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Abstract

In this paper we study some classes of Wiener functionals whose elements can be composed with a non-linear, non-absolutely continuous transformation of the form of perturbation of identity in the direction of Cameron-Martin space. We show that under certain conditions the image of the Wiener measure under the above transformation induces a generalized Wiener functional on certain Sobolev spaces generalizing the Radon-Nikodym relation to non absolutely continuous transformations. A series representation for the generalized Radon-Nikodym derivative is presented and conditional expectations of some generalized random variables are considered.

1 Introduction

Let (C, H, μ) denote the classical Wiener space, i.e. $C = C_o([0, 1])$, H is the Cameron-Martin space and μ is the Wiener measure on $(C, \mathcal{B}(C))$. Let $u(\omega)$ be an H -valued random variable. Set $T\omega = \omega + u(\omega)$, the transformation T from C to C induces a measure R on $\mathcal{B}(C)$ via $R(A) = \mu(T^{-1}A) = \mu\{\omega : T\omega \in A\}$. Let $f(\omega)$ be a Wiener functional, assume that the measure R is absolutely continuous with respect to μ then $f(T\omega)$ is a well defined Wiener functional. More precisely, $f(\omega)$ is the equivalence class of $\mathcal{B}(C)$ measurable function: $f(\omega) = \{f_\theta(\omega), \theta \in \Theta\}$ such that for every $\theta_1, \theta_2 \in \Theta$, it holds that $f_{\theta_1}(\omega) = f_{\theta_2}(\omega)$ a.s. μ . Now, since $f_\theta(\omega)$ are defined for all $\omega \in C$, $f_\theta(T\omega)$ are well defined functions of ω for every $\theta \in \Theta$. Let θ_1, θ_2 be fixed, consider

$$A_1 = \{\omega : f_{\theta_1}(\omega) \neq f_{\theta_2}(\omega)\}, \quad A_2 = \{\omega : f_{\theta_1}(T\omega) \neq f_{\theta_2}(T\omega)\}$$

then $A_2 = T^{-1}A_1$ and if the measure R is absolutely continuous with respect to μ , then $\mu(A_1) = 0$ implies $\mu(T^{-1}A_1) = 0$ and consequently $f(T\omega)$ is a random variable on (C, H, μ) . If, R is not absolutely continuous with respect to μ then $f(T\omega)$ need not be a (C, H, μ) random variable, however, if the equivalence class $(f_\theta(\omega), \theta \in \Theta)$ contains an element $f_c(\omega)$ which is a continuous function on C then it is natural to define $f(T\omega)$ by the μ -equivalence class of functions induced by $f_c(T\omega)$. Note that if $R \ll \mu$ then this extension is consistent with the previously defined extension.

In this paper we consider the following problems. Let u be a nice H -valued random variable; however, $R = \mu \circ T^{-1}$ is not assumed to be absolutely continuous with respect to μ . The question arises whether $f(T\omega)$ can be defined “in a reasonable way” for some classes of random variables which are not necessarily continuous. This problem is considered in section 3. In section 4 we introduce a class of functionals defined by

$$\sup_{|h| \leq \alpha} |\nabla^k f(\omega + h)| \in L^p(\mu), \quad p \geq 0,$$

$h \in H$, for $\alpha > 0$ and a given integer k , ∇^k is the weak H -derivative of order k . The class of functionals is called k -Rademacher functionals. It is shown that this class “behaves nicely” under composition with certain transformations T . Explicit results for the gradient and divergence of the composition of Rademacher functionals with certain shifts are presented, these results extend the earlier results of Kuo ([5]), Buckdahn, Üstünel and Zakai

and Kusuoka. Moreover it is shown that solutions of stochastic differential equations with smooth coefficients are Rademacher functionals. Turning to section 5, let $\mathcal{F} = \{f(\omega)\}$ be a linear collection of random variables which forms a Banach space, let φ denote this Banach space; assume that for every $f \in \varphi$, $f \circ T$ is well defined and $E|f \circ T| < \infty$. The question arises whether φ' the dual to φ , contains an element, say $\Lambda_T \in \varphi'$ such that for every f in φ , $E f \circ T = \varphi' \langle \Lambda_T, f \rangle_\varphi$. Such a relation can be considered as a weak extension of the notion of absolute continuity and the Radon-Nikodym relation and is considered in section 5. The conditioning of generalized random variables and random variables composed with shifts with respect to the class of Wiener chaos stable subsigma fields is also considered in the same section .

2 Notation and preliminaries

Let (C, H, μ) be the classical Wiener space. The weak H-derivative will be denoted ∇ and the Ornstein-Uhlenbeck operator will be denoted by L (cf. e.g. [2] or [10]). Let E be a separable Hilbert space. $\mathcal{D}_{p,s}(E)$ will denote the completion of the E -valued polynomials on C completed with respect to the norm

$$\|G\|_{p,s} = \|(1 + L)^{s/2} G\|_{L^p(\mu, E)} ,$$

$p > 1, -\infty < s < \infty$. The elements of C will be denoted ω . $H^{\otimes n}$ will denote the *symmetrization* of the tensor product $H \otimes \cdots \otimes H$; for any $g_n \in H^{\otimes n}$, the $H^{\otimes n}$ norm of g_n will be denoted $\|g_n\|_{H^{\otimes n}}$ or just $\|g_n\|$. Recall the extension of the classical multiple Wiener-Itô integral representation to the representation of E -valued Wiener functionals (cf. e.g. [2] or [9]): let g_n be an element of $E \otimes H^{\otimes n}$, then $I_n(g_n)$ is an E -valued random variable such that for every $e \in E$

$$(I_n(g_n), e)_E = I_n((g_n, e)_E) \quad (2.1)$$

where the right hand side is a classical multiple Wiener-Itô integral. Let G be an E -valued r.v. and $E \|G\|_E^2 < \infty$, then there exists a sequence $g_n \in E \otimes H^{\otimes n}$ and

$$G = \sum I_n(g_n) \quad ; \quad E \|G\|_E^2 = \sum n! \|g_n\|_{E \otimes H^{\otimes n}}^2 .$$

In addition to the Sobolev spaces of Wiener functionals $\mathcal{D}_{p,s}(E)$, we will also need the following Sobolev spaces of E -valued Wiener functionals. Let, $n = 0, 1, 2, \dots$, $g_n \in E \otimes H^{\otimes n}$;

for any λ in $(0, \infty)$ define the classes $\mathcal{G}_\lambda$ of formal sums $\Sigma I_n(g_n)$:

$$\mathcal{G}_\lambda = \left\{ \sum_0^\infty I_n(g_n) : \sum_0^\infty n! \lambda^{2n} \|g_n\|_{H^{\otimes n}}^2 < \infty \right\} . \quad (2.2)$$

For $\lambda \geq 1$, the elements of $\mathcal{G}_\lambda$ are square integrable E -valued random variables. Let $\rho > 0$ and $G \in \mathcal{G}_\rho$ then $G \in \mathcal{G}_\lambda$ for any λ , $0 < \lambda < \rho$. Define

$$\begin{aligned} \mathcal{G}_\lambda^-(E) &= \bigcap_{0 < \theta < \lambda} \mathcal{G}_\theta(E) \quad ; \mathcal{G}_\lambda^+ = \bigcup_{\theta > \lambda} \mathcal{G}_\theta(E) \\ \mathcal{G}_0(E) &= \mathcal{G}_0^+(E) \quad ; \mathcal{G}_\infty(E) = \mathcal{G}_\infty^-(E) . \end{aligned} \quad (2.3)$$

For any $G \in \mathcal{G}_0(E)$ and λ in $(0, \infty)$ with $G = \Sigma I_n(g_n)$, set $G_\lambda = \Sigma \lambda^n I_n(g_n)$, then $G_\lambda \in \mathcal{G}_0(E)$ and if $G \in \mathcal{G}_{\lambda_1}(E)$ then $G_\lambda \in \mathcal{G}_{\lambda_1/\lambda}(E)$. Obviously

$$\mathcal{G}_1^- \supset \bigcup_{s \in \mathcal{R}} \mathcal{D}_{2,s} \supset \bigcap_{s \in \mathcal{R}} \mathcal{D}_{2,s} \supset \mathcal{G}_1^+ \quad (2.4)$$

It follows from the hypercontractivity theorem that

$$\mathcal{G}_\infty \subset \mathcal{D}_\infty = \bigcap_{\substack{p > 1 \\ s \in \mathcal{R}}} \mathcal{D}_{p,s} \quad ,$$

$\mathcal{G}_\infty$ is called the class of analytic Wiener functionals (cf. also [7]). The following lemma, whose proof is immediate, will be useful in the sequel:

Lemma 2.1 *For $\lambda > 1$, $(\phi_n; n \in \mathbb{N}) \subset \mathcal{G}_\lambda(E)$ converges to zero in $\mathcal{G}_\lambda(E)$ if and only if there exists some $(\psi_n : n \in \mathbb{N}) \subset L^2(\mu, E)$ which converges to zero in $L^2(\mu, E)$, such that $\phi_n = P_{\log \lambda}(\psi_n)$ for all $n \in \mathbb{N}$, where P_t denotes the semigroupe of Ornstein-Uhlenbeck on the Wiener space.*

We conclude this section with the construction of the derivation and divergence operations and integration by parts operation in $\mathcal{G}_\lambda$ spaces. Given $g_n \in E \otimes H^{\otimes n}$, $n \geq 1$, we can consider g_n as an element of $(E \otimes H) \otimes H^{\otimes n-1}$, we will denote by $\tilde{g}_n$ this interpretation of g_n . For any $G \in \mathcal{G}_0(E)$, $G = \Sigma I_n(g_n)$, define the derivation

$$DG = \sum_{n=1}^{\infty} n I_{n-1}(\tilde{g}_n)$$

Then DG is a continuous mapping from $\mathcal{G}_\lambda^-(E)$ to $\mathcal{G}_\lambda^-(E \otimes H)$ and from $\mathcal{G}_\lambda^+(E)$ to $\mathcal{G}_\lambda^+(E \otimes H)$. $D^2G, D^3G, \dots$ are defined recurrently. Since $\mathcal{D}_\infty \supset \mathcal{G}_\infty$, it follows that $\mathcal{D}_{-\infty} =$

$\cup_{p>1, s \in \mathbf{R}} \mathcal{D}_{p,s} \subset \mathcal{G}_0$. Note that if $G \in \mathcal{G}_0(E) \cap \mathcal{D}_{-\infty}(E)$, then DG and ∇G possess the same formal multiple Wiener-Itô representation, from now on both will be denoted by ∇G . Let $G \in \mathcal{G}_0(H)$, then $g_n \in H \otimes H^{\otimes n}$, let $\tilde{g}_n$ denote the symmetrization of g_n over $H^{\otimes n+1}$ and set

$$\delta G = \sum_{n=0}^{\infty} I_{n+1}(\tilde{g}_{n+1})$$

Then $\delta G \in \mathcal{G}_0(\mathbb{R})$ and δG is a continuous mapping from $\mathcal{G}_{\lambda}^+(H)$ to $\mathcal{G}_{\lambda}^+(\mathbb{R})$ and from $\mathcal{G}_{\lambda}^-(H)$ to $\mathcal{G}_{\lambda}^-(\mathbb{R})$.

For every $G^a \in \mathcal{G}_{\lambda}(E)$, $G^b \in \mathcal{G}_{\lambda-1}(E)$, define

$$\begin{aligned} \langle G^a, G^b \rangle &= E(G_{\lambda}^a, G_{\lambda-1}^b)_E \\ &= \sum n! (g_n^a, g_n^b)_{E \otimes H^{\otimes n}} \end{aligned} \quad (2.5)$$

We will denote $\langle G_a, G_b \rangle$ by $E(G^a, G^b)_E$. The integration by parts formula in this setup is as follows: for any $G^a \in \mathcal{G}_{\lambda-1}(E)$, $G^b \in \mathcal{G}_{\lambda}^+(E \otimes H)$

$$E(G^a, \delta G^b) = E(\nabla G^a, G^b)_{E \otimes H} \quad .$$

For any $G = \sum I_n(g_n) \in \mathcal{G}_0(E)$, set $EG = I_0(g_0) = g_0$. Note that for any $G \in \mathcal{G}_0(E)$ ([10])

$$g_n = \frac{1}{n!} E(\nabla^n G) \quad . \quad (2.6)$$

3 The extension of Wiener functionals

Definition 1 Let $u(\omega)$ be an H -valued random variable and $T\omega = \omega + u(\omega)$. Denote by A and B any two topological vector spaces of Wiener functionals in which the (cylindrical) polynomials are dense and that for any polynomial f , $f \circ T$ is in B . We say that T is (A, B) -extendable if the map $f \mapsto f \circ T$ has a continuous extension to A .

Remarks:

(a) Note that the above definitions are consistent in the sense that if A' is continuously injected into A , then any (A, B) extendable shift is (A', B) -extendable and if B is continuously injected into B' then any (A, B) extendable shift is (A, B') -extendable.

In order to point out an example of nonconsistency, let $f = \delta_0(W(1))$ where δ denotes the

Dirac measure at zero, then as pointed out by D. Ocone, $f \in \mathcal{D}_{2,-1}$ but any sequence of polynomials f_n converging to f in $\mathcal{D}_{2,-1}$ converges to zero in L^0 and therefore even for the case $u(\omega) = 0, T\omega = \omega$, there is no consistency.

(b) In the next section we will introduce a definition of $f(T\omega)$ as a limit of $\varphi(T_n\omega)$ for certain approximations T_n to T .

(c) Let $p > 1, k \geq 1$, and denote by $c_{p,k}$ the capacity associated to the $\mathcal{D}_{p,k}$ norm (c.f., for example [2]). Suppose that for any cylindrical set M , such that $c_{p,k}(M) = 0$, we have $\mu(T^{-1}(M)) = 0$. Then it is easy to see that T is $(\mathcal{D}_{p,k}, L^0(\mu))$ -extendable. We will give a concrete example of this case after the proof of Lemma 4.2 .

The first result deals with the multiple Wiener-Itô integrals:

Proposition 3.1 (*Nilüfer's equality*): *Let $T\omega = \omega + u(\omega)$ where $u(\omega)$ is an H -valued random variable. Then T is $(\mathcal{G}_\infty, L^0(\mu))$ -extendable. In particular, for any multiple Wiener-Itô integral $\varphi = I_n(h_n), h_n \in H^{\otimes n}$, we have*

$$(I_n(h_n))(\omega + u(\omega)) = \sum_{i=0}^n \binom{n}{i} (I_{n-i}(h_n), u^{\otimes i})_{H^{\otimes i}} \quad (3.1)$$

Proof: Let φ be in $\mathcal{G}_\infty$ and (f_n) be a sequence of polynomials converging to φ in $\mathcal{G}_\infty$. Let $\epsilon > 0$ be an arbitrary number and choose $a > 0$ such that $\mu\{|u| > a\} \leq \epsilon/2$. Then we have

$$\mu\{|f_n \circ T - f_m \circ T| > a\} \leq \mu\{|f_n \circ T - f_m \circ T| > a, |u| \leq a\} + \frac{\epsilon}{2}.$$

There exists, from the Lemma 2.1, a sequence of polynomials (g_n) converging in L^2 such that $f_n = P_t g_n$ for any $n = 1, 2, \dots$, for some $t > 0$. Hence, on the set $\{|u| \leq a\}$

$$\begin{aligned} |f_n \circ T - f_m \circ T| &\leq \sup_{|h| \leq a} |f_n(\omega + h) - f_m(\omega + h)| \\ &\leq \sup_{|h| \leq a} |P_t g_n(\omega + h) - P_t g_m(\omega + h)| \\ &= \sup_{|h| \leq a} \left| \int (g_n - g_m)(e^{-t}(\omega + h) + \sqrt{1 - e^{-2t}}y) \mu(dy) \right| \\ &= \sup_{|h| \leq a} \left| \int (g_n - g_m)(e^{-t}(\omega) + \sqrt{1 - e^{-2t}}y) k_t(h)(y) \mu(dy) \right| \\ &\leq \sup_{|h| \leq a} \|k_t(h)\|_{L^2(\mu)} (P_t(|g_n - g_m|)^2)^{1/2} \mapsto 0 \end{aligned}$$

in probability, where

$$k_t(h) = \exp\left[\frac{e^{-t}}{\sqrt{1 - e^{-2t}}} \delta h - 1/2 \frac{e^{-2t}}{1 - e^{-2t}} |h|^2\right],$$

and this term comes from the Cameron-Martin formula. The rest of the proof follows from the Taylor formula, applied first to smooth kernels and a limiting procedure.

In fact the above hypothesis can be weakened as in the following corollary whose proof is straightforward from the proof of the proposition:

Corollary 3.1 *For any $\lambda > 1$, T is $(\mathcal{G}_\lambda, L^0)$ -extendable, if furthermore $|u| \leq a$ almost surely, then T is $(\mathcal{G}_\lambda, L^2)$ -extendable.*

Example: Consider the transformation $T\omega = \omega - \langle \omega, e \rangle \cdot e$, with $e \in H$, $\|e\|_H = 1$ (e.g. the Brownian bridge $(T\omega)_t = \omega_t - t\omega_1$). In this case μ and $\mu \circ T^{-1}$ are mutually singular. However, from the above results, T is $(\mathcal{G}_\lambda, L^0)$ -extendable, for any $\lambda > 1$.

4 Wiener functionals of Rademacher class

Definition 2 $F : W \rightarrow \mathbb{R}$ is said to belong to the Rademacher class $R_{\alpha,k}^p$ if $F \in \mathcal{D}_{q,r}$ for some $q > 1, r \geq 1$ and

$$\sup_{|h| \leq \alpha} |\nabla^k F(\omega + h)| \in L^p(\mu) \quad .$$

If $p = 0$ we denote this class by $R_{\alpha,k}^0$. We say $F \in R_{\infty,k}^p$ if the above condition is true $\forall \alpha > 0$ and $F \in R_{\infty,\infty}^p$ if $F \in R_{\infty,k}^p$ for all $k \in \mathbb{N}$. If $F \in R_{\alpha,k}^p$ for any $p \geq 0$, we write $F \in R_{\alpha,k}^\infty$.

Remarks:

o) Let us note that, since $F \in \mathcal{D}_{p,k}, p > 1, k \geq 1$, $h \mapsto F(\omega + h)$ is continuous in $L^1(\mu)$, hence it is separable and consequently the supremum is well-defined.

i) Note that $R_{\alpha,k}^p \supset R_{\alpha,k+1}^p$. In fact

$$\nabla^k F(\omega + h) = \nabla^k F(\omega) + \int_0^1 (\nabla^{k+1} F(\omega + \lambda h), h) d\lambda \quad ,$$

consequently we have

$$\sup_{|h| \leq \alpha} |\nabla^k F(\omega + h)| \leq |\nabla^k F(\omega)| + \alpha \sup_{|h| \leq \alpha} |\nabla^{k+1} F(\omega + h)|$$

almost surely.

ii) If $\nabla F \in L^\infty(\mu; H)$, then we have

$$|F(\omega + h)| \leq |F(\omega)| + |h|_H \|\nabla F\|_\infty \quad \text{a.s.}$$

hence $F \in R_{\infty,0}^p$ for any $p > 1$ i.e., $F \in R_{\infty,0}^\infty$. In fact in this case we have $E[\exp cF^2] < \infty$ for some $c > 0$ (c.f.[11]).

We have

Proposition 4.1 *Suppose that $F \in \mathcal{G}_\lambda$ with $\lambda > 1$. Then $F \in R_{\infty,\infty}^2$.*

Proof: From the Lemma 2.1, we know that there exists some $G \in L^2(\mu)$ such that $F = P_t G$ where $t = \log \lambda$. For any $h, k \in H$, we have, using the integration by parts and Cameron-Martin formulae,

$$\begin{aligned} \nabla_k P_t G(\omega + h) &= a_t \int \delta k(y) G(e^{-t}(\omega + h) + \sqrt{1 - e^{-2t}}y) \mu(dy) \\ &= a_t \int [\delta k(y + a_t h) - a_t(h, k)] G(e^{-t}\omega + \sqrt{1 - e^{-2t}}(y + a_t h)) \mu(dy) \\ &= a_t \int [\delta k(y + a_t h) - a_t(h, k)] \varepsilon(a_t \delta h(y)) G(e^{-t}\omega + \sqrt{1 - e^{-2t}}y) \mu(dy) \\ &\leq a_t E[(\delta k(y + a_t h) - a_t(h, k))^2 \varepsilon(a_t \delta h(y))^2]^{1/2} (P_t(G^2))^{1/2}, \end{aligned}$$

where $a_t = e^{-t}/\sqrt{1 - e^{-2t}}$ and $\varepsilon(\delta h(y))$ denotes the exponential martingale associated to the first order Wiener integral δh . Consequently, we have

$$\sup_{|h| \leq \alpha} |\nabla F(\omega + h)| \leq \sup_{|h| \leq \alpha, |k| \leq 1} a_t E[(\delta k(y + a_t h) - a_t(h, k))^2 \varepsilon(a_t \delta h(y))^2]^{1/2} (P_t(G^2))^{1/2} \in L^2(\mu).$$

The iteration of this technique completes the proof.

Lemma 4.1 *Suppose that $F \in R_{\infty,1}^0 \cap L^\infty(\mu)$. Then there exists a sequence of measurable subsets $(\Omega_n)_n$ of W with $\cup \Omega_n = W$ almost surely, a sequence of functionals (F_n) such that $\nabla F_n \in L^\infty(\mu, H)$ and $F = F_n$ on Ω_n , i.e., each element of $R_{\infty,1}^0$ has a locally bounded derivative.*

Proof: Let $\Omega_{n,m} = \{\omega : |\nabla F(\omega + h)| \leq n, \quad \forall |h| \leq m\}$. By the hypothesis, we have $\cup \Omega_{n,m} = W$ a.s. Let

$$q_{nm}(\omega) = \inf(|h|_H : h \in (\Omega_{n,m} - \omega) \cap H) \quad (4.1)$$

$$= \infty \text{ if } \{\cdot\} = \emptyset, \quad (4.2)$$

$\varphi \in C_0^\infty(\mathbb{R})$, $\varphi(0) = 1$, $\varphi(t) = 0$ for $|t| > \frac{1}{4}$ and define $\rho_{nm}(\omega) = \varphi\left(\frac{1}{m}q_{nm}(\omega)\right)$.

Define F_{nm} as $F\rho_{nm}$. We have

$$|\nabla F_{nm}| \leq |\nabla F| \cdot \rho_{nm} + |F| |\nabla \rho_{nm}| \leq |\nabla F| \rho_{nm} + \|F\|_\infty \frac{1}{m} \|\varphi'\|_\infty, \quad ,$$

for $\rho_{nm} \neq 0$, we should have some $\ell \in (\Omega_{nm} - \omega) \cap H$ with $|\ell| < \frac{m}{4} + \epsilon$ for some $\epsilon > 0$. Hence $\omega = \tilde{\omega} - \ell$ with $\tilde{\omega} \subset \Omega_{nm}$ and $|\ell| \leq m$, consequently, by the definition of Ω_{nm} , we should have $|\nabla F(\omega)| \leq n$.

We turn now to the definition of $F \circ T$: Suppose, first, that $u = \sum_{i=1}^n \mathbf{1}_{A_i} h_i$ with $h_i \in H$, $A_i \cap A_j = \emptyset$ ($i \neq j$). Suppose that (F_k) is any sequence of smooth cylindrical functions. We have

$$F_k \circ T(\omega) = \sum_{i=1}^n F_k(\omega + h_i) \mathbf{1}_{A_i}(\omega)$$

If (F_k) is Cauchy in $L^p(\mu)$ ($p > 0$), then

$$E [|F_k \circ T - F_e \circ T|^r] \leq \sum_{i=1}^n E [|F_k - F_e|^r \cdot \mathcal{E}(\delta h_i)] \rightarrow 0,$$

where $\mathcal{E}(\delta h) = \exp \delta h - \frac{1}{2} |h|^2$, for any $r < p$, for $p = 0$, $(F_k \circ T)$ is also Cauchy in $L^0(\mu)$. Consequently, any simple shift is (L^p, L^r) -extendable and (L^0, L^0) -extendable, for any $p > r$ (in fact T is absolutely continuous).

Suppose now that F is of class $R_{1,1}^0$, $u_n(\omega) \in H$, $|u_n(\omega)| \leq 1$ a.s. and $u_n \rightarrow u$ in $L^0(\mu; H)$, where (u_n) are H -valued step functionals as above. We then have

$$|F \circ T_n(\omega) - F \circ T_m(\omega)| \leq \sup_{|h| \leq 1} |\nabla F(\omega + h)| |u_n(\omega) - u_m(\omega)| \xrightarrow{L^0} 0, \quad ,$$

obviously, the limit is independent of the choice of (u_n) provided that $|u_n| \leq 1$. Consequently we can give the following definitions **for this section and this section only** :

Definition 3 1. Suppose that $F \in R_{\alpha,1}^0$, $\alpha > 0$, $u : C \mapsto H$ with $|u| \leq \alpha$ almost surely.

We define $F \circ T$ to be the limit in probability of $(F \circ T_n; n \in \mathbb{N})$ where $|u_n| \leq \alpha$ are H -valued step processes, converging to u in $L^0(\mu, H)$.

2. If $F \in R_{\infty,1}^0$, then we define $F \circ T$ as the limit in L^0 of any step shifts $T_n(\omega) = \omega + u_n(\omega)$, where $u_n \mapsto u$ in $L^0(\mu, H)$.

Evidently the second part of this definition requires to be justified:

Lemma 4.2 *Suppose that $F \in R_{\infty,1}^0$ and that $u : C \mapsto H$ is any random variable. Then $F \circ T$ is well-defined.*

Proof: Let $(u_n; n \in \mathbb{N})$ be a sequence of step functions converging to u in $L^0(\mu, H)$. Then the double sequence $(u_n, u_m; n, m \in \mathbb{N})$ converges also in $L^0(\mu, H \times H)$. Consequently, for any $\epsilon > 0$, there exists a compact subset of H , say K_ϵ , and $N_\epsilon > 0$ such that, for any $m, n \geq N_\epsilon > 0$, we have

$$\mu\{(u_n, u_m) \in K_\epsilon \times K_\epsilon\} > 1 - \frac{\epsilon}{2}.$$

Therefore

$$\begin{aligned} \mu\{|F \circ T_n - F \circ T_m| > c\} &\leq \frac{\epsilon}{2} + \mu\{|u_n - u_m| \sup_{h \in K_\epsilon} |\nabla F(\omega + h)| > c\} \\ &\rightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$ for any $c > 0$. In particular this result shows that $F \circ T$ is independent of the choice of $(u_n; n \in \mathbb{N})$.

QED

Let $\{e_1, \dots, e_n\}$ be an orthonormal set in H and suppose that $u(\omega)$ belongs to the unit ball of the n -dimensional subspace of H spanned by $\{e_1, \dots, e_n\}$. If $\varphi \in \mathcal{D}$, we know that the map

$$x \mapsto \varphi(\omega + \sum_1^n x_i e_i)$$

is C^∞ in $\mathbb{R}^n$ after a modification on a null set. Suppose Ω is a bounded domain in $\mathbb{R}^r$, if $mp > n$, then the Sobolev injection theorem thus says that the injection $W^{m+j,p}(\Omega) \hookrightarrow C_B^j(\Omega)$ is continuous for any $j \geq 0$. Consequently, we have, for any $|\alpha| \leq j$, with $K_1(n) = \{h \in H_n : |h| \leq 1\}$:

$$\sup_{x \in K_1(n)} \left| \partial_x^\alpha \varphi \left(\omega + \sum_1^n x_i e_i \right) \right|^p \leq \left(\int_{K_1(n)} \sum_{|\beta| \leq m+j} \left| \partial^\beta \varphi \left(\omega + \sum_1^n x_i e_i \right) \right|^p dx \right) C(p, n, m) .$$

Since $\frac{\partial}{\partial x_k} \varphi \left(\omega + \sum_1^n x_i e_i \right) = \nabla_{e_k} \varphi \left(\omega + \sum_1^n x_i e_i \right)$, we have

$$\sup_{x \in K_1(n)} \left| \partial_x^\alpha \varphi \left(\omega + \sum_1^n x_i e_i \right) \right|^p \leq \tilde{C}(p, n, m) \int_{K_1(n)} \sum_{k=1}^{m+j} \left| \nabla^k \varphi \left(\omega + \sum_1^n x_i e_i \right) \right|^p dx .$$

Taking the expectation of both sides, we see that

$$\begin{aligned}
E \left[\sup_{x \in K_1(n)} \left| \partial_x^\alpha \varphi \left(\omega + \sum_1^n x_i e_i \right) \right|^p \right] &\leq \tilde{C}(p, n, m) E \int_{K_1(n)} \sum_{b=1}^{m+j} \left| \nabla^b \varphi \left(\omega + \sum_1^n x_i e_i \right) \right|^p dx \\
&= \tilde{C}(p, n, m) E \int_{K_1(n)} \sum_1^{m+j} |\nabla^k \varphi(\omega)|^p \mathcal{E} \left(\delta \sum_1^n x_i e_i \right) dx \leq \\
&\leq \tilde{C}(p, n, m) \cdot E \left[\sum_1^{m+j} |\nabla^k \varphi|^{p(1+\epsilon)} \right]^{1/1+\epsilon} \left(\int_{K_1(n)} E \left[\mathcal{E} \left(\sum x_i \delta e_i \right)^{\frac{1+\epsilon}{\epsilon}} \right] dx \right)^{\frac{\epsilon}{1+\epsilon}}
\end{aligned}$$

Therefore we have

$$E[|\varphi(\omega + u(\omega))|^p] \leq \tilde{C}(p, n, m) \|\varphi\|_{p(1+\epsilon), m+j}^p$$

provided that $mp > n = \dim \text{range}(u)$. With the hypothesis of (2), supposing further that $u \in \mathcal{D}_{p,k}(H)$ for some $p > 1$, we have

$$\nabla(\varphi \circ T) = (I + \nabla u)^* \nabla \varphi \circ T$$

for any $\varphi \in \mathcal{D}$. In fact this can be extended to some $\mathcal{D}_{r,k}$ with careful utilization of the Hölder inequality.

Remark: Suppose that $F \in R_{1,2}^0 \cap \mathcal{D} \cap L^\infty(\mu)$, $u : W \rightarrow H$, $|u|_W \leq 1$ a.s. and $u \in \bigcap_p D_{p,1}(H)$. Then for any $\epsilon > 0$, there exists a set A_ϵ s.t. $\mu(A_\epsilon) < \epsilon$ and outside A_ϵ we have $\nabla(F \circ T) = (I + \nabla u)^* \nabla F \circ T$. In fact this result follows from the Egorov theorem. However we cannot in general prove that $A_\epsilon = \emptyset$ almost surely since, in general we define $\nabla(F \circ T)$ in the sense of distributions, however $\nabla F \circ T$ is defined in $L^0(\mu)$ and as already mentioned earlier these two convergences are not compatible (even in the finite dimensional case).

To overcome this difficulty we will define:

Definition 4 We will say that $F : W \rightarrow \mathbb{R}$ belongs to $\mathcal{D}_{0,1}$ if, $\forall h \in H$

i) $t \mapsto F(\omega + th)$ is absolutely continuous w.r. to Lebesgue measure.

ii) $\exists G : W \rightarrow H$ s.t. $\frac{1}{t}[F(\omega + th) - F(\omega)] - (G(\omega), h) \rightarrow 0$ in probability.

Then we denote G by ${}^\circ \nabla F$.

Remark: Suppose that $F \in \mathbb{D}_{p,1}$ for some $p > 1$. Then, choosing a smooth sequence $(F_n) \rightarrow F$ in $\mathbb{D}_{p,1}$, we have

$$F(\omega + th) - F(\omega) = \lim_n (F_n(\omega + th) - F_n(\omega)) = \lim_n \int_0^t (\nabla F_n(\omega + sh), h) ds$$

on the other hand

$$E \int_0^t |\nabla F_n(\omega + sh) - \nabla F(\omega + sh)| ds = E \left[|\nabla F_n - \nabla F| \int_0^1 \mathcal{E}(sh) ds \right]_n \rightarrow 0 \quad .$$

Therefore $\mathbb{D}_{p,1} \subset \mathbb{D}_{0,1}$ for any $p > 1$ and $\nabla F =^o \nabla F$. We have

Theorem 4.1 *Suppose that $F \in R_{\infty,2}^0 \cap \mathbb{D}$ and that $u \in \mathbb{D}_{p,2}(H)$ for some $p > 1$. Then we have*

$$^o \nabla(F \circ T) = (I + \nabla u)^* \nabla F \circ T \quad a.s.$$

Proof: Suppose T_n is a sequence of cylindrical shifts s.t. $u_n \rightarrow u$ in $\mathbb{D}_{p,2}(H)$. Then we know that

$$\nabla(F \circ T_n) = (I + \nabla u_n)^* \nabla F \circ T_n \quad .$$

Since $F \in R_{\infty,2}^0$, note that $\nabla F \circ T_n \xrightarrow{L^0} \nabla F \circ T$. We will show that $\nabla(F \circ T_n)$ converges to $^o \nabla(F \circ T)$ in probability. First, from the above relation, we see that $\nabla(F \circ T_n)$ converges in probability. By a theorem of Kusuoka (cf. [4], Theorem 3.1), it is enough to show that there exists a random variable β such that $\inf_{t \in [-a,a]} \beta(\omega + th) > 0$ almost surely for any $h \in H$ and $a > 0$ with

$$E[\beta \cdot |\nabla_h(F \circ T_n) - (I + \nabla u)^* \nabla_h F(T)|] \rightarrow 0$$

and that $E[\beta |I + \nabla u| \cdot |\nabla F \circ T|] < \infty$. But it is sufficient to choose β as (taking a subsequence of (u_n) if necessary)

$$\beta(\omega) = \frac{1}{1 + \sup_n |\nabla u_n(\omega)| \sup_{|h| \leq \alpha} |\nabla F(\omega + h)|} \quad .$$

To see that $\inf_{t \in [-a,a]} \beta(\omega + th) > 0$ almost surely for any $h \in H$ (the negligible set may depend on h) it is sufficient to show that $\sup_{t \in [-a,a]} \sup_n |\nabla u_n(\omega + th)| < \infty$ and $\sup_{t \in [-a,a]} \sup_{|h| \leq \alpha} |\nabla F(\omega + h + th)| < \infty$ almost surely. The second inequality is evident from the hypothesis. To prove the first, without loss of generality, we can take $a = 1$ and, if necessary, choose and restrict ourselves to a subsequence (u_{n_k}) such that $\nabla u_{n_k} \nabla^2 u_{n_k}$ and

$$\int_{[-1,1]} |\nabla^2 u_{n_k}(\omega + sh) - \nabla^2 u_{n_l}(\omega + sh)|_{H^{\otimes 3}} ds$$

converge also almost surely to, respectively, ∇u , $\nabla^2 u$ and to zero. We have then

$$\sup_{n_k} \sup_{t \in [-1, 0]} |\nabla u_{n_k}(\omega + th)| \leq \sup_k |\nabla u_{n_k}(\omega)| + \sup_k \int_{-1}^0 |\nabla^2 u_{n_k}|_{H^{\otimes 3}} |h| ds < \infty,$$

we have also a similar inequality on the interval $[0, 1]$ and this completes the proof.

QED

Suppose now that $F \in D_{r,1}$ for some $r > 1$ and that $\nabla F \in L^\infty(\mu, H)$ (in particular this implies that $F \in \cap_q D_{q,1}$ cf.[11]). If $(h_k) \subset W^*$ is a complete orthonormal basis for H , let $V_n = \sigma\{\delta h_1, \dots, \delta h_k\}$ and denote by F_k the functional $E[P_{1/k} F | V_k]$. Then we have

$$\sup_k \|\nabla F_k\|_{L^\infty(\mu, H)} < \infty.$$

Hence for almost all $w \in W$, the set of functionals $(F_k(w + \cdot); k \in \mathbb{N})$ is uniformly equicontinuous on H . Since $u \in L^0(\mu, H)$, for every $\epsilon > 0$, there exists a compact subset K_ϵ of H such that $\mu\{u \in K_\epsilon^c\} < \epsilon/2$. Hence

$$E[|F_k \circ T - F_l \circ T|^p] \leq \|\nabla F\|_\infty^p \epsilon + E[|F_k \circ T - F_l \circ T|^p 1_{u \in K_\epsilon}] \quad (4.3)$$

$$\leq \|\nabla F\|_\infty^p \epsilon + E[\sup_{h \in K_\epsilon} |F_k(w + h) - F_l(w + h)|^p 1_{u \in K_\epsilon}], \quad (4.4)$$

but from the equicontinuity, the convergence of $F_k(w + h)$ to $F(w + h)$ is uniform with respect to h when h runs in the compact subsets of H . Therefore $F_k \circ T \rightarrow F \circ T$ in $L^p(\mu)$ for any $p > 1$. Suppose now that $u \in D_{p,1}(H)$ for some $p > 1$. Then we have

$$\|\nabla(F_k \circ T)\|_{L^p} = \|(I + \nabla u)^* \nabla F_k \circ T\|_{L^p} \quad (4.5)$$

$$\leq (1 + \|\nabla u\|_{L^p}) \|\nabla F\|_{L^\infty(\mu, H)}, \quad (4.6)$$

consequently, taking a subsequence, we can suppose that

$$\nabla(F_k \circ T) \rightarrow \nabla(F \circ T)$$

in the weak topology $\sigma(L^p, L^q)$, hence $F \circ T$ is in $D_{p,1}$. However, in general, we can not write $\nabla(F \circ T) = (I + \nabla u)^* \nabla F \circ T$, since $\nabla F \circ T$ is not defined apriori. Even in the finite dimensional case we have a similar difficulty. However, if the operator valued random function $I_H + \nabla u$ is almost surely invertible, then we define a random variable $\nabla F \star T$ as $(I_H + \nabla u)^{\star-1} \nabla(F \circ T)$. Note that, in the finite dimensional case the hypothesis of invertibility

implies that the image of the measure μ under the map $T = I_W + u$ is absolutely continuous with respect to the (finite dimensional) Gaussian measure. In infinite dimension, this is not in general true, but, for $F \in R_{\infty,2}^0$ we have

$$\nabla(F \circ T) = {}^0 \nabla(F \circ T) = (I + \nabla u)^* \nabla F \star T,$$

hence

$$\nabla F \star T = \nabla F \circ T$$

almost surely.

Proposition 4.3 *Suppose that $\nabla \delta \xi \in L^\infty(\mu; H)$, $\nabla \xi \in L^\infty(\mu; H \otimes H)$, u in some $\mathcal{D}_{p_0,1}(H)$, $p_0 > 1$ with $I + \nabla u$ a.s. invertible. Then we have*

$$(\delta \xi) \circ T = \delta(\xi \circ T) + (\xi \circ T, u) + \text{trace} \quad (\nabla u^* \nabla \xi \star T) \quad .$$

Proof: Let ξ_k be defined as $E[\pi_k P_{1/k} \xi \mid V_k]$, where π_k is the orthogonal projection associated to $\text{span}\{h_1, \dots, h_k\}$ and P_t represents the Ornstein-Uhlenbeck semigroup on W . Then we have, trivially

$$(\delta \xi_k) \circ T = \delta(\xi_k \circ T) + (\xi_k \circ T, u) + \text{trace} \quad (\nabla u^* \nabla \xi_k \star T) \quad .$$

We have $E[\delta \xi \mid V_n] = E[\delta \pi_n \xi \mid V_n]$ and $\nabla E[\delta \xi \mid V_n] = E[\pi_n \nabla \delta \xi \mid V_n]$,

$$\begin{aligned} \|\nabla \delta \xi_k\|_{L^\infty} &= \left\| \nabla \cdot E[\delta P_{1/k} \xi \mid V_k] \right\|_{L^\infty} = \left\| E[\pi_k \nabla \delta P_{1/k} \xi \mid V_k] \right\|_{L^\infty} \\ &\leq \left\| \pi_k \nabla \delta P_{1/k} \xi \right\|_{L^\infty} \leq \left\| \nabla \delta P_{1/k} \xi \right\|_{L^\infty} \leq \|\nabla \delta \xi\|_{L^\infty}, \end{aligned}$$

consequently, using the same equicontinuity argument as above and the triangle inequality, we see that $\delta \xi_k \circ T \xrightarrow{L^p} \delta \xi \circ T$, $\forall p > 1$ and $\xi_k \circ T \rightarrow \xi \circ T$ in all the L^p -spaces. Since $\{\nabla(\xi_k \circ T); k \in \mathbb{N}\}$ is bounded in $L^{p_0}(\mu)$ (p_0 is chosen such that $u \in \mathcal{D}_{p_0,1}(H)$), choosing a subsequence we may suppose that $\nabla(\xi_k \circ T) \rightarrow \nabla(\xi \circ T)$ in the weak topology $\sigma(L^{p_0}, L^{q_0})$. Since $(\nabla \xi_k \circ T; k \in \mathbb{N})$ is bounded in L^∞ we can choose a subsequence which converges weakly to some $\eta \in L^\infty(\mu, H \otimes H)$ in any L^p and we know that $\nabla(\xi_k \circ T)$ converges weakly to $\nabla(\xi \circ T)$. Hence we have

$$(I + \nabla u)^* \nabla \xi_k \circ T \rightarrow (I + \nabla u)^* \eta,$$

weakly in some L^p . On the other hand we know that $\nabla(\xi_k \circ T)$ converges also weakly to $\nabla(\xi \circ T)$, consequently, we have $\eta = \nabla \xi \star T$.

QED

Theorem 4.4 Suppose that $\xi \in R_{\infty,2}^0(H)$ and that $\delta\xi \in R_{\infty,1}^0$; suppose that $I + \nabla u$ is almost surely invertible and $u \in \mathcal{D}(H)$. Then we have

$$\delta\xi \circ T = \tilde{\delta}(\xi \circ T) + (\xi \circ T, u) + \text{trace}(\nabla u^* \nabla \xi \circ T)$$

where $\tilde{\delta}$ denotes the local version of δ .

Proof: Let $\Omega_{m,n} = \{\omega : |\nabla \xi(w + h)| \leq n, |\xi(w + h)| \leq n, |\nabla \delta \xi(w + h)| \leq n, \forall |h| \leq m\}$. Define $q_{n,m}(\omega)$ as $\inf(|h| : h \in (\Omega_{n,m} - \omega) \cap H)$, $f \in C_0^\infty(\mathbb{R})$, $f(0) = 1$, $f = 0$ for $|t| > \frac{1}{4}$; $\rho_{nm}(\omega) = f(\frac{1}{m}q_{n,m}(\omega))$. Note that $\bigcup_{m,n} \Omega_{m,n} = W$ a.s. Let $\xi_{nm} = \rho_{nm}\xi$. Then ξ_{nm} satisfies the hypothesis of the preceding proposition, therefore we have

$$\delta\xi_{nm} \circ T = \delta(\xi_{nm} \circ T) + (\xi_{nm} \circ T, u) + \text{trace}(\nabla u^* \nabla \xi_{nm} \star T) \quad .$$

Note that $\xi_{nm} \circ T = \xi \circ T$ on $T^{-1}(\Omega_{m,n})$ and $\nabla \xi_{nm} \star T = \nabla \xi \circ T$ on $T^{-1}(\Omega_{m,n})$, hence to prove the theorem it is sufficient to show that $\bigcup_{m,n} T^{-1}(\Omega_{m,n}) = W$ μ -a.s. Since T is not absolutely continuous, this is not a priori evident. We have

$$\begin{aligned} \mu\left(\bigcap T^{-1}(\Omega_{m,n}^c)\right) &\leq \mu\{|\nabla \xi(w + u(w) + h)| > n, \forall n, \text{ for some } |h| \leq m\} \\ &+ \mu\{|\xi(w + u(w) + h)| > n, \forall n, \text{ for some } |h| \leq m\} \\ &+ \mu\{|\nabla \delta \xi(w + u(w) + h)| > n, \forall n, \text{ for some } |h| \leq m\}. \end{aligned}$$

We have, for instance for the first term at the right hand side of this inequality, choosing $K_\epsilon \subset H$ compact with $\mu\{u \in K_\epsilon^c\} < \frac{\epsilon}{4}$,

$$\begin{aligned} \mu\{ & \mid |\nabla \xi(w + u(w) + h)| > n, \forall n, \text{ for some } |h| \leq m\} \leq \\ &\leq \mu(\{|\nabla \xi(w + u(w) + h)| > n, \forall n, \text{ for some } |h| \leq m\} \cap \{u \in K_\epsilon\}) + \frac{\epsilon}{4} \leq \\ &\leq \mu\left\{\sup_{k \in K_\epsilon} |\nabla \xi(w + k + h)| > n, \forall n, \text{ for some } |h| \leq m\right\} + \frac{\epsilon}{4} \leq \\ &\leq \sum_m \mu\left\{\sup_{k \in K_\epsilon} |\nabla \xi(w + k + h)| > n, \forall n, \text{ for some } |h| \leq m\right\} + \frac{\epsilon}{4}. \end{aligned}$$

Each element of the sum can be majorated with

$$\mu\left\{\sup_{k \in K_\epsilon, |h| \leq m} |\nabla \xi(w + k + h)| > n, \forall n\right\} = 0,$$

by hypothesis. The other terms are treated similarly and the proof is completed.

QED

Remark: Let T be a transformation as above and denote by $[\phi]$ the equivalence class of Wiener functionals such that ϕ and ψ are in the same equivalence class if and only if $\phi \circ T = \psi \circ T$ μ -almost surely. Then theorems 4.1 and 4.4 say that the equivalence classes are preserved under the gradient and divergence operators provided that $I + \nabla u$ is μ -almost surely invertible.

Example: Suppose that (X_t) is a diffusion process given by

$$\begin{aligned} dX_t(x) &= \xi_i(X_t) \circ dW_t^i + \xi_o(X_t) dt & (\text{Stratonovitch}) \\ X_o(x) &= x \in \mathbb{R}^d, \end{aligned}$$

where ξ_i, ξ_o are smooth vector fields, ξ_i 's are of compact support and ξ_o has bounded derivatives. If $h \in H$, then $X_t(\omega + h, x)$ satisfies

$$\begin{aligned} dX_t(\omega + h) &= \xi_i(X_t(\omega + h)) \circ dW_t^i + \xi_i(X_t(\omega + h)) \dot{h}^i(t) dt + \xi_o(X_t(\omega + h)) dt \\ X_o(\omega + h) &= x. \end{aligned}$$

Let us denote by $\pi_t(\omega, x)$ the stochastic process with values in $C^\infty(\mathbb{R}^d) \tilde{\otimes} \mathbb{R}^d$ such that, for fixed $x \in \mathbb{R}^d$, $\pi_t(\omega, x) = X_t(\omega, x)$ a.s. Let $Z_t^h(\omega, x)$ be the solution of the following ordinary differential equation:

$$dZ_t = \left[D\pi(Z_t)^{-1} \right] \xi_i(\pi_t(Z_t)) \dot{h}_t^i dt; \quad Z_0 = x.$$

From the Itô-Ventzell formula, we know that (c.f. [1] and [12]):

$$\pi_t(\omega, Z_t(\omega)) = X_t(\omega + h, x) \quad \text{a.s.}$$

Consequently, in order to control $\sup_{|h| \leq \alpha} |X_t(\omega + h)|$, it is sufficient to control $\sup_{|h| \leq \alpha} |Z_t^h(\omega)|$. Because of the compact support property of the vector fields ξ_i , the ordinary differential

equation written above is uniformly non-explosive and using the Gronwall inequality, we have

$$|Z_t^h(\omega) - Z_t^k(\omega)| \leq K_T(\omega) |h - k|_H^2 \exp \cdot C(\omega) t |h|_H^2 \wedge |k|_H^2$$

almost surely, where

$$K_T(\omega) = \sup_{t \leq T, x \in \mathbb{R}^d} \left[\sum_i \left| \partial_i (D\pi_t)^{-1}(\omega, x) \right| + \left| D\pi_t^{-1}(\omega, x) \right| \right]$$

which is finite by a result of Bismut ([1]).

We have

$$\begin{aligned} \nabla_k X_t(\omega + h) &= \left. \frac{d}{d\lambda} X_t(\omega + h + \lambda k) \right|_{\lambda=0} = \left. \frac{d}{d\lambda} \pi_t(\omega, Z_t^{h+\lambda k}) \right|_{\lambda=0} \\ &= D\pi_t(\omega, Z_t^h) \cdot \partial_k Z_t^h \end{aligned}$$

where $\partial_k Z_t^h$ is defined as

$$\left. \frac{d}{d\lambda} Z_t^{h+\lambda k} \right|_{\lambda=0}.$$

and D represents the derivation on the finite dimensional Euclidean space. Note that $h \mapsto Z^h$ is infinitely differentiable from H into $C([0, 1], \mathbb{R}^d)$ (c.f. [1]). In fact we will show that these derivatives are in H .

$\partial_k Z_t^h$ satisfies the following equation:

$$\begin{aligned} \partial_k Z_t^h &= \int_0^t D(\pi_s^{*-1} \xi_i)(Z_s^h) \cdot \partial_k Z_s^h \dot{h}_i(s) ds + \\ &+ \int_0^t (\pi_s^{*-1} \xi_i)(Z_s^h) \cdot \dot{k}^i(s) ds \end{aligned}$$

where, following Bismut ([1]), we use the notation $\pi_s^{*-1} \xi_i(x) = [D\pi_s(x)]^{-1} \xi_i(\pi_t(x))$. Hence using the Gronwall lemma, we have

$$\begin{aligned} \sup_{r \leq t} |\partial_k Z_r^h| &\leq D_t(\omega) \cdot \int_0^t |\dot{k}_s| \cdot ds \cdot \exp \cdot \left[\int_0^t |\dot{h}_s| ds \cdot C_t(\omega) \right] \\ &\leq D_t(\omega) \cdot |k|_H \exp C_t(\omega) |h| \end{aligned}$$

where $C_t(\omega)$ and $D_t(\omega)$ are some (finite) random variables. Hence $k \mapsto \partial_k Z_t^h(\omega)$ is a linear, continuous functional on H . Let us look at the second derivative $\partial_{k\ell}^2 Z_t^h$: it satisfies the following (ordinary) differential equation:

$$\partial_{k\ell}^2 Z_t^h = \int_0^t D^2(\pi_s^{*-1} \xi_i)(Z_s^h) \cdot \partial_\ell Z_s^h \cdot \partial_k Z_s^h \dot{h}^i(s) ds +$$

$$\begin{aligned}
& + \int_0^t D(\pi_s^{*-1}\xi_i)(Z_s^h) \cdot \partial_{\ell k}^2 Z_s^h \dot{h}^i(s) ds + \\
& + \int_0^t D(\pi_s^{*-1}\xi_i)(Z_s^h) \cdot \partial_k Z_s^h \dot{\ell}^i(s) ds + \\
& + \int_0^t D(\pi_s^{*-1}\xi_i)(Z_s^h) \cdot \partial_\ell Z_s^h \dot{k}^i(s) ds \quad .
\end{aligned}$$

Denoting with $\|\cdot\|$ the Hilbert-Schmidt norm in $H \tilde{\otimes}_2 H$, and with $(e_k : k \in \mathbb{N})$ a complete orthonormal basis in H and also with $\partial_k = \partial_{e_k}$, we have

$$\|\partial^2 Z_t^h\|^2 = \sum_{k,\ell=1}^{\infty} |\partial_{k\ell}^2 Z_t^h|^2 \quad .$$

Let us control each of the terms: for the first term we have:

$$\begin{aligned}
& \sum_{k,\ell=1}^{\infty} \left| \int_0^t D^2(\pi_s^{*-1}\xi_i)(Z_s^h) \partial_\ell Z_s^h \partial_k Z_s^h \dot{h}^i(s) ds \right|^2 \leq \\
& \leq \sum_{k,\ell} \int_0^t |D^2(\pi_s^{*-1}\xi_i)(Z_s^h)|^2 |\partial_\ell Z_s^h|^2 |\partial_k Z_s^h|^2 |\dot{h}^i(s)|^2 ds \\
& = \sum_\ell \int_0^t |D^2(\pi_s^{*-1}\xi_i)(Z_s^h)|^2 |\partial_\ell Z_s^h|^2 |\partial Z_s^h|_H^2 |\dot{h}^i(s)|^2 ds \\
& = \int_0^t |D^2(\pi_s^{*-1}\xi_i)(Z_s^h)|^2 |\partial Z_s^h|_H^4 |\dot{h}^i(s)|^2 ds \\
& \leq \sup_{s \leq t} |\partial Z_s^h|_H^4 \cdot \sup_{s \leq t, x \in \mathbb{R}^d} |D^2(\pi_s^{*-1}\xi_i)(x)|^2 \cdot |h|_H^2
\end{aligned}$$

where ∂Z_s^h is the element of H defined by $k \rightarrow \partial_k Z_s^h$. For the second term we have immediately:

$$\begin{aligned}
& \sum_{k,\ell=1}^{\infty} \left| \int_0^t D(\pi_s^{*-1}\xi_i)(Z_s^h) \partial_{\ell k}^2 Z_s^h \dot{h}^i(s) ds \right|^2 \leq \\
& \leq \int_0^t |D(\pi_s^{*-1}\xi_i)(Z_s^h)|^2 \|\partial^2 Z_s^h\|^2 \cdot |\dot{h}^i(s)|^2 ds \quad .
\end{aligned}$$

Let us now try to control the third (hence also the fourth) term. First, denote by $\partial_r Z_t^h$ the kernel defined by

$$\int_0^1 \partial_r Z_t^h \cdot \dot{k}_r dr = \partial_k Z_t^h.$$

From the equation of $\partial_k Z_t^h$ written above, we find that $\partial_r Z_t^h$ satisfies:

$$\partial_r Z_t^h = \int_0^t D(\pi_s^{*-1}\xi_i)(Z_s^h) \cdot \partial_r Z_s^h \dot{h}_s^i ds + \mathbf{1}_{[0,t]}(r) (\pi_r^{*-1}\xi_i)(Z_r^h) \quad .$$

Consequently, using the Gronwall lemma and the fact that

$$\sup_{t,x \in [0,1] \times \mathbb{R}^d} |(\pi_t^{*-1}\xi_i)(x)| < +\infty \quad \text{a.s.} \quad ,$$

we have

$$|\partial_r Z_t^h| \leq C(\omega) \cdot \exp \left(D(\omega) \int_0^t |\dot{h}(s)|_{\mathbb{R}^d} ds \right) \quad ;$$

i.e. $(k, f) \mapsto \int_0^1 \int_0^1 \dot{k}(r) f(t) \partial_r Z_t^h dr dt$ is a Hilbert-Schmidt kernel which belongs to $H \tilde{\otimes} L^2([0, 1]) \tilde{\otimes} \mathbb{R}$

Now, taking into account this observation, we find that:

$$\begin{aligned} & \sum_{k, \ell} \left| \int_0^t D(\pi_s^{*-1} \xi_i)(Z_s^h) \partial_k Z_s^h \cdot \dot{e}_\ell^i(s) ds \right|^2 \leq \\ & \leq \sum_k \left| D(\pi^* \xi_i)(Z^h) \partial_k Z^h \right|_{L^2([0, 1]) \otimes \mathbb{R}^d}^2 \\ & \leq \sum_k A^2(\omega) \cdot \left| \partial_k Z^h \right|_{L^2([0, 1]) \otimes \mathbb{R}^d}^2 \\ & = A^2(\omega) \cdot \left\| \partial Z^h \right\|_{L^2([0, 1]) \tilde{\otimes} H \tilde{\otimes} \mathbb{R}^d}^2 \end{aligned}$$

since $\sup_{0 \leq t \leq 1, x \in \mathbb{R}^d} |D(\pi_t^{*-1} \xi_i)(x)| = A(\omega)$ is finite.

Summing all these inequalities, we see that

$$\begin{aligned} \sup_{r \leq t} \left\| \partial^2 Z_r^h \right\|^2 & \leq A_1(\omega) e^{2C_1(\omega)|h|} + A_2(\omega) \int_0^t \left\| \partial^2 Z_s^h \right\|^2 |\dot{h}(s)|^2 ds + \\ & + A_3(\omega) \exp \left(C_2 \int_0^t |\dot{h}(r)| dr \right) \quad , \end{aligned}$$

hence, the Gronwall inequality gives

$$\sup_{r \leq t} \left\| \partial^2 Z_r^h \right\|^2 \leq \left[A_1(\omega) e^{2C_1|h|} + A_3(\omega) e^{C_2|h|} \right] \exp A_2(\omega) \int_0^t |\dot{h}(s)|^2 ds$$

where A_1, A_2, A_3 are positive (finite) random variables independent of h . Consequently, we have

Theorem 4.5 $X_t \in \mathbb{R}_{\infty, \infty}^0$ for any $t \geq 0$.

Proof: We have $X_t(\omega + h, x) = \pi_t(\omega, Z_t^h)$ and

$$\begin{aligned} \sup_{|h| \leq \alpha} |X_t(\omega + h, x)| & = \sup_{|h| \leq \alpha} |\pi_t(\omega, Z_t^h)| \leq \\ & \leq \sup \{ |\pi_t(\omega, x)| : x \in \{Z_t^h(\omega); |h| \leq \alpha\} \} \end{aligned}$$

but the set $\{Z_t^h(\omega); |h| \leq \alpha\}$ is almost surely bounded in $\mathbb{R}^d$ and $(t, x) \mapsto \pi_t(\omega, x)$ is continuous almost surely.

Moreover

$$\begin{aligned} \sup_{|h| \leq \alpha} |\nabla X_t(\omega + h)| &\leq \sup_{|h| \leq \alpha} \left(|D\pi_t(\omega, Z_t^h)| \left| \partial Z_t^h \right|_H \right) \\ &\leq \sup \{ |D\pi_t(\omega, x)| : x \in \{Z_t^h : |h| \leq \alpha\} \} \cdot \\ &\quad \sup_{|h| \leq \alpha} \left| \partial Z_t^h \right|_H < +\infty, \end{aligned}$$

from the preceding calculations.

Since we have also, for any $\alpha \geq 0$,

$$\sup_{|h| \leq \alpha} \sup_{r \leq t} \left\| \partial^2 Z_r^h \right\|^2 < \infty \quad \text{a.s.}$$

we deduce as above, that

$$\sup_{|h| \leq \alpha} \left\| \nabla^2 X_t(\omega + h) \right\|^2 < +\infty \quad \text{a.s.}$$

The theorem is proved then by iteration of these techniques.

QED

5 A generalization of the Radon-Nikodym relation

Let $T\omega = \omega + u(\omega)$, $T\omega$ is said to induce a $\mathbb{D}_{p,s}$ or $\mathcal{G}_\lambda$ distribution if for every polynomial functional f , $E|f \circ T| < \infty$ and there exists a constant c depending of T only such that $|E f \circ T| \leq c \|f\|_{q,-s}$, $q^{-1} + p^{-1} = 1$ or $E|f \circ T| \leq c \|f_{1/\lambda}\|_{L^2(\mu)}$ respectively. Consequently, T induces a $\mathbb{D}_{p,s}$ (or $\mathcal{G}_\lambda$) distribution if and only if every G in $\mathbb{D}_{q,-s}$ (or $\mathcal{G}_{\lambda^{-1}}$) is T extendable in L^1 and there exists an element Λ in $\mathbb{D}_{p,s}$ (or $\mathcal{G}_\lambda$) such that for every G in $\mathbb{D}_{q,-s}$ ($\mathcal{G}_{\lambda^{-1}}$) $E G \circ T = "E\Lambda G"$ where " $E\Lambda G$ " denotes the pairing between $\mathbb{D}_{p,s}$ and $\mathbb{D}_{q,-s}$ and for the $\mathcal{G}_\lambda$ case

$$E G \circ T = \langle \Lambda, G \rangle = E\Lambda_\lambda \cdot G_{\lambda^{-1}} \quad (5.1)$$

The following lemma will be needed later

Lemma 5.1 *Let $u(\omega)$ be an H -valued random variable.*

(a) *If $E \|u\|_H^{2i} < \infty$ and $n > i$ then*

$$\left\| E \nabla^{n-i} u^{\otimes i} \right\|_{H^{\otimes n}} \leq ((n-i)!)^{1/2} \cdot E^{1/2} \|u\|_H^{2i} \quad (5.2)$$

(b) Assume, furthermore, that $E \exp \beta \|u\|_H^2 < \infty$ for some $\beta > 0$, let $\lambda_0 = 1$ and for $M = 1, 2, \dots$

$$\lambda_M = \frac{1}{M!} \sum_{j=0}^M \binom{M}{j} E \nabla^{M-j} u^{\otimes j} \quad (5.3)$$

or, symbolically

$$\lambda_M = \frac{1}{M!} E (\nabla + u^{\otimes})^M \quad (5.3a)$$

Then

$$\|\lambda_M\|_{H^{\otimes M}} \leq \left(\frac{(1 + \beta^{-2})^n}{n!} \right)^{1/2} E \exp \beta \|u\|_H^2 \quad (5.4)$$

and consequently the formal sum $\sum I_n(\lambda_n)$ defines an element in $\mathcal{G}_0$, more precisely

$$\Lambda(\omega) = 1 + \sum_{n=1}^{\infty} I_n(\lambda_n) \in \mathcal{G}_{(\beta/1+\beta)^{1/2}}^- \quad (5.5)$$

Proof: Let $h \in H$ and $\|h\| = 1$, integration by parts yields

$$\begin{aligned} |E(h^{\otimes n}, \nabla^{n-i} u^{\otimes i})_{H^{\otimes n}}|^2 &= |E \nabla_{h^{\otimes n-i}}^{n-i} (u^{\otimes i}, h^{\otimes i})_{H^{\otimes i}}|^2 \\ &= |E(\delta^{n-i} h^{\otimes n-i} \cdot (u^{\otimes i}, h^{\otimes i})_{H^{\otimes i}})|^2 \\ &\leq E(\delta^{n-i} h^{\otimes n-i}) \cdot E(u^{\otimes i}, h^{\otimes i})_{H^{\otimes i}}^2 \\ &\leq n-i! E(u^{\otimes i}, h^{\otimes i})_{H^{\otimes i}}^2 \end{aligned}$$

and (5.2) follows. Turning now to the proof of (5.4),

$$\begin{aligned} \|\lambda_n\|_{H^{\otimes n}} &= \left\| E \sum_{i=1}^n \frac{1}{(n-i)! \cdot i!} \nabla^{n-i} u^{\otimes i} \right\|_{H^{\otimes n}} \\ &\leq \sum \frac{1}{i!(n-i)!} \|E \nabla^{n-i} u^{\otimes i}\|_{H^{\otimes n}} \end{aligned}$$

applying (5.2) yields

$$\begin{aligned} \|\lambda_n\|_{H^{\otimes n}} &\leq \sum_{i=0}^n \frac{\beta^{-i}}{i!((n-i)!)^{1/2}} \cdot E^{1/2} \|u\|_H^{2i} \cdot \beta^i \\ &\leq \sum_{i=0}^n \left(\frac{\beta^{-2i}}{i!(n-i)!} \right)^{1/2} \cdot E^{1/2} \frac{\|u\|_H^{2i}}{(i!)^{1/2}} \cdot \beta^i \\ &\leq \left(\sum_{i=0}^n \frac{\beta^{-2i}}{i!(n-i)!} \right)^{1/2} \cdot \left(E \sum \frac{\beta^i \|u\|_H^2}{i!} \right)^{1/2} \\ &\leq \left(\frac{1}{n!} (1 + \beta^{-2})^n \right)^{1/2} \cdot E \exp \beta \|u\|_H^2 \end{aligned}$$

which is (5.4) and (5.5) follows directly.

Theorem 5.1 Let $T\omega = \omega + u(\omega)$ and for some $\beta > 0$, $E \exp \beta \|u\|_H^2 < \infty$, then for every G in $\mathcal{G}_{(1+\beta^{-2})^{1/2}}^+$

$$E G(T\omega) = \langle \Lambda, G \rangle = E(\Lambda_\gamma \cdot G_{\gamma^{-1}}) \quad (5.6)$$

where Λ is as defined by (5.5) and (5.3) and $\gamma = (1 + \beta^{-1})^{-1/2}$.

Remark: Note that $E \exp \beta \|u\|_H^2 < \infty$ or even $\|u\|_H^2 \in L_\infty(\mu)$ is not sufficient to imply the absolute continuity of $\mu \circ T^{-1}$ with respect to μ_0 . If, however, the Λ of (5.6) satisfies $\Lambda \in \mathcal{G}_0 \cap L_0(\mu)$ then under the conditions above $\mu \circ T^{-1}$ is absolutely continuous.

Proof: Let $f_M(\omega) = \varphi(W(e_1), \dots, W(e_k))$ where $\varphi(\cdot, \dots, \cdot)$ is a polynomial of order M or less and $e_i \in H$, then for any $K \geq M$:

$$\begin{aligned} f_M(\omega + u(\omega)) &= f_M(\omega) + \sum_{n=1}^M \frac{1}{n!} (\nabla^n f_M(\omega), u^{\otimes n})_{H^{\otimes n}} \\ &= \sum_0^K \frac{1}{n!} (\nabla^n f_M(\omega), u^{\otimes n})_{H^{\otimes n}} \end{aligned}$$

where for $n = 0$, $u^{\otimes n} = 1$. Integration by parts yields

$$E f_M(\omega + u) = E \sum_{n=0}^{\infty} \frac{1}{n!} f_M(\omega) \delta^n u^{\otimes n} \quad (5.7)$$

where for $n = 0$, $(n!)^{-1} \delta^n u^{\otimes n} = 1$.

Let $Q_K = \sum_{n=0}^K \frac{1}{n!} \delta^n u^{\otimes n}$, consider the Wiener chaos expansion of Q_K , $Q_K = \sum I_n(q_{K,n})$ where

$$q_{K,n} = \frac{1}{n!} \sum_{m=0}^K \frac{1}{m!} E \nabla^m \delta^n u^{\otimes n}$$

Now for $n > m$

$$\begin{aligned} E (\nabla^m \delta^n u^{\otimes n}, h_1 \otimes \dots \otimes h_m)_{H^{\otimes m}} &= E (\delta^n u^{\otimes n} \cdot \delta^m (h_1 \otimes \dots \otimes h_m)) \\ &= E (u^{\otimes n}, \nabla^n \delta^m (h_1 \otimes \dots \otimes h_m))_{H^{\otimes m}} \\ &= 0 \end{aligned}$$

therefore, for $K > m$, $q_{K,m} = q_{m,m}$. Therefore

$$Q_K = \sum_0^K I_m(g_{m,m})$$

where

$$q_{m,n} = \frac{1}{m!} \sum_{n=0}^m \frac{1}{n!} E \nabla^m \delta^n u^{\otimes n} \quad (5.8)$$

Now, for $M \geq n$, for any $\tilde{u} \in \mathcal{D}_{-\infty}(H^{\otimes n})$ we have the following identity (cf.[7]):

$$\nabla^M \delta^n \tilde{u} = \sum_{i=0}^n \binom{n}{i} \binom{M}{i} i! \delta^{n-i} \nabla^{M-i} \tilde{u}$$

where δ operates on the variables of $\tilde{u}$ (i.e. not those introduced by differentiation). Hence in our case

$$q_{M,M} = \frac{1}{M!} E \sum_{n=0}^M \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} \binom{M}{i} i! \delta^{n-i} \nabla^{M-i} u^{\otimes n}$$

Note that

$$E \delta^{n-i} \nabla^{M-i} u^{\otimes n} = 0, \quad \text{for } n > i. \quad (5.9)$$

Since by a straightforward Fubini type argument ($e_i, h_j \in H$):

$$\begin{aligned} & \left(\left(\delta^{n-i} \nabla_{e_1, \dots, e_{M-i}}^{M-i} u^{\otimes n} \right), \quad h_1 \otimes \dots \otimes h_i \right)_{H^{\otimes i}} = \\ & = \delta^{n-i} \left(\nabla_{e_1, \dots, e_{M-i}}^{M-i} (u^{\otimes n}), \quad h_1 \otimes \dots \otimes h_i \right)_{H^{\otimes i}} \end{aligned}$$

and (5.9) follows since $E(1 \cdot \delta^{n-i} \alpha) = E(\nabla^{n-i} 1, \alpha)_{H^{n-i}} = 0$. Therefore

$$q_{M,M} = \frac{1}{M!} \sum_{n=0}^M \binom{M}{n} E \nabla^{M-n} u^{\otimes n}$$

Comparing this with (5.3) yields that Q_K is the projection of Λ on the K -th Wiener chaos and $E f_M \circ T = E((f_M)_{\gamma^{-1}} \cdot \Lambda_\gamma)$ and the result follows by completion.

5.1 The conditioning of $\mathcal{G}_\lambda$ functionals with respect to certain subsigma fields

Let $\mathcal{F}$ denote, throughout this section, a Wiener chaos stable subsigma field of $\mathcal{B}(C)$ ([4]) i.e., for every n and $g_n \in H^{\otimes n}$, there exists a $q_n \in H^{\otimes n}$ such that

$$E(I_n(g_n) \mid \mathcal{F}_n) = I_n(q_n)$$

and consequently $\|q_n\| \leq \|g_n\|$. Otherwise stated, if π^n denotes the projection of the n -th Wiener chaos then π^n and $E(\cdot \mid \mathcal{F})$ commute. For this class of subsigma fields we can define $E(G \mid \mathcal{F})$ for all $G \in \mathcal{G}_0$ by

$$E(G \mid \mathcal{F}) = \sum_{n=0}^{\infty} E(I_n(g_n) \mid \mathcal{F}) \quad (5.1)$$

and since q_n is a contraction of g_n in $H^{\otimes n}$, if $G \in \mathcal{G}_\lambda$ so is $E(G \mid \mathcal{F})$.

Let $\eta = \mu \circ T^{-1}$ and let $E_\eta(\cdot \mid \mathcal{F})$ denote the conditional expectation with respect to the η measure. $E_\mu(\cdot \mid \mathcal{F})$ will denote the generalized conditional expectation defined by (5.1).

Remark: Note that in general $E_\eta(X(\omega) \mid \mathcal{F}) \neq E_\mu(X(T\omega) \mid \mathcal{F})$ since for the case where $\mathcal{F} = \mathcal{B}(C)$, $E_\eta(X(\omega) \mid \mathcal{F}) = X(\omega)$ and $E_\mu(X(T\omega) \mid \mathcal{F}) = X(T\omega)$.

As defined in the previous section, the measure η induces a $\mathcal{G}_0$ distribution Λ if and only if all polynomial functionals are in $L_1(\eta)$ and there exists constants c and $\gamma > 0$ such that all $F \in \mathcal{G}_{\gamma^{-1}}$ are in $L_1(\eta)$ and moreover

$$\left| \int_C F(\omega) d\eta(\omega) \right| \leq c \|F\|_{\gamma^{-1}} \quad . \quad (5.2)$$

Let $G(\omega)$ be a bounded random variable on $(C, \mathcal{B}(C))$, then it follows from (5.2) that under the above assumptions, $G \cdot \Lambda$ defined through

$$g_\infty \langle F, G \cdot \Lambda \rangle_{g_0} = \int_C F(\omega) G(\omega) \eta(d\omega) \quad (5.3)$$

is also an element of $\mathcal{G}_\gamma$.

Proposition 5.1 *Assume that $\mathcal{F}$ is Wiener chaos stable, that the conditions of Theorem 5.1 are satisfied and moreover $|G(\omega)| \leq K$ then*

$$E_\eta(G \mid \mathcal{F}) \cdot E_\mu(\Lambda \mid \mathcal{F}) = E_\mu(G\Lambda \mid \mathcal{F}) \quad (5.4)$$

in the sense of equality in $\mathcal{G}_0$.

Proof: Let $f(\omega) \in \mathcal{G}_\infty$ be $\mathcal{F}$ -measurable, in view of (5.3)

$$\begin{aligned} E_\eta(f(\omega)G(\omega)) &= {}_\lambda \langle f(\omega)G(\omega), \Lambda \rangle_{\lambda^{-1}} \\ &= {}_\lambda \langle f(\omega), G(\omega)\Lambda \rangle_{\lambda^{-1}} \\ &= {}_\lambda \langle f(\omega), E_\mu G(\omega)\Lambda(\omega) \mid \mathcal{F} \rangle_{\lambda^{-1}} \end{aligned} \quad (5.5)$$

On the other hand

$$\begin{aligned} E_\eta(f(\omega)G(\omega)) &= E_\eta(f(\omega)E_\eta(G(\omega) \mid \mathcal{F})) \\ &= {}_\lambda \langle f(\omega)E_\eta(G(\omega) \mid \mathcal{F}), \Lambda \rangle_{\lambda^{-1}} \\ &= {}_\lambda \langle f(\omega)E_\eta(G \mid \mathcal{F}), E_\mu(\Lambda \mid \mathcal{F}) \rangle_{\lambda^{-1}} \\ &= {}_\lambda \langle f, E_\eta(G \mid \mathcal{F})E_\mu(\Lambda \mid \mathcal{F}) \rangle_{\lambda^{-1}} \end{aligned} \quad (5.6)$$

Comparing (5.5) with (5.6) yields (5.4).

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