OPTIMAL REPLACEMENT UNDER A MINIMAL
REPAIR STRATEGY - A GENERAL FAILURE MODEL

by

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Abstract

In this paper we generalize the minimal repair, replacement model introduced by Barlow and Hunter (1960).

We assume that there is an underlying stochastic "condition", representing e.g. wear, accumulated damage, affecting the failure rate of the system and the costs involved.

At a random time T , based on the "condition", the system is replaced. Intervening failures are rectified through minimal repair after which the failure rate is the same as before the failure. The problem is to find a T which minimizes the total expected discounted cost.

Under appropriate conditions on the failure rate and the costs an optimal T is found. Some generalizations and special cases are given.

REPLACEMENT, MINIMAL REPAIR, FAILURE RATE, STOCHASTIC WEAR,
MINIMUM DISCOUNTED COST

1. INTRODUCTION

Barlow and Hunter (1960) introduced a so-called minimal repair policy for a complex system, which is stochastically deteriorating and at any instant of time can be in one of two states, the working state and the failure state.

Under this repair policy, the system is replaced or overhauled at age T (a constant) at a cost c_0 . Intervening failures are rectified at a cost c_1 through minimal repair after which the failure rate of the system is the same as before the failure.

The problem is to select T so as to minimize the expected average long run cost per unit time or the total expected discounted cost.

In this paper we shall generalize the model by Barlow and Hunter (1960).

We assume that there is an observable stochastic underlying "condition" which is influencing the deteriorating process. This "condition" represents wear, accumulated damage etc. At time $t = 0$ we assume the system is "new". Based on continuous observation of the "condition" of the system we replace the system by a new and identical one at the random time T . The replacement times are assumed to be of zero length. A random cost is incurred. Intervening failures are rectified (without delay) at a random cost through minimal repair, abbr. MR, after which the failure rate of the system is as before failure. Note that the failure rate and the costs depend on the "condition".

The problem is to determine the random time T so as to minimize the total expected discounted cost.

The minimal repair policy discussed in this paper will be abbreviated MRP.

2. THE MODEL AND RESULTS

Let (Ω, Σ, P) be a probability space and let $\{F_t\}_{t \geq 0}$ be a non-decreasing family of sub- σ -fields of Σ . The σ -field F_t represents the "condition" of the system at time t . In most applications $F_t = \sigma(X(s), 0 \leq s \leq t)$ where $\{X(t), t \geq 0\}$ is a stochastic process defined on (Ω, Σ) . ($\sigma(W)$, where W is a set of random variables, means the smallest σ -algebra with respect to which all the variables in W are measurable.) We shall denote by F the σ -field generated by all F_t , $t \geq 0$, i.e.

$$F = \bigvee_{t \geq 0} F_t$$

(A and B sub- σ -fields of Σ , then $A \vee B$ is the smallest σ -field of Σ which contains both A and B .)

Now let $\{\lambda(t), t \geq 0\}$ be a measurable non-negative stochastic process adapted to $\{F_t\}$, i.e. the map $\lambda: \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+ = [0, \infty)$ is $\mathcal{B} \otimes \Sigma$ -measurable ($\mathcal{B}$ is the Borel σ -field of $\mathbb{R}^+$) and $\lambda(t)$ is F_t -measurable for all $t \geq 0$. (See Appendix B, Remark.) The random variable $\lambda(t)$ represents the failure rate of the system at time t .

Define

$$\Lambda(t) = \int_0^t \lambda(s) ds$$

and assume $\Lambda(t) < \infty \quad \forall t < \infty$.

Let $\{N(t), t \geq 0\}$ be conditional Poisson process (or a double stochastic Poisson process), see Serfozo (1972) (or Grandell (1975) with intensity $\lambda(\cdot)$, i.e. $\{N(t)\}$ is conditionally a non-homogenous Poisson process with mean value function $\Lambda(t)$, given the σ -field F . That is $N(0) = 0$ a.s. and for any $s_1 < t_1 < \dots < s_n < t_n$ and $x_1, \dots, x_n$

$$\begin{aligned} & P\{N(t_1)-N(s_1) \leq x_1, \dots, N(t_n)-N(s_n) \leq x_n | F\} \\ &= \prod_{k=1}^n P\{N(t_k)-N(s_k) \leq x_k | F\} \quad \text{a.s.} \end{aligned}$$

and for any $0 \leq s \leq t$ and $x = 0, 1, \dots$

$$(2.0) \quad P\{N(t)-N(s) = x | F\} = \frac{(\Lambda(t)-\Lambda(s))^x}{x!} e^{-(\Lambda(t)-\Lambda(s))} \quad \text{a.s.}$$

The MRP is now defined by letting $N(t)$ be the number of MRs in $[0, t]$.

Let E be a sub- σ -field of Σ independent of F and $\{N(t), t \geq 0\}$. We shall denote by T' the set of extended non-negative random variables measurable with respect to $F \vee E$. Clearly T' includes all stopping times relative to $\{F_t\}$, (T is called a stopping time relative to $\{F_t\}$ if the event $\{T \leq t\} \in F_t$). The σ -field E can be thought to represent a randomizing experiment independent of everything else (see Nummelin (1980)).

Let $\{c_1(t), t \geq 0\}$ and $\{c_0(t), t \geq 0\}$ be non-negative measurable processes adapted to the family $\{F_t\}$ (see Appendix B, Remark). The random variable $c_1(t)$ represents the cost of a MR at t whereas the random variable $c_0(t)$ represents the cost of a replacement at t . We will assume that every trajectory of $\{c_0(t)\}$ is absolutely continuous w.r.t. Lebesgue measure.

Let $\alpha > 0$ be a fixed discount factor. We shall study the problem of minimizing the total expected discounted cost, B_α^T , $T \in T'$.

It follows from Appendix A that (we assume independence between the systems)

$$(2.1) \quad B_\alpha^T = \frac{E V_\alpha^T}{E[1-e^{-\alpha T}]}, \quad T \in T'$$

where V_{α}^T is the discounted cost in one cycle of length T .
The following expression for B_{α}^T shows that our model may be included in the general set-up described in Aven (1981).

Theorem 1.

For any $T \in T'$,

$$(2.2) \quad B_{\alpha}^T = \frac{E\left[\int_0^T a(t)d\mu(t) + c_0(0)\right]}{E\left[\int_0^T d\mu(t)\right]}$$

where

$$d\mu(t) = \alpha e^{-\alpha t} dt$$

$$a(t) = \frac{1}{\alpha} [c_1(t)\lambda(t) + \frac{d}{dt} c_0(t) - \alpha c_0(t)]$$

Proof. Clearly

$$E[1 - e^{-\alpha T}] = E\int_0^T d\mu(t),$$

hence it suffices to consider the numerator of (2.2) (see (2.1)).

Since the discounted cost in one cycle of length T is given by

$$V_{\alpha}^T = \int_{[0,T]} c_1(t)e^{-\alpha t} dN(t) + c_0(T)e^{-\alpha T}$$

(see Appendix B),

it follows from the model-assumptions and Appendix B that

$$\begin{aligned} EV_{\alpha}^T &= E[EV_{\alpha}^T | F \vee E] \\ &= E\left[\int_{[0,T]} c_1(t)e^{-\alpha t} d\Lambda(t) + c_0(T)e^{-\alpha T}\right] \\ &= E\left[\int_0^T c_1(t)e^{-\alpha t} \lambda(t) dt + \int_0^T \left(\frac{d}{dt} c_0(t) - \alpha c_0(t)\right) e^{-\alpha t} dt + c_0(0)\right] \\ &= E\left[\int_0^T a(t)d\mu(t) + c_0(0)\right] \quad \square \end{aligned}$$

We now make the following assumptions:

$a(\cdot)$ is non-decreasing and right-continuous, (we may assume that almost all paths of $\{a(t)\}$ have the appropriate properties, however, it is then necessary to complete (Ω, Σ, P) and add all null sets to $\mathcal{F}_t$, see Aven (1981)), $0 < E c_0(0) < \infty$ and $\inf_{t>0} [\text{ess inf}_{\omega} a(t)] > -\infty$. The conditions stated in Aven (1981) are easily verified and the general theory applies. (See also Nummelin (1980) and Bergman (1980)). Before we state the main results we need some definitions:

Let the stopping times (rel. to $\{\mathcal{F}_t\}$) $T_\lambda, \lambda \in \mathbb{R}$ be defined by

$$T_\lambda = \inf\{t: a(t) \geq \lambda\}.$$

λ is called the threshold value. (It can be shown that T_λ minimizes $E V_\alpha^T - \lambda E(1 - e^{-\alpha T})$). Furthermore let

$$B(\lambda) = B_\alpha^T \lambda$$

Results:

R1 $T_{\lambda^*} = \inf\{t: a(t) \geq \lambda^*\}$, where λ^* is the unique solution of the equation $\lambda = B(\lambda)$, minimizes B^T , $T \in \mathcal{T}'$. Furthermore $\lambda > B(\lambda)$ if $\lambda > \lambda^*$, $B(\lambda)$ is non-increasing in λ for $\lambda \leq \lambda^*$ and non-decreasing for $\lambda \geq \lambda^*$ and $B(\cdot)$ is left-continuous.

R2 Choose any λ_1 such that $P\{T_{\lambda_1} > 0\} > 0$, and set iteratively

$$\lambda_{n+1} = B(\lambda_n), \quad n=1, 2, \dots$$

Then

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda^*$$

Fig. 1. Principle features of $B(\lambda)$

We shall now give some useful formulae for computing $B(\lambda)$. Since the events $\{T_\lambda > t\}$ and $\{a(t) < \lambda\}$ are equivalent, it follows from (2.2) and Fubini that

$$B(\lambda) = \frac{\int_0^\infty E[I(a(t) < \lambda)a(t)] d\mu(t) + Ec_0(0)}{\int_0^\infty E[I(a(t) < \lambda)]d\mu(t)}$$

$$\text{where } I(a(t) < \lambda) = I((t, \omega); a(t, \omega) < \lambda) = \begin{cases} 1 & \text{if } a(t, \omega) < \lambda \\ 0 & \text{if } a(t, \omega) \geq \lambda \end{cases}$$

Hence if $F_t = \sigma(X(s), 0 \leq s \leq t)$, $a(t, \omega) = \tilde{a}(X(t, \omega))$ and $Q_t(\cdot)$ is the distribution of $X(t)$, we may write

$$(2.3) \quad B(\lambda) = \frac{\int_0^\infty \int I(\tilde{a}(x) < \lambda) \tilde{a}(x) Q_t(dx) d\mu(t) + Ec_0(0)}{\int_0^\infty \int I(\tilde{a}(x) < \lambda) Q_t(dx) d\mu(t)}$$

Note that if $X(\cdot)$ is a vector process, then one of the components may be the time.

We end this section by giving an economic interpretation of $a(\cdot)$. Consider a time point t . Let F be given and define

$G_t(h)$ = Expected discounted cost associated with
waiting an additional short time h and
then replace - expected cost of a replacement now at t .

Thus $G_t(h)$ represents the discounted maintenance cost in $[t, t+h]$ (i.e. the discounted cost in $[t, t+h]$ due to minimal repairs) plus the difference between the replacement costs at $t+h$ and t . It is assumed that all costs are discounted to time t . We assert that

$$(2.4) \quad \lim_{h \rightarrow 0^+} \frac{G_t(h)}{h} = \alpha a(t) \quad \text{a.s.}$$

This follows since we have

$$\begin{aligned} G_t(h) &= e^{\alpha t} \left\{ \int_t^{t+h} c_1(s) e^{-\alpha s} \lambda(s) ds + c_0(t+h) e^{-\alpha(t+h)} - c_0(t) e^{-\alpha t} \right\} \\ &= e^{\alpha t} \int_t^{t+h} a(s) e^{-\alpha s} \alpha ds \quad \text{a.s.} \end{aligned}$$

Now in view of (2.4), the form of the optimal rule seems very reasonable under the assumption of a non-decreasing $a(\cdot)$.

The reader is referred to Berg (1980) for a similar analysis under age and block replacement policies.

3. GENERALIZATIONS

1. "Occupancy cost"

Let $\{b(t), t \geq 0\}$ be a measurable stochastic process adapted to $\{F_t\}$. For each t the random variable $b(t)$ represents an occupancy cost reflecting the burden of routine maintenance.

The expected discounted cost may now be written (see the proof of Theorem 1)

$$\begin{aligned} EV_{\alpha}^T &= E \left[\int_{[0,T]} e^{-\alpha t} c_1(t) dN(t) + \int_0^T e^{-\alpha t} b(t) dt + c_0(T) e^{-\alpha T} \right] \\ &= E \left[\int_0^T a(t) d\mu(t) + c_0(0) \right] \end{aligned}$$

where

$$\begin{aligned} \alpha a(t) &= [c_1(t) \lambda(t) + b(t) + \frac{d}{dt} c_0(t) - \alpha c_0(t)] \\ d\mu(t) &= \alpha e^{-\alpha t} dt \end{aligned}$$

We can now proceed as before.

2. "n failure modes"

Consider a system with n failure modes ($1 \leq n \leq \infty$) which we denote by $1, 2, \dots, n$.

Let $\{N_i(t), t \geq 0\}$, $i = 1, 2, \dots, n$ be conditional Poisson processes with intensity $\lambda_i(t)$. $\{\lambda_i(t), t \geq 0\}$ is a measurable stochastic process adapted to $\{F_t\}$ and represents the failure rate to mode i .

A minimal repair policy is defined by letting $N_i(t)$ be the number of minimal repairs in $[0, t]$ from mode i . A cost function $c_i(t)$, $i = 1, 2, \dots, n$, a measurable stochastic process adapted to $\{F_t\}$, is associated with a minimal repair from mode i .

The expected discounted cost may for this model be written

$$\begin{aligned} EV_{\alpha}^T &= E \left[\sum_{i=1}^n \int_{[0, T]} e^{-\alpha t} c_i(t) dN_i(t) + c_0(T) e^{-\alpha T} \right] \\ &= E \left[\int_0^T a(t) d\mu(t) + c_0(0) \right] \end{aligned}$$

where

$$\alpha a(t) = \sum_{i=1}^n c_i(t) \lambda_i(t) + \frac{d}{dt} c_0(t) - \alpha c_0(t)$$

$$d\mu(t) = \alpha e^{-\alpha t} dt$$

3. "Non-negligible replacement times"

We have assumed that the replacements always happen instantaneously. This assumption may be dropped.

Let $\{z(t), t \geq 0\}$ be a measurable stochastic process adapted to the family $\{F_t\}$.

$z(t)$ represents the amount of time required for the replacement given that the replacement starts at t . We assume that for all $\omega \in \Omega$, the function $t \rightarrow z(t)$ is absolutely continuous w.r.t. Lebesgue measure.

We will show that the expected discounted cost may be written in a form similar to that of (2.2).

It follows from Appendix A that

$$B_{\alpha}^T = \frac{EV_{\alpha}^T}{E[1 - e^{-\alpha[T+z(T)]}]}$$

(We assume the same cost-structure as before.)

From (2.2) we have

$$EV_{\alpha}^T = E\left[\int_0^T a(t) d\mu(t) + c_0(0)\right],$$

hence it suffices to consider the denominator.

We get

$$\begin{aligned} & E[1 - e^{-\alpha(T+z(T))}] \\ &= E\left[1 + \int_0^T \alpha\left(1 + \frac{d}{dt} z(t)\right) e^{-\alpha(t+z(t))} dt - e^{-\alpha z(0)}\right] \\ &= E\left[\int_0^T d\hat{\mu}(t) + 1 - e^{-\alpha z(0)}\right] \end{aligned}$$

where

$$\begin{aligned} d\hat{\mu}(t) &= \left(1 + \frac{d}{dt} z(t)\right) e^{-\alpha z(t)} d\mu(t) \\ &\stackrel{\text{def}}{=} \frac{d\hat{\mu}}{d\mu}(t) d\mu(t) \end{aligned}$$

Thus introducing

$$\hat{a}(t) = \left(\frac{d\hat{\mu}}{d\mu}(t)\right)^{-1} a(t), \text{ we may write}$$

$$(3.1) \quad B_{\alpha}^T = \frac{E\left[\int_0^T \hat{a}(t) d\hat{\mu}(t) + c_0(0)\right]}{E\left[\int_0^T d\hat{\mu}(t) + 1 - e^{-\alpha z(0)}\right]}$$

We can now proceed as before.

Note that (3.1) equals (2.2) if $z(t) \equiv 0$ (we may drop the $\wedge$ notation).

Some special cases

Example 1.

If we let $F_t = \{\Omega, \emptyset\} \forall t$, $E = \{\Omega, \emptyset\}$ and $c_i(t) \equiv c_i$, $i = 0, 1$, we have the "discounted version" of the model of Barlow and Hunter (1960).

Example 2.

Assume the systems (units) installed after replacements are of different types (in Bergman (1978) a model with two types of systems are considered). Let the failure rate of the system be $\lambda(t) = h(Z, t)$, where Z is a random variable with distribution $G(\cdot)$. After the replacements the type is immediately identified, i.e. Z is observed.

As a simple illustration we will determine the optimale rule when

$$h(Z, t) = Z,$$

Z has a uniform distribution on $[0, a]$, $a > 0$ and $c_i(t) = c_i > 0$, $t \geq 0$, $i = 0, 1$.

Let $\tilde{a}(Z) = a(t) = \frac{1}{\alpha}[c_1 Z - \alpha c_0]$ and $k_\lambda = \alpha(\lambda + c_0) \frac{1}{c_1} > 0$.

Then it follows that

$$T_\lambda(\omega) = \begin{cases} 0 & \text{if } Z(\omega) \geq k_\lambda \\ \infty & \text{if } Z(\omega) < k_\lambda \end{cases}$$

From (2.2) we easily get

$$B(\lambda) = \begin{cases} \frac{c_1 a}{2\alpha} & \text{if } a < k \\ \frac{1}{\alpha} \left[\frac{c_1 k_\lambda}{2} - \alpha c_0 \right] + \frac{c_0 a}{k_\lambda} & \text{if } a \geq k \end{cases}$$

Let $\alpha = 1$, $a = 100$, $c_1 = 1$, $c_0 = 10$. Then we find the optimal threshold value $\lambda^* = 34.7$.

Thus we replace the systems immediately until we find a system with failure rate less than $k_{\lambda*} = 44.7$. The system is then never replaced; only minimal repairs are performed.

Example 3

Let $F_t = \sigma(X(u), 0 \leq u \leq t)$ and assume $X(\cdot)$ is a Poisson process with intensity s . Furthermore we shall assume $c_i(t) = c_i > 0$, $i = 0, 1$ and that $\lambda(t) = k_1 + k_2 X(t)$, where $k_1 \geq 0$ and $k_2 > 0$.

From the definition of the stopping times T_λ it is clear that it suffices to consider the threshold values

$$d_n = \frac{1}{\alpha} [c_1(k_1 + k_2 n) - \alpha c_0], \quad n = 1, 2, \dots$$

Let $p_x(s) = \frac{e^{-s}}{x!} s^x$. Then using (2.3) and formula 1.2-22 on page 4 in Haight (1967) we find the following expressions for $B(d_n)$:

$$\begin{aligned} B(d_n) &= \frac{\frac{1}{\alpha} \int_0^\infty \sum_{x=0}^{n-1} [c_1(k_1 + k_2 x) - \alpha c_0] p_x(st) d\mu(t) + c_0}{\int_0^\infty \sum_{x=0}^{n-1} p_x(st) d\mu(t)} \\ &= \frac{\frac{c_1 k_1 - \alpha c_0}{s + \alpha} \sum_{x=0}^{n-1} \left(\frac{s}{s + \alpha}\right)^x + \frac{c_1 k_2 s}{(s + \alpha)^2} \sum_{x=0}^{n-2} (x+1) \left(\frac{s}{s + \alpha}\right)^x + c_0}{\frac{\alpha}{s + \alpha} \sum_{x=0}^{n-1} \left(\frac{s}{s + \alpha}\right)^x} \\ &= \frac{\frac{1}{\alpha} (c_1 k_1 - \alpha c_0) \left(1 - \left(\frac{s}{s + \alpha}\right)^n\right) + \frac{c_1 k_2 s}{\alpha^2} \left(1 - n \left(\frac{s}{s + \alpha}\right)^{n-1} \frac{\alpha}{s + \alpha} - \left(\frac{s}{s + \alpha}\right)^n\right) + c_0}{1 - \left(\frac{s}{s + \alpha}\right)^n} \end{aligned}$$

As an illustration, let $\alpha = k_1 = c_1 = s = 1$, $k_2 = 10$ and $c_0 = 10$.

Then

$$B(d_n) = \frac{11 - \left(\frac{1}{2}\right)^n - 10n\left(\frac{1}{2}\right)^n}{1 - \left(\frac{1}{2}\right)^n}, \quad d_n = 10n - 9$$

The optimal threshold value is easily found to be 11 (or 7.6) i.e. $n = 2$. (See Fig. 2 and 3.)

Fig. 2

Fig. 3

Example 4.

Consider a coherent system or more generally a monotonic system, see Barlow and Proschan (1975), page 6 and 84, of $n(< \infty)$ independent components. Define the process $X(t) = (X_1(t), \dots, X_n(t))$ where $X_i(t)$, $i = 1, 2, \dots, n$ represents the state of the i -th component at time t , i.e.

$$X_i(t) = \begin{cases} 1 & \text{if comp. } i \text{ is functioning at } t \\ 0 & \text{otherwise} \end{cases}$$

Let $\varphi(X(\cdot))$ be the state of the system defined as 1 if the system is functioning, and 0 if the system is not functioning.

The MRP is characterized in the following way: If the system fails at t , the component which causes system failure is minimal repaired such that the failure rate of this component is as before failure. (A component which fails without causing system failure is not minimal repaired.)

We say that the system fails to mode i if the failure of component i coincides with the failure of the system.

Now let $\{C_t^i\}_{t \geq 0}$, $i = 1, 2, \dots, n$ be non-decreasing families of sub- σ -fields of Σ . The σ -field C_t^i represents the wear etc. of the i -th component at time t .

Furthermore let $\{\tilde{\lambda}_i(t), t \geq 0\}$, $i = 1, 2, \dots, n$ be measurable stochastic processes adapted to $\{C_t^i\}_{t \geq 0}$. The random variable $\tilde{\lambda}_i(t)$ represents the failure rate of component i at time t .

Denote by

$$F_t = \bigvee_{i=1}^n C_t^i \vee \sigma(X_i(t), t \geq 0, i = 1, 2, \dots, n)$$

A cost function $c_i(t)$, $i = 1, 2, \dots, n$, a measurable stochastic process adapted to F_t , is associated with a minimal repair of component i .

The failure rate to mode i is given by

$$\lambda_i(t) = \tilde{\lambda}_i(t) X_i(t) (1 - \varphi(0_i, X(t)))$$

where $\varphi(0_i, x) = \varphi(x_1, \dots, x_{i+1}, 1, x_{i+1}, \dots, x_n)$.

From generalization 2 we get

$$\alpha a(t) = \sum_{i=1}^n c_i(t) \lambda_i(t) + \frac{d}{dt} c_0(t) - \alpha c_0(t)$$

Now since $X_i(t)(1-\varphi(0_i, X(t))) = 1$ implies $X_i(s)(1-\varphi(0_i, X(s))) = 1$ for $s \geq t$, we see that $X_i(\cdot)(1-\varphi(0_i, X(\cdot)))$ is non-decreasing (possibly a.s.).

Assuming $c_i(\cdot)\tilde{\lambda}_i(\cdot)$ is non-decreasing and $c_0(t) \equiv c_0$, it follows that $a(\cdot)$ is non-decreasing (possibly a.s.), and the form of the optimal rule is found.

Note that the failure rate of the system is given by $\sum_{i=1}^n \lambda_i(t)$. Observe also that if the i -th component is in series with the rest of the system, then $X_i(t) \equiv 1$.

In view of formula (2.3) it is of interest to find the distribution of $X(t) = (X_1(t), \dots, X_n(t))$. Let T_i , $i = 1, 2, \dots, n$ be the lifelength of component i and let $F_i(\cdot)$ be the distribution of T_i , i.e. $F_i(t) = 1 - Ee^{-\int_0^t \tilde{\lambda}_i(s) ds}$.

Furthermore, let $Q_t(x) = P\{X_i(t) = x_i, i = 1, 2, \dots, n\}$ where $x_i \in \{0, 1\}$.

Now since the process $X(\cdot)$ is determined by $(T_1, \dots, T_n)$, it follows that

$P\{X_i(t) = x_i, i = 1, 2, \dots, n | T_1 = t_1, \dots, T_n = t_n\}$ is either 0 or 1.

Note that the probability that two components fail at the same time is zero.

Writing

$$Q_t(x) = \int_0^\infty \dots \int_0^\infty P\{X_i(t) = x_i, i = 1, 2, \dots, n | T_1 = t_1, \dots, T_n = t_n\} dF_1(t_1) \dots dF_n(t_n),$$

we find $Q_t(x)$ expressed by $F_1(\cdot), \dots, F_n(\cdot)$.

As an illustration, consider a parallel system of 2 components.

Then

$$Q_t(x) = \begin{cases} 0 & x_1 = 0 & x_2 = 0 \\ \int_0^t \bar{F}_1(u) dF_2(u) & x_1 = 1 & x_2 = 0 \\ \int_0^t \bar{F}_2(u) dF_1(u) & x_1 = 0 & x_2 = 1 \\ \bar{F}_1(t) \bar{F}_2(t) & x_1 = 1 & x_2 = 1 \end{cases}$$

where $\bar{F}_i(\cdot) = 1 - F_i(\cdot)$.

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Appendix A

Let $\{\tau_n\}_{n=1}^{\infty}$ be a renewal process and let $V_{\alpha}^n(\tau_n)$ be the discounted cost in the n -th cycle when α is a positive discount factor (a cost c at t has a value $ce^{-\alpha t}$ at 0).

If we denote by B_{α} the total expected discounted cost, it is seen by conditioning on the time for the first renewal that

$$B_{\alpha} = EV_{\alpha}^1(\tau_1) + Ee^{-\alpha\tau_1} B_{\alpha}$$

Hence

$$B_{\alpha} = \frac{EV_{\alpha}^1(\tau_1)}{E(1-e^{-\alpha\tau_1})}$$

Appendix B

Definition

Let (Ω, Σ) , $(Y, \mathcal{B})$ be measurable spaces. A non-negative Σ -measurable measure on $\mathcal{B}$ is a function N from $\mathcal{B} \times \Omega$ to $\mathbb{R}^+$ such that

- i) for each $\omega \in \Omega$, $N(\cdot, \omega)$ is a finite ($N(Y, \omega) < \infty$) measure on $\mathcal{B}$,
- ii) for each $B \in \mathcal{B}$, $N(B, \cdot)$ is Σ -measurable.

B.1

Let (Ω, Σ, P) be a probability space and A a sub- σ -field of Σ . Furthermore let $(Y, \mathcal{B})$ be a measurable space, and $N(\cdot, \cdot)$ a non-negative Σ -measurable measure on $\mathcal{B}$.

We will assume that we can find a non-negative A measurable measure on $\mathcal{B}$, $\Lambda(\cdot, \cdot)$ such that

$$\Lambda(B) = E(N(B)|A) \text{ a.s.}$$

Let f be a $\mathcal{B} \otimes A$ -measurable function from $Y \times \Omega$ to $R^+ \cup \{\infty\}$.

Then we have

$$(B.1) \quad E \int f(y, \cdot) N(dy, \cdot) = E \int f(y, \cdot) \Lambda(dy, \cdot)$$

Proof

The expectations are well defined since $Z(\omega) \stackrel{\text{def}}{=} \int f(y, \omega) N(dy, \omega)$ is Σ -measurable and $Z'(\omega) \stackrel{\text{def}}{=} \int f(y, \omega) \Lambda(dy, \omega)$ is A -measurable. This follows from a generalized version of Fubini.

(See e.g. Torgersen and Lindquist (1975), Appendix B.)

It suffices to prove that (B.1) holds whenever f is an indicator function since (B.1) then follows by standard extension to simple functions and monotone limits of simple functions.

Let

$$\mathcal{D} = \{D \in \mathcal{B} \otimes A, E \int I_D N(dy, \cdot) = E \int I_D \Lambda(dy, \cdot)\}$$

We will first prove that $\mathcal{D}$ is a λ -system, i.e.

- i) $\emptyset \in \mathcal{D}$,
- ii) $D \in \mathcal{D} \Rightarrow D^c \in \mathcal{D}$,
- iii) $D_1, D_2 \in \mathcal{D}$, $D_1 \cap D_2 = \emptyset \Rightarrow D_1 \cup D_2 \in \mathcal{D}$,
- iv) $D_1 \subseteq D_2 \subseteq \dots \Rightarrow \bigcup_{i=1}^{\infty} D_i \in \mathcal{D}$ whenever $D_1, D_2, \dots \in \mathcal{D}$.

i) is trivially satisfied.

ii) holds since

$$\begin{aligned} E[\int I_{D^c} N(dy, \cdot)] &= E[\int (1 - I_D) N(dy, \cdot)] = E[N(Y, \cdot) - \int I_D N(dy, \cdot)] \\ &= E[\Lambda(Y, \cdot) - \int I_D \Lambda(dy, \cdot)] = E[\int I_{D^c} \Lambda(dy, \cdot)] \end{aligned}$$

(We have used that $EN(Y, \cdot) = E[E(N(Y, \cdot) | A)] = E\Lambda(Y, \cdot)$.)

iii) follows since $D_1, D_2 \in \mathcal{D}$, $D_1 \cap D_2 = \emptyset \Rightarrow I_{D_1 \cup D_2} = I_{D_1} + I_{D_2}$.

iv) Assume $D_1 \subseteq D_2 \subseteq \dots$ and that $D_i \in \mathcal{D}$, $i = 1, 2, \dots$.

Set $D = \bigcup_{i=1}^{\infty} D_i$. Then $I_{D_i} \uparrow I_D$. Applications of monotone convergence theorem yields

$$\begin{aligned} E[\int I_D N(dy, \cdot)] &= E[\lim_i \int I_{D_i} N(dy, \cdot)] \\ &= \lim_i E \int I_{D_i} N(dy, \cdot) = \lim_i E \int I_{D_i} \Lambda(dy, \cdot) = E \int I_D \Lambda(dy, \cdot) \end{aligned}$$

Now let $\mathcal{C}$ be the set of rectangles $B \times A$, $B \in \mathcal{B}$, $A \in \mathcal{A}$.

Then since $\mathcal{C}$ is closed under finite intersections, it remains to prove that $\mathcal{C} \subseteq \mathcal{D}$ (see Billingsley (1979) page 33-34 or Torgersen and Lindquist (1975), Appendix B).

Let $C = B \times A$, $B \in \mathcal{B}$, $A \in \mathcal{A}$. Then since $A \in \mathcal{A}$,

$$\begin{aligned} E[\int I_{B \times A} N(dy, \cdot)] &= E[I_A \int I_B N(dy, \cdot)] \\ &= E[E[I_A \int I_B N(dy, \cdot)] | A] = E[I_A E[\int I_B N(dy, \cdot)] | A] \\ &= E I_A E[N(B, \cdot) | A] = E I_A \Lambda(B, \cdot) = E \int I_{B \times A} \Lambda(dy, \cdot) \end{aligned}$$

Hence $\mathcal{C} \subseteq \mathcal{D}$.

□

Remark

Consider the model stated in Section 2. Let

$$f(t, \omega) = I((t, \omega), t \leq T(\omega)) c_1(t, \omega) e^{-\alpha t}$$

$$\text{where } I((t, \omega), t \leq T(\omega)) = \begin{cases} 1 & t \leq T(\omega) \\ 0 & t > T(\omega) \end{cases}$$

We will show that

$$E \int_{[0, \infty)} f(t, \cdot) N(dt, \cdot) \stackrel{(B.2)}{=} E \int_{[0, \infty)} f(t, \cdot) \Lambda(dt, \cdot) = E \int_{[0, \infty)} f(t, \cdot) \lambda(t, \cdot) dt$$

($\int_{[0, \infty)} f(t, \omega) N(dt, \omega)$ and $\int_{[0, \infty)} f(t, \omega) \Lambda(dt, \omega)$ are Lebesgue Stieltjes integrals for each $\omega \in \Omega$.)

(B.2) will be established by showing for each $s < \infty$ (using B.1) that

$$E \int_{[0, s]} f(t, \cdot) N(dt, \cdot) = E \int_{[0, s]} f(t, \cdot) \Lambda(dt, \cdot).$$

The details follow:

It is tacitly understood that the process $\{N(t)\}$ is a "regular" conditional Poisson process in that sense that $N(\cdot, \omega)$ is non-decreasing and right-continuous and $0 \leq N(s, \omega) < \infty$ $0 \leq s < \infty$ for all $\omega \in \Omega$.

We shall sketch how a "regular" process can be constructed.

Let $\tilde{N}(\cdot, \cdot)$ be a conditional Poisson process (defined as in Section 2) with intensity $\lambda(\cdot, \cdot)$, and let for each $r \in Q$ (= the set of non-negative rational numbers

$$N(r, \omega) = \tilde{N}(r, \omega).$$

Using (2.0) (and the fact that $\tilde{N}(0) = 0$ a.s.) it follows that for all ω (after a possible redefinition on a null set)

- (i) $N(r_1, \omega) \leq N(r_2, \omega) \quad r_1 \leq r_2 \quad r_1, r_2 \in \mathbb{Q}$
- (ii) $N(r + \frac{1}{n}, \omega) \downarrow N(r, \omega) \quad \text{as } n \rightarrow \infty \quad r \in \mathbb{Q}$
- (iii) $0 \leq N(r, \omega) < \infty \quad r \in \mathbb{Q}$

Putting $N(t, \omega) = \inf_{\substack{r \geq t \\ r \in \mathbb{Q}}} N(r, \omega)$ for each non-negative real number t ,

we see that $N(\cdot, \cdot)$ is "regular", and since $N(t) = \tilde{N}(t)$ a.s. for all $t \geq 0$, it follows that $N(\cdot, \cdot)$ is a conditional Poisson process with intensity $\lambda(\cdot, \cdot)$. ■

For each $\omega \in \Omega$ we now generate a Lebesgue-Stieltjes measure $N(\cdot, \omega)$ on $[0, \infty)$ by $N((a, b], \omega) = N(b, \omega) - N(a, \omega)$. We claim that $N(\cdot, \cdot)$ is a non-negative Σ -measurable measure on $\mathcal{B}[0, s]$ where $\mathcal{B}[0, s]$ denotes the σ -field of Borel sets on $[0, s]$, $s < \infty$. Since $N(\cdot, \omega)$ is a finite measure on $\mathcal{B}[0, s]$ for each ω , it suffices to show that $N(B, \omega)$ is Σ -measurable for each $B \in \mathcal{B}[0, s]$. As the class of sets B , $B \in \mathcal{B}[0, s]$, such that $N(B, \cdot)$ is Σ -measurable is a λ -system (see the proof of B.1) and contains the system of intervals $\{[0, t], t \leq s\}$, we find that $N(B, \cdot)$ is Σ -measurable for all $B \in \mathcal{B}[0, s]$ (see Billingsley (1979) page 33-34).

Now let for each ω , $\Lambda(\cdot, \omega)$ be the Lebesgue-Stieltjes measure on $[0, \infty)$ generated by $\Lambda((a, b], \omega) = \Lambda(b, \omega) - \Lambda(a, \omega) = \int_{(a, b]} \lambda(t, \omega) dt$ (remember that $\Lambda(t, \omega) < \infty$, $0 \leq t < \infty$ by assumption).

We will show that $\Lambda(t, \cdot)$ is $\mathcal{F}_t$ -measurable for all $0 \leq t < \infty$.

The following results from Meyer (1966), Chapter IV, page 68-70 will be used:

- (i) If $\{X(t)\}$ is a measurable real-valued (non-negative) process adapted to $\{\mathcal{F}_t\}$ (i.e. the mapping $(t, \omega) \rightarrow X(t, \omega)$ is $\mathcal{B} \otimes \Sigma$ -measurable and $X(t)$ is $\mathcal{F}_t$ -measurable $\forall t$, $\mathcal{B}$ is the σ -field of Borel sets on $[0, \infty)$), having right-continuous

paths, then $\{X(t)\}$ is progressively measurable with respect to $\{F_t\}$, i.e. the map $(t, \omega) \mapsto X(t, \omega)$ from $[0, s] \times \Omega$ to $R(R^+)$ is $B[0, s] \otimes F_s$ -measurable for all $s < \infty$.

(ii) If $\{X(t)\}$ is a measurable real-valued (non-negative) process adapted to $\{F_t\}$, then there always exists a modification $\tilde{X}(\cdot)$ of $X(\cdot)$, i.e. $\tilde{X}(t) = X(t)$ a.s., such that $\{\tilde{X}(t)\}$ is progressively measurable w.r.t. $\{F_t\}$.

((i) and (ii) also hold if $\{X(t)\}$ is extended real valued)

Thus $\{\lambda(t)\}$ may be assumed progressively measurable w.r.t. $\{F_t\}$. Using Fubini it is seen that $\Lambda(t, \cdot) = \int_0^t \lambda(u, \cdot) du$ is F_t -measurable, and it follows that $\Lambda(\cdot, \cdot)$ is a non-negative F_s -measurable measure on $B[0, s]$.

As the class of sets B , $B \in B[0, s]$, such that $\Lambda(B)$ is a version of $E(N(B)|F)$ (or $E(N(B)|F \vee E)$) is a λ -system and contains the system of intervals $\{[0, t], t \leq s\}$, it follows that $\Lambda(B)$ is a version of $E(N(B)|F)$ (or $E(N(B)|F \vee E)$) for all $B \in B[0, s]$.

Now since the process $\{c_1(t)\}$ may be assumed progressively measurable w.r.t. $\{F_t\}$ and since $I((t, \omega), t \leq T(\omega))e^{-\alpha t}$ is progressively measurable w.r.t. $F \vee E$, we may assume that $f(t, \omega) = c_1(t, \omega)I((t, \omega), t \leq T(\omega))e^{-\alpha t}$ is progressively measurable w.r.t. $F \vee E$.

From (B.1) we now get for each $0 < s < \infty$

$$E \int_{[0, s]} f(t, \cdot) N(dt, \cdot) = E \int_{[0, s]} f(t, \cdot) \Lambda(dt, \cdot)$$

Letting $s \rightarrow \infty$

$$E \int_{[0, \infty)} f(t, \cdot) N(dt, \cdot) = E \int_{[0, \infty)} f(t, \cdot) \Lambda(dt, \cdot)$$

(we have used monotone convergence theorem)

By a well-known chain rule for Radon-Nikodym derivatives we have for each $\omega \in \Omega$

$$\int_{[0, \infty)} f(t, \omega) \Lambda(dt, \omega) = \int_{[0, \infty)} f(t, \omega) \lambda(t, \omega) dt$$

Thus

$$E \int_{[0, \infty)} f(t, \cdot) N(dt, \cdot) = E \int_{[0, \infty)} f(t, \cdot) \lambda(t, \cdot) dt$$

Finally we note that the above results also hold if $c_1(t)$ takes values in $\mathbb{R}^+ \cup \{\infty\}$.

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