GEOMETRY OF THE FANO THREEFOLD OF DEGREE 10
OF THE FIRST TYPE

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§ 1. Preliminaries.

(1.1) DEFINITION. The smooth irreducible projective variety X of dimension three over the field of complex numbers is called a Fano threefold, if the anticanonical sheaf $-K_X$ (resp. the anticanonical divisor $-K_X$) is ample.

Let X be a Fano threefold and let K_X be the canonical divisor of X . From the Riemann-Roch formula and from the Kodaira vanishing theorem it follows, that:

(a) $h^i(X, \mathcal{O}_X(-m.K_X)) = 0$, for $i=1,2$ and for any m - integer; or, for $i > 0$, $m \geq 0$; or, for $i < 3$, $m < 0$;

(b) $h^0(X, \mathcal{O}_X(-m.K_X)) = (m(m+1)(2m+1).(K_X)^3)/12 + 2m + 1$.

In particular, $h^i(X, \mathcal{O}_X) = 0$ for $i > 0$, and $h^0(X, -K_X) = \dim | -K_X | + 1 = -(K_X)^3/2 + 1$.

(1.2) DEFINITION. The integer $g = g(X) = -(K_X)^3/2 + 1$ is called a genus of the Fano threefold X .

The Fano variety X is called a Fano variety of the main series, if the anticanonical divisor $-K_X$ is very ample. Therefore the linear system $| -K_X |$ gives an embedding of X in P^{g+1} as a subvariety of degree $2.g - 2$, ($g \geq 3$). The intersections of X with codimension 2 subspaces $P^{g-1} \subset P^{g+1}$ are canonical curves of degree $2.g - 2$ and of arithmetical genus $p_a = g(X)$.

The Fano variety X is called a (Fano) variety of the 1st kind, if $\text{Pic}(X) \simeq \mathbb{Z}$, where $\text{Pic}(X)$ is the Picard group of X . The Fano varieties of the 1st kind (i.e., with $\text{Pic} \simeq \mathbb{Z}$) are classified by V.A.Iskovskikh ([I1], [I2], [I4]) after G.Fano ([F1], [F2]) and L.Roth ([R]).

In the papers [MM1] , [MM2] of S.Mori and S.Mukai is given a list of the Fano threefolds of $\text{rank Pic} \geq 2$ (i.e., the Fano 3-folds of the 2nd kind). From a birational point of view both kinds of Fano threefolds are interesting, but non-trivial questions, connected with the problem of non-rationality, arise from the Fano 3-folds with $\text{Pic} \simeq \mathbb{Z}$. Such non-trivial examples are the cubic hypersurface in P^4 ([CG] , [B1]) , the quartic double solid ([W] , [C] , [T1] , [T2] , [V]) , the complete intersection of three quadrics in P^6 ([B] , [TN] , [D]) , the two-sheeted covering of the cone over the Veronese surface ([T3] , [H]) , the quartic hypersurface in P^4 ([IM] , [I3] , [L]) , etc.

(1.3) The Fano threefolds of the main series and of the 1st kind have genus $g \leq 12$. The 3-folds of $g = 7$ and of $g \geq 9$ are rational . The variety of $g = 8$ is interpreted as an intersection of the Grassmannian of the lines in P^5 , embedded by Plücker in P^{14} , with a codimension 5 subspace $P^9 \subset P^{14}$; the last is birational to a smooth cubic hypersurface in P^4 . The variety $X_8 \subset P^6$ of genus 5 is a complete intersection of three quadrics in P^6 , etc. (see [I1] , [B2]). In particular, the Fano varieties $X_{10} \subset P^7$ of genus 6 are the last non-trivial examples of Fano threefolds for which $\text{Pic} \simeq \mathbb{Z}$ and which have very ample anticanonical class. They are divided into two types (see [G]) :

1st type. An intersection of the Grassmannian of the lines in P^4 , embedded by Plücker in P^9 , with a codimension 2 subspace $P^7 \subset P^9$ and with a quadric ;

2nd type. An intersection of a cone (in P^7) over the threefold $X_5 \subset P^6$ with a quadric, where $X_5 \subset P^6$ is an intersection of the Grassmannian of the lines in P^4 , embedded by Plücker in P^9 , with a codimension 3 subspace $P^6 \subset P^9$.

It is known, that the two types of Fano threefolds of genus $g = 6$ belong to one type of deformations ($[G], [I4]$).

The Fano threefold of genus $g = 6$ (resp. of degree $d = 2.g - 2 = 10$) of the first of the abovementioned types is studied in the present paper.

(1.4) The Fano threefold of degree 10 of the 1st type is represented as a 3-dimensional family of lines, lying in P^4 (see for ex. §(1.5) and § 2). In § 2 are derived some elementary incidences, which are fulfilled for the lines $l \subset P^4$, belonging to $X = X_{10}$ (see Propositions (2.2.2), (2.2.3), (2.2.4)). It is shown also the existence of a special hypersurface $R = R_6 \subset P^4$ such that the lines $l \subset P^4$, which belong to X , are at least three-tangents to R_6 (see (2.2.5) and (2.2.6)). In § 3 is considered a two-dimensional analogue of the threefold X_{10} . In fact, this is the Del Pezzo surface $S = S_{2.2}$ in P^4 , represented as an intersection of the Grassmannian $G(2,4) = G(1:P^3)$, embedded as a Plücker quadric in P^5 , with a hyperplane $P^4 \subset P^5$ and with a quadric (see Example 3.2.). In this case, the lines $l \subset P^3$, which belong to S , are at least bi-tangents to a Kummer surface in P^3 (see (3.2.7)).

In the rest of § 3 is shown, that the propositions from § 2 can be extended in an appropriate way (see Cor. (3.3.3)).

§ 5, 6, 7 are devoted to a detailed study of the family $\mathcal{C}_e^3$ of rational normal cubic curves, lying on the three-fold $X = X_{10}$. First, in § 5 a geometrical investigation of the possible bundles of lines in P^4 , which represent the cubics $C \in \mathcal{C}_e^3(X)$, is made (see § 5.2.). It is shown also the existence of a well-defined involution

$\sigma : \mathcal{C}_e^3 \longrightarrow \mathcal{C}_e^3$, inducing a natural embedding of the factorfamily $\mathcal{C}_0^3 = \mathcal{C}_e^3 / \sigma$ in the Grassmannian $G = G(2,5)$ (see Cor. (5.3.2)). It is shown in § 6, that the factorfamily $\mathcal{C}_0^3$ can be embedded in a special projective bundle over the Grassmannian $G = G(2,5)$ as a degeneration locus (more precisely, as a second determinantal) of an appropriate net of quadrics over the Grassmannian (see § 6.3., § 6.4. and Prop. (6.4.4)). The essential results in § 7 are the Corollaries (7.2.4) and (7.2.6), the Proposition (7.3.2) and their geometrical description in the Corollary (7.3.9). They are technical consequences from the results in § 6, but under some additional open conditions on the variety X_{10} (see (6.3.5) (i), (ii) and (7.1.2) (iii)). In particular, it is shown, that the family $\mathcal{C}_e^3$ of rational normal cubic curves on X_{10} can be embedded naturally in the projectivized bundle $P_Z(\sum^*) = G_Z(4, \sum)$ as a zero-scheme of some section (see Prop. (7.3.2)) ; see

also (6.3.2) and (6.3.3) for the definitions of Z and $\cdot$.

The geometrical description of the embedding

$\mathcal{C}^3 \hookrightarrow P_Z(\Sigma^*)$, given in Corollary (7.3.3) is used in the formulation and in the proof of the Tangent Bundle Theorem (T.B.T.) for the family $\mathcal{C}^3$, stated and proven in § 8 (see Proposition (8.1.9) and Theorem (8.2.12)).

The last describes the tangent bundle of the family $\mathcal{C}^3$ by means of some standard bundles, defined over $\mathcal{C}^3$ (see (8.2.12) and Remark (8.2.10)). In fact, the T.B.T., in its formulation presented in the paper, describes the disposition of the tangent spaces at the points of the Abel-Jacobi image of the family $\mathcal{C}^3$, inside the Intermediate Jacobian $J(X)$ of X .

In addition, some known results on the family of conics on X_{10} are followed out in § 4 and in § 9. The original approach to the family of conics, which inspired the author to go on further and to apply an appropriate technique to the family of rational normal cubics on X_{10} , is due to D. Logachev (see [L1], [L2] and also (9.2.1), (9.2.5), (9.3.6), (9.3.8), (9.3.10), (9.3.11)).

The use and the possible applications of the T.B.T. to the study of the principally polarized Intermediate Jacobian of X (see for ex. [CG]) are made in the author's comments in § 9.1.

(1.5) Let $V = V_5$ be the five-dimensional space $\mathbb{C}^5$ and let $G = G(2,5) = G(2,V) = G(1:P(V))$ be the Grassmannian of the two-dimensional subspaces of V (resp. of lines in $P(V) = P^4$). There is a standard Plücker embedding:

$$Pl: G = G(2,V) \hookrightarrow P^9 = P(\wedge^2 V),$$

which can be described as follows:

Let $0 < L < V$ be a 2-dimensional subspace of V (i.e. $L \in G$), and let $L = \mathbb{C} \cdot u + \mathbb{C} \cdot v$ for some basis $\{u,v\}$ of L . Then, by definition, $Pl: L \mapsto$ (the 1-dim. subspace $0 < \mathbb{C} \cdot u \wedge v < \wedge^2 V$), i.e. $Pl(L) \in G(1, \wedge^2 V) = P(\wedge^2 V) = P^9$. It is easy to see that the definition of $Pl(L)$ is independent of the choice of the basis $\{u,v\}$ of L .

Let $P^4 = P(V)$ has homogeneous coordinates $x_0, x_1, \dots, x_4$, and let $e_0, e_1, \dots, e_4$ the corresponding dual basis of vectors in V . Let $e_{ij} = e_i \wedge e_j$, $0 \leq i < j \leq 4$ and let $\{x_{ij}\}$ be the corresponding to $\{e_{ij}\}$ dual basis. The embedded Grassmannian $G = G(2,V) \subset P(\wedge^2 V)$ is described as the set of all classes of decomposable 2-forms (bi-vectors) in $\wedge^2 V$ modulo a multiplication with an element of $\mathbb{C}^*$. Therefore, the ideal $I(G)$ of $G \subset P^9$ is generated by the five Plücker quadrics $Pl_m = x_{ij} \cdot x_{kl} - x_{ik} \cdot x_{jl} + x_{il} \cdot x_{jk}$, $0 \leq m \leq 4$, $0 \leq i < j < k < l \leq 4$ and $i, j, k, l \neq m$. The Plücker embedding represents the Grassmannian G as a smooth subvariety $G \subset P^9$ of degree 5 and of dimension 6.

Let $P^7 \subset P^9$ be a codimension 2 subspace of $P^9 = P(\wedge^2 V)$, which is in general position with the embedded Grassmannian G , let H_1 and H_2 be two hyperplanes in P^9 such that $H_1 \cap H_2 = P^7$ and let Q be a general (esp. smooth) quadric in P^9 (or, equivalently, in P^7). According to the

choice of the P^7 and Q , the variety $X = X_{10} = G \cap P^7 \cap Q = G \cap H_1 \cap H_2 \cap Q$ is smooth, and it is easy to see, that X is a Fano variety of dimension 3, of degree 10 and of genus 6; the anticanonical divisor $-K_X$ represents the hyperplane section of the embedded $X = X_{10} \subset P^7$.

§ 2. Some elementary incidences, connected with the Schubert calculus on the Grassmannian of the lines in P^4 .

2.1. It is well known, that the integer homologies of the Grassmannian have no torsion and are generated freely by the homology classes of the Schubert cycles (see for ex. [GH] or [F]). In particular, the Schubert classes $\sigma_{a,b}$, $3 \geq a \geq b \geq 0$ (of real codimension $2 \cdot (a+b)$) of the Grassmannian $G = G(2,5) = G(2,V)$ are described as follows:

Let $P^4 = P(V)$ for $V = \mathbb{C}^5$ and let $P^0 \subset P^1 \subset P^2 \subset P^3 \subset P^4 = P(V)$ be a flag in $P(V)$. Using the natural isomorphism: $G(2,V) \simeq G(1:P(V))$, which identify the two-dimensional subspace $0 < L \subset V$ and the projective line $l = P(L) \subset P(V)$, we define $\sigma_{a,b} = \{ l : l \text{ is a line in } P^4 \text{ and } l \cap P^{3-a} \neq \emptyset, l \subset P^{4-b} \}$, where $3 \geq a \geq b \geq 0$.

The intersections of cycles in the Chow ring $A_*(G)$ can be described by the Pieri-formula (see for ex. [GH, ch.I, §5]):

$$\sigma_{a,b} \cdot \sigma_{c,0} = \sum_{d+e=a+b+c, d \geq a \geq e \geq b} \sigma_{d,e}.$$

The intersection of general cycles $\sigma_{a,b} \cdot \sigma_{c,d}$ can be obtained from the Pieri-formula by the additive properties of the intersections of cycles (see for ex. the Giambelli formula in [GH, ch.I, §5]).

2.2.

(2.2.1) Let $V = V_5$, $P^4 = P(V)$, $P^7 \subset P^9 = P(\wedge^2 V)$, H_1 , H_2 ($H_1 \cap H_2 = P^7$), $G = G(2, V) = G(1: P(V))$, Q and $X = X_{10} = G \cap P^7 \cap Q = G \cap H_1 \cap H_2 \cap Q$ be as above.

Let $W = W_5 = G \cap P^7 = G \cap H_1 \cap H_2$. The fourfold W is represented by the homology class $\sigma_{1,0} \cdot \sigma_{1,0}$ in the Chow ring $A.(G)$, i.e. $W = \sigma_{1,0} \cdot \sigma_{1,0} = \sigma_{2,0} + \sigma_{1,1}$ (here $=$ means the homology equivalence of cycles). In fact, the homology cycle of $\sigma_{1,0}$ represents the hyperplane section according to the Plücker embedding, so the quadrics are represented by the cycle $2 \cdot \sigma_{1,0}$.

Let x be a general point of P^4 . Using the fact, that the set of the lines in P^4 , which pass through x , is represented by the cycle $\sigma_{3,0}(x) = \{l \in G : x \in l\}$ (x is the P^0 in the flag, see above), and the homological equivalence $Q = 2 \cdot \sigma_{1,0}$, we derive the following:

(2.2.2) PROPOSITION.

(1) The set of the lines in P^4 , which belong to W and which pass through a general point $x \in P^4$ is homologous to the cycle $\sigma_{3,2}$, which represents a plane bundle of lines with a center the point $x \in P^4$ (i.e. $\sigma_{3,2}$ is a Grassmann line in G);

(2) There are exactly two lines in P^4 , which belong to X and which pass through the general point $x \in P^4$.

Proof. It rests only to apply the formulae for the intersections of cycles on G .

Let now $P^3 \subset P^4$ be a general hyperplane. Using the fact, that the set (the Grassmannian) of all the lines in the

chosen P^3 is represented by the cycle $\sigma_{1,1}(P^3) = \{1 \in G : 1 \subset P^3\}$ (P^3 is the P^3 in the flag, see above), we obtain in a similar way:

(2.2.3) PROPOSITION.

(1) The set of the lines, which belong to W and which lie in a fixed generally chosen hyperplane $P^3 \subset P^4$, is homologous to the cycle $\sigma_{3,1} + \sigma_{2,2}$ and represents a two-dimensional quadric $Q(P^3) = S(P^3)$, embedded in G ;

(2) The set of the lines, which belong to X and which lie in a fixed generally chosen $P^3 \subset P^4$ is homologous to the cycle $4 \cdot \sigma_{3,2}$ and represents an elliptic curve $C(P^3)$ of degree 4 in G . The curve $C(P^3)$ is an intersection of the quadric $Q(P^3)$ with the quadric Q in the 3-dimensional projective space $Q(P^3) = \text{Span } Q(P^3)$.

From another point of view, the quadric surface $Q(P^3)$ is an intersection of the four-dimensional Plücker quadric $G(1:P^3)$, embedded in $G(1:P^4) \subset P^9$ by the natural embedding, corresponding to the embedding $P^3 \subset P^4$, with the subspace $P^7 \cap P^9$. From the last it follows immediately, that, if P^3 is chosen sufficiently general, then the quadric $Q(P^3)$ and the curve $C(P^3)$ are smooth subvarieties of G . As a set of lines in $P^3 \subset P^4$, the quadric $Q(P^3)$ is described as the set of all the lines in P^3 , which intersect the both of a given pair of lines l_1 and l_2 in P^3 , which does not intersect between them.

Let now $l \subset P^4$ be a generally chosen line, and let $\sigma_{2,0}(l) = \{m \in G : m \cap l \neq \emptyset\}$ (l is the P^1 in the flag, as above). In a similar way, we can prove the analogues

of the propositions above, namely:

(2.2.4) PROPOSITION. Let l be a general line in P^4 . Then:

(1) The set of all the elements of W , which intersect as lines in P^4 the line l , is homologous to the Grassmann cycle $2 \cdot \sigma_{3,1} + \sigma_{2,2}$ and represents a linearly normal surface S_1 of degree 3; more precisely, the surface S_1 is embedded as a rational normal ruled surface $S_1 = F_1$ in G by the linear system $|C_0 + 2 \cdot f|$, where C_0 is the (-1) -section and f is the fiber of F_1 ;

(2) The set of the elements of X , which intersect as lines in P^4 the line l , is homologous to the cycle $6 \cdot \sigma_{3,2}$ and represents a linearly normal curve C_1 of degree 6 and of (arithmetic) genus 2.

(2.2.5) REMARKS.

(1) It is clear, that the curve C_1 is an intersection of the surface S_1 with the quadric Q in the four-dimensional projective space $P^4(l) = \langle S_1 \rangle = \text{Span } S_1$. The hyperelliptic linear system g_2^1 on the curve C_1 is described in concrete geometrical terms as follows:

Let x be a general point of l . There are exactly two lines $l'(x)$ and $l''(x)$, which are elements of X and which pass through x (see Prop. (2.2.2)), in particular $l'(x)$ and $l''(x)$ belong to C_1 . The linear system $g_2^1 = |l'(x) + l''(x)|$ maps C_1 twice on the projective line $l \subset P^1$; the point $x \in l$ has as preimages the two lines $l'(x)$ and $l''(x)$ as elements of C_1 .

(2) As in the previous case, it is easy to see that the curve C_1 is non-singular under the general choice of the line $l \subset P^4$.

(3) Let l be chosen so that the curve C_1 is smooth. Then the hyperelliptic projection g_2^1 for C_1 has exactly 6 ramification points: $x_1, x_2, \dots, x_6$; the last means, that the corresponding lines $l'(x_i)$ and $l''(x_i)$ (see (1)) coincide, $i = 1, 2, \dots, 6$.

The arose situation outlines the existence of a hypersurface $R = R_6 \subset P^4$ of degree 6, which is defined as the closed completion of the set of all the points x_i , $i = 1, 2, \dots, 6$ of ramification of the hyperelliptic systems $g_2^1 = g_2^1(l)$ (see (3)) corresponding to the lines $l \in X$, for which C_1 is smooth.

(2.2.6) DEFINITION - COROLLARY. The closed completion $R = \overline{R^0}$ of the set $R^0 = \{x \in P^4 : \text{there is exactly one line (a "double" line); which pass through } x \text{ and which belongs to } X \text{ as an element of } G\}$ is a hypersurface of degree 6 in P^4 . We call $R = R_6$ the ramification hypersurface for X .

§ 3. The elements of X as three-tangents

to the ramification hypersurface $R \subset P^4$.

3.1. Let $l \subset P^4$ be a line, which belongs, as a point of the Grassmannian, to X . We shall show, that (when $l \in X$ is chosen sufficiently general) the line l has three points of a simple contact with the ramification hypersurface R (see (2.2.6)). The point here is not in proving this fact (by the way proving the threetangency is not so difficult), but in the suggestion that the geometrical interpretation of the elements of the subvariety X of the Grassman-

nian as lines, which are totally tangent to some hypersurface, is somewhat natural. The following example clarifies the situation in one more simple case.

3.2. Example.

Let $P^3 = P(\mathbb{C}^4)$ be the projective 3-space, and let $G(2,4) = G(2, \mathbb{C}^4) = G(1:P^3)$ be the Grassmannian of the lines in P^3 , embedded as a Plücker quadric in the projective 5-space $P^5 = P(\wedge^2 \mathbb{C}^4)$. Let $S = G(1:P^3) \cap H \cap Q$ be a smooth intersection of the embedded Grassmannian with the hyperplane $H = P^4 \subset P^5$ and with the quadric $Q \subset P^5$. Using, as in the case of X , the Schubert representation of S , we obtain easily:

(3.2.1) PROPOSITION.

(1) There are exactly two projective lines which pass through the general point $x \in P^3$ and which, as points of the Grassmannian $G(1:P^3)$, belong to S .

(2) There are exactly two lines, which lie in the generally chosen plane $P^2 \subset P^3$ and which belong to S as points of the $G(1:P^3)$.

(3) Let $l \subset P^3$ be a general line. Then the set $C_1 = \{ m \subset P^3 \text{ -a line : } m \in S \text{ and } m \cap l \neq \emptyset \}$, is an elliptic curve of degree 4 (the curve C_1 is, in the general case, a non-singular, though special element of the anticanonical system $|-K_S|$).

(3.2.2) The four points of ramification x_1, x_2, x_3 and x_4 of the hyperelliptic linear system $g_2^1 = \{ l'(x) + l''(x) \}$ (here $l'(x)$ and $l''(x)$ are the two elements of S , which pass through the point $x \in l$, see Prop.(3.2.1) (1)),

define, just as in the case of X (see Remark (2.2.5) (3)), a surface $R = R_4 \subset P^3$ of degree 4 - the ramification (hyper)surface for S .

(3.2.3) Let now $P^2 \subset P^3$ be a sufficiently general plane. Let l' and l'' be the two lines in P^2 , which belong to S as Grassmann points (see Prop. (3.2.1) (2)). It is easy to see, that there exists a rational map $\varphi : S \dashrightarrow P^2$ such that:

(1) If $x \in P^2 \setminus \{l' \cup l''\}$, and $x \notin R \cap P^2$, then $\varphi^{-1}(x) = \{l'(x), l''(x)\}$, where $l'(x)$ and $l''(x)$ are as above;

(2) If $x \in P^2 \cap R$ and $x \notin l' \cup l''$, then $\varphi^{-1}(x) = l(x)$, where $l(x) = l'(x) = l''(x)$ is the unique element of S , which passes through x as a line in P^3 ;

(3) If $x \in l'$ (or if $x \in l''$), but $x \neq p = l' \cap l''$, then $\varphi^{-1}(x) = \{l', l''(x)\}$ (or resp. $\varphi^{-1}(x) = \{l'', l'(x)\}$);

(4) If $x = p = l' \cap l''$, then $\varphi^{-1}(x) = \{l', l''\}$.

(3.2.4) It is clear, that there exists a regularization π of the rational map φ , which makes the following diagram commutative:

$$\begin{array}{ccc} \tilde{S} & & \\ \sigma \downarrow & \searrow \pi & \\ S & \xrightarrow{\varphi} & P^2 \end{array}$$

(3.2.5) In the diagram above the π -preimage of $l' \cup l''$ is an union of curves $l'_1 + l'_2 + l''_1 + l''_2$ such that :

(i) If $\pi^{-1}(p) = \{p_1, p_2\}$, then $l'_1 \cap l''_1 = p_1$, $l'_2 \cap l''_2 = p_2$, $l'_1 \cap l''_2 = \emptyset$, $l''_1 \cap l'_2 = \emptyset$;

(ii) The map $\sigma : \tilde{S} \longrightarrow S$ is a product of two σ -processes, which blow down the one of the two pairs of

non-intersecting lines (for ex. the lines l'_1 and l''_2) to a pair of non-singular points ; moreover, the lines l'_1 and l''_2 are, of course, (-1) -curves on S .

It is clear now, that the lines l' and l'' are bi-tangents to the ramification curve $R \subset P^2$ of π .

(3.2.6) There are exactly 56 lines on the surface $\tilde{S}$, which are (-1) -curves and which are also the 56 preimages of the 28 bi-tangent lines to the plane quartic curve of ramification $R \subset P^2$; that is, the surface S is a Del Pezzo surface, which is obtained from the projective plane after blowing-up of 7 points in the general position with respect to the lines, conics and cubics on the plane. On the other hand, there are exactly 16 lines ((-1) -curves) lying on the surface S .

Let $y_1, y_2, \dots, y_7$ be the seven points of the projective plane $P = P^2$, after blowing-up of which is obtained the surface $\tilde{S}$ and let the lines l'_1 and l''_2 (see above) be the preimages of the points y_6 and y_7 . Then the 16 lines on S correspond to: (1) the conic, which pass through the points $y_1, y_2, \dots, y_5$; (2) the exceptional divisors over the points $y_1, y_2, \dots, y_5$; (3) the lines through the 10 pairs of points (y_i, y_j) , $1 \leq i < j \leq 5$.

(3.2.7) The interpretation of the surface S as a subvariety of the Grassmannian $G(1:P^3)$ permits to get some interesting conclusions about the ramification surface $R \subset P^3$.

(1) Let l_0 be a line on the surface S . As a set of projective lines in P^3 , the grassmann line l_0 is described by a plane pencil of lines through a fixed point $x_0 = x(l_0)$. The plane of this pencil $P^2_0 = P^2(l_0) =$

$= (\bigcup \lambda : \lambda \subset P^3 \text{ - a line \& } \lambda \in l_0)$ stands in a special position with respect to the ramification surface $R \subset P^3$. As the line $l_0 \subset S$ intersects five other lines $l_1, l_2, \dots, l_5$ lying on the Del Pezzo surface $S = S_{2.2} = S_4 \subset P^4$, so there are five points $x_1, x_2, \dots, x_5$ in the plane $P_0^2 = P^2(l_0)$ each of them being the centre of the corresponding plane pencil of lines. In this case the analogue of the map φ from above $\varphi : S \dashrightarrow P_0^2$ is a birational morphism. More concretely, by definition, the formal preimage $\varphi^{-1}(x)$ of the point $x \in P_0^2$ is defined to be the set of all the elements of S , which pass through x as lines in P^3 . The regularization π of the rational map φ can be given by the commutative diagram

$$\begin{array}{ccc}
 \tilde{S} & & \\
 \sigma \downarrow & \searrow \pi & \\
 S & \xrightarrow{\varphi} & P_0^2 = P^2(l_0)
 \end{array}
 ,$$

where the regular map π is a composition of six blow-ups over the points x_i , $i = 0, 1, \dots, 5$, and the map $\sigma : \tilde{S} \rightarrow S$ blows down the line $l \subset S$, which is the preimage of the conic C in P_0^2 , passing through the five points $x_1, x_2, \dots, x_5$.

(2) The plane $P_0^2 = P^2(l_0)$ is totally bi-tangent to the ramification surface R , along the conic C . It follows immediately from the description above that the projective lines, which are elements of $S \subset G(1:P^3)$ are bi-tangents to the ramification surface R ; but not every bi-tangent to R belongs to S . For example, there are 28 bi-tangent

lines to R , which lie in the general plane $P^2 \subset P^3$, but only 2 of them belong to the surface $S \subset G(1:P^3)$. There are 16 planes, corresponding to the 16 lines on S , each plane being totally tangent to the surface $R \subset P^3$; moreover, the 16 points, which are centers of the corresponding pencils of lines in P^3 , are singular points for the quartic $R \subset P^3$ and there are exactly 6 singular points of R , which lie on a fixed totally tangent plane, etc.

(3.2.8) The facts, stated till now, give us a sufficient reason to claim, that the surface R is a Kummer surface in P^3 ; for example, there are 16 singular points on the quartic R , but the quartic surface with a maximal number of singular points is a surface of Kummer and the 16 singular points are simple nodes (see [B3, ch.VIII, ex.]).

The family of the bi-tangent lines to a Kummer surface splits into 28 (rational) components; 16 of them are isomorphic to the projective plane, and they are the dual planes to the 16 totally tangent planes to R ; the rest 12 components are isomorphic to Del Pezzo surfaces of degree 4 (as the surface S). The described set of 28 surfaces is a degeneration of the surface of the bi-tangents to a general quartic surface in P^3 , the last has been studied by Welters (see [W, part 1, §3]) and Tikhomirov (see [T1]).

3.3. The elements of X as three-tangent lines to R .

(3.3.1) Let l be a line in P^4 , which belongs, as a point of the Grassmannian $G = G(1:P^4)$, to X , and let $P^2 \subset P^4$ be a sufficiently general plane through the line l . It follows from the adjunction formula, that the surface $S(P^2) = \{m \subset P^4 - \text{a line} : m \cap P^2 \neq \emptyset\}$ is a K3-surface of degree 10, which represents a (special) hyperplane section of the threefold $X = X_{10}$. It is easy to see, that we can choose the line l and then the plane P^2 through l sufficiently general, such that the surface $S(P^2)$ be non-singular and moreover, such that the line l be the unique element of X , which lies on the plane P^2 . Let $\varphi : S = S(P^2) \dashrightarrow P^2$ be, as in § 3.2., the natural rational map; that is, the formal preimage $\varphi^{-1}(x)$ of the point $x \in P^2$ is equal, by definition, to the set of all the lines in P^4 , which pass through x and which belong, as elements of the Grassmannian, to X . In the above-described case the rational map φ can be regularized by the commutative diagram

$$\begin{array}{ccc} \tilde{S} & \searrow \pi & \\ \downarrow \sigma & \varphi & P^2 \end{array},$$

where σ is a blowing-down of one of the two preimages of the line l , to a non-singular point of S and where π is a regular two sheeted covering with a ramification curve $R \cap P^2$ of degree 6. It is clear, that the two π -preimages of the line l are (-1) -curves on $\tilde{S}$ and hence, the line l is a three-tangent to the ramification curve $R \cap P^2$; in particular, the line l is a threetangent line to R .

(3.3.2) Let $P^5 \subset P^7$ be any subspace of codimension 2 ; the intersection $C = X \cap P^5$ is a canonical curve of degree 10 and of arithmetical genus $p_a(C) = 6$.

Let now $l \subset P^3$ be a pair of a line l and a hyperplane P^3 through l . The curve $C = X \cap G'_{2,0}(l) + X \cap G_{1,1}(P^3) = C_1 + C(P^3)$ (see § 2.2.) is an intersection of X with a special codim. 2 subspace of P^7 . In particular, the curve C is a can. curve of $\deg(C) = 10$ and of $p_a(C) = 6$. The Schubert calculus gives, that $\deg(C_1) = 6$, $\deg(C(P^3)) = 4$ and $(C_1, C(P^3)) = 4$. As $p_a(C_m) = 2$ for the general line $m \subset P^4$ (see Prop.(2.2.4) (2)) and $p_a(C(P^3)) = 1$ for the general hyperplane $P^3 \subset P^4$ (see Prop.(2.2.3) (2)), then we derive from the numerical equalities above that $p_a(C_1) = 2$ and that $p_a(C(P^3)) = 1$ for any line $l \subset P^4$ and for any $P^3 \subset P^4$. In fact, $p_a(C_1) \geq 2$ and $p_a(C(P^3)) \geq 1$ follow from the corresponding equalities, mentioned just above. The equality $(C_1, C(P^3)) = 4$ for $l \subset P^3$ implies that $p_a(C) = p_a(C_1) + p_a(C(P^3)) + 3$. But $p_a(C) = 6$, since C is a canonical curve of degree 10. Therefore, the strong inequalities are impossible, that is $p_a(C_1) = 2$ and $p_a(C(P^3)) = 1$ for any line $l \subset P^4$ and for any hyperplane $P^3 \subset P^4$.

(3.3.3) COROLLARY. Let $l \subset P^4$ be any line, and let $P^3 \subset P^4$ be any hyperplane. Let $C_1 = \{ m \in X - \text{a line} : m \cap l \neq \emptyset \}$, $C(P^3) = \{ m \in X - \text{a line} : m \subset P^3 \}$ (see (2.2.3) (2) and (2.2.4) (2)). Then:

(1) C_1 is a curve of degree 6 and of arithm. genus 2 ;

(2) $C(P^3)$ is a curve of degree 4 and of arithm. genus 1 .

Moreover, if $l \subset P^3$, then the curves C_1 and $C(P^3)$

intersect between themselves in four points (with multiplicities) and the curve $C = C_1 + C(P^3)$ is a canonical curve of degree 10.

(3.3.4) In particular, let the line l be a three-tangent to the ramification hypersurface $R = R_6 \subset P^4$. Then the normalization of the curve C_1 is P^1 and the line l is a triple point of the curve C_1 ; moreover, the tangents to the three branches at the point $l \in C_1$ do not lie in one plane (see also the Corollary above).

§ 4. A description of the conics on $X = X_{10}$.

4.1. The family of the conics on the Fano threefold X_{10} of the 1st type (see (1.3)) is studied in the papers of P. Puts [P] and D. Logachev [L1], [L2]. In many respects the analysis of the family $\mathcal{C}^3$ of the rational normal cubics in the present paper is similar to Logachev's study of the family of conics on X_{10} ; because of that we shall give a brief expose of some results about the family of conics on $X = X_{10}$ "in the sense of Logachev".

4.2. Let $X = G \cap H_1 \cap H_2 \cap Q = G \cap P^7 \cap Q \subset P^7 \subset P^9 = P(\wedge^2 V)$, $V = V_5$ be as before. The hypersurfaces H_1 and H_2 , regarded as elements of $P((\wedge^2 V)^*) = P(\wedge^2 V^*)$ (i.e., considered as one-dimensional subspaces of the vector space $\wedge^2 V^*$), determine a 2-dimensional subspace $E \subset \wedge^2 V^*$. There exists a natural mapping $S: E \otimes V \rightarrow V^*$ defined as follows:

Let $H \in E$ and let $v \in V$; then $S(H \otimes v) = S_H(v) \in V^*$, where $S_H: V \rightarrow V^*$ is the skew-symmetric linear mapping, corresponding to the element $H \in \wedge^2 V^*$. The condition -

- the variety $W = G \cap H_1 \cap H_2$ to be smooth - means that the equality $\text{rank } S_H = 4$ is valid for any $H \in E$.

Then the 2-dimensional subspace $E \subset \wedge^2 V^*$ defines the projective line $P(E)$ and the embedding $\psi_0 : P(E) \hookrightarrow P(V)$, where ψ_0 is defined by the rule: $\psi_0(H) = \text{Ker } S_H$, $H \in E$. In fact, $\text{Ker } S_H$ is one-dimensional subspace of V for any $H \in E$, since W is smooth (see above); hence, the definition of ψ_0 is correct. The image $C_0 = \psi_0(P(E))$ is a conic in $P^4 = P(V)$. The projective plane $P_0^2 \subset P^4$, spanned on the conic C_0 corresponds to some fixed 3-dimensional subspace $U \subset V$, such that $P_0^2 = P(U)$. The "dual" plane $(P_0^2)^* = P(U^*)$ represents the set of the projective lines, lying in the plane P_0^2 , that is, $(P_0^2)^*$ is a $\wp$ -plane, lying in the fourfold $W \subset P^7 = H_1 \cap H_2$; moreover, the $(P_0^2)^*$ is the unique $\wp$ -plane in W (see [P] or [L1]).

Let $l_0 = P((V/U)^*) \subset P(V^*)$ be the "ortogonal" line to the plane $P_0^2 = P(U) \subset P(V)$, that is, $l_0 = \{V_4 \subset V - \text{a four-dimensional subspace} : U \subset V_4 \subset V\}$. There exists a natural isomorphism $s : C_0 \xrightarrow{\sim} l_0$, defined by the rule: $s(v) = S(E \otimes v) \in P(V^*)$, for $v \in C_0$.

The element $v \in C_0$ defines the Schubert cycle $\sigma(v) = \sigma_{3,1}(v, s(v)) = \{l \in G = G(1:P^4) : v \in l \subset s(v)\}$. The Schubert cycle $\sigma(v)$ represents the set of all the projective lines, passing through the point $v \in C_0 \subset P(V)$ and lying in the three-dimensional embedded subspace $P^3(v) = s(v) \subset P^4 = P(V)$, that is, $\sigma(v)$ is a σ -plane in the Grassmannian G (ibid.).

(4.2.1) CLAIM (Logachev, [L1]). The set $\{\mathcal{G}(v) : v \in C_0\}$ describes all the $\mathcal{G}$ -planes, lying in the fourfold $W = G \cap H_1 \cap H_2$. Moreover, for any $v \in C_0$, the set $l(v) = \mathcal{G}(v) \cap (P_0^2)^* = \mathcal{G}(v) \cap (\text{Span } C_0)^*$ is a line, which is tangent to the dual conic $C_0^* \subset (P_0^2)^*$. For any $V_4 \in l_0$ the surface $S(P(V_4)) = W \cap \mathcal{G}_{1,1}(P(V_4))$ is a degenerated quadric, which splits into two planes: $S(P(V_4)) = (P_0^2)^* \cup \mathcal{G}(s^{-1}(V_4))$. The opposite is also true, namely:

If $S(P(V_4))$ splits into two planes, then $V_4 \in l_0$ (for the definition of $S(P(V_4))$ see (2.2.3)).

4.3. Let now $q \subset X$ be a conic, lying on the Fano three-fold $X = G \cap H_1 \cap H_2 \cap Q$. As a subvariety of the Grassmannian $G = G(1:P(V))$, the conic q represents an one-dimensional bundle of projective lines in $P^4 = P(V)$, which is of degree 2 with respect to the cycle $\mathcal{G}_{1,0}$ corresponding to the hyperplane section of the embedded by Plücker Grassmannian $G \subset P^9$. Consequently, the surface $S_q = (\bigcup l, l - \text{a line in } P^4 \text{ \& } l \in q) \subset P^4$ is one of the following:

(1) $S_q \subset \text{Span } S_q = P_q^3 \subset P^4$ is a non-singular quadric surface and q corresponds to one of the two bundles of lines on the quadric S_q , that is, q is a $\mathcal{V}$ -conic on X (see [P]);

(2) $S_q \subset \text{Span } S_q = P_q^3 \subset P^4$ is a quadratic cone and q corresponds to the one-dimensional family of lines, lying on the cone S_q , that is, q is a $\mathcal{G}$ -conic on X (ibid.);

(3) $\text{Span } S_q = P_q^2 \subset P^4$ is a plane and q corresponds to

the set of all the tangent lines to a fixed conic in P_q^2 , that is, q is a ρ -conic on X (ibid.).

Since $X = W \cap Q$ is a general intersection of W with a quadric $Q \subset P^7 = H_1 \cap H_2$, we can suppose that the above cases of possible conics, lying on X , are realized "in general position" that is, in particular, we can suppose, that the ρ -conics (resp., the σ -conics) on $X = W \cap Q$ are obtained as intersections of the ρ -planes (resp., of the σ -planes) on W with the quadric Q . In the last context, the opposite is also true (see [P] or [L1]). In fact, the general conic on X is of type τ ; the σ -conics on X correspond to the points of the curve C_0 (see Claim (4.2.1)), and there is an unique ρ -conic on X , which corresponds to the unique ρ -plane $(P_0^2)^*$ on W .

4.4. Let now q be a τ -conic or a σ -conic on X (that is, $q \neq$ the unique ρ -conic $q_0 = (P_0^2)^* \cap Q$ on X); in particular, $q \subset G(1 : P_q^3 = \text{Span } S_q)$. As we know (see (2.2.3) and (3.3.3)), the surface $S(P_q^3) = W \cap \sigma_{1,1}(P_q^3)$ is a quadric and the curve $C(P_q^3) = S(P_q^3) \cap Q \cap X$ is a space curve of degree 4 and of arithmetical genus 1. As q is a component of the curve $C(P_q^3)$, there exists an additional conic $\bar{q} \subset X$, such that $C(P_q^3) = q + \bar{q}$ and $(q, \bar{q}) = 2$. In particular, $P_q^3 = P_{\bar{q}}^3 = P^3(q, \bar{q})$. The conic $\bar{q}$ can be τ -conic, σ -conic, or ρ -conic. In fact, if $\bar{q} = q_0$, then the symbol $P_{\bar{q}}^3$ does not make sense. Then, under $P_{\bar{q}}^3$ we mean the 3-space P_q^3 . The incidence $\bar{q} = q_0$ is possible

iff q is a $\mathcal{G}$ -conic (see (4.2.1)). From the Claim (4.2.1) we derive also that in the case when q is a $\mathcal{G}$ -conic, the quadric $S(P_q^3)$ splits into two planes : $S(P_q^3) = (P_{q_0}^2)^* + \mathcal{G}(x_0(q))$ (here $x_0(q) \in C_{q_0}$ is the center of the cone $(\bigcup l : l \subset P^4 - \text{a line} \ \& \ l \in q)$, i.e. $\bar{q} = q_0$ is the unique $\mathcal{P}$ -conic on X . In the case, when q is a $\mathcal{U}$ -conic, $\bar{q}$ is also a $\mathcal{U}$ -conic and the surface $S(P_q^3) = S(P_{\bar{q}}^3)$ is a non-singular quadric surface in W .

(4.4.1) DEFINITION. The set of conics :

$$F_c = F_c(X) = \{q - \text{a conic} : q \subset X\}$$

is called a surface of the geometrical conics on X .

The adjective "geometrical" is necessary in the context of the following investigations ; see, for example, the difficulties, arising from the attempts to define correctly the natural involution on the surface of the conics on X .

(4.4.2) DEFINITION. We call the set of pairs :

$$F = F(X) = \{(q, V_4) : q \text{ is a conic on } X, \text{ and } V_4 \text{ is a subspace of dim. 4 of } V, \text{ such that } S_q \subset P(V_4)\}$$

an extended surface of conics on X (or, simply, a surface of conics for X) .

4.5. The mapping $q \mapsto \bar{q}$, $q \neq q_0$ defines a birational isomorphism of F_c , which can be completed to an involution i on the surface $F = F(X)$ in view of the following considerations :

(4.5.1) It follows immediately from the definitions of F and F_c that the projection $(q, V_4) \mapsto q$ on the first factor defines naturally a morphism $r_F : F \longrightarrow F_c$, which is one-to-one over $F_c \setminus \{q_0\}$. The curve $r_F^{-1}(q_0) \subset F$

is described by the set of all the four-dimensional subspaces $V_4 \subset V$, for which $P(V_4)$ contains the projective plane $P_{q_0}^2 = P_o^2$. Evidently, the curve $r_F^{-1}(q_0)$ is naturally isomorphic to the curve of $\mathcal{G}$ -conics $(= \{q : q \text{ is a } \mathcal{G}\text{-conic on } X\} \subset F_c)$; moreover, the morphism r_F is a $\mathcal{G}$ -process over the point $q_0 \in F_c$. In this way the mapping $\bar{\cdot} : F_c \longrightarrow F_c$ defines correctly the involution $i : F(X) \longrightarrow F(X)$.

(4.5.2) DEFINITION. We define $F_o = F_o(X)$ to be the factorsurface F/i of F under the involution i ; that is F_o is identified with the set of all (non-ordered) pairs of involutive elements of F .

4.6. The projection on the second factor $(q, V_4) \longmapsto V_4$ defines a morphism $\varphi : F \longrightarrow (P^4)^* := P(V^*)$.

Let $p_F : F \longrightarrow F_o$ be the natural two-sheeted covering, defined by the involution i . Clearly, there exists a commutative diagram:

$$\begin{array}{ccc} F & & \\ p_F \downarrow & \searrow \varphi & \\ F_o & \xrightarrow{\varphi_o} & P(V^*) \end{array},$$

because the involutive elements of F have same φ -images.

It is easy to see also that the mapping φ_o provides an embedding of the factorsurface F_o in the projective space $P(V^*)$.

§ 5. The family $\mathcal{C}^3$ of rational normal
normal cubic curves on X .

5.1. PROPOSITION (see [G , §2. Prop. 2.1 and 2.2]).

($X = X_{10}$ is, as usual, a general Fano threefold of degree
10 and of the 1st type , see (1.3))

(1) Let C be a rational normal cubic curve on X . Then
the normal sheaf $N_{C/X}$ is one of the following :

$$N_{C/X} = \begin{cases} \mathcal{O}_C \oplus \mathcal{O}_C(1) \\ \mathcal{O}_C(-1) \oplus \mathcal{O}_C(2) \\ \mathcal{O}_C(-2) \oplus \mathcal{O}_C(3) \end{cases} ;$$

(2) There are rational normal cubic curves on X ;

(3) Let T be an irreducible reduced component of the
family $\mathcal{C}^3$ of all the rational normal cubic curves on X ,
and let x be a sufficiently general point of X . Then :

(a) $\dim \{ C \in T : C \text{ passes through } x \} = 1$,

(b) The normal sheaf $N_{C/X}$ of the general $C \in T$ is
isomorphic to $\mathcal{O}_C \oplus \mathcal{O}_C(1)$.

5.2. A geometrical representation of the rational normal
cubics on X as cubic scrolls.

(5.2.1) Let $C \subset X$ be a rational normal cubic, lying on
the Fano threefold $X = G \cap H_1 \cap H_2 \cap Q$. The points of the
curve C , regarded as lines in $P^4 = P(V)$, sweep out a
surface $S = S_C$ of degree 3 in P^4 . The ruled surface
 $S_C \subset P^4$ is one of the following :

Case 1. $S_C \subset \text{Span } S_C = P^4$, and :

(1.a) $S = \mathbb{F}_1$ is a rational normal cubic scroll embedded
in P^4 by means of the linear system $|s + 2.f|$, where s

is the (-1) -section and f is the fiber of the ruled surface $\mathbb{F}_1 = P(0_{P^1} \oplus 0_{P^1}(1))$,

(1.b) S_C is a cone over a rational normal cubic curve Z of degree 3, with a center outside of the "ambient" space $\text{Span } Z$;

Case 2. $S_C \subset \text{Span } S_C = P^3 \subset P^4$, and:

(2.a) S_C is a projection of a cubic scroll $\mathbb{F}_1$ (see (1.a)) on $P^3 \subset P^4$,

(2.b) S_C is a projection of a rational cubic cone (see (1.b)) on P^3 .

(5.2.2) CLAIM. The cases (1.b) and (2.b) do not occur for the general X .

Proof. We shall consider simultaneously the cases (1.b) and (2.b).

Let $S = S_C$ be a rational cubic cone (in P^4 , or resp. in some $P^3 \subset P^4$), corresponding to some rational normal cubic $C \subset X$. As we know (see 4.2) the set of all $\mathcal{C}$ -conics on X describes a rational curve on the surface of geometrical conics F_C (see Def. (4.4.1)). The lines, which are points of such a conic, sweep out a quadratic cone in P^4 . The rational curve on F_C , which points corresponds to the $\mathcal{C}$ -conics on X , is naturally isomorphic to the conic $C_0 = \Psi_0(P(E))$ (see Claim (4.2.1)). Therefore, the set of the centers of the corresponding quadratic cones describes some rational curve $\hat{C}_0$ in P^4 . In fact, the curve $\hat{C}_0$ is a non-singular conic, lying on the plane

P_0^2 (see § 4.2 and § 4.3); it is not hard to see also that the curve $\hat{C}_0$ coincides with the conic C_0 (see Claim (4.2.1)), but we don't need such a precision.

Let x_0 be the center of the cubic cone, corresponding to the cubic $C \subset X$ and let y be a center of some quadratic cone, corresponding to some $\mathcal{G}$ -conic q on X . Let $l = \langle x_0, y \rangle$ be the line through the points x_0 and y . As we know, the curve $C_1 = \mathcal{G}_{2,0}(l) \cap X$ has a degree 6 and an arithmetical genus 2, independently of the choice of the line $l \subset P^4$ (see Corollary (3.3.3) (1)). Consequently, the curve C_1 has an additional component L of degree 1, such that $C_1 = C + q + L$, where C and q are the cubic and the conic from above (all the lines of C pass through $x_0 \in l$, and all the lines of q pass through $y \in l$). As the curve C_1 is connected, and all the projective lines, corresponding to the points of the "Grassmann" line L , intersect the line $l \subset P^4$, the center z of the plane pencil of lines, which describes L , lies on the line $l = \langle x_0, y \rangle$. Obviously, the point z is the unique point on the line $\langle x_0, y \rangle$ with the property, that z is a center of some plane pencil of lines in P^4 , corresponding to some line, lying on X .

The last considerations show, that the existence of a cubic $C \subset X$ of the type (1.b) or (2.b) implies an existing of an isomorphism between the rational curve C_0 and some component of the curve Γ (see $[P]$ or $[M]$), parametrizing the lines, lying on the Fano threefold X . But, as it is known ($[P, \S 8]$), if X is general, then

the family Γ has only one component ; moreover, the geometrical genus of Γ is 71 , which leads us into a contradiction. Therefore, the cases (1.b) and (2.b) do not occur, if X is sufficiently general .

(5.2.3) COROLLARY. Let C be a rational normal cubic curve, lying on X . Then, the corresponding to C bundle of lines in P^4 describes either the ruled surface F_1 , realized as a cubic scroll in P^4 , or some projection of a cubic scroll F_1 on some $P^3 \subset P^4$.

(5.2.4) In both the cases (1.a) and (2.a) there exists an unique line $l = l(C)$ in P^4 , which coincides geometrically with the (-1) -section of the corresponding F_1 (or of its projection) and such that the cubic curve C is a component of the curve $C_1 = \sigma_{2,0}(l) \cap X$.

(5.2.5) On the other hand, $\deg C_1 = 6$ and $p_a(C_1) = 2$. The last implies that there exists an additional rational normal cubic $\bar{C}$ on X , such that $C_1 = C + \bar{C}$ and

$(C, \bar{C}) = 3$. It is possible curve $\bar{C}$ to be singular, which in our particular case means "a degenerated rational normal cubic" ; because of this, as in the case of the family of conics, we introduce the following refinement of the definition of the family of rational normal cubics on X :

(5.2.6) DEFINITION. We call the set

$\mathcal{C}^3 = \mathcal{C}^3(X) = \{ C \subset X : C \text{ is a rational normal (possibly, degenerate) cubic curve} \}$ a family of the rational normal cubic curves (or , simply r.n. cubics) on X .

5.3. The rational mapping $C \mapsto \bar{C}$ defines an involution on the family of rational normal cubics on X . To be precise, we need to consider separately the special cases of "degenerate" rat. normal cubics on X .

Let $C \in \mathcal{C}_r^3$ be a degenerate r.n. cubic on X . We can separate the following possible cases for C :

(i) $C = q + L$, where $q \in F_C(X)$ is a conic on X and $L \subset X$ is a line, such that $(q, L) = 1$;

(ii) $C = L_1 + L_2 + L_3$, where L_1, L_2 and L_3 are lines on X and $(L_1, L_2) = (L_2, L_3) = 1$, $(L_1, L_3) = 0$.

We shall find the involutive curve $\bar{C}$ in the cases (i) and (ii).

Case (i') Let q be a τ -conic. Then the surface $S_q = (\bigcup l, l \subset P^4 - \text{a line \& } l \in q)$ is a non-singular quadric and the set of lines $\{l : l \in q\}$ describes one of the two pencils of lines on S_q . Let P_L^2 be the plane $(\bigcup l, l \subset P^4 - \text{a line \& } l \in L)$, and let $y \in P_L^2$ be the center of the plane pencil of lines $\{l : l \in L\}$. The equality $(q, L) = 1$ means that there exists a line $l_0 \subset P^4$, which belongs to both of the pencils; evidently, $y \in l_0$. Let $l' \subset S_q$ be the line from the second pencil of lines on S_q , which passes through the point y . Then the conic q and the line L are components of the curve $C_{1'} = \mathcal{G}_{2,0}(l') \cap X$. Since $\deg C_{1'} = 6$ and $p_a(C_{1'}) = 2$, then the residue component $\bar{C} = C_{1'} - q - L$ belongs to the family

$\mathcal{C}_r^3(X)$; moreover $(q + L, \bar{C}) = 3$, i.e. $\bar{C} = \overline{q + L}$ is the involutive of $C = q + L$.

Case (i'') Let q be a $\mathcal{G}$ -conic. Then the surface $S_q = \left(\bigcup l, l \subset P^4 - \text{a line} \ \& \ l \in q \right)$ is a quadratic cone and the equality $(q, L) = 1$ means that there is a line l_0 , which is common for the pencils $\{l : l \in q\}$ and $\{l : l \in L\}$. Let x_0 be the singular point (the center) of the cone S_q and let y be the center of the plane pencil of lines, corresponding to L (see (i')). Since $(q, L) = 1$, then the plane of the second pencil P_L^2 does not lie in the subspace $\text{Span } S_q \subset P^4$. In the opposite case the plane P_L^2 intersects the cone S_q along a pair of lines l_0 and l'_0 . Both l_0 and l'_0 are common members of the pencils $\{l : l \in q\}$ and $\{l : l \in L\}$, i.e. $(q, L) = 2$ - a contradiction. Moreover, if the points y and x_0 coincide, then the configuration $S_q + P_L^2$ corresponds to a degeneration of the case (1.b) (see (5.2.1)), which does not occur (see Claim (5.2.2)); the proof of the "degenerate" variant of the Claim (5.2.2) is, obviously, the same. Consequently, the points x_0 and y determine the line $l_0 = \langle x_0, y \rangle$. As in the case (i'),

$C_{l_0} = q + L + \bar{C}$, where $(q + L, \bar{C}) = 3$, and the involutive curve $\bar{C} = \overline{q + L}$ belongs to $\mathcal{C}^3$.

Case (i''') Let $q = q_0$ be the unique $\mathcal{G}$ -conic on X . Then, the center y of the bundle $\{l : l \in L\}$ lies on the plane $P_0^2 = \text{Span } C_0$ (see § 4.2). The points $v \in C_0$ are centers of the bundles of lines in P^4 corresponding to the Schubert cycles ($\mathcal{G}$ -planes) $\mathcal{G}(v) = \mathcal{G}_{3,1}(v, s(v))$ (see § 4.2) on the fourfold $W = \mathcal{G} \cap H_1 \cap H_2 \subset H_1 \cap H_2 = P^7$. The "Grassmann"

quadric Q intersects each $\mathcal{G}$ -plane $\mathcal{G}(v)$ in the

$\mathcal{G}$ -conic $q(v) = \mathcal{G}(v) \cap Q$; the center of the corresponding cone $S_{q(v)}$ coincides with the point v , i.e.

$x_0(v) = v$, $v \in C_0$. Let now $l_0 \subset P_0^2$ be a line through the center y of the bundle $\{l : l \in L\}$. The line l_0 intersects the conic C_0 in two points: v_1 and v_2 ; the last are the centers of the cones, corresponding to the $\mathcal{G}$ -conics $q(v_1)$ and $q(v_2)$. Therefore, the curve C_{l_0} has as components the conics $q(v_1)$, $q(v_2)$, q_0 and the line L (the centers v_1, v_2 and y lie on l_0 and l_0 lies in the plane P_0^2 of the dual conic q_0), hence, $6 = \deg C_{l_0} \geq \deg q(v_1) + \deg q(v_2) + \deg q_0 + \deg L = 7$ - a contradiction. Therefore, the case (i''') does not occur.

(5.3.1) REMARK. As $q_0 = (P_0^2)^* \cap Q$, the "Grassmann" conic q_0 is represented by the set of all lines in P_0^2 which are lines of intersection of the cones $S_{q(v)}$, $v \in C_0$ and the plane P_0^2 . For every $v \in C_0$ the cone $S_{q(v)}$ intersects P_0^2 in a pair of lines through the point v ; the set of all these lines coincides with the set of the tangent lines to some fixed conic $C'_0 \subset P_0^2$. In particular, every $\mathcal{G}$ -conic $q(v)$, $v \in C_0$ intersects its "involutive" $\overline{q(v)} = q_0$ in a pair of points (or, in a double point, if v is a point of an intersection of C_0 and C'_0), which correspond to the pair of tangents to the conic C'_0 through the point $v \in C_0$.

Case (ii) Let the degenerate rational normal cubic C splits into a chain of three lines: $C = L_1 + L_2 + L_3$,

$(L_1, L_2) = (L_2, L_3) = 1$, $(L_1, L_3) = 0$. Let P_1^2 , P_2^2 and P_3^2 be the planes, and y_1 , y_2 and y_3 be the centers of the corresponding to the "Grassmann" lines L_1 , L_2 and L_3 plane pencils of lines in P^4 . The conditions for the intersections give, that the points y_1 and y_3 lie in the plane P_2^2 . The case $y_1 = y_2 = y_3$ represents a special degeneration of the case (1.b) or (2.b) (see (5.2.1)), which does not occur for the general X (see Claim (5.2.2)).

If $y_1 = y_3 \neq y_2$ or if y_1 , y_2 and y_3 are collinear, then the line $l = \langle y_1, y_2 \rangle = \langle y_3, y_2 \rangle$ is a common point of the "Grassmann" lines L_1 and L_3 ; but the last two do not intersect between them - a contradiction. Therefore, the line $l_0 = \langle y_1, y_3 \rangle$ is correctly defined and moreover l_0 does not pass through the point y_2 . Just as in the case (i), $C_{l_0} = L_1 + L_2 + L_3 + \bar{C}$, where the curve $\bar{C}$ belongs to the family $\mathcal{C}^3$, and

$(L_1 + L_2 + L_3, \bar{C}) = 3$, i.e. $\bar{C} = \overline{L_1 + L_2 + L_3}$ is the correctly defined involutive of $C = L_1 + L_2 + L_3$.

(5.3.2) COROLLARY. Let $\mathcal{C}^3 = \mathcal{C}^3(X)$ be the family of the rational normal cubics on X . Then, there is a correctly defined involution $\sigma: \mathcal{C}^3 \longrightarrow \mathcal{C}^3$, such that

$\sigma(C) = \bar{C}$, $C \in \mathcal{C}^3$. The factorscheme $\mathcal{C}_0^3 = \mathcal{C}^3 / \sigma = \{ \text{the non-ordered pairs } (C, \bar{C}) \text{ of involutive elements of } \mathcal{C}^3 \}$ is naturally embedded in the Grassmannian $G = G(1: P(V))$ by the rule:

$$(C, \bar{C}) \longmapsto l,$$

where the line $l \subset P^4 = P(V)$ is defined as above (see

cases (5.2.1) (1.a), (2.a) and § 5.3 (i), (ii)); i.e.

l is the unique line in $P^4 = P(V)$, such that $C_1 = \mathcal{G}_{2,0}(l) \cap X = C + \bar{C}$. The involutive pair of rational normal cubics C and $\bar{C}$ is defined equivalently by the numerical condition $(C, \bar{C})_X = 3$.

To prove the Corollary, it remains to note that the uniqueness of the line l is demonstrated separately at every of the cases above; from the last it follows immediately that $\bar{\bar{C}} = C$ for every $C \in \mathcal{C}_0^3(X)$.

5.4. COMMENTS.

(5.4.1) Let $(C, \bar{C})$ be an involutive pair of $\mathcal{C}_0^3$ and let $C \cdot \bar{C} = t_1 + t_2 + t_3$, where t_1, t_2 and t_3 are points of $P^7 = H_1 \cap H_2$, possibly with multiplicities. As C and $\bar{C}$ are rational normal cubic curves, the intersection $C \cdot \bar{C}$ determines correctly an unique plane (the plane, "spanned on the intersection points of C and $\bar{C}$ ").

Let $l \subset P^4$ be the line, such that $C_1 = C + \bar{C}$, and let $p_i = t_i \cap l$, $i = 1, 2, 3$ (the points t_1, t_2 and t_3 are regarded here as lines in P^4). It is evident that the line l is a three-tangent to the ramification hypersurface $R = R_6 \subset P^4$ (see Def.-Cor. (2.2.6)), that is $l \cdot R = 2 \cdot p_1 + 2 \cdot p_2 + 2 \cdot p_3$.

(5.4.2) The involutive pairs of the factorfamily $\mathcal{C}_0^3$ as sections of the rational cubic scrolls $S_1 \subset W$ with the quadric Q .

As above, let $W = G \cap H_1 \cap H_2$ and let l be a line in P^4 . The intersection $S_1 = \mathcal{G}_{2,0}(l) \cdot W$ is homolo-

gous to the cycle $2 \cdot \sigma_{3,1} + \sigma_{2,2}$ and represents a surface of degree 3; moreover, for the general line $l \subset P^4$, the surface S_l coincides with a surface

$\mathbb{F}_1 = P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(1))$, embedded in some four-dimensional subspace of $P^7 = H_1 \cap H_2$ as a rational normal cubic scroll (see Proposition (2.2.4) (1)). As $X = W \cap Q$, where Q is a (sufficiently general) quadric, the curve $C_1 = \sigma_{2,0}(1) \cdot X$ is an intersection of the surface S_l with Q ; in the notations of Prop. (2.2.4), the curve C_1 belongs to the linear system $|2.C_0 + 4.f|$ (C_0 is the (-1) -section, f is the fiber of $\mathbb{F}_1$). The splitting $C_1 = C + \bar{C}$ means that the restriction of the quadric Q on the surface $S_l (= \mathbb{F}_1$, in the general case) can be represented as a sum of two hyperplane sections (elements of the linear system $|C_0 + 2.f|$) of the cubic scroll $\mathbb{F}_1$. In the degenerate cases the surface S_l is a degeneration of a family of $\mathbb{F}_1$'s; we need to be careful with the symbols, because the surfaces S_C in the cases (1.a) and (2.a) (see § 5.2) and also the surfaces $S_Q + P_L^2$ and $P_1^2 + P_2^2 + P_3^2$ (see § 5.3), which are cubic scrolls or their degenerations, represent cubic surfaces in P^4 , which are swept out by lines in P^4 corresponding to the points of the curves $C \in \mathcal{C}^3$. We shall study the conditions for the splitting $C_1 = C + \bar{C}$ in detail in the next paragraph.

§ 6. The factorfamily $\mathcal{C}_0^3$
as a determinantal variety .

6.1. Tautological sequences on $P(V^*)$ and on $G(2, V)$.

In the present section we need some details, concerning the well-known standard tautological sequences on the Grassmannians.

(6.1.1) We shall regard the "dual" projective space $(P^4)^* = P(V^*)$ as a Grassmannian : $P(V^*) = G(4, V) = \{ V_4 : V_4 \text{ - a four-dimensional subspace of } V \}$. In this interpretation the standard tautological sequence on $P(V^*)$:

$$0 \longrightarrow \tau_{4,V} \longrightarrow P(V^*) \times V \longrightarrow \tau_{1,V^*}^* \longrightarrow 0$$

parametrizes the family of embeddings

$$\{ 0 \longrightarrow V_4 \longrightarrow V : V_4 \in G(4, V) = P(V^*) \} .$$

Let $f = {}^t(f_1, \dots, f_4)$ and $g = {}^t(g_1, \dots, g_4)$ be the vector-columns of two given bases of the subspace

$V_4 \subset V$ and let $f = A \cdot g$ be a change of the basis

f by the basis g . Then $\bigwedge^4 f = f_1 \wedge f_2 \wedge f_3 \wedge f_4 = \det A \cdot g_1 \wedge g_2 \wedge g_3 \wedge g_4 = \det A \cdot \bigwedge^4 g$. On the other hand, the Plucker embedding $Pl(\bigwedge^4) : G(4, V) \longrightarrow P^4$ provides the natural isomorphism: $G(4, V) \simeq P(V^*)$;

$Pl(\bigwedge^4) : V_4 \longrightarrow \bigwedge^4 f$ (mod. proportionality by elements of $\mathbb{C}^*$) , $V_4 \in G(4, V)$. From here we get immediately, that

$\det \tau_{4,V} = \mathcal{O}_{G(4,V)}(-1)$, according to the embedding above. Keeping in mind the last, we replace $G(4, V)$ with $P(V^*)$; in particular, we write $\det(\tau_{1,V^*}^*) = \tau_{1,V^*}^* = \mathcal{O}_{G(4,V)}(1) = \mathcal{O}_{P(V^*)}(1)$.

(6.1.2) Let now $G = G(2, V) = \{ V_2 : V_2 \text{ is a 2-dim. subspace of } V \}$ be the Grassmannian and let

$$0 \longrightarrow \mathcal{T}_{2,V} \longrightarrow G(2,V) \times V \longrightarrow \mathcal{T}_{3,V}^* \longrightarrow 0$$

be the standard tautological sequence on G . Over the element $V_2 \subset V$ of $G(2,V)$ the embedding of the left side coincides with the natural embedding $0 \longrightarrow V_2 \longrightarrow V$.

If we perform a base change $f = A \cdot g$ in the fiber V_2 ($f = {}^t(f_1, f_2)$ and $g = {}^t(g_1, g_2)$ are two bases of V_2 , as above), we obtain a change of the second exterior powers $\bigwedge^2 f = \det A \cdot \bigwedge^2 g$.

On the other hand, the Plücker embedding

$$Pl(\bigwedge^2) : G(2, V) \longrightarrow P(\bigwedge^2 V) = P^9$$

maps the element $V_2 \in G$ to the class $\bigwedge^2 f \pmod{\text{proportionality by elements of } \mathbb{C}^*}$. In view of the last, the former change means that $\det(\mathcal{T}_{2,V}) = \mathcal{O}_{G(2,V)}(-1)$, according to the Plücker embedding $Pl(\bigwedge^2)$.

6.2. Pfaff ideals on $P(V^*)$ and $G(2, V)$.

(6.2.1) We shall define a subbundle Pf of the bundle of quadrics $S^2 \bigwedge^2 \mathcal{T}_{4,V}$ over $P(V^*) = G(4, V)$.

Let $V_4 \subset V$ be an element of $P(V^*) = G(4, V)$, let $f = {}^t(f_1, \dots, f_4)$ and $g = {}^t(g_1, \dots, g_4)$ be, as above, bases of V_4 and let $y = (y_1, \dots, y_4)$ and $z = (z_1, \dots, z_4)$ be the corresponding bases of coordinates on V_4 . Assigned to the base f , we define the fibre of Pf to be: $Pf(V_4)_f =$ the Plücker quadric $y_{12} \cdot y_{34} - y_{13} \cdot y_{24} + y_{14} \cdot y_{23}$, where y_{ij} , $1 \leq i < j \leq 4$ are the coordinates, which correspond to the

basis $f_{ij} = f_i \wedge f_j$, $1 \leq i < j \leq 4$ of $\wedge^2 V_4$
 (the same for the coordinates z_{ij} and for the base vec-
 tors g_{ij} of $\wedge^2 V_4$). Let $f = A \cdot g$ be a base
 change of V_4 ; the change of the coordinates is $y \cdot A = z$,
 i.e. $y = B \cdot z$, where $B = A^{-1}$. The last allows to
 compute the change of Pf : $\text{Pf}(V_4)_f = y_{12} \cdot y_{34} - y_{13} \cdot y_{24} +$
 $+ y_{14} \cdot y_{23} = \dots = \det B \cdot (z_{12} \cdot z_{34} - z_{13} \cdot z_{24} + z_{14} \cdot z_{23}) =$
 $= (\det A)^{-1} \cdot \text{Pf}(V_4)_g$; but A changes the bases of the
 bundle $\tau_{4,V}$ (see (6.1.1)), hence $\text{Pf} = (\det \tau_{4,V})^{-1} =$
 $= \mathcal{O}_{P(V^*)} (1)$. Here we shall describe one geometrical inter-
 pretation of the bundle of quadrics Pf .

Taking the second exterior power of the embedding
 $0 \longrightarrow \tau_{4,V} \longrightarrow P(V^*) \times V$ from the tautological sequ-
 ence, we obtain the embedding

$$0 \longrightarrow \wedge^2 \tau_{4,V} \longrightarrow P(V^*) \times \wedge^2 V ,$$

which parametrizes the family of embeddings $\wedge^2 V_4 \hookrightarrow \wedge^2 V$,
 $V_4 \in P(V^*)$. The fiber $\text{Pf}(V_4)$ coincides with the
 one-dimensional vector space, spanned on the equation of the
 embedded Grassmannian $G(2, V_4) = \mathcal{G}_{1,1}(P(V_4)) \subset$
 $\subset \text{Span } \mathcal{G}_{1,1}(P(V_4)) = P(\wedge^2 V_4)$ as a subvariety
 of $G(2, V) \subset P(\wedge^2 V) = P^9$.

(6.2.2) Let we take the exterior product of the embedding
 $0 \longrightarrow \tau_{2,V} \longrightarrow G(2, V) \times V$ with the constant bundle
 $G(2, V) \times V$. The obtained embedding

$$0 \longrightarrow \tau_{2,V} \wedge V \longrightarrow G(2, V) \times \wedge^2 V$$

can be interpreted geometrically as follows:

Let $V_2 \subset V$ be an element of $G(2, V)$ and let $\sigma_{2,0}(P(V_2))$ be the Schubert cycle $\{L \subset P(V) - \text{a line} : L \cap P(V_2) \neq \emptyset\} = \{L \subset V - \text{a subspace of dim.} = 2 : \dim(L \cap V_2) \geq 1\}$, embedded in $P(\wedge^2 V)$ as a subvariety of the embedded Grassmannian $G(2, V)$. We can check directly that $\text{Span } \sigma_{2,0}(P(V_2)) = P(V_2 \wedge V)$. Now we can define the "ideal" subbundle $\mathbb{I} \subset S^2(V_2 \wedge V)$, namely:

$\mathbb{I}(V_2) = \left[\text{the set of all the quadrics in } P(V_2 \wedge V) \right.$
 (regarded as elements of $S^2(V_2 \wedge V)^*$) which vanish on
 the subvariety $\sigma_{2,0}(P(V_2)) =$
 $= H^0(P(V_2 \wedge V), \mathcal{O}(2 - \sigma_{2,0}(P(V_2))) \left. \right] \subset S^2(V_2 \wedge V)^*$
 As in (6.2.1) we compute the cocycle of the base changes
 of the bundle $\mathbb{I}$; as a result we obtain that $\mathbb{I}$ is
 isomorphic to the bundle $\tau_{3,V}^*$.

(6.2.3) COROLLARY.

(i) Let $Pf \subset S^2 \wedge^2 \tau_{4,V}^*$ be the sheaf of quadrics
 over $P(V^*)$ with a fibre $Pf(V_4) =$
 $= H^0(P(\wedge^2 V_4), \mathcal{O}(2 - \sigma_{1,1}(P(V_4))))$
 over the element $V_4 \subset V$ of $P(V^*) = G(4, V)$. Then
 there is a natural isomorphism:

$$Pf \simeq \mathcal{O}_{P(V^*)}(1);$$

(ii) Let $\mathbb{I} \subset S^2(\tau_{2,V} \wedge V)$ be the bundle of quadrics
 over $G = G(2, V)$ with a fibre $\mathbb{I}(V_2) =$
 $= H^0(P(V_2 \wedge V), \mathcal{O}(2 - \sigma_{2,0}(P(V_2))))$

over the element $V_2 \subset V$ of $G(2, V)$. Then there is a natural isomorphism :

$$\mathbb{I} \simeq \tau_{3,V}^*$$

where $\tau_{3,V}^*$ is the tautological factorbundle over $G(2, V)$.

(6. 1-2) COMMENTS. The bundles of quadrics $\text{Pf} \longrightarrow P(V^*)$ and $\mathbb{I} \longrightarrow G(2,V)$ are the components of degree 2 in the graded bundles of ideals of the families of the embedded Schubert cycles $\{ \sigma_{1,1}(P(V_4)) \subset P(\wedge^2 V_4) : V_4 \in P(V^*) \}$ and $\{ \sigma_{2,0}(P(V_2)) \subset P(V_2 \wedge V) : V_2 \in G(2, V) \}$ respectively. Obviously, the components Pf and $\mathbb{I}$ generate the corresponding ideals.

6.3. The factorfamily $\mathcal{Q}_0^3$ as a set of quadrics.

(6.3.1) Let $Q \subset S^2(\tau_{2,V} \wedge V)^*$ be the bundle of quadrics over $G = G(2,V)$, which parametrizes the family of restrictions of the quadric $Q \in S^2(\wedge^2 V)^*$ on the subspaces of the form $V_2 \wedge V$, $V_2 \in G(2,V)$. As the quadric Q is sufficiently general, the corresponding bundle of restrictions $Q(V_2) = [\text{the } \mathbb{C}^* \text{-class of the equation of the restriction of the quadric hypersurface } Q = 0 \text{ in } P(V_2 \wedge V)]$ is correctly defined.

(*) NOTE. We use the same symbol Q for the quadric $Q \in S^2 \wedge^2 V^*$, for the surface $(Q = 0)$ in $P(\wedge^2 V)$ and for the bundle "of quadrics" Q over $G(2,V)$.

Obviously, the bundle Q , defined above, is trivial, i.e. the corresponding sheaf Q is isomorphic to the structure sheaf $\mathcal{O}_{G(2,V)}$ over $G = G(2,V)$.

(6.3.2) PROPOSITION. Let $Z \subset P_G (S^2 (\tau_{2,V} \wedge V)^*)$ be the set :

$$Z = P \left\{ (V_2, q) : V_2 \in G, q \in H^0(P(V_2 \wedge V), \mathcal{O}(2 - \sigma_{2,0}(P(V_2)) \cdot X)) \right\}$$

Then Z is naturally isomorphic to the projectivized bundle $P_G (\tau_{3,V}^* \oplus \mathcal{O}_G)$ over $G = G(2,V)$.

Proof. The proposition follows immediately from the Corollary (6.2.3) (ii) and from (6.3.1). It remains to see that the fiber of the natural projection $Z \longrightarrow G(2,V)$ over the "point" $V_2 \in G(2,V)$ coincides with the space

$$\text{Span} \left\{ \mathbb{I}(V_2) \cup Q(V_2) \right\} = \mathbb{I}(V_2) \oplus Q(V_2).$$

(6.3.3) Let now $X = G(2,V) \cap H_1 \cap H_2 \cap Q$ and let $V_8 \subset V_{10} = \wedge^2 V$ be the subspace, such that $P^7 = H_1 \cap H_2 = P(V_8)$. We define a bundle Σ over $G(2,V)$ with fibers $\Sigma(V_2) = (V_2 \wedge V) \cap V_8$, $V_2 \in G(2,V)$. We can suppose that the subspace $P^7 = P(V_8)$ is chosen a "sufficiently general" in such a way, that all the intersections $(V_2 \wedge V) \cap V_8$ are transversal (and, hence, are vector spaces of dimension 5) for every $V_2 \in G(2,V)$. Considerations on the level "intersection of Schubert cycles" give that the last requirement is fulfilled for the elements V_8 of an open subset of the Grassmannian $G(8, V_{10})$.

(6.3.4) Let $V_2 \in G(2,V)$. As we know, $S_P(V_2) = \sigma_{2,0}(P(V_2)) \cap P(V_8) = \left\{ l \subset P^4 - \text{a line} : l \in W, l \cap P(V_2) \neq \emptyset \right\}$ (see Proposition (2.2.4) (1) and (5.4.2)). Let $V_8 \subset V_{10} =$

$= \bigwedge^2 V$ be chosen general. Then, taking into account the considerations above, we have that all the restriction maps :

$$\begin{aligned} H^0 (P(V_2 \wedge V) , \mathcal{O} (2 - \mathcal{G}_{2,0} (P(V_2)))) &\longrightarrow \\ &\longrightarrow H^0 (P (\sum (V_2)) , \mathcal{O} (2 - S_{P(V_2)})) , \end{aligned}$$

are isomorphisms. So we obtained

(6.3.5) COROLLARY. Let $V_8 \subset V_{10} = \bigwedge^2 V$ and Q are chosen sufficiently general, such that :

- (i) The variety $X = G \cap Q \cap P(V_8)$ is smooth ;
- (ii) $\dim (V_8 \cap (V_2 \wedge V)) = 5$, for every

$$V_2 \in G = G(2,V) .$$

Let $\mathbb{I} \longrightarrow G$ and $Q \longrightarrow G$ be the bundles, as in (6.2.3) (ii) and (6.3.1) . Then we can consider that

$\mathbb{I}$ and Q are embedded in $S^2 \sum^*$ in such a way that

$$\mathbb{I}(V_2) = H^0 (P (\sum (V_2)) , \mathcal{O} (2 - S_{P(V_2)})) ,$$

$$Q(V_2) = H^0 (P(\sum (V_2)) , \mathcal{O}_{P(\sum(V_2))} \otimes \mathbb{C}.Q ,$$

where $V_2 \in G(2,V)$; here $S_{P(V_2)} = \mathcal{G}_{2,0}(P(V_2)) \cdot W$, $V_2 \in G$.

(6.3.6) Let now $C \subset X$ be a rational normal cubic curve. As we know (see Corollary (5.3.2)) , there exists a line $l \subset P^4 = P(V)$ (resp., a subspace $V_2 \subset V$, such that $P(V_2) = l$) , such that $C_1 = \mathcal{G}_{2,0}(l) \cdot X = C + \bar{C}$ ($\bar{C}$ is the involutive of C) . But $C_1 = S_1 \cdot Q$, where S_1 , $l = P(V_2)$, is as in the Corollary above. The last means that there exists a quadric q , which is a point of the space $\mathbb{I}(V_2) \oplus Q(V_2) \subset S^2 \sum^*(V_2)$ and which splits, as a subvariety of $P(\sum(V_2))$ into

two hyperplanes : $(q = 0) = \langle C \rangle + \langle \bar{C} \rangle \subset P(\sum (V_2))$.

The last is equivalent, in the interpretation of the Corollary (6.3.5), to the existence of a quadric of rank ≤ 2 in the vector-space "of quadrics" $(\mathbb{I} \oplus Q)(V_2)$. Obviously, the opposite is also true : the existence of a quadric

$q \in (\mathbb{I} \oplus Q)(V_2)$ of rank ≤ 2 , $q = H \cdot \bar{H}$ means that the curves $C = S_{P(V_2)} \cdot P(H)$ and $\bar{C} = S_{P(V_2)} \cdot P(\bar{H})$ are an involutive pair of rational normal cubics such that

$$C_{P(V_2)} = C + \bar{C}.$$

(6.3.7) COROLLARY. The factorfamily $\mathcal{C}_0^3 = \mathcal{C}^3 / \sigma$ (see (5.3.2)) is embedded naturally in the projectivized vector bundle $P_G(\mathbb{I} \oplus Q) \subset G(2, V) \times P(S^2 \Sigma^*)$ as the set :

$$D_2 = P\left\{ (V_2, q) : V_2 \in G, q \in (\mathbb{I} \oplus Q)(V_2) \subset S^2 \Sigma^*(V_2), \right. \\ \left. , \text{rank}(q) \leq 2 \right\}.$$

Postponing the comments of the Corollary till later on, we shall explain in brief some facts about the degeneration loci (determinantal) (see [F, ch. 14]).

6.4. The factorfamily $\mathcal{C}_0^3$ as a determinantal.

(6.4.1) DEFINITION.

(i) Let $\varphi : E \longrightarrow F$ be a homomorphism of vector bundles (a vector bundle map) over the variety Y ; let $\text{rank } E = e$, $\text{rank } F = f$. Let k be a non-negative integer, such that $k \leq \min(e, f)$. The locus

$$D_k = D_k(\varphi) = \{ y \in Y : \text{rank } \varphi(y) \leq k \} \subset Y$$

is called the k^{th} degeneration locus (the k^{th} determinantal) of φ .

(ii) In particular, let $F = E^* \otimes L$ for some invertible sheaf L on Y . Multiplying by L^{-1} , we obtain the map $\varphi \otimes L^{-1} : E \otimes L^{-1} \longrightarrow E^*$. Taking the dual $(\varphi \otimes L^{-1})^* : E^{**} = E \longrightarrow E^* \otimes L$, we obtain another vector bundle map from E to $F = E^* \otimes L$. The vector bundle map $\varphi : E \longrightarrow E^* \otimes L$ is called symmetric, if φ is "selfdual", i.e. if

$\varphi = (\varphi \otimes L^{-1})^*$. Using the natural correspondence between so defined symmetric maps and the "nets of quadrics" $\hat{\varphi} : L^{-1} \longrightarrow S^2 E^*$, we define the k -th degeneration locus (the k -th determinantal) of the net of quadrics $\hat{\varphi} : L^{-1} \longrightarrow S^2 E^*$ to be the k -th determinantal of the corresponding (symmetric) vector bundle map

$$\varphi : E \longrightarrow E^* \otimes L.$$

(6.4.2) As we know, $\mathbb{I} \oplus Q \simeq \tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)}$, that is $Z = P_{G(2,V)}(\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)}) \simeq P_{G(2,V)}(\mathbb{I} \oplus Q)$ (see Corollary (6.2.3)).

Let $\pi : Z = P_G(\tau_{3,V^*}^* \oplus \mathcal{O}_G) \longrightarrow G = G(2,V)$ be the natural projection and let

$$\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)} \xrightarrow{\sim} \mathbb{I} \oplus Q \hookrightarrow S^2 \Sigma^*$$

be the abovedescribed embedding of vector bundles. Taking the tautological sequence for the projectivized vector bundle $\pi : Z = P_G(\tau_{3,V^*}^* \oplus \mathcal{O}_G) \longrightarrow G(2,V)$:

$$0 \longrightarrow \mathcal{O}_\pi(-1) \xrightarrow{j} \pi^*(\tau_{3,V^*}^* \oplus \mathcal{O}_G) \longrightarrow R \longrightarrow 0,$$

we obtain the composition of natural embeddings :

$$0 \longrightarrow \mathcal{O}_{\pi}(-1) \xrightarrow{j} \pi^*(\tau_{3,V^*}^* \oplus \mathcal{O}_G) \xrightarrow{i} \pi^*(S^2 \Sigma^*)$$
 of vector bundles over Z .

(6.4.3) By the construction of the tautological sequences, the projectivization of the embedding of the sheaf

$\mathcal{O}_{\pi}(-1)$ (= the relative $\mathcal{O}(-1)$, or the tautological subbundle) :

$$P(\mathcal{O}_{\pi}(-1)) \hookrightarrow P(\pi^*(\tau_{3,V^*}^* \oplus \mathcal{O}_G))$$

represents the embedding of the points of the fibers of

$P_G(\tau_{3,V^*}^* \oplus \mathcal{O}_G)$, regarded, in view of the isomorphism

$$\tau_{3,V^*}^* \oplus \mathcal{O}_G \simeq \mathbb{I} \oplus Q, \text{ as quadratic hypersurfaces}$$

in the corresponding projective spaces $P(\Sigma(V_2))$,

$V_2 \in G(2,V)$, lifted to Z . From the last and from the Corollary (6.3.7) we obtain :

(6.4.4) PROPOSITION. The factorfamily $\mathcal{C}_0^3 = \mathcal{C}^3 / \sigma$ is embedded naturally in $Z = P_{G(2,V)}(\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)})$ as a second determinantal of the net of quadrics :

$$0 \longrightarrow \mathcal{O}_{\pi}(-1) \xrightarrow{j} \pi^*(\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)}) \xrightarrow{i} \pi^*(S^2 \Sigma^*);$$

i.e. $\mathcal{C}_0^3 \simeq D_2(\hat{\varphi})$, where $\hat{\varphi} = i \cdot j$ (see (6.4.2)).

§ 7. The families $\mathcal{C}^3$ and $\mathcal{C}_0^3$ as zero-schemes.

7.1. We shall use the same symbol q for the quadratic form $q \in S^2 \Sigma^*(V_2)$ and for the corresponding symmetric operator $q : \Sigma(V_2) \longrightarrow \Sigma^*(V_2)$. Then Proposition (6.4.4) means that $\mathcal{C}_0^3 = \{(V_2, q) \in Z : \text{rank}(q) \leq 2\} = D_2(\hat{\varphi})$, (the q in the pair is regarded as a form, the q after the colon is regarded as an operator). The condition $\text{rank}(q) = k$ means that $\dim \text{Im}(q) = k$ ($\text{Im}(q)$ is a subspace of $\Sigma^*(V_2)$).

Let $0 \leq k \leq 5 = \dim \Sigma(V_2)$ and let $\tilde{D}_k = D_k(\hat{\varphi}) = \{(z; U_k) \in G_Z(k, \Sigma^*) : \text{Im}(q) \subseteq U_k, \text{ where } Z \ni z = (V_2, q)\}$.

The natural projection $\pi_k : (z; U_k) \longmapsto z$ provides a map: $\pi_k : \tilde{D}_k \longrightarrow D_k$, which is an isomorphism outside the locus $D_{k-1} \subset D_k$, $1 \leq k \leq 5 = \dim(\text{fibers of } \Sigma)$. The natural embedding $\tilde{D}_k \hookrightarrow G_Z(k, \Sigma^*)$ can be included in the commutative diagram:

$$(7.1.1) \quad \begin{array}{ccc} \tilde{D}_k & \hookrightarrow & G_Z(k, \Sigma^*) \\ \pi_k \downarrow & & \downarrow \wp \\ D_k & \hookrightarrow & Z \end{array},$$

where $\wp : G_Z(k, \Sigma^*) \longrightarrow Z$ is the natural projection.

(7.1.2) REMARK. Let $Q \in S^2 V_8^*$ (resp. $(Q = 0) \subset \mathbb{C} P(V_8)$) be a quadric. It is not hard to check that the set of all the quadrics Q in $P(V_8)$ (resp. $Q \in S^2 V_8^*$),

such that some of the spaces $\mathbb{I}(V_2) \oplus Q(V_2)$ contains a quadric of rank 1 (a double hyperplane in the corresponding $P(\sum(V_2))$ (resp., of $S^2 \sum^*(V_2)$), has a codimension > 1 inside the space of quadrics in $P(V_8)$ (resp., of $S^2 V_8^*$). Therefore, for the general quadric Q in V_8 the first determinantal $D_1(\hat{\varphi})$ vanishes. Because of that we can suppose in addition that the quadric Q is chosen such that the open condition

$$(iii) \quad D_1(\hat{\varphi}) = \emptyset$$

is fulfilled (see also the conditions (i) and (ii) in the Corollary (6.3.5)).

Taking into account (iii) from the Remark, we obtain that the projection $\pi_2: \tilde{D}_2 \longrightarrow D_2$ is an isomorphism.

7.2.

(7.2.1) Let

$$0 \longrightarrow \tau_{2, \sum^*} \xrightarrow{\alpha} G_Z(2, \sum^*) \times \sum^* \xrightarrow{\beta} \tau_{3, \sum}^* \longrightarrow 0$$

be the standard tautological sequence on the grassmannization $\wp: G_Z(2, \sum^*) \longrightarrow Z$. The embedding

$$\alpha: \tau_{2, \sum^*} \hookrightarrow G_Z(2, \sum^*) \times \sum^* \quad \text{defines an embedding}$$

$$s^2 \alpha: s^2 \tau_{2, \sum^*} \hookrightarrow G_Z(2, \sum^*) \times s^2 \sum^* \quad . \quad \text{The embedding}$$

$s^2 \alpha$ can be included in the exact sequence :

$$0 \longrightarrow s^2 \tau_{2, \sum^*} \xrightarrow{s^2 \alpha} G_Z(2, \sum^*) \times s^2 \sum^* \xrightarrow{\delta^*} \text{Coker}(2) \longrightarrow 0 ,$$

where $\text{Coker}(2)$ is the factor-bundle $s^2 \sum^* / s^2 \alpha(s^2 \tau_{2, \sum^*})$ on $G_Z(2, \sum^*)$.

(7.2.2) On the other hand, the net of quadrics $\hat{\varphi}$ de-

defines the embedding $0 \rightarrow 0_{\pi}(-1) \xrightarrow{\hat{\varphi}} s^2 \Sigma^*$ of bundles over Z . Taking into account the projection $\wp : G_Z(2, \Sigma^*) \rightarrow Z$ we can look at the sheaves $0_{\pi}(-1)$ and Σ as sheaves on $G_Z(2, \Sigma^*)$ (here we omit, for simplicity, the symbol $\wp^*$).

Composing the morphisms in the sequences above, we obtain the sequence

$$0 \rightarrow 0_{\pi}(-1) \xrightarrow{\hat{\varphi}} G_Z(2, \Sigma^*) \times s^2 \Sigma^* \xrightarrow{\gamma^*} \text{Coker}(2)$$

of natural maps of sheaves (resp., of bundles) over

$G_Z(2, \Sigma^*)$. Multiplying by the sheaf $(0_{\pi}(-1))^* = 0_{\pi}(1)$, we obtain the composition :

$$0 \rightarrow 0_{G(2, \Sigma^*)} \xrightarrow{\hat{\varphi} \otimes 0_{\pi}(1)} 0_{\pi}(1) \otimes s^2 \Sigma^* \rightarrow \\ \xrightarrow{\gamma^* \otimes 0_{\pi}(1)} 0_{\pi}(1) \otimes \text{Coker}(2) ;$$

the last defines a section s_0 of the sheaf

$0_{\pi}(1) \otimes \text{Coker}(2)$ over the variety $G_Z(2, \Sigma^*)$,

$$s_0 = (\gamma^* \otimes 0_{\pi}(1)) \cdot (\hat{\varphi} \otimes 0_{\pi}(1)).$$

(7.2.3) By definition the zero scheme $Z(s_0) \subset G_Z(2, \Sigma^*)$

of the section $s_0 \in H^0(G_Z(2, \Sigma^*), 0_{\pi}(1) \otimes \text{Coker}(2))$

coincides with the 0^{th} determinantal $D_0(s_0)$ of the

section s_0 , regarded as a vector bundle map

$$s_0 : 0_{G(2, \Sigma^*)} \rightarrow 0_{\pi}(1) \otimes \text{Coker}(2). \text{ Consequently,}$$

$$Z(s_0) = D_0(s_0) =$$

$$= \left\{ (z; U_2) \in Z \times G(2, \Sigma^*(V_2)) : z = (V_2, q) \text{ \& } s_0(z; U_2) = 0 \right\}$$

So we have:

$$(1) \quad \hat{\varphi} = i \cdot j, \text{ where the embedding } j : 0_{\pi}(-1) \rightarrow$$

$\longrightarrow \pi^*(\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)})$ represents the closed embedding of the "point" q in the fiber $\pi^{-1}(V_2) \subset Z$ and the embedding $i : \pi^*(\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)}) \hookrightarrow S^2 \Sigma^*$ corresponds to the natural representation of the points in the fibers of $\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)}$ as quadrics in the fibers $\Sigma(V_2)$ of Σ (here we use the isomorphism $\tau_{3,V^*}^* \oplus \mathcal{O}_{G(2,V)} \xrightarrow{\sim} \mathbb{I} \oplus \mathcal{Q}$, see (6.2.3) (ii) and (6.3.1));

(2) The maps

$$\gamma(z) = \gamma(V_2, q) : S^2 \Sigma^*(V_2, q; U_2) \longrightarrow \text{Coker}(2),$$

as factor-maps, send the quadrics $q \in S^2 \Sigma^*(V_2)$ to the corresponding classes $q(\text{mod. } S^2 U_2)$, over the elements $(z; U_2) = (V_2, q; U_2) \in G_Z(2, \Sigma^*)$.

From (1) and (2) we derive, that

$$\begin{aligned}
 Z(s_0) &= \{(V_2, q; U_2) \in G_Z(2, \Sigma^*) : q(\text{mod. } S^2 U_2) = 0; \\
 &\quad q \text{ regarded as an element of } S^2 \Sigma^*(V_2)\} = \\
 &= \{(V_2, q; U_2) \in G_Z(2, \Sigma^*) : U_2 \subseteq \text{Image}(q) \subset \Sigma^*(V_2); \\
 &\quad q \text{ regarded as an operator } q \in \text{Sym}(\Sigma(V_2), \Sigma^*(V_2))\}.
 \end{aligned}$$

Consequently, $Z(s_0) = \widetilde{D}_2(\widehat{\varphi}) \subset G_Z(2, \Sigma^*)$,
 (see 7.1 and (7.1.1)).

But, by the condition (iii) (see Remark (7.1.2)),

$D_1(\widehat{\varphi}) = \emptyset$, that is, there are no double hyperplanes in the fibers $H^0(P(\Sigma(V_2)), \mathcal{O}(2 - \sigma_{2,0}(P(V_2)).X_{10}))$,
 $V_2 \in G(2, V)$ of the bundle "of quadrics" $\mathbb{I} \oplus \mathcal{Q}$.

Consequently, the natural projection $\pi_2 : \tilde{D}_2(\hat{\varphi}) \longrightarrow D_2(\hat{\varphi})$ (cf. (7.1.2)) is an isomorphism.

Summing up the obtained incidences, we derive :

(7.2.4) COROLLARY. Let $V_8 \subset V_{10} = \wedge^2 V$ and Q fulfil the open conditions (i), (ii) from the Corollary (6.3.5) and (iii) from the Remark (7.1.2). Let $\mathcal{C}_0^3 = \mathcal{C}^3 / \sigma$ be the factorfamily of the family of rational normal cubics on $X = X_{10} = G(2, V) \cap P(V_8) \cap Q$ and let $s_0 \in H^0(G_Z(2, \Sigma^*), \mathcal{O}_\pi(1) \otimes \text{Coker}(2))$ be the section, described in (7.2.2). Then $\mathcal{C}_0^3$ embeds naturally in $G_Z(2, \Sigma^*)$ as a zero scheme of the section s_0 , that is :

$$\mathcal{C}_0^3 \xrightarrow{\sim} Z(s_0) \subset G_Z(2, \Sigma^*).$$

(7.2.5) NOTE. By convention, the symbol $\wp^*$, where $\wp : G_Z(2, \Sigma^*) \longrightarrow Z$, is dropped, i.e. with some abuse of the notations we write $\wp^* \mathcal{O}_\pi(1) \otimes \text{Coker}(2) = \mathcal{O}_\pi(1) \otimes \text{Coker}(2)$.

There exists a natural isomorphism of duality :

$$\delta : G_Z(2, \Sigma^*) \xrightarrow{\sim} G_Z(3, \Sigma),$$

induced by the natural duality of the fibers

$$\delta(V_2) : G(2, \Sigma^*(V_2)) \xrightarrow{\sim} G(3, \Sigma(V_2)), \quad V_2 \in G(2, V).$$

The composition $s_0^* = \delta^* \cdot s_0$ defines a section of (the isomorphic δ -preimage of) the sheaf

$\mathcal{O}_\pi(1) \otimes \text{Coker}(2)$ on $G_Z(3, \Sigma)$. So we obtained :

(7.2.6) COROLLARY. In the conditions of the Corollary(7.2.4) there is a natural embedding $\mathcal{C}_0^3 \hookrightarrow G_Z(3, \Sigma)$, such that :

$$(1) \quad \mathcal{C}_o^3 \xrightarrow{\sim} Z(s_o^*) \subset G_Z(3, \Sigma),$$

where $s_o^* = \delta \cdot s_o$ is the section, defined just above;

(2) Let $(C, \bar{C})$ be an element of $\mathcal{C}_o^3$ (i.e. $(C, \bar{C})$ is a pair of involutive rational normal cubics on $X = X_{10}$). Let $V_2 \subset V$ be the subspace (of $\dim. = 2$), such that

$C_{P(V_2)} = \sigma_{2,0}(P(V_2)) \cdot X_{10} = C + \bar{C}$ (see Corollary (5.3.2)) and let H_C and $H_{\bar{C}} \in \Sigma^*(V_2)$ be the hyperplanes in $\Sigma(V_2)$ such that $\langle C \rangle = \text{Span}(C) = P(H_C)$

and $\langle \bar{C} \rangle = \text{Span}(\bar{C}) = P(H_{\bar{C}})$. Then the embedding

$\mathcal{C}_o^3 \hookrightarrow G_Z(3, \Sigma)$ sends the pair $(C, \bar{C})$ to the triple $(V_2, q = H_C \cdot H_{\bar{C}}; H_C \cap H_{\bar{C}}) \in (V_2, q = H_C \cdot H_{\bar{C}}) \times G(3, \Sigma(V_2)) \subset G_Z(3, \Sigma)$.

Proof. Point (1) follows immediately from the definition of the section $s_o^* = \delta \cdot s_o$ and from the previous Corollary.

To prove (2) it is sufficient to see that the third member $H_C \cap H_{\bar{C}}$ of the triple corresponds, by the duality δ , to the subspace $U_2 := \text{Image}(q) \subset \Sigma^*(V_2)$, where

$q = H_C \cdot H_{\bar{C}}$ is regarded as an operator from $\text{Sym}(\Sigma(V_2), \Sigma^*(V_2))$. Moreover, $\dim(\text{Image}(q = H_C \cdot H_{\bar{C}})) = 2$, because of the fact, that $D_1(\hat{\varphi}) = \emptyset$. In particular, H_C and $H_{\bar{C}}$ are not proportional, that is

$U_2 = \text{Image}(q = H_C \cdot H_{\bar{C}})$ belongs to the Grassmannian $G(2, \Sigma(V_2))$.

7.3.

(7.3.1) Let

$$0 \longrightarrow \tau_{4,\Sigma} \longrightarrow P_Z(\Sigma^*) \times \Sigma \longrightarrow \tau_{1,\Sigma^*}^* \longrightarrow 0$$

be the standard tautological sequence on the projectivization $P_Z(\Sigma^*) = G_Z(4, \Sigma)$. Taking the dual, we obtain the natural surjection $P_Z(\Sigma^*) \times \Sigma^* \longrightarrow \tau_{4,\Sigma}^* \longrightarrow 0$, hence, the surjection

$$\epsilon : P_Z(\Sigma^*) \times s^2 \Sigma^* \longrightarrow s^2 \tau_{4,\Sigma}^* \longrightarrow 0.$$

Let now $\tau : P_Z(\Sigma^*) \longrightarrow Z$ be the natural projection. Taking the τ -preimages of the sheaves, we lift the map $\hat{\varphi}$ onto $P_Z(\Sigma^*)$:

$$0 \longrightarrow \mathcal{O}_{\pi}(-1) \xrightarrow{\hat{\varphi}} P_Z(\Sigma^*) \times s^2 \Sigma^* ;$$

as usual the symbol τ^* is dropped.

Just as in (7.2.2), after multiplying by $\mathcal{O}_{\pi}(1)$, we obtain the composite map :

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{P_Z(\Sigma^*)} \xrightarrow{\hat{\varphi} \otimes \mathcal{O}_{\pi}(1)} P_Z(\Sigma^*) \times \mathcal{O}_{\pi}(1) \otimes s^2 \Sigma^* \longrightarrow \\ \xrightarrow{\epsilon \otimes \mathcal{O}_{\pi}(1)} \mathcal{O}_{\pi}(1) \otimes s^2 \tau_{4,\Sigma}^* , \end{aligned}$$

hence, we obtain a well-defined section s of the sheaf

$$\begin{aligned} \mathcal{O}_{\pi}(1) \otimes s^2 \tau_{4,\Sigma}^* \text{ over } P_Z(\Sigma^*) , \quad s = \\ = (\epsilon \otimes \mathcal{O}_{\pi}(1)) \cdot (\hat{\varphi} \otimes \mathcal{O}_{\pi}(1)) . \end{aligned}$$

As in § 7.2 we shall prove the following

(7.3.2) PROPOSITION. In the conditions of the Corollary (7.2.4), the family $\mathcal{C}^3$ of the rational normal cubics on X is embedded naturally in the projectivized bundle

$P_Z(\Sigma^*)$ as a zero scheme of the section s , that is :

$$\mathcal{C}^3 \xrightarrow{\sim} Z(s) \subset P_Z(\Sigma^*) .$$

Proof. (for more details see the proof of the Corollary (7.2.4)). As in (7.2.3) by the definition of the tautological sequence the map $\epsilon : P_Z(\Sigma^*) \times S^2 \Sigma^* \longrightarrow S^2 \tau_{4,\Sigma}^*$ sends the element $(V_2, q ; Y_4)$ of the fibre $\tau^{-1}(V_2, q)$ to the element $(V_2, q ; q|_{Y_4}) \in S^2 \tau_{4,\Sigma}^*$ (here $q|_{Y_4}$ is the restriction of the quadric $q \in S^2 \Sigma^*(V_2)$ to the subspace $Y_4 \subset \Sigma(V_2)$, $\dim(Y_4) = 4$) .

Consequently, the elements of the zero scheme of s are defined by the condition: $Y_4 \subseteq \text{Kernel}(q)$; here the quadric q is regarded as an operator from $\text{Sym}(\Sigma(V_2), \Sigma^*(V_2))$. In other words, the zero scheme

$Z(s) = \{ (V_2, q ; Y_4) : (V_2, q) \in Z, Y_4 \subset \Sigma(V_2) \text{ is an element of } G(4, \Sigma(V_2)) \text{ such that } Y_4 \subseteq \text{Kernel}(q) \}$

But the symmetric operator $q : \Sigma(V_2) \longrightarrow \Sigma^*(V_2)$, $\dim \Sigma(V_2) = 5$, contains four-dimensional subspaces in its kernel, if and only if $\text{rank}(q) \leq 2$. By the

way, there are no quadrics q of rank = 1, since

$D_1(\hat{\varphi}) = \emptyset$ (see for example (7.2.3) - the proof of the Corollary (7.2.4)). Because of that $\text{rank}(q) \leq 2$

iff $\text{rank}(q) = 2$, that is iff $q = H \cdot \bar{H}$, for some non-proportional linear functions (non-coincident hyperplanes) H and $\bar{H}$ of $\Sigma^*(V_2)$. But the projective hyperplanes $P(H)$ and $P(\bar{H})$ cut out

on X (in the described situation) a pair of involutive rational normal cubics $C = X \cap P(H)$ and

$\bar{C} = X \cap P(\bar{H})$. Because of that the zero scheme $Z(s)$

describes the set of the components of the quadrics q of rank = 2, q being the second element of the pair $(V_2, q) \in D_2(\hat{\mathcal{V}})$ (see Proposition (6.4.4), § 7.1 and § 7.2). Equivalently, the zero scheme $Z(s)$ describes the set of the rational normal cubics (the intersections of $X = X_{10}$ with the corresponding components $P(H)$ and $P(\bar{H})$ of the quadrics $q = H \cdot \bar{H}$ of rank = 2, as above). The Proposition is proved.

From the proof of the Proposition (7.3.2), just as for the Corollary (7.2.6) (2), we obtain the following geometrical description of the embedding $\mathcal{C}^3 \hookrightarrow P_Z(\Sigma^*) = G_Z(4, \Sigma)$:

(7.3.3) COROLLARY. Let $C \in \mathcal{C}^3$ be a rational normal cubic curve on $X = X_{10}$. Let $V_2 \subset V$ be the subspace (of dim. = 2), such that $C_{P(V_2)} = \mathcal{G}_{2,0}(P(V_2)) \cdot X = C + \bar{C}$, where $\bar{C}$ is the involutive of C (see Corollary (5.3.2)) and let H_C and $H_{\bar{C}} \in \Sigma^*(V_2)$ be the hyperplanes in $\Sigma(V_2)$, such that $\langle C \rangle = \text{Span}(C) = P(H_C)$ and $\langle \bar{C} \rangle = \text{Span}(\bar{C}) = P(H_{\bar{C}})$. Then the embedding $\mathcal{C}^3 \hookrightarrow P_Z(\Sigma^*) = G_Z(4, \Sigma)$ sends the element $C \in \mathcal{C}^3$ to the triple $(V_2, q = H_C \cdot H_{\bar{C}}; H_C) \in (V_2, q = H_C \cdot H_{\bar{C}}) \times G(4, \Sigma(V_2)) \subset G_Z(4, \Sigma) = P_Z(\Sigma^*)$.

§ 8. Tangent Bundle Theorem for the family
of rational normal cubics $\mathcal{C}^3$.

8.1. Before formulating and proving the Tangent Bundle Theorem (T.B.T.), we shall carry out some constructions, which clarify the local situation. From now on we shall suppose that the curve $C \in \mathcal{C}^3$ is chosen sufficiently general, in particular, we can suppose that the curve C and its involutive $\bar{C}$ are smooth rational normal cubics with normal bundles $N_{C/X}$ and $N_{\bar{C}/X}$, isomorphic to $\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(1)$ (see Proposition 5.1), etc.

(8.1.1) Let C and $\bar{C}$ be a pair of involutive rational normal cubics on X , let $V_2 \subset V$ be the corresponding subspace of $\dim = 2$, such that $C_1 = C + \bar{C}$, where $1 = P(V_2)$ (see Corollary (5.3.2)). Let $W = G \cdot P(V_8)$, $X = X_{10} = W \cdot Q = G \cdot P(V_8) \cdot Q$ and $S_1 = \mathcal{G}_{2,0}(1) \cdot W$ are defined as before.

Examine the pairs of successive embeddings:

$$C \subset X \subset W \quad \text{and} \quad C \subset S_1 \subset W$$

and the corresponding exact sequences of normal sheaves:

$$(1) \quad 0 \rightarrow N_{C/X} \rightarrow N_{C/W} \rightarrow N_{X/W} \otimes \mathcal{O}_C \rightarrow 0 \quad \text{and}$$

$$(2) \quad 0 \rightarrow N_{C/S_1} \rightarrow N_{C/W} \rightarrow N_{S_1/W} \otimes \mathcal{O}_C \rightarrow 0.$$

Multiplying the sequences (1) and (2) by the sheaf $\mathcal{O}(-1) = \mathcal{O}_{P(V_8)}(-1)$ and identifying the middle members, we obtain the diagonal map:

$$(3) \quad \gamma_C : N_{C/S_1} \otimes \mathcal{O}(-1) \rightarrow N_{C/W} \otimes \mathcal{O}(-1) \rightarrow N_{X/W} \otimes \mathcal{O}(-1) \otimes \mathcal{O}_C$$

Since $\deg(C) = 3$, then $\mathcal{O}(-1) \otimes \mathcal{O}_C \simeq \mathcal{O}_C(-3)$,
 $C \simeq P^1$.

From $X = W \cdot Q$, where Q is a quadric in $P^7 = P(V_8)$, it follows that $N_{X/W} = O_X(2)$. Therefore,

$$N_{X/W} \otimes O(-1) \otimes O_C \simeq O(1) \otimes O_C \simeq O_C(3).$$

(8.1.2) On the other hand, $N_{C/S_1} = O(1) \otimes O_C$, since C is a hyperplane section of the rat. norm. cubic scroll $S_1 \subset \text{Span}(S_1) = P(\sum (V_2))$, $1 = P(V_2)$; (see (5.4.2)). Therefore:

$$(3') \quad \varphi_C: O_C \longrightarrow O(1) \otimes O_C = O_C(3).$$

Taking into account the identifications above, the sequence $(2) \otimes O(-1)$ takes the form:

$$(2') \quad 0 \longrightarrow O_C \longrightarrow N_{C/W} \otimes O(-1) \longrightarrow N_{S_1/W} \otimes O(-1) \otimes O_C \longrightarrow C$$

(8.1.3) LEMMA.

Let $C \in \mathcal{C}^3(X)$ be a general r. n. cubic. Then

$$N_{S_1/W} \otimes O_C \simeq O_C(2) \oplus O_C(2).$$

The proof of the lemma will be given later on (see (8.1.10)).

(8.1.4) From (8.1.3) we derive the identification:

$$(4) \quad N_{S_1/W} \otimes O(-1) \otimes O_C \simeq O_C(-1) \oplus O_C(-1),$$

therefore the long exact sequence of cohomologies, associated to the sequence $(2')$, defines the natural isomorphisms:

$$(5) \quad H^i(C, O_C) \xrightarrow{\sim} H^i(C, N_{C/W} \otimes O(-1)), \quad i = 0, 1, \dots$$

In fact, all the cohomology groups of the sheaf $N_{S_1/W} \otimes O(-1) \otimes O_C$ vanish.

(8.1.5) Let now look at the sequence $(1) \otimes O(-1)$.

Since $N_{C/X} \otimes O(-1) \simeq O_C(-3) \oplus O_C(-2)$, then

$$H^0(C, N_{C/X} \otimes O(-1)) = 0.$$

From $H^1(C, N_{C/X}) = H^1(C, \mathcal{O}_C \oplus \mathcal{O}_C(1)) = 0$,
we obtain that the tangent space $T_{\mathcal{C}^3(C)}$ is isomorphic
to $H^0(C, N_{C/X})$.

Let $\mathcal{R}_{\mathcal{C}^3} = T_{\mathcal{C}^3}^*$ be the cotangent sheaf of
and let ω_X be the sheaf of higher differential forms
on X (the canonical sheaf of X). Using the Serre
duality, we can write the following sequence of identities:
$$\begin{aligned} \mathcal{R}_{\mathcal{C}^3(C)} &= T_{\mathcal{C}^3(C)}^* = H^0(C, N_{C/X})^* = \\ &= H^1(C, N_{C/X}^* \otimes \omega_C) = H^1(C, N_{C/X} \otimes \det N_{C/X}^* \otimes \omega_C) = \\ &= H^1(C, N_{C/X} \otimes \mathcal{O}_C(-1) \otimes \omega_C) = H^1(C, N_{C/X} \otimes \mathcal{O}(-1)) = \\ &= H^1(C, N_{C/X} \otimes \omega_X), \text{ since } \omega_X = \mathcal{O}(-1) \otimes \mathcal{O}_X. \end{aligned}$$

From the last we obtain that the long exact cohomology
sequence, associated to the exact sequence $(1) \otimes \mathcal{O}(-1)$
takes the form:

$$\begin{aligned} (1') \quad 0 &\longrightarrow (H^0(C, N_{C/X} \otimes \omega_X) = 0) \longrightarrow H^0(C, N_{C/W} \otimes \mathcal{O}(-1)) \longrightarrow \\ &\longrightarrow H^0(C, \mathcal{O}(1) \otimes \mathcal{O}_C) \longrightarrow (H^1(C, N_{C/X} \otimes \omega_X) = \mathcal{R}_{\mathcal{C}^3(C)}) \longrightarrow \\ &\longrightarrow H^1(C, N_{C/W} \otimes \mathcal{O}(-1)) \longrightarrow \dots \end{aligned}$$

Using (5) we obtain the sequence (see also (8.1.2) (3')):

$$\begin{aligned} (1'') \quad 0 &\longrightarrow H^0(C, \mathcal{O}_C) \xrightarrow{H^0(\varphi_C)} H^0(C, \mathcal{O}(1) \otimes \mathcal{O}_C) \longrightarrow \\ &\longrightarrow \mathcal{R}_{\mathcal{C}^3(C)} \longrightarrow (H^1(C, \mathcal{O}_C) = 0) \end{aligned}$$

(8.1.6) If we suppose for a while that we have performed
a globalization of the constructions above, simultaneously
for all the elements C of the family of rational normal
cubics $\mathcal{C}^3 = \mathcal{C}^3(X)$, then the members of the sequence

(1'') will be induced by restriction on the fibers from sheaves, defined globally on the family $\mathcal{C}^3$ (as the fiber $\mathcal{H}_{\mathcal{C}^3(C)}$).

(6) REMARK. In fact, the diagonal map $\varphi_C : \mathcal{O}_C \longrightarrow \mathcal{O}(1) \otimes \mathcal{O}_C$ (see (3')) corresponds to the choice of the divisor

$C \cap \bar{C}$ on C ; the last divisor represents the hyperplane section of C in the projective space $\langle C \rangle = \text{Span}(C) \subset P^3$ (see (5.4.1) and (5.4.2)).

(8.1.7) On the other hand, as we know, the family $\mathcal{C}^3$ is embedded in $P_Z(\Sigma^*) = G_Z(4, \Sigma)$ as a zero scheme of the section s , and the factorfamily $\mathcal{C}_0^3 = \mathcal{C}^3/\sigma$ is embedded in $G_Z(3, \Sigma)$ as a zero scheme of the section s_0^* (see Cor. (7.2.6), Prop. (7.3.2) and Cor. (7.3.3)).

From the last we conclude, that the tautological sheaves

$\tau_{3, \Sigma}$ and $\tau_{2, \Sigma}^*$ from the standard tautological sequence :

$$(a) \quad 0 \longrightarrow \tau_{3, \Sigma} \longrightarrow G_Z(3, \Sigma) \times \Sigma \longrightarrow \tau_{2, \Sigma}^* \longrightarrow 0$$

are well-defined on the family $\mathcal{C}_0^3$, and the tautological sheaves $\tau_{4, \Sigma}$ and $\tau_{1, \Sigma}^*$ from the sequence :

$$(b) \quad 0 \longrightarrow \tau_{4, \Sigma} \longrightarrow P_Z(\Sigma^*) \times \Sigma \longrightarrow \tau_{1, \Sigma}^* \longrightarrow 0$$

are well-defined on the family $\mathcal{C}^3$.

But the existence of the natural 2-sheeted covering

$$\pi : \mathcal{C}^3 \longrightarrow \mathcal{C}_0^3, \text{ induced from the involution}$$

$$\sigma : \mathcal{C}^3 \longrightarrow \mathcal{C}^3 \quad (\sigma(C) = \bar{C}, \quad \sigma^2 = \text{id})$$

shows that the (π -preimages of the) sheaves $\tau_{3, \Sigma}$ and $\tau_{2, \Sigma}^*$ are defined also on the family $\mathcal{C}^3$;

as usual we write $\tau_{3,\Sigma}$ and $\tau_{2,\Sigma}^*$ instead of $\pi^* \tau_{3,\Sigma}$ and $\pi^* \tau_{2,\Sigma}^*$. Then, the (lifted by π) exact sequence (a) on , together with the exact sequence (b) , define an embedding $j: \tau_{3,\Sigma} \hookrightarrow \tau_{4,\Sigma}$ of sheaves on $\mathbb{C}^3$ (see also (7.2.6), (7.3.2) and (7.2.6)).

Let $R = \tau_{4,\Sigma} / \tau_{3,\Sigma}$ be the factor-sheaf of the embedding j . Hence, we obtain the exact sequence:

$$(c) \quad 0 \longrightarrow \tau_{3,\Sigma} \xrightarrow{j} \tau_{4,\Sigma} \xrightarrow{\beta} R \longrightarrow 0$$

of sheaves on the family $\mathbb{C}^3$.

(8.1.8) From the cited corollaries and proposition we derive also that the embedding $j: \tau_{3,\Sigma} \hookrightarrow \tau_{4,\Sigma}$ corresponds to the embeddings $H_C \cap H_{\bar{C}} \hookrightarrow H_C$ of subspaces of $\sum (V_2)$; remember that $P(H_C) = \text{Span}(C) \subset P(\sum (V_2))$, $P(H_{\bar{C}}) = \text{Span}(\bar{C}) \subset P(\sum (V_2))$, and $C + \bar{C} = C_1 = \sigma_{2,0}(1) \cdot X_{10} \subset \text{Span}(C_1) = P(\sum (V_2))$, where $1 = P(V_2)$. (ibid.).

But the last means exactly that the epimorphism

$$\beta: \tau_{4,\Sigma} \longrightarrow R = \tau_{4,\Sigma} / \tau_{3,\Sigma} \text{ coincides with the dual of the map}$$

$$\begin{aligned} H^0(\varphi_C): H^0(C, \mathcal{O}_C) &\longrightarrow H^0(C, \mathcal{O}(1) \otimes \mathcal{O}_C) \simeq \\ &\simeq H^0(C, \mathcal{O}_{P(\sum (V_2))}(1) \otimes \mathcal{O}_C) \end{aligned}$$

(see the sequence (1'') and the description of the map

φ_C in the beginning of the Remark). In fact, the description of the epimorphism $\beta: \tau_{4,\Sigma} \longrightarrow R$ corresponds to the description of the global sections of

the linear system $|P(H_C \cap H_{\bar{C}})|_C| = |C \cap \bar{C}|$ on C ,
defined by the map φ_C . Therefore, the exact sequence (1'')
takes the form:

$$(1''') \quad 0 \longrightarrow (\tau_{4,\Sigma} / \tau_{3,\Sigma})^*(C) \xrightarrow{H^0(\varphi_C) = \beta^*} \tau_{4,\Sigma}(C) \longrightarrow \mathcal{R}_{\mathcal{C}^3(C)} \longrightarrow 0,$$

therefore $\mathcal{R}_{\mathcal{C}^3(C)} \simeq \tau_{3,\Sigma}^*(C).$

So, we formulate the following

(8.1.9) PROPOSITION (The open Tangent Bundle Theorem for
the family $\mathcal{C}^3$ of rational normal cubic curves on X).

Let $U = \{C \in \mathcal{C}^3 : C \text{ and } \bar{C} \text{ are smooth rational normal cubic curves on } X = X_{10} = G \cap P(V_8) \cap Q, \text{ such that } N_{C/X} \text{ and } N_{\bar{C}/X} \text{ are isomorphic to } \mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(1) \text{ (see } \S 5.1 \text{), and also, such that the surfaces } S_1 \text{ and } S_C \text{ are rational normal cubic scrolls (see (2.2.4) (1) and } \S 5.2 \text{) and } N_{S_1/W} \otimes \mathcal{O}_C \simeq N_{S_1/W} \otimes \mathcal{O}_{\bar{C}} \simeq \mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(2) \} \subset \mathcal{C}^3.$

Then, there exists a naturally defined isomorphism

$$\tau_{3,\Sigma}|_U \longrightarrow \mathcal{R}_{\mathcal{C}^3}|_U.$$

To prove the Tangent Bundle Theorem (8.1.9) we need:

(1). to prove Lemma (8.1.3), and (2). to globalize the local constructions, leading to the sequence (1''').

(8.1.10) Proof of Lemma (8.1.3).

We shall suppose in addition that the surface $S_C = (\bigcup l : l - \text{a line in } P^4 \text{ \& } l \in C)$ is a rational normal cubic scroll (see (5.2.1) ; by the way, the surface S_C cannot be a cone (see Corollary (5.2.3))). Though this superfluous restriction will be not used in the proof, the author supposes that the normal sheaves $N_{C/Y}$ for the curves C of the type (5.2.1) (2.a) (see also Corollary (5.2.3)), where $Y = Y_5$ is the threefold defined below (see (1) , (5) and (7)) , are isomorphic to $O_{P^1}(1) \oplus O_{P^1}(3)$. In any case, the last does not prevent us from proving (8.1.3).

(1) It is well known (see [F , (B.5.8)]) that the tangent bundle $T(G(2,V))$ is isomorphic to the bundle of homomorphisms $\text{Hom} (\tau_{2,V} , \tau_{3,V}^*) = \tau_{2,V}^* \otimes \tau_{3,V}^*$.

Let $P^3 = P(V_4)$, where $0 \subset V_4 \subset V$. Then, the embedded cycle $\sigma_{1,1}(P^3)$ carries out, in practice, an embedding of the Grassmannian $G(2,V_4)$ into $G(2,V)$; in particular, we have the exact sequence :

$$\begin{aligned} 0 \longrightarrow \text{Hom} (\tau_{2,V_4} , \tau_{2,V_4}^*) &\longrightarrow \text{Hom} (\tau_{2,V} , \tau_{3,V}^*) \otimes O_{\sigma_{1,1}(P^3)} \\ &\longrightarrow N (\sigma_{1,1}(P^3) , G) \longrightarrow 0 . \end{aligned}$$

We can derive from here that the normal sheaf $N (\sigma_{1,1}(P^3) , G)$ equals to the restriction $\tau_{2,V}^* \otimes O_{\sigma_{1,1}(P^3)}$.

Let now $Y = Y_5 = G \cap P^6$, and let $P^3 \subset P^4$ is choosen in such a way that $\sigma_{1,1}(P^3)$ intersects Y transversely; we shall suppose also that Y is choosen to be a smooth intersection. The intersection $q = \sigma_{1,1}(P^3) \cap Y$ is a conic; we shall suppose in addition that the surface $S_q = (\bigcup l : l - \text{a line in } P^4 \text{ \& } l \in q)$ is a smooth quadric (see § 4.3 (1)). Then we shall have:

$$\begin{aligned} N_{q/Y} &=: N(q, Y) = N(\sigma_{1,1}(P^3) \cap P^6, G \cap P^6) = \\ &= N(\sigma_{1,1}(P^3), G) \otimes O_q = \tau_{2,Y}^* \otimes O_q = O_q(1) \oplus O_q(1). \end{aligned}$$

(2) Let now l be a line lying on $Y = Y_5$. It is easy to see that $N_{l/Y}$ is one of the sheaves $O_l \oplus O_l$ or $O_l(-1) \oplus O_l(1)$. In [I1, (5.2)] (see also [I4, (7.2)]) was proven that $N_{l/Y} = O_l \oplus O_l$ for the general line $l \subset Y_5$ (see also [FN]).

(3) Let now q and l be a conic and a line, lying on $Y = Y_5$ and let q and l intersect in a point. Let $H = P^5 \subset P^6$ be a sufficiently general hyperplane through $q \cup l$, such that the surface $S = S_5 = Y \cap H$ is smooth. The surface S is a Del Pezzo surface of degree 5 in $P^5 = H \subset P^6$. The surface $S = S_5$ is obtained from the projective plane P^2 after blowing-up of four points x_1, x_2, x_3 and $x_4 \in P^2$. As S_5 is choosen to be smooth, then no three of the points x_1, x_2, x_3 and x_4 are collinear.

(4) Let now $z \in P^2$ be a point, which does not lie on the union of the lines $l_{ij} = \langle x_i, x_j \rangle$, $1 \leq i < j \leq 4$.

Let $\sigma = \bigcap_{1 \leq k \leq 4} \sigma(x_k) : S \longrightarrow P^2$ be the map described in (3). The inverse of σ is the rational map

φ given by the non-complete linear system

$$\left| \mathcal{O}_{P^2} (3 - x_1 - x_2 - x_3 - x_4) \right| : P^2 \dashrightarrow P^5.$$

Let now Q be a conic in P^2 through the points x_1, x_2, x_3 and z . It is easy to see that the proper σ -preimage $C \subset S = S_5$ of the conic $(Q = 0) \subset P^2$, $Q \in H^0(\mathcal{O}_{P^2}(2 - x_1 - x_2 - x_3 - z))$ is a rational normal cubic curve.

The line l (see (3)) is one of the ten lines, which lie on the Del Pezzo surface $S = S_5 \subset P^5$. Obviously, we can suppose that the morphism σ is chosen in such a way that $l = \sigma^{-1}(x_4) \subset S_5$.

Let $(t) = (t_1 : t_2)$ be the homogeneous coordinates of $P^1 = \left| \mathcal{O}_{P^2}(2 - x_1 - x_2 - x_3 - z) \right|$. Evidently, the proper preimage $C_{(t)}$ of the conic $Q_{(t)} \in P^1 = P^1(t)$ splits iff $Q_{(t)}$ passes through the point x_4 . Let $(1) = (1 : 1)$, and let $x_4 \in Q_{(1)}$. Then $C_{(1)} = q \cup l$, where q is the conic and l is the line $l = \sigma^{-1}(x_4)$; obviously, $(q, l)_S = 1$.

(5) Let now $\beta_{(t)} : Y_{(t)} \longrightarrow Y = Y_5$, $(t) \neq (1)$, be the blowing-up of the curve $C_{(t)} (\subset S_5) \subset Y = Y_5$; let also $S_{(t)} = \beta_{(t)}^{-1}(C_{(t)})$ be the ruled surface over the curve $C_{(t)}$, $(t) \neq (1)$. It is easy to see that the normal sequence for $C_{(t)} \subset S_5 \subset Y$ is :

$$0 \rightarrow \mathcal{O}_{C(t)}(1) \rightarrow N_{C(t)/Y} \rightarrow \mathcal{O}_{C(t)}(3) \rightarrow 0.$$

Therefore, $N_{C(t)/Y} = \mathcal{O}_{C(t)}(a) \oplus \mathcal{O}_{C(t)}(b)$, where $a + b = 4$ and $1 \leq a \leq b \leq 3$.

$$\text{Let } \mathcal{S} = \overline{(t) \neq (1) \cup S_{(t)}} \subset \overline{(t) \neq (1) \cup Y_{(t)}} = \mathcal{Y}.$$

Obviously, $\mathcal{S}$ (resp. $\mathcal{Y}$) is a bundle of rational surfaces (resp. of rational threefolds) over the projective line $P^1 = P^1(t)$.

(6) Let the general rational normal cubic $C \subset Y_5$ has a normal bundle $\mathcal{O}_C(1) \oplus \mathcal{O}_C(3)$. Then we can choose the hyperplane H (and, hence, the surface $S_5 = Y_5 \cap H$) through the given degenerate rational normal cubic $C_{(1)} = q \cup l$ (see (3)), and the point $z \in P^2$ (see (4)) in such a way that $N_{C(t)/Y} = \mathcal{O}_{C(t)}(1) \oplus \mathcal{O}_{C(t)}(3)$ for any $(t) \in P^1(t)$ in some neighbourhood of $(t) = (1)$. We can also suppose that the normal sheaves of q and l are as in (1) and (2), i.e. $N_{q/Y} = \mathcal{O}_q(1) \oplus \mathcal{O}_q(1)$ and $N_{l/Y} = \mathcal{O}_l \oplus \mathcal{O}_l$. The exceptional surfaces $S_{(t)} = \beta_{(t)}^{-1}(C_{(t)}) \subset Y_{(t)}$ are isomorphic to the ruled surface

$\mathbb{F}_2 = P(N_{C(t)/Y}) = P(\mathcal{O}(1) \oplus \mathcal{O}(3))$, for all $(t) \neq (1)$, around (1), The surface $S_{(1)} = \mathcal{S} \setminus [(t) \neq (1) \cup S_{(t)}]$ is a degeneration of $\mathbb{F}_2$; on the other hand, $S_{(1)}$ is an union of the two quadrics - the quadric A over q and the quadric B over l .

The smooth quadrics A and B intersect along the common generator $A \cap B$, because $(q, 1)_S = 1$. But then the union $A \cup B$ cannot be a degeneration of the family of $\mathbb{F}_2$'s, described above. Consequently, the general rational normal cubic curve $C \subset Y = Y_5$ has a normal bundle $N_{C/Y} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$ (see (5)).

(7) Let now $C \subset X = X_{10}$ be a general rational normal cubic curve. In particular, C is embedded in the rational normal cubic scroll $S_1 \subset W$ as a hyperplane section (see Prop. (2.2.4) (1) and Cor. (5.3.2)).

Let $H = P^6 \subset P^7$ be a sufficiently general hyperplane through the curve C , such that the intersection $Y = Y_5 = W \cap H$ is smooth. In particular, let H intersect S_1 transversely along the curve C . Then

$$N_{S_1/W} \otimes \mathcal{O}_C = N_{(S_1 \cap H)/(W \cap H)} \otimes \mathcal{O}_C = N_{C/Y}.$$

Clearly, we can also suppose that the curve C is sufficiently general in $Y = Y_5$, i.e. $N_{C/Y} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$. But then $N_{S_1/W} \otimes \mathcal{O}_C = N_{C/Y} = \mathcal{O}_C(2) \oplus \mathcal{O}_C(2)$. Lemma (8.1.3) is proved.

8.2. Proof of the Tangent Bundle Theorem (Prop.(8.1.9)).

(8.2.1) The proof of the T.B.T. consists in the repeating of the local constructions from the beginning globally over the whole family $\mathcal{C}^3 = \mathcal{C}^3(X)$. For this purpose we introduce the following subvarieties of the projectivized bundle $P_{\mathcal{C}^3}(\bar{\Sigma})$ on the places of the curve $C \subset X = X_{10}$ and of the surface $S_1 \subset W$:

$$D_{\mathcal{C}^3} = \left\{ (C, x) \in \mathcal{C}^3 \times X : x \in C \right\} \quad \text{and}$$

$$D_{\mathcal{C}^3}(G) = \left\{ (C, x) \in \mathcal{C}^3 \times W : x \in S_1 \right\} ;$$

remind that $C + \bar{C} = C_1 = S_1 \cdot Q \subset S_1 \subset P(\bar{\Sigma}(V_2))$ where $1 = P(V_2)$.

(8.2.2) The pairs of successive embeddings

$$D_{\mathcal{C}^3} \subset \mathcal{C}^3 \times X \subset \mathcal{C}^3 \times W \quad \text{and}$$

$$D_{\mathcal{C}^3} \subset D_{\mathcal{C}^3}(G) \subset \mathcal{C}^3 \times W$$

globalize respectively the pairs of embeddings $C \subset X \subset W$ and $C \subset S_1 \subset W$. Correspondingly to the normal sequences (8.1.1) (1) and (8.1.1) (2) they induce the sequences of normal sheaves:

$$(1, \mathcal{C}) \quad 0 \rightarrow N(D_{\mathcal{C}^3}, \mathcal{C}^3 \times X) \rightarrow N(D_{\mathcal{C}^3}, \mathcal{C}^3 \times W) \rightarrow \\ \rightarrow N(\mathcal{C}^3 \times X, \mathcal{C}^3 \times W) \otimes \mathcal{O}_{D_{\mathcal{C}^3}} \rightarrow 0 \quad \text{and}$$

$$(2, \mathcal{C}) \quad 0 \rightarrow N(D_{\mathcal{C}^3}, D_{\mathcal{C}^3}(G)) \rightarrow N(D_{\mathcal{C}^3}, \mathcal{C}^3 \times W) \rightarrow \\ \rightarrow N(D_{\mathcal{C}^3}(G), \mathcal{C}^3 \times W) \otimes \mathcal{O}_{D_{\mathcal{C}^3}} \rightarrow 0.$$

(8.2.3) Let $\mathcal{O}(1)$ be the antitautological sheaf on the projective bundle $P_{G(2,V)}(\bar{\Sigma}) \rightarrow G = G(2,V)$.

According to the definition of the bundle Σ (see (6.3.3)), the sheaf Σ on G is embedded in a natural way in the sheaf $V_8 \otimes \mathcal{O}_G$ (corresponding to the constant bundle $G \times V_8$ over $G = G(2,V)$). In fact, for every $V_2 \in G = G(2,V)$, the embedding of the corresponding fibers

$\Sigma(V_2) \subset (V_8 \otimes \mathcal{O}_G)(V_2)$ coincides with the natural embedding $(V_2 \wedge V) \cap V_8 \subset V_8$. Therefore, the sheaf $\mathcal{O}(1)$ coincides with the restriction of the sheaf

$$\mathcal{O}_{P(V_8)}(1) = \mathcal{O}_{P(\wedge^2 V)} \otimes \mathcal{O}_{P(V_8)} \quad \text{on the subvariety}$$

$P_G(\Sigma) \subset P_G(V_8 \otimes \mathcal{O}_G) = G \times P(V_8)$, (the "old" $\mathcal{O}(1)$, see above).

(8.2.4) As in the local case we multiply the sequence

$(2, \mathcal{L})$ by the sheaf $\mathcal{O}(-1)$:

$$\begin{aligned} (2', \mathcal{L}) \quad 0 &\longrightarrow N(D_{\mathbb{C}^3}(G)) \otimes \mathcal{O}(-1) \longrightarrow \\ &\longrightarrow N(D_{\mathbb{C}^3}, \mathbb{C}^3 \times W) \otimes \mathcal{O}(-1) \longrightarrow N(D_{\mathbb{C}^3}(G), \mathbb{C}^3 \times W) \otimes \mathcal{O}_{\mathbb{C}^3} \otimes \mathcal{O}(-1) \longrightarrow 0 \end{aligned}$$

Similarly, we obtain the sequence $(1, \mathcal{L}) \otimes \mathcal{O}(-1)$:

$$\begin{aligned} (1', \mathcal{L}) \quad 0 &\longrightarrow N(D_{\mathbb{C}^3}, \mathbb{C}^3 \times X) \otimes \mathcal{O}(-1) \longrightarrow \\ &\longrightarrow N(D_{\mathbb{C}^3}, \mathbb{C}^3 \times W) \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}(1) \longrightarrow 0, \end{aligned}$$

since $N(\mathbb{C}^3 \times X, \mathbb{C}^3 \times W) = \mathcal{O}(2)$.

(8.2.5) Let, for the brevity, denote by B the sheaf

$N(D_{\mathbb{C}^3}, \mathbb{C}^3 \times W) \otimes \mathcal{O}(-1)$ and let α and β are the

natural projections for the Fano family $D_{\mathbb{C}^3}$:

$$\begin{array}{ccccc}
 D_{\mathcal{C}^3} & \hookrightarrow & \mathcal{C}^3 \times X & \hookrightarrow & \mathcal{C}^3 \times W \\
 & \searrow \beta & \downarrow \alpha & \swarrow \alpha & \\
 & & \mathcal{C}^3 & &
 \end{array}$$

As before, using the equality $R^1_{\beta} N(D_{\mathcal{C}^3}, \mathcal{C}^3 \times X) = 0$ which can be obtained on the level of fibers from the local equality $H^1(C, N_{C/X}) = 0$, and also the relative duality of Serre, we obtain :

$$\mathcal{R}_{\mathcal{C}^3} = R^1_{\beta} (N(D_{\mathcal{C}^3}, \mathcal{C}^3 \times X) \otimes \mathcal{O}(-1))$$

(here $R^i_{\bullet}$ is the operation of taking the i -th direct image) .

(8.2.6) In a similar way, using the local equalities :

$$H^i(N_{S_1/W} \otimes \mathcal{O}_C \otimes \mathcal{O}(-1)) = 0, \quad i = 1, 2, \dots$$

we derive the identities :

$$\begin{aligned}
 (5, \mathcal{C}) \quad R^i_{\beta} (N(D_{\mathcal{C}^3}(G), \mathcal{C}^3 \times W)) \otimes \mathcal{O}_{D_{\mathcal{C}^3}} \otimes \mathcal{O}(-1) &= 0, \\
 i &= 1, 2, \dots
 \end{aligned}$$

$$(8.2.7) \quad \text{As is already clear, } N(\mathcal{C}^3 \times X, \mathcal{C}^3 \times W) = \mathcal{O}(2).$$

Using the obtained just above and taking the direct β -images of the sequences $(1', \mathcal{C})$ and $(2', \mathcal{C})$, we obtain :

$$\begin{aligned}
 (2'', \mathcal{C}) \quad & 0 \longrightarrow \beta_*(N(D_{\mathcal{C}^3}, D_{\mathcal{C}^3}(G)) \otimes \mathcal{O}(-1)) \longrightarrow \beta_* B \longrightarrow C \\
 & 0 \longrightarrow R^1_{\beta} (N(D_{\mathcal{C}^3}, D_{\mathcal{C}^3}(G)) \otimes \mathcal{O}(-1)) \longrightarrow R^1_{\beta} B \longrightarrow 0
 \end{aligned}$$

and :

$$\begin{aligned}
 (1'', \mathcal{C}) \quad & 0 \longrightarrow \beta_*(N(D_{\mathcal{C}^3}, \mathcal{C}^3 \times X) \otimes \mathcal{O}(-1)) \longrightarrow \beta_* B \longrightarrow \\
 & \longrightarrow \beta_* \mathcal{O}(1) \longrightarrow \mathcal{R}_{\mathcal{C}^3} \longrightarrow R^1_{\beta} B \longrightarrow R^1_{\beta} \mathcal{O}(1) \longrightarrow 0,
 \end{aligned}$$

where $\beta_* = R^0_{\beta}$.

(8.2.8) Now, we introduce the variety

$$D_{\mathbb{C}^3}(\langle \cdot \rangle) = \left\{ (C, x) : C \in \mathbb{C}^3, x \in \langle C \rangle = \text{Span}(C) \right\}.$$

As $\text{Span}(C) \subset P(\sum(C)) = P(\sum(V_2))$, then

$D_{\mathbb{C}^3}(\langle \cdot \rangle)$ is embedded naturally in the projectivized bundle

$P_{\mathbb{C}^3}(\sum)$. In fact, the bundle $\sum$ is defined, originally, on the Grassmannian $G = G(2, V)$, but it is defined also on the subsets of $G(2, V)$ and on the preimages of these subsets under morphisms; as usual, we drop the symbols of the maps. From the Remark (6) (see (8.1.6)), (8.1.7) and (8.1.8), we conclude that

$$D_{\mathbb{C}^3}(\langle \cdot \rangle) = P_{\mathbb{C}^3}(\tau_{4, \sum}),$$

(see also the Corollary (7.3.3)).

The Fano family $D_{\mathbb{C}^3}$ is embedded identically in the

projectivized bundle $D_{\mathbb{C}^3}(\langle \cdot \rangle) = P_{\mathbb{C}^3}(\tau_{4, \sum})$;

in particular, the projection $\beta : D_{\mathbb{C}^3} \longrightarrow \mathbb{C}^3$ is a

restriction of the projection $\alpha : D_{\mathbb{C}^3}(\langle \cdot \rangle) = P_{\mathbb{C}^3}(\tau_{4, \sum}) \longrightarrow \mathbb{C}^3$. From the properties of the antitautological sheaf $\mathcal{O}(1)$ on $P_{\mathbb{C}^3}(\tau_{4, \sum})$ we have:

$$\beta_* \mathcal{O}(1) = \tau_{4, \sum}^* \quad \text{and} \quad R^1_{\beta} \mathcal{O}(1) = 0.$$

(8.2.9) Let us denote the sheaf $N(D_{\mathbb{C}^3}, D_{\mathbb{C}^3}(G)) \otimes \mathcal{O}(-1)$

by A . Taking into account the identity

$$\beta_*(N(D_{\mathbb{C}^3}, \mathbb{C}^3 \times X) \otimes \mathcal{O}(-1)) = 0,$$

which follows from the corresponding local equality, we

conclude, just as in the local case, that the exact sequence

$(1'', \mathcal{C})$ can be written in the form :

$$(1''', \mathcal{C}) \quad 0 \longrightarrow \beta_* A \longrightarrow \tau_{4, \Sigma}^* \longrightarrow \mathcal{R}_{\mathcal{C}^3} \longrightarrow R_{\beta}^1 A \longrightarrow 0,$$

(see the "local" sequences (8.1.5) $(1'')$ and (8.1.8) $(1''')$).

According to the comments in the Remark (8.1.6) (6) , the sheaf $\beta_* A$ coincides with the factorsheaf $R =$

$$= \left(\tau_{4, \Sigma} / \tau_{3, \Sigma} \right)^* . \quad \text{If, moreover, the curve } C$$

($C \in \mathcal{C}^3$) is general (in the sense of the conditions of Proposition (8.1.9)) , then the first direct image $R_{\beta}^1 A$

vanishes over C , since the fiber of $R_{\beta}^1 A$ over

$$C \in U \subset \mathcal{C}^3 \quad \text{is} \quad H^1(N_{C/S_1} \otimes \mathcal{O}(-1)) = H^1(C, \mathcal{O}_C) = 0$$

($C \simeq P^1$) . The open T.B.T. (Proposition (8.1.9)) is proved

(8.2.10) REMARK.

The correct formulation of the T.B.T. must take into account a joint geometrical interpretation of the sequences $(1, \mathcal{C})$ and $(2, \mathcal{C})$, taking into consideration also the disposition of the locus $\mathcal{C}_{ex}^3 \subset \mathcal{C}^3$, where the local constructions fail ; it is already clear that $\mathcal{C}_{ex}^3 \subseteq \mathcal{C}^3 \setminus U$ (see (8.1.4) (5) and (8.1.3)).

(8.2.11) The geometry of the Fano threefold $X = X_{10}$ can provide an additional useful information by means of the geometrical analogue of the just proven open T.B.T.. First, the exact sequence $(1''', \mathcal{C})$ induces a natural isomorphism

$$\psi : \tau_{3, \Sigma}^*|_U \longrightarrow \mathcal{R}_{\mathcal{C}^3}|_U \quad \text{over the open subset } U \subset \mathcal{C}^3 .$$

The definition of $H^0(\psi_C)$ in (8.1.5) $(1'')$

shows that the fibre $\tau_{3, \Sigma}(C)$ coincides with the

embedded plane $H_C \cap H_{\bar{C}} \subset \Sigma(C) (= \Sigma(V_2) =$
 $= \Sigma(\bar{C}))$; see also the Corollary (7.2.6) (2). Here, as
usual, $C + \bar{C} = C_{P(V_2)}$, $P(H_C) = \langle C \rangle$, $P(H_{\bar{C}}) = \langle \bar{C} \rangle$

The dual map ψ^* defines the isomorphism:

$$\psi^*: T_{\mathcal{C}^3|U} \longrightarrow \tau_{3,\Sigma}|U \quad \text{over } U \subset \mathcal{C}^3$$

(here $T_{\mathcal{C}^3} = \mathcal{R}_{\mathcal{C}^3}^*$ is the tangent sheaf of the family $\mathcal{C}^3$).

So, we can formulate the following:

(8.2.12) **THEOREM** (Geometric (open) T.B.T. for $\mathcal{C}^3$).

There is a naturally defined isomorphism (see (8.1.9)):

$$\psi^*: T_{\mathcal{C}^3|U} \longrightarrow \tau_{3,\Sigma}|U,$$

such that ψ^* coincides with the natural map

$$\psi^*(C) : T_{\mathcal{C}^3}(C) \xrightarrow{\sim} H_C \cap H_{\bar{C}} \subset \Sigma(V_2) \subset V_8$$

on the fiber $T_{\mathcal{C}^3}(C)$; (see Corollary (5.3.2) and Corollary (7.2.6) for the definitions of V_2 , $\bar{C}$, H_C and $H_{\bar{C}}$).

§ 9. Tangent Bundle Theorem for the family

of conics on the Fano threefold $X = X_{10}$.

9.1. In the present section we shall follow some results of the Thesis of D. Logachev [L1] (see also [L2] and [P]) on the family of conics on $X = X_{10} = G \cap P(V_8) \cap Q$. The original idea of representing the family of conics on X_{10} as a degeneration locus, resp. as a zero scheme of a section of some vector bundle, and the corresponding statement of the tangent bundle theorem, belongs to Logachev.

The main observations in the present work are :

(1). The use of the simple fact that the conics and the rational normal cubics on X_{10} correspond to the standard splittings of the codim. 2 -cycles on X_{10} of the types

$$\tilde{\sigma}_{1,1} \cap X_{10} \quad \text{and} \quad \tilde{\sigma}_{2,0} \cap X_{10} \quad \text{respectively ; the}$$

last generate the subring of the ring of cycles of codimension 2 on $X = X_{10}$, which are induced from cycles on the Grassmannian $G(2,V)$;

(2). The possibility to perform the constructions of Logachev also to the family of the rational normal cubic curves on X_{10} .

The two marked families are, probably, the only families of curves on X_{10} , for which the mentioned methods and representations are adequate.

In particular, (1), together with the geometrical interpretations of the tangent bundle theorems for both families (see the T.B.T. (8.2.12) and the T.B.T.^{-S} (9.4.1,3,5) below), grounds us to carry out a geometrical approach to the Abel-Jacobi maps for some families of curves, naturally arising on X (see for ex. [CG]). Let, for example :

$$\mathcal{C}_1^5 = \{ C - \text{a curve} : C \subset X, \quad p_a(C) = 1, \quad \deg(C) = 5 \}$$

be the family of the linearly normal elliptic quintics on X ;

the last corresponds to the set of components of the degenerate canonical curves of degree 10 on X_{10} into pairs of elliptic curves. The T.B.T.^{-S} for the families $F =$

$$= \{ \text{conics on } X_{10} \} \quad \text{and} \quad \mathcal{C}^3 = \{ \text{rat. norm. cubics on } X_{10} \},$$

enables us to perform a geometrical approach to the

$$\text{Abel-Jacobi map of the family } \mathcal{W} = \mathcal{C}^3 + F =$$

$$= \{ C + q : C \in \mathbb{C}^3, q \in F \}.$$

$$\text{Let } \mathcal{C} = \mathcal{C}_1^5 \cap W = \{ C + q : C \in \mathbb{C}^3, q \in F, (C, q) = 2 \}$$

be the three-dimensional family, the elements of which are, by definition, degenerate elliptic quintics on X , which split into a sum of a rat.norm. cubic and a conic, intersecting between them in two points. The results in the paper [V] of Claire Voisin on the Quartic Double Solids give us a hope to suppose that the general points z of the two-dimensional subvariety Z of the Intermediate Jacobian

$J(X_{10})$ of X_{10} , which is the Abel-Jacobi image of the "diagonal" family $\mathcal{C}$, are double points of the theta-divisor $(H)(X_{10}) \subset J(X_{10})$. The author's suggest is that the tangent cones $TC_z((H))$ to the theta divisor (H) at the singular points z of (H) , are quadrics of rank = 6. The last will provide us with a geometrical proof of the non-rationality of the Fano threefold $X = X_{10} = G(2,5) \cap P^7 \cap Q$ (see [V, § 4]).

9.2. In (6.2.1) was defined the bundle $Pf \rightarrow P(V^*)$; the fiber $Pf(V_4)$ over the element $V_4 \in P(V^*)$ was defined to be the Plücker quadric, corresponding to the 4-space V_4 (for more details, see § 6.2). It was proven that the corresponding invertible sheaf Pf over $P(V^*)$ is isomorphic to the sheaf $(\det \tau_{4,V})^{-1} = \mathcal{O}_{P(V^*)}(1)$.

Let now V_8 , $V_8 \subset V_{10} = \wedge^2 V$ be choosen sufficiently general, such that:

- (i) the variety $X = G \cap P(V_8) \cap Q$ is smooth ;
- (ii) $\dim (V_8 \cap (\wedge^2 V_4)) = 4$, for every $V_4 \in P(V^*) = G(4, V)$.

Then we can define correctly a bundle M over $P(V)$ with 4-dimensional fibers $M(V_4)$ such that

$$M(V_4) = \wedge^2 V_4 \cap V_8, \quad V_4 \in P(V^*) = G(4, V).$$

On the other hand, the restriction of the Plücker bundle $Pf \longrightarrow P(V^*)$ (regarded as a bundle of quadrics on $\wedge^2 \tau_{4,V} \longrightarrow P(V^*)$) to the subbundle

$$M = \wedge^2 \tau_{4,V} \cap V_8 \subset \wedge^2 \tau_{4,V}, \text{ represents}$$

the bundle Pf as a subbundle of $S^2 M^* \longrightarrow P(V^*)$; moreover Pf has fibers

$$Pf(V_4) = H^0(P(\wedge^2 V_4 \cap V_8), \mathcal{O}(2 - (\sigma_{1,1}(P(V_4)) \cap P(V_8))))$$

(see Corollary (6.2.3)).

We can also define an analogue to the bundle $Q \rightarrow G(2, V)$ which was regarded as a subbundle of $S^2 \Sigma^* \longrightarrow G(2, V)$, namely:

Let $X = G \cap P(V_8) \cap Q$, where Q is a quadric. We define the bundle (the invertible sheaf) Q over $P(V)$ such that:

- (1) Q is a subbundle of $S^2 (\wedge^2 \tau_{4,V})^*$;
- (2) $Q(V_4) = \mathbb{C} \cdot Q|_{P(\wedge^2 V_4)} \subset H^0(P(\wedge^2 V_4), \mathcal{O}(2))$,
i.e. $Q(V_4)$ is the restriction of the equation of the quadric Q , etc.

From the conditions (i) and (ii) we derive that we can look at the bundle $Q \longrightarrow P(V^*)$ also as a bundle of quadrics (a "quadric") on the bundle M . Moreover , it is evident that the corresponding sheaf Q is isomorphic to the structure sheaf $\mathcal{O}_{P(V^*)}$. Just as in § 6.3

(see Prop. (6.3.2)) we can prove the following :

(9.2.1) PROPOSITION (Logachev , [L1]) .

Let $Z \subset P_{P(V^*)} (S^2 (\wedge^2 \tau_{4,V})^*)$ be the set :

$$Z = P \left\{ (V_4, q) : V_4 \in P(V^*), q \in H^0(P(\wedge^2 V_4), \mathcal{O}(2 - Q \cap \sigma_{1,1}(P(V_4))) \right\}$$

Then Z is naturally isomorphic to the projectivized bundle $P_{P(V^*)} (\mathcal{O}_{P(V^*)} (1) \oplus \mathcal{O}_{P(V^*)})$.

Moreover, from (i) and (ii) (see above) , follows, that

$$\begin{aligned} Z &= P \left\{ (V_4, q) : V_4 \in P(V^*), q \in H^0(P(\wedge^2 V_4 \cap V_8), \mathcal{O}(2 - \sigma_{1,1}(P(V_4)) \cap P(V_8) \cap Q) \right\} \\ &= P \left\{ (V_4, q) : V_4 \in P(V^*), q \in H^0(P(M(V_4)), \mathcal{O}(2 - \sigma_{1,1}(P(V_4)) \cap X)) \right\} \subset \\ &\subset P_{P(V^*)} (S^2 M^*) \quad (\text{ see Proposition (6.3.2) }) . \end{aligned}$$

(9.2.2) From the definition of the variety Z follows

that the fiber $\eta^{-1}(V_4) = Z(V_4)$ of the projective bundle $\eta : Z \longrightarrow P(V^*)$ is isomorphic to the projective line $\langle Pf(V_4), Q(V_4) \rangle = \text{Span} \{ Pf(V_4) \cup Q(V_4) \}$ in the projective space $P(S^2 M^*(V_4))$ of quadrics in $P(M(V_4))$.

The surface F_c of the geometrical conics on $X = X_{10}$, the surface F (of "conics on X ") and the factorsurface $F_0 = F/i = \{ (q, \bar{q}; V_4) : q \in F_c, (\bar{q}, V_4) = i(q, V_4) \}$ were defined in § 4.4 and in § 4.5 (here i is the involution, which is correctly defined on F). The non-correctness of the defining the rational map $(\bar{\cdot})^{-1} : F_c \dashrightarrow F_c$ arises from the fact that the linear Span of the set $S_{q_0} = \left(\bigcup l : l \subset P(V) - \text{a line}, l \in q_c \right)$ where q_0 is the unique $\mathcal{P}$ -conic on X , is contained in an one-dimensional family of P^3 's; the last coincides with the line $P((V/U)^*) =$
 $= \{ P^3 : P^3 \subset P^4 = P(V) \text{ \& (the plane) } P(U) = S_{q_0} \subset P^3 \}$
 (see § 4.2 and § 4.3) .

(9.2.3) Let now $V_4 \in P(V^*)$ and let W denotes, as usual, the intersection $W = G \cap P(V_8) \subset P(V_8) \subset P(\wedge^2 V)$. As we know (see Proposition (2.2.3) (2)) the cycle $Q(P(V_4)) = \mathcal{G}_{1,1}(P(V_4)) \cap P(V_8) \subset W$ is a two-dimensional quadric. It follows from the definition of the bundle $M \longrightarrow P(V^*)$ (see the introduction of § 9.2) that $\langle Q(P(V_4)) \rangle = \text{Span}(Q(P(V_4))) = P(M(V_4))$. By definition, $W \cap Q = X$. Hence, the elliptic quartic $C(P(V_4))$ (see Corollary (3.3.3)), which is, by definition, an intersection of the cycle $\mathcal{G}_{1,1}(P(V_4))$ with X , coincides with the intersection of the quadrics $Q(P(V_4))$ and Q

in the space $P(M(V_4)) = P(\wedge^2 V_4 \cap V_8)$.

The quadric $Q(P(V_4))$ defines a section of the Pfaff bundle $Pf \rightarrow P(V^*)$; the restrictions of the quadric Q on the fibers of the bundle $M \rightarrow P(V^*)$ define a section of the bundle $Q \rightarrow P(V)$, defined above. Therefore, the condition of splitting of the curve $C(P(V_4))$ into a pair of involutive conics is equivalent to the existence of a quadric of rank ≤ 2 on the line $\langle Pf(V_4) \cup Q(V_4) \rangle$ in the space of quadrics in $P(M(V_4))$. The last arguments are correct on the subset $F_c \setminus \{q_0\} \subset F_c$ (q_0 is the unique $\wp$ -conic on X), but it is not hard to see that the "exceptional" quadrics of F , which lie on the exceptional line $r_F^{-1}(q_0)$ (see (4.5.1)) correspond to degenerate quadrics in the bundle $Pf \oplus Q$ (see for ex. § 4.3, § 4.4, Claim (4.2.1) and (4.5.1)). The involution i on F identifies the elements of the exceptional curve $r_F^{-1}(q_0)$ with the elements of the curve of the $\wp$ -conics on X (ibid.).

(9.2.4) The tautological sequence :

$$0 \rightarrow \mathcal{O}_Z(-1) \rightarrow \mathcal{O}_Z^*(\mathcal{O}_{P(V^*)}(1) \oplus \mathcal{O}_{P(V^*)}) \rightarrow \mathcal{O}_Z(1) \rightarrow 0$$

on the projective bundle

$$\gamma : Z = P(\mathcal{O}_{P(V^*)}(1) \oplus \mathcal{O}_{P(V)}) \rightarrow P(V^*)$$

defines, just as in (6.4.4) (see also (6.4.2)), a net of quadrics on Z . It follows from the properties of the tautological sequence that the projectivization of the embedding

$$0 \longrightarrow \mathcal{O}_Z(-1) \xrightarrow{j} \gamma^*(\mathcal{O}_{P(V^*)}(1) \oplus \mathcal{O}_{P(V^*)})$$

represents the "embedded" points of the fibers of the projective bundle Z ; these points are interpreted also as quadrics in the corresponding spaces $P(M(V_4))$.

Because of that, as in § 6, Proposition (6.4.4), we can conclude the following:

(9.2.5) CLAIM (Logachev, [L1]). The factorfamily (of pairs of involutive conics) $F_0 = F/i$ is embedded naturally in the projective bundle

$$Z = P_{P(V^*)}(\mathcal{O}_{P(V^*)}(1) \oplus \mathcal{O}_{P(V^*)})$$

as a second determinantal of the net of quadrics:

$$0 \longrightarrow \mathcal{O}_Z(-1) \xrightarrow{j} \gamma^*(\mathcal{O}_{P(V^*)}(1) \oplus \mathcal{O}_{P(V^*)}) \xrightarrow{i} \gamma^*(S^2 M^*)$$

i.e. $F_0 = D_2(\hat{\varphi})$, where $\hat{\varphi} = i \cdot j$.

9.3. The remaining constructions and results are carried out parallel to the corresponding ones for the family of the rational normal cubics on X (see § 7).

$$(9.3.1) \text{ Let } Z \text{ and } \hat{\varphi} : \mathcal{O}_Z(-1) \longrightarrow \gamma^*(S^2 M^*)$$

be as above. We define:

$$\begin{aligned} \tilde{D}_k &= \tilde{D}_k(\hat{\varphi}) = \{ (z; U_k) = (V_4, q; U_k) \in G_Z(k, M^*) : \\ &: \text{Im}(q) \subseteq U_k \} \subset G_Z(k, M^*), \quad 0 \leq k \leq 4. \end{aligned}$$

(9.3.2) It is not hard to prove that the first determinantal $D_1(\hat{\varphi}) \subset Z$ vanishes for the general intersection

$X = G \cap P(V_8) \cap Q$, hence, the natural projection

$$\pi_2 : \tilde{D}_2(\hat{\varphi}) \longrightarrow D_2(\varphi) \quad (\text{see (7.1.1)}) \text{ is an}$$

isomorphism for the general X (see Remark (7.1.2)).

(9.3.3) From Corollary (3.3.3) (2) we have that the curve $C(P(V_4)) \subset X$ is a linearly normal curve of arithmetical genus 1 and of degree 4; the linear normality follows from the fact that the curve $C(P(V_4))$ is a component of the canonical curve $C(P(V_4)) + C_1$, $1 = P(V_2)$ (see (3.3.2) and Cor. (3.3.3)), where $V_2 \in G(2, V)$ is chosen in such a way that $V_2 \subset V_4$. In particular, in the case, when $C(P(V_4))$ splits into a sum of two plane sections: $C(P(V_4)) = q + \bar{q}$, we obtain that $(q, \bar{q}) = 2$, hence, the intersection $\langle q \cap \bar{q} \rangle$ of the spaces $\langle q \rangle = \text{Span}(q)$ and $\langle \bar{q} \rangle = \text{Span}(\bar{q})$ is a line in $P(V_8)$. In the case when the two points of intersection of q and $\bar{q}$ coincide, the line $\langle q \cap \bar{q} \rangle$ is equal, by definition, to their common tangent line (see also (5.4.1) and (5.4.2)).

(9.3.4) The tautological sequence on $G_Z(2, M)$, together with the injection $\hat{\varphi} : \mathcal{O}_\eta(-1) \hookrightarrow \gamma^* S^2 M^*$, define, as in (7.2.2), a section s_0 of the sheaf $\mathcal{O}_\eta(1) \otimes \text{Coker}(2)$ over $G_Z(2, M)$, where $\text{Coker}(2) = \text{Coker}(S^2 \tau_{2, M^*} \longrightarrow \gamma^* S^2 M^*)$.

(9.3.5) The same arguments, as in Corollary (7.2.4), give us a ground to impose the following open conditions on the subspace $V_8 \subset V_{10} = \wedge^2 V$ ($\dim(V_8) = 8$) and on the quadric Q , where $X = G \cap P(V_8) \cap Q$ (see also Corollary (6.3.5) and Remark (7.1.2)).

- (i) $X = G \cap P(V_8) \cap Q$ is smooth ;
- (ii) $\dim (V_8 \cap (\wedge^2 V_4)) = 4$, for every $V_4 \in P(V^*) = G(4, V)$;
- (iii) $D_1(\hat{\psi}) = \emptyset$.

(9.3.6) CLAIM (Logachev , [L1]). Let $X = G \cap P(V_8) \cap Q$ where the subspace $V_8 \subset \wedge^2 V$ and the quadric Q fulfil the open conditions (i) , (ii) and (iii). Let $Z, M, \hat{\psi}, s_0$ and $F_0 = F/i$, where F is the surface of conics on X (see Definition (4.4.2)), be as above.

Then F_0 is embedded in $G_Z(2, M^*)$ as a zero scheme of the section s_0 , i.e.

$$F_0 \xrightarrow{\sim} Z(s_0) \subset G_Z(2, M^*) .$$

(9.3.7) The section s_0 , together with the duality $G_Z(2, M^*) \simeq G_Z(2, M)$ define a section s_0^* with zeros on $G_Z(2, M)$ (see (7.2.5)). So we can formulate

(9.3.8) COROLLARY. In the conditions of the Claim (9.3.6) the "dual" natural embedding $F_0 \hookrightarrow G_Z(2, M)$ represents the isomorphic image of F_0 in $G_Z(2, M)$ as a zero scheme of the section s_0^* , i.e.

$$F_0 \xrightarrow{\sim} Z(s_0^*) \subset G_Z(2, M) .$$

Moreover, this embedding can be described as follows :

Let (q, V_4) and $(\bar{q}, V_4) = i(q, V_4)$ be a pair of involutive elements (conics) of F . Let H_q and $H_{\bar{q}} \in M^*(V_4)$ be the hyperplanes in $M(V_4)$, such that $\langle q \rangle = \text{Span}(q) = P(H_q)$ and $\langle \bar{q} \rangle = \text{Span}(\bar{q}) = P(H_{\bar{q}})$.

Then the embedding $F_0 \hookrightarrow G_Z(2, M)$ sends the pair

$((q, V_4), (\bar{q}, V_4))$ to the triple

$$(V_4, H_q \cdot H_{\bar{q}}; H_q \cap H_{\bar{q}}) \in G_Z(2, M).$$

(9.3.9) REMARK. Note that $H_q \cap H_{\bar{q}}$ is always a plane since $P(H_q \cap H_{\bar{q}})$ coincides with the projective line $\langle q \cap \bar{q} \rangle$ (see (9.3.3)).

Similarly, as in the "rational normal cubic" case (see (7.3.1)), the tautological sequence on $P_Z(M^*)$ together with the injection $\hat{\gamma}$ define a section s of the sheaf $\mathcal{O}_{\gamma}(1) \otimes S^2 \gamma_{4,M}^*$ over $P_Z(M^*)$. The same arguments, as in (7.3.1) and (7.3.2), give

(9.3.10) CLAIM (Logachev, [L1]). In the conditions of the Claim (9.3.6), the family F of the conics for the Fano threefold $X = X_{10} = G(2, V) \cap P(V_8) \cap Q$ is embedded naturally in the projective bundle $P_Z(M^*)$ as a zero scheme of the section s , defined just above, that is:

$$F \xrightarrow{\sim} Z(s) \subset P_Z(M^*).$$

Using the duality $P_Z(M^*) \simeq G_Z(3, M)$, we can formulate the following:

(9.3.11) COROLLARY. The embedding $F \hookrightarrow P_Z(M^*) \simeq G_Z(3, M)$ sends the element $(q, V_4) \in F$ to the triple $(V_4, H_q \cdot H_{\bar{q}}; H_q) \in G_Z(3, M)$ (the hyperplanes H_q and $H_{\bar{q}}$ are defined as in the Corollary (9.3.8)).

9.4. Let now $p_F : F \longrightarrow F_0 = F/i$ be the two-sheeted covering, which corresponds to the involution $i: F \longrightarrow F$ (see (4.5.1) and (4.5.2)). The embeddings $F_0 \hookrightarrow G_Z(2, M)$ and $F \hookrightarrow P_Z(M^*) = G_Z(3, M)$, together with p_F define natural maps $F \longrightarrow F_0 \longrightarrow G_Z(2, M)$ and $F \longrightarrow G_Z(3, M)$. Therefore, the standard tautological bundles $\tau_{2, M}$ on $G_Z(2, M)$ and $\tau_{3, M}$ on $G_Z(3, M) = P_Z(M^*)$ can be lifted on the family F . Just as in the "rational normal cubic" case, there is a representation of the tangent bundle T_F by means of some bundle on F , which can be described in geometrical terms.

(9.4.1) CLAIM (Logachev) The open Tangent Bundle Theorem for the family F of "conics" for X ; see [14]).

Let $l_\sigma = \{ (q, V_4) : q \text{ is a } \sigma\text{-conic on } X \} \subset F$ be the smooth rational curve of the σ -conics in the family of conics F (see § 4.2); let also Z , M , $G_Z(2, M)$, $P_Z(M^*)$ and the bundles $\tau_{2, M}$ and

$\tau_{3, M}$ be as above. Let $\mathcal{R}_F$ be the cotangent sheaf (resp., the cotangent bundle) of F .

Let $U = F \setminus (l_\sigma \cup i(l_\sigma)) \subset F$. Then there exists a naturally defined isomorphism of sheaves over the open subset $U \subset F$:

$$\psi^*: \tau_{2, M}^*|_U \longrightarrow \mathcal{R}_F|_U.$$

(9.4.2) REMARK. The curve $i(l_\sigma)$ coincides with the

exceptional curve $r_F^{-1}(q_0)$ (the curve of the exceptional "conics" of F (see § 4.2 , or the comments (9.2.2)).
The curves l_σ and $i(l_\sigma)$ are mutually disjoint (ibid.)
Moreover, there is an exact global version of the T.B.T. for the family of conics, proven by Logachev:

(9.4.3) THEOREM (Logachev , [L1]). There is a naturally defined exact sequence of sheaves over the surface of conics F for X :

$$0 \longrightarrow \tau_{2,M}^* \longrightarrow r^* \mathcal{R}_{F_m} \longrightarrow \mathcal{O}_{l_\sigma} \vee \mathcal{O}_{i(l_\sigma)} \longrightarrow 0$$

where $r: F \longrightarrow F_m$ is the morphism onto the minimal model F_m of F . In fact, the morphism r is a composition of $r_F: F \longrightarrow F_c$ (which blows-down the curve $r_F^{-1}(q_0)$ of the exceptional conics in F), and of the blowing-down of the r_F -image of its involutive curve (the curve of the σ -conics on X in F_c). The both contractions blow-down (-1) -curves to (non-singular) points of the smooth minimal surface F_m (see also [P]).

(9.4.4) COMMENTS.

(1) Note that the statement of the global T.B.T. (9.4.3) is, in fact, a correct treatment of the analogue of the sequence $(1''', \mathcal{C}_Q)$ for the family F (see (8.2.9), seq. $(1''', \mathcal{C}_Q)$ and Remark (8.2.10)).

(2) The factor-sheaf $\mathcal{O}_{l_\sigma} \vee \mathcal{O}_{i(l_\sigma)}$ has a support on the closet subset $l_\sigma \cup i(l_\sigma)$ of codim. = 1 in F .

(9.4.5) COROLLARY (see (8.2.12)).

Let $\psi^*: T_F|_U \longrightarrow \tau_{2,M}|_U$ be the

dual of ψ (see Claim (9.4.1)). Then ψ^* coincides with the natural map

$$\psi^*(q, V_4) : T_F(q, V_4) \longrightarrow H_q \cap H_{\bar{q}} \subset M(V_4) \subset V_8$$

on the fiber $T_F(q, V_4)$, where $(q, V_4) \in U$ and the hyperplanes H_q and $H_{\bar{q}}$ are defined as in Corollary (9.3.8).

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