & Kleinschmidt, E. 1957
Dynamic Meteorology, in Handbuch der Physik, ed. by S. Flügge, Springer-Verlag 1957, XLVIII, 1-154.
- Emanuel, K.A. 1978
Inertial stability and mesoscale convective systems. Ph.D. Thesis, Massachusetts Institute of Technology.
- Kuo, H.-L. 1954
Symmetrical disturbances in a thin layer of fluid subject to a horizontal temperature gradient and rotation. J.Meteor. 11, 399-411.
- Faller, A.J. 1963
An experimental study of the instability of the laminar Ekman boundary layer. J.Fluid Mech. 15, 560-576.
- Faller, A.J. 1965
Large eddies in the atmospheric boundary layer and their possible role in the formation of cloud rows. J.Atmos.Sci., 22, 176-184.

- Faller, A.J. 1971 Oceanic turbulence and the Langmuir circulations. *Ann.Rev.Ecology & Systematics*, 2, 201-236.
- Foster, T.D. 1972 Haline convection in polynyas and leads. *J.Phys.Oceanogr.* 2, 462-469.
- Gammelsrød, T. 1975 Instability of Couette flow in a rotating fluid and origin of Langmuir circulations. *J.Geophys. Res.* 80, 5069-5075.
- Kuettner, J.P. 1971 Cloud bands in the earth's atmosphere. *Tellus*, 23, 404-425.
- Kuo, M.L. 1963 Perturbations of plane Couette flow in a stratified fluid and origin of cloud streets. *Phys.Fluids*, 6, 195-211.
- Lilly, D.K. 1966 On the stability of Ekman boundary flow. *J.Atmos.Sci.* 23, 481-494.
- McIntyre, M.E. 1970 Diffusive destabilization of the baroclinic vortex. *Geophys.Fluid Dyn.*, 1, 19-57.
- Stone, P.H. 1966 On non-geostrophic baroclinic stability. *J.Atmos.Sci.* 23, 390-400.
- Walton, I.C. 1975 The viscous nonlinear symmetric baroclinic instability of a zonal shear flow. *J.Fluid Mech.*, 68, 757-768.
- Weber, J.E. 1978 On the stability of thermally-driven shear flow heated from below. *J.Fluid Mech.*, 87, 65-84.

Figure legends

Figure 1

The critical baroclinicity parameter B_c as a function of E for small values of S and $Pr = 1$. The curve $S = 0$ represents the parameter $F_c = (1+Pr)B_c/2$ and are valid for all values of Pr . The dashed curve is the analytical result (33) and the open circles are values from Emanuel (1978) for $S = 0$.

Figure 2

The critical Richardson number Ri^c as a function of E for moderate and large values of S and $Pr = 1$. The dashed curves represent the analytical result (38). The open circles and crosses are values for free and rigid boundaries, respectively, from Emanuel (1978) for $S \gg 1$.

Figure 3

The critical wavenumber l_c times $(S+1)^{\frac{1}{2}}$ vs. E for a wide range of S and $Pr = 1$. The dashed curve is the analytical result (39). Open circles are from Emanuel (1978) for $S = 0$ and free boundaries.

Figure 4

Dimensionless critical wavelength $L = (2\pi/l_c)(2/Ri^{\frac{1}{2}}S^{\frac{1}{2}})$ for $S \gg 1$. Solid curve: present result; dashed curve: Emanuel (1978) for free boundaries.

Figure 1.

Figure 2.

Figure 3.

Figure 4.