A proof of Higman's embedding theorem,
using Britton extensions of groups.

by

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Abstract. A new proof is given of Higman's theorem: Let R be a finitely generated group, then (1) G can be embedded in a finitely presented group iff (2) R is recursively presented. To prove (1) $\Rightarrow$ (2) is trivial; and we shall only deal with the problem of proving (2) $\Rightarrow$ (1). In order to prove (2) $\Rightarrow$ (1), we start with a Turing machine Z_E , which recognizes the recursive enumerable set E of defining relation for R . From Z_E we construct a semigroup T_E by Post's construction and then a group G_E , by Boone's (and Britton's), method. Then we construct three Britton extensions, starting with the free product $G_E * R$. Finally we delete redundant relations and we obtain a finite presented group in which R is embedded. By using Britton's lemma, we avoid the use of free product with amalgamations. Higman's proof as well as Shoenfield's proof gives an indirect construction by developing a theory about benign subgroups. Our construction is simple and explicit and we avoid the notion "benign subgroup" altogether.

§ 1. Introduction. The aim of this paper is to give a new proof of the following theorem due to Higman. (See Higman 1961 Theorem 1, p.456).

Theorem 1. A finitely generated group can be embedded in a finitely presented group iff it is recursively presented.

Shoenfield 1967 pp.321-326 presents a variant of Higman's proof.

As pointed out by Higman, one-half of theorem 1 is trivial. Hence, from now on we shall concentrate on the non-trivial half of the theorem which we shall state as a lemma:

Lemma 1. Every finitely generated and recursively presented group can be embedded in a finitely presented group.

Definition 1. A positive group presentation is a presentation $(S|D)$ where each relation in D is of the form $w = 1$, for some positive word over S .

Note that every group presentation $(S|D)$ can be turned into a positive presentation of the same group by replacing S by $S' = \{s' | s \in S\} \cup S$ and D by D' , where D' is obtained from D by replacing each occurrence of s^{-1} by s' and by adding the relation $ss' = 1$ for every s in S .

Definition 2. A group is recursively presented if it has a positive presentation of the form

$$(u_1, u_2, \dots, u_m | w = 1, w \in E)$$

where E is a recursive enumerable set of positive words on $\{u_1, u_2, \dots, u_m\}$.

§ 2. Outline of the proof.

We shall start by some remarks on Britton extension and Britton's lemma in § 3 . Then in § 4 , we shall outline the construction of the Turing machine Z_E and the corresponding semigroup T_E . In § 5 we shall study the Boone-Britton group G . (See Britton 1963 pp.22-23) The final aim is to prove lemma 11 which is the only lemma in § 5 which is used in § 6 . Lemma 11 corresponds to lemma 5.1 p.473 in Higman 1961, and to lemma 18 p.335 in Shoenfield 1967. Lemma 11 shows that the group G_E so to speak semicomputes or recognizes the enumerable set of words E .

In § 5 we first form the free product of G_E and the group R , which we want to embed. Then we construct three Britton extensions, which by using a finite number of new generators and new defining relations transfer the information given in group G_E about E to generate relations which make all but a finite number of defining relations in $G * R$ superfluous.

§ 3. Remarks on Britton extension.

We shall assume that the reader is familiar with Britton's lemma, which is the main tool in this paper. (See lemma 4 p.20 of Britton 63.) We shall not state the lemma here since the reader, in order to understand the details of § 4 , must have the paper of Britton before him anyhow. We would only like to define some new concepts and to make some remarks.

Definition 3. A presentation $(S^*|D^*)$ is a Britton extension of $(S|D)$ with stable letters P , if the following two conditions are satisfied:

1. $(S^*|D^*)$ has stable letters P and corresponding basis $(S|D)$.
2. The isomorphism conditions are satisfied.

All unexplained concepts are as in Britton 1963. We shall not always state explicitly what the basis and the stable letters consist of when it should be obvious from the context.

Moreover, we shall not always define the mapping which is used to verify that the isomorphism condition holds. Note that the isomorphism condition is satisfied iff for all $v \in V$ and all pairs of words w, w' where w is a word on the set $\{A_i | i \in J_v\}$ and w' is the corresponding word on $\{B_i | i \in J_v\}$, then $w = 1$ in $(S|D)$ iff $w' = 1$ in $(S|D)$. We may also verify the isomorphism conditions by proving that $A(v)$ and $B(v)$ have equivalent presentations, where A_i corresponds to B_i , $i \in J_v$.

We shall sometimes deviate a little from Britton's notations in order to avoid subsubscripts, and sub-to-superscripts. Instead of denoting the stable letters by $p_v (v \in V)$ or $\{p_v | v \in V\}$ we shall use the set P . Then p without subscript may be used to denote an arbitrary element of P . We shall often use the same letter to denote the group and its presentation.

$G_1 = (S_1|D_1)$ means that G_1 may be used both to denote the group defined by the presentation $(S_1|D_1)$. We shall use $(G_1; S_2|D_2)$ as an abbreviation for $(S_1 \cup S_2 | D_1, D_2)$ if $G_1 = (S_1|D_1)$.

$G_2 = (G_1; S_2|D_2)$ is called an extension of the presentation G_1 . Note that in most cases used in this paper G_1 is embedded in $(G_1; S_2|D_2)$, but in some cases this is not true. We shall not explicitly define homomorphism and isomorphism when the reader

should have no difficulty in finding out the mapping we have in mind which will fit into the proof. Talking about the homomorphism (or isomorphism) of the group $G_1 = (S_1 | D_1)$ into $G_2 = (S_2 | D_2)$ when $S_1 \cap S_2 \neq \emptyset$, we often mean that the mapping

$$\varphi(s) = \begin{cases} s & \text{if } s \in S_1 \cap S_2 \\ 1 & \text{if } s \in S_1 - S_2 \end{cases}$$

can be extended to a homomorphism of G_1 into G_2 .

Definition 4. Let W be a word. We shall say that W is trivially (or freely) reduced if W contains no subword of the form $s^{-1}s$ or ss^{-1} .

Definition 5. Let $(S^* | D^*)$ be a Britton extension of $(S | D)$ with stable letters P . A pinch is a word of one of the following forms:

- (i) $p^{-1}Cp$, where C is a word on S and $C \in A(p)$, and $p \in P$
- or
- (ii) pCp^{-1} , where C is a word on S and $C \in B(p)$, and $p \in P$.

Definition 6. Given $(S^* | D^*)$ with basis $(S | D)$ and stable letters P , then a word on S^* is p-reduced (for a fixed $p \in P$) if W contains no pinches of the forms $p^{-1}Cp$ or pCp^{-1} as subwords. Moreover, W is p-reduced if W is p-reduced for all $p \in P$.

Boone 1966 p.57 defines p-reduction of a word. We shall use p-reduction in a special case.

Definition 7. Let $(S^*|D^*)$ be a Britton extension of $(S|D)$ with stable letters P .

Let $p \in P$ and suppose all relations in D^* involving p are of the form $p^{-1}Xp = X$. Let W and W' be words on S^* . Then W' is a p-reduction of W iff W contains a pinch $p^{-e}Cp^e$, ($e = \pm 1$) and W' is the result of replacing $p^{-e}Cp^e$ by C .

Definition 8. A set X is called a free set of generators in G if the subgroup F generated by X is free on X .

Lemma 2. Let G^* be a Britton extension of G with stable letters P . Let F be a free subgroup of G free on the set X . Let H be a subgroup of G such that $F \cap H = \{1\}$. Let $P_1 \subseteq P$ and suppose $F \cap A(p) = F \cap B(p) = \{1\}$ for all $p \in P_1$. Let F^* be the subgroup of G^* generated by $X \cup P_1$. Then F^* is a free group on the set $X \cup P_1$, and $F^* \cap H = \{1\}$. Moreover, suppose H' is a group generated by $Y = \{h_i p_i g_i \mid 1 \leq i \leq l\}$, where $h_i, g_i \in G$, $p_i \in P$, $p_i \notin P_1$ and $p_i \neq p_j$ if $i \neq j$, ($i = 1, 2, \dots, l$; $j = 1, 2, \dots, l$), then H' is a free group on Y and $F^* \cap H' = \{1\}$.

Proof. Let w be a trivially reduced word on $P_1 \cup X$ and suppose $w = h$ in G^* for some $h \in H$. We shall prove that this implies $w = 1$, which implies that F^* is free on the set $X \cup P_1$ and that $F^* \cap H = \{1\}$. Suppose that w is a word which involves at least one stable letter $p \in P_1$. Since $wh^{-1} = 1$ in G^* , and since p does not occur in h , we have that w contains a pinch of the form $p^{-e}Cp^e$, ($e = \pm 1$) where $p \in P_1$

and $C \in A(p) \cap F = \{1\}$ or $C = B(p) \cap F = \{1\}$. Hence $C = 1$ and $p^{-e} C p^e = 1$, which contradicts the fact that w is trivially reduced. Hence w involves no letter $p \in P_1$. That is $w \in F$ since w is a word on X . But $F \cap H = \{1\}$ which shows that $w \equiv 1$ and $h = 1$ in G^* . To prove the last part of the theorem, let w be trivially reduced word on $X \cup P_1$ and let u be a trivially reduced word on Y , and let $w = u$ in G^* . Then $w^{-1}u = 1$ in G^* and exactly as before we can prove that w is a word on X and if u contains a pinch of the form $p_i^{-1} C p_i$ then u contains a subword of the form $g_i^{-1} p_i^{-1} h_i^{-1} h_i p_i g_i$, which contradicts the assumption that u is trivially reduced. Moreover, by a similar argument we can prove that u contains no pinch of the form $p_i C p_i^{-1}$. Hence $u \equiv 1$ and $w = 1$ in G^* . This proves the last part of the lemma.

§ 4. The Turing machine and the corresponding semigroups

(Readers who are not familiar with Turing machines and recursive theory, may jump to lemma 4.)

We shall use the convention that s_0 is blank.

Definition. T is a Turing machine, its alphabet is the set $\{s_0, s, \dots, s_{M-1}\}$ of all s -letters occurring in its quadruples. If w is a positive word on the non-blank symbols, $\{s_1, s_2, \dots, s_{M-1}\}$, then T accepts w if there is a computation of T beginning with $q_1 w$. (We use "computation of T " in the sense of Davis 1958, def. 1.9 p.7.)

Informally, we regard the machine T as being started in state q_1 scanning the leftmost symbol in w , then T accepts

w iff it eventually stops. We shall also call w an input to the Turing machine T in the case we are studying the (finite or infinite) sequence of instantaneous descriptions $q_1 w \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \dots$. We shall say that a set of positive words are recursively enumerable iff the set of its gödel numbers is recursively enumerable.

Lemma 3. Let E be a recursively enumerable set of positive words on the letters $\{a_1, a_2, \dots, a_m\}$. Then there exists a Turing machine Z_E with alphabet $S' = \{s_0, s_1, s_2, \dots, s_{M-1}\}$ and states $Q' = \{q_1, q_2, \dots, q_{N-2}\}$ such that

1. $\{a_1, a_2, \dots, a_m\} \subseteq \{s_1, s_2, \dots, s_{M-1}\}$
2. If w is a positive word on the letters $\{a_1, a_2, \dots, a_m\}$, then the Turing machine Z_E accepts w iff $w \in E$.

Remark. Note that here, and from now on E is a recursively enumerable set of words on $\{a_1, a_2, \dots, a_m\}$, while in the abstract, and in definition 2, E was a r.e. set of positive words on $\{u_1, u_2, \dots, u_m\}$.

Proof. Let ω be the set of non-negative integers. Numbers are represented by a sequence of the symbol 1, on its tape. An input is numerical iff it consists of a sequence of 1's, non-negative integers. A subset A of ω is recursively enumerable (abbreviated r.e.) iff $A = \emptyset$ or $A = \text{range } f$ for some recursive function f . A basic theorem on r.e. sets says that A is r.e. iff $A = \text{domain } \psi$ for some partial recursive function ψ . Since Turing computable is the same as recursive, we have that A is r.e. iff there exists a Turing machine

which accepts the numerical input n iff $n \in A$. Since a Turing machine can compute the gödel number of a word w given as input, we obtain lemma 3.

By Post's construction (see Post 1947) or better by Davis modification of Post's construction we obtain a semigroup $T = T_E$ which corresponds to the r.e. set E . T is generated by $S \cup Q$, where $S = S' \cup \{s_M\}$ and $Q = Q' \cup \{q_0, q_{N-1}, q_N\}$. We shall write q for q_0 . A word Σ on the letters $S \cup Q$ is special iff either $\Sigma \equiv q_0 \equiv q$ or $\Sigma \equiv wq_jw'$ for some $j \in \{0, 1, 2, \dots, N\}$ and for some positive words w and w' on $S = \{s_0, s_1, \dots, s_M\}$.

Lemma 4. Let E be a recursively enumerable set of words on the letters $\{a_1, a_2, \dots, a_m\}$. Then there exists a semigroup $T = T_E$ generated by the set $S \cup Q$ ($S = \{s_0, s_1, \dots, s_M\}$, $Q = \{q_0, q_1, \dots, q_N\}$) and defining relations

$$\Sigma_i = \Gamma_i \quad (i = 1, 2, \dots, P)$$

where each Σ_i and Γ_i is a special word, such that for each positive word w on the letters $\{a_1, a_2, \dots, a_m\}$ we have that

$$(*) \quad hq_1wh = q \text{ in } T \text{ iff } w \in E.$$

Moreover

$$(**) \text{ if } W = q \text{ in } T \text{ then } W \equiv q \text{ or } W \equiv h\beta q_j \beta' h \text{ where } j \in \{0, 1, 2, \dots, N\} \text{ and } \beta \text{ and } \beta' \text{ are positive words on } \{s_0, s_1, \dots, s_{M-1}\}.$$

§ 5. Properties of the Boone-Britton Group.

By Britton's modification of Boone's construction we obtain a group $G = G_E$ such that

Lemma 5. Let E be a r.e. set of positive words on $\{a_1, a_2, \dots, a_m\}$. Then there exists a finitely presented group $G = G_E$ such that

1. G_E is generated by $S \cup Q \cup \{r_1, r_2, \dots, r_p, l_1, l_2, \dots, l_p, t, k\}$.
2. $\{a_1, a_2, \dots, a_m\} \subseteq \{s_1, s_2, \dots, s_M\}$.
3. If Σ is a special word then
 $k^{-1}\Sigma^{-1}t\Sigma k = \Sigma^{-1}t\Sigma$ in G_E iff $\Sigma = q$ in T_E .
4. If w is a positive word on $\{a_1, a_2, \dots, a_m\}$ then
 $k^{-1}(hq_1wh)^{-1}t(hq_1wh)k = (hq_1wh)^{-1}t(hq_1wh)$ iff $w \in E$.

It turns out that we need to know more about Boone's group. In order to prove lemma 11 we start with the following lemma.

Lemma 6. The subgroup of G generated by the set $S \cup Q \cup \{t\}$ is free on this set.

Proof. The detail of the proof is left to the reader, since by using lemma 2, the proof becomes easy, although somewhat lengthy, since many details have to be checked.

We first prove that the group generated by $S \cup Q$ is free on this set, by repeated application of lemma 2. Then we prove that the group generated by $S \cup \{p_0, p_1, \dots, p_N, z\}$ in H_2 is free on this set. (See page 24 of Britton 63, next to the last line.) Hence $S \cup Q$ is a free set of generators in G_2 . By application of lemma 2 again we can complete the proof of lemma 6.

Lemma 7. The subgroup of G generated by the set $S \cup Q \cup \{k\}$ is free on this set.

Proof. Let $G'_1 = (G_2, k | k^{-1} x k = x, k^{-1} r_i k = r_i, (i=1, 2, \dots, P))$
Then we have that $G = (G_1, t | t^{-1} y t = y, t^{-1} l_i t = l_i,$
 $t^{-1} (q k q^{-1}) t = q k q^{-1}, (i=1, 2, \dots, P))$.

Hence G'_1 is a Britton extension of G_2 with stable letter k
and G is a Britton extension of G_1 with stable letter t .
Hence lemma 7 can be proved in the same way as lemma 6. This
completes the proof of lemma 7.

By modification of Britton's proof we can prove.

Lemma 8. Let Δ be a word on $s_0, s_1, \dots, s_M, q, q_0, q_1, \dots, q_N$
and let Δ be trivially reduced. Then $k^{-1} \Delta^{-1} t \Delta k = \Delta^{-1} t \Delta$ im-
plies that Δ is a special word.

Proof. First we shall show that Δ contains exactly one
occurrence of a symbol from the alphabet $\{q_0, q_1, \dots, q_N\}$. Exactly
as in equation (4) on p.25 in Britton 1963 we can prove that

$$(4') \quad \Delta = N_0 z^{i_1} N_1 z^{i_2} \dots N_{b-1} z^{i_b} N_b = L p z R$$

where $N_0, N_1, \dots, N_b$ are words on $s_0, s_1, \dots, s_M$, $p_0, p_1, \dots, p_N$,
and where L is a word on $l_1, l_2, \dots, l_P, y$ and where R is a
word on $r_1, r_2, \dots, r_P, x$.

Let K be the group generated by $s_0, s_1, \dots, s_M, p_0, p_1, \dots, p_N$.
Then $K \cap A(z) = 1$ and $K \cap B(z) = 1$ by lemma 2, (see the
proof of lemma 6). Since (4') implies

$$(4'') \quad R^{-1} z^{-1} p^{-1} L^{-1} N_0 z^{i_1} N_1 z^{i_2} \dots N_{b-1} z^{i_b} N_b = 1$$

we have by Britton's lemma that $p^{-1} L^{-1} N_0 \in A(z)$. Hence for
some $C \in B(z)$ we have

$$(4''') \quad R^{-1} C N_1 z^{i_2} \dots N_{b-1} z^{i_b} N_b = 1 .$$

But since $K \cap A(z) = 1$, and $K \cap B(z) = 1$ and since $K, A(z)$ and $B(z)$ are free groups and Δ is freely reduced, we have that $b = 1$ and $i_1 = 1$.

Hence we have that

$$(4^{iv}) \quad \Delta \equiv Fq_s G = Fp_s z G = Lp z R \text{ in } H_2.$$

where F and G are word on $\{s_0, s_1, \dots, s_M\}$.

It remains to prove that F and G are positive words. To prove that we change lemma 8 on page 26 in Britton 1963 to read as follows:

"Lemma 8'. Let F and G be reduced words on $\{s_0, s_1, \dots, s_M\}$ and let $\Delta \equiv Fq_s G$. If $v_1, \dots, v_n$ and $e_1, \dots, e_n$ and words L (in l_i, y), R (in r_i, x) exist such that (6) and (7) hold, then Δ is a positive word (and hence a special word), moreover, in the semigroup T , Δ can be transformed into q by a sequence of at most n elementary transformations".

Proof. The proof is by induction on n as before. The first place we have to change the proof is in part (s) on page 27. Change the second paragraph of part (s) to read as follows:

"The reader is reminded that, in (8), F and H_i are trivially reduced words (possibly empty) in the s_b , and H_i is even a positive word; and Y is a word in y only. We shall prove that F has the form $F \equiv UH_{v_1}$ (i.e., the form UH_i), for some reduced word U . Now reduced words W, F', H'_i in the s_b certainly exist such that W and H'_i are positive and such that $F \equiv F'W$, $H_i \equiv H'_i W$, and suppose W has maximal length, so it is sufficient to prove that $H'_i \equiv 1$. Assume that $H'_i \neq 1$.

The rest of part (s) may be left almost unchanged. We shall here quote some basic facts and add some new facts:

Since $\Delta \equiv Fq_s G \equiv UH_i q_{g_i} K_i V$, the word Δ can be transformed into the word $\Delta^* \equiv UF_i q_{s_i} G_i V$ by one elementary transformation, as before. By (6*)^{*)} and (7*)^{*)} we obtain that $\Delta^* \equiv F^* q_{s_i} G^* \equiv UF_i q_{s_i} G_i V$ is a special word, and Δ^* can be transformed into q in at most $n-1$ steps. Hence U and V are positive words, and since H_i and K_i are positive words, we obtain that $UH_i q_{g_i} K_i V \equiv \Delta$ is a special word, moreover Δ can be transformed into q in at most n steps. When $e_1 = -1$ only slight changes have to be made as before. This completes the proof of the revised lemma 8'.

Hence our lemma 8 is proved.

Remark. A shorter proof of our lemma 8, may be possible by applying lemma 26, p.74 in Boone 1966, and some other results in his paper.

Lemma 9. Let $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ be reduced words on $s_0, s_1, \dots, s_M, q, q_0, q_1, \dots, q_N$. Suppose $k^{-1} \Delta_1^{-1} t \Delta_2 k = \Delta_3^{-1} t \Delta_4$, in the group G_E . Then $\Delta_1 \equiv \Delta_2 \equiv \Delta_3 \equiv \Delta_4$ and Δ_1 is a special word.

Proof. Since the group obtained by adding the relation $k = 1$ to G is isomorphic to G_1 , the group obtained by deleting k as generator and deleting all relations containing k , we have that $k^{-1} \Delta_1^{-1} t \Delta_2 k = \Delta_3^{-1} t \Delta_4$ implies $\Delta_1^{-1} t \Delta_2 = \Delta_3^{-1} t \Delta_4$. Hence $\Delta_1 \equiv \Delta_3$ and $\Delta_2 \equiv \Delta_4$ since the letters in $\Delta_1^{-1} t \Delta_2$ and $\Delta_3^{-1} t \Delta_4$ form a free set of generators, by lemma 7. In the same way, by adding the relation $t = 1$ to G which is isomorphic to the

*) (See Britton 1963 p.29)

group G'_1 , obtained by deleting t from the generators and deleting all relations in the presentation of G which contain t we obtain that $k^{-1}\Delta_1^{-1}t\Delta_2k = \Delta_3^{-1}t\Delta_4$ and $t = 1$ implies $k^{-1}\Delta_1^{-1}\Delta_2k = \Delta_3^{-1}\Delta_4$. Hence $\Delta_1 \equiv \Delta_2$ and $\Delta_3 \equiv \Delta_4$. Hence $\Delta_1 \equiv \Delta_2 \equiv \Delta_3 \equiv \Delta_4$ and $k^{-1}\Delta_1^{-1}t\Delta_1k = \Delta_1^{-1}t\Delta_1$, hence Δ_1 is a special word by lemma 8. This proves lemma 9.

Lemma 10. Let H_1 be the subgroup of $G = G_E$ generated by

$$(1) \quad S \cup Q \cup \{t, k\}$$

Then H has the presentation:

$$(2) \quad (S \cup Q \cup \{t, k\} \mid k^{-1}\Sigma^{-1}t\Sigma k = \Sigma^{-1}t\Sigma, \text{ where } \Sigma \text{ is a special word and } \Sigma = q \text{ in } T.)$$

Proof. Let H'_1 be a group having the presentation above. Let w be a word on the letters (1). Suppose $w = 1$ in H'_1 , then $w = 1$ in H_1 by lemma 7 on page 23 of Britton 63. Suppose $w = 1$ in H_1 and $w \neq 1$ in H'_1 . We shall prove that this leads to a contradiction. We may choose w to have as few occurrences of k as possible. From lemma 6 and 7 follows that w must contain at least two occurrences of t and two occurrences of k .

Since $w = 1$ in H_1 we also have $w = 1$ in G . By Britton's lemma we have that w must contain a pinch of the form $k^{-\epsilon}Ck^{\epsilon}$ ($\epsilon = \pm 1$), where $C \in A(k) = B(k)$. Hence $k^{-\epsilon}Ck^{\epsilon} = C$ in H_1 . Suppose $k^{-\epsilon}Ck^{\epsilon} = C$ is true in H'_1 . Then $w = w'$ in H'_1 where w' is obtained from w by replacing $k^{-\epsilon}Ck^{\epsilon}$ in w by C . Then $w' = 1$ in H_1 and $w' = w \neq 1$ in H'_1 .

Since w' contains a smaller number of occurrences of k than w , this contradicts the fact that w is minimal. Hence $k^{-1}Ck = C$ in H_1 but $k^{-1}Ck \neq C$ in H'_1 .

Let $w'' = k^{-1}CkC^{-1}$. Then $w'' = 1$ in H_1 and $w'' \neq 1$ in H'_1 , moreover w'' contains just two occurrences of k . The group G of Britton (see p.28 of Britton 63), is obtained from G_2 by Britton extension as follows: $G_1 = (G_2; t \mid t^{-1}l_i t = l_i, t^{-1}yt = y, (i = 1, 2, \dots, P))$ $G = (G_1; k \mid k^{-1}r_i k = r_i, k^{-1}xk = x, k^{-1}(q^{-1}tq)k = q^{-1}tq) \quad i = 1, 2, \dots, P)$. Here t is first added as stable letter and then k is added as stable letter in the second extension. But we may as well obtain G from G_2 by first adding k as stable letter and then t in the next extension;

$$\begin{aligned} G'_1 &= (G_2; k \mid k^{-1}r_i k = r_i, k^{-1}xk = x, (i = 1, 2, \dots, P)) \\ G_2 &= (G_1; t \mid t^{-1}l_i t = l_i, t^{-1}yt = y, t^{-1}(qkq^{-1})t = qkq^{-1} \\ &\quad (i = 1, 2, \dots, P)) \end{aligned}$$

Among the words w on the letters $S \cup Q \cup \{t, q\}$ choose a w such that

1. $w = 1$ in H_1
2. $w \neq 1$ in H'_1
3. w contains just two occurrences of k .
4. w contains as few occurrences of t as possible.

We can now prove that w contains just two occurrences of t , by using the fact that G is a Britton extension of G'_1 . The proof is analogous to the proof above. Hence w contains no more than two occurrences of k and two occurrences of t . By using Britton's lemma and lemma 9 it is easy to prove that $w = 1$ in H'_1 , also. Hence we have a contradiction which proves

lemma 10.

Lemma 11. Let $k_0 = hkh^{-1}$ and let $t_0 = q_1^{-1}h^{-1}thq_1$. Then the subgroup of G generated by $a_1, a_2, \dots, a_m, t_0, k_0$ has the presentation:

$$(a_1, a_2, \dots, a_m, t_0, k_0 \mid k_0^{-1}(w^{-1}t_0w)k_0 = w^{-1}t_0w \\ w \text{ positive word on } \{a_1, \dots, a_m\} \text{ and } w \in E.)$$

Proof. $hq_1wh = q$ in $T = T_E$ by lemma 4 we obtain lemma 11 from lemma 10, since $k_0^{-1}w^{-1}t_0wk_0 = w^{-1}t_0w$ iff $k_0^{-1}(hq_1wh)^{-1}t_0(hq_1wh) = (hq_1wh)^{-1}t_0(hq_1wh)$.

Note that our lemma 11 corresponds to lemma 5.1 p.473 in Higman 1961 and to lemma 18 p.335 in Shoenfield 1967.

§ 6. Completion of the proof.

Given a word w on the letters $\{a_1, a_2, \dots, a_m\}$, let w_u be the result of replacing all occurrences of a_i in w by u_i for all $i = 1, 2, \dots, m$, simultaneously. Moreover w_b is defined to be a word on the letters $\{b_1, b_2, \dots, b_m\}$ in the same way. (Example $(a_1a_2a_3^{-1})_u = u_1u_2u_3^{-1}$, and $(a_1a_2a_3^{-1})_b = b_1b_2b_3^{-1}$). Let

$$R = (u_1, u_2, \dots, u_m \mid w_u = 1, (w \in E))$$

Let G_4 be the free product of $G = G_E$ and R . i.e.

$$G_4 = G * R.$$

We shall now consider a sequence of extensions of G_4 and prove that we have a sequence of Britton extensions.

$$G_5 = (G_4; b_1, b_2, \dots, b_m \mid b_i^{-1} u_j b_i = u_j, b_i^{-1} a_j b_i = a_j, \\ b_i^{-1} k_0 b_i = k_0 u_i^{-1}, \text{ for all } i, j = 1, 2, \dots, m)$$

$$G_6 = (G_5; d \mid d^{-1} k_0 d = k_0, d^{-1} a_i b_i d = a_i, \text{ all } i = 1, 2, \dots, m)$$

$$G_7 = (G_6; p \mid p^{-1} t_0 p = t_0 d, p^{-1} k_0 p = k_0, p^{-1} a_i p = a_i \\ \text{all } i = 1, 2, \dots, m)$$

Lemma 12. The subgroups $\langle a_1, \dots, a_m, k_0 \rangle_4$ and $\langle a_1, \dots, a_m, t_0 \rangle_4$ of G_4 are free on the displayed generating sets.

Remark. The subscript 4 indicates that the subgroups are computed in G_4 , i.e., only the relations of G_4 are used.

Proof. Lemma 12 is an immediate consequence of lemma 11.

Lemma 13. G_5 is a Britton extension of G_4 .

Proof. By lemma 12 we have

$$A(b_i) = \langle u_1, \dots, u_m, a_1, \dots, a_m, k_0 \rangle_4 \\ = \langle u_1, \dots, u_m, a_1, \dots, a_m \rangle_4 * \langle k_0 \rangle_4$$

hence the mapping $\varphi_i(u_j) = u_j, \varphi_i(a_j) = a_j$ and $\varphi_i(k_0) = k_0 u_i^{-1}$ may be extended to a homomorphism of $A(b_i)$ onto $B(b_i)$. Since $B(b_i) = A(b_i)$ and the map $\psi_i(u_j)$ given by $\psi_i(u_j) = u_j, \varphi_i(a_j) = a_j$ and $\psi_i(k_0) = k_0 u_i$ can be extended to a homomorphism of $B(b_i)$ into $A(b_i)$, and since the extension of ψ_i is the inverse of the extension φ_i we have that φ_i is an isomorphism. This shows that the isomorphism condition holds and

lemma 13 is proved.

Lemma 14. G_6 is a Britton extension of G_5 .

Proof. Let $G'_5 = (G_5 \mid b_i = u_i = 1)$. Then G'_5 is isomorphic to G_4 . Since $\langle a_1, a_2, \dots, a_m, k_0 \rangle_4$ is a free group on the displayed letters, $\langle a_1, a_2, \dots, a_m, k_0 \rangle$ and $\langle a_1 b_1, a_2 b_2, \dots, a_m b_m, k_0 \rangle$ are also free groups on the displayed letters in G'_5 and then also in G_5 . This makes it easy to prove that the isomorphism condition holds. The rest of the proof of lemma 14 is left to the reader.

Definition. Let G'_4, G'_5, G'_6, G'_7 be the result of deleting all the relations of the form $w_u = 1$, $w \in E$ in the presentations of G_4, G_5, G_6, G_7 , respectively.

Lemma 15. Let the presentation of H be an extension of the presentation of G'_6 . Let w be a word on the letters $\{a_1, a_2, \dots, a_m\}$. Then $w_u = 1$ in H iff $k_0^{-1} w_b k_0 = w_b$ in H .

Proof. Since $b_i^{-1} k_0 b_i = k_0 u_i^{-1}$ is a defining relation in G'_5 , and hence also in G'_6 and H , we have that $k_0^{-1} b_i k_0 = b_i u_i$. Since b_i and u_j commute, we also have $k_0^{-1} w_b k_0 = w_b w_u$. Hence if $w_u = 1$ in H then $k_0^{-1} w_b k_0 = w_b$ in H . Moreover if $k_0^{-1} w_b k_0 = w_b$ then $w_b = w_b w_u$, hence $w_u = 1$. This proves lemma 15.

Lemma 16. Let the presentation of H be an extension of the presentation of G'_6 . Let w be a word on the letters

$\{a_1, a_2, \dots, a_m\}$, and suppose $k_o^{-1} w^{-1} t_o w k_o = w^{-1} t_o w$ in H . Then $k_o^{-1} w^{-1} t_o d w k_o = w^{-1} t_o d w$ in H iff $w_u = 1$ in H .

Proof. Since a_i and b_j commute in H , and since $da_i = a_i b_i d$ in H , we have that

$$(8) \quad w^{-1} t_o d w = w^{-1} t_o w w_b d .$$

Here k_o commutes with d , and we have supposed that k_o commutes with $w^{-1} t_o w$. Hence

$$\begin{aligned} k_o^{-1} w^{-1} t_o d w k_o &= w^{-1} t_o d w && \text{(by (8))} \\ &\Downarrow \\ k_o^{-1} (w^{-1} t_o w) w_b d k_o &= w^{-1} t_o w w_b d && \text{(by the hypothesis)} \\ &\Downarrow \\ (w^{-1} t_o w) k_o^{-1} w_b k_o d &= (w^{-1} t_o w) w_b d && \text{(by cancelation)} \\ &\Downarrow \\ k_o^{-1} w_b k_o &= w_b && \text{by lemma 15.} \\ &\Downarrow \\ w_u &= 1 . \end{aligned}$$

This proves lemma 16.

Lemma 17. G_7 is a Britton extension of G_6 .

Proof. The group $A(p)$ has the presentation

$$(9) \quad (a_1, a_2, \dots, a_m, t_o, k_o \mid k_o^{-1} w^{-1} t_o w k_o = w^{-1} t_o w, w \in E)$$

by lemma 11, since G is embedded in G_6 .

In order to prove that the isomorphism condition holds, it is sufficient to prove that the group $B(p) = \langle a_1, a_2, \dots, a_m, t_d, k_o \rangle_6$

(where $t_d = t_o d$) has the presentation

$$B' = (a_1, a_2, \dots, a_m, t_d, k_o | k_o^{-1} w^{-1} t_d w k_o = w^{-1} t_d w, \quad w \in E)$$

Let $w \in E$. Then $k_o^{-1} w^{-1} t_o w k_o = w^{-1} t_o w$ in G_6 by lemma 11 and $w_b = 1$ in G_6 by definition. Since the presentation of G_6 is an extension of the presentation of G'_6 , we have by lemma 16 that $k_o^{-1} w^{-1} t_d w k_o = w^{-1} t_d w$ in G_6 and hence also in $B(p)$. Let W be a word on the generators of B' . Suppose $W = 1$ in B' . Then $W = 1$ in $B(p)$ since $B(p)$ satisfies the defining relation of B' . Suppose $W = 1$ in $B(p)$. Let

$$G''_6 = (G_6 \mid u_i = 1, \quad b_i = 1, \quad d = 1, \quad t_d = t_o)$$

Then G''_6 is a homomorphic image of G_6 , and G'' is isomorphic to G . Hence $W = 1$ in G''_6 . Let W' be the result of replacing t_d by t_o in W . Then $W' = 1$ in G''_6 , but W' is a word on the generators of G and G is isomorphic to G''_6 . Hence $W' = 1$ in G and so $W' = 1$ in $A(p)$. But $A(p)$ and B' are isomorphic by definition. Hence $W = 1$ in B' . This proves lemma 17.

Lemma 18. G'_7 is isomorphic to G_7 , G'_7 is finitely presented and R is embedded in G'_7 .

Proof. Recall that G'_7 is the result of deleting all the relations of the form $w_u = 1$, ($w \in E$) in G_7 . We shall prove that all the relations $w_u = 1$, ($w \in E$) can be proved in G'_7 . Let $w \in E$. Then

$$(10) \quad k_o^{-1} w^{-1} t_o w k_o = w^{-1} t_o w_o$$

Hence

$$(11) \quad p^{-1}(k_o^{-1}w^{-1}t_o w k_o)p = p^{-1}(w^{-1}t_o w_o)p$$

$$(12) \quad k_o^{-1}w^{-1}t_o d w k_o = w^{-1}t_o d w, \text{ since } p \text{ commutes with all letters in (10) except } t_o \text{ and } p^{-1}t_o p = t_o d.$$

The presentation of G'_7 is an extension of the presentation of G'_6 . Hence by (10), (12) and lemma 16 we have $w_u = 1$ in G'_7 , as desired. Hence G'_7 and G_7 are isomorphic and G'_7 is obviously finitely presented. Moreover, since G_7 is obtained from G_4 by a sequence of Britton extensions, we have that R is embedded in G_7 . Hence R is embedded in G'_7 . This completes the proof of lemma 18. Since R was an arbitrary recursively presented group, this also proves lemma 1.

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