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Abstract: Let (α, u) denote a cocycle-crossed action of a discrete group G on a unital C^* -algebra $\mathcal{A}$ and $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ the associated reduced (twisted) C^* -crossed product. We discuss the following problem: When is $\mathcal{B}$ a simple C^* -algebra with a unique trace? To this aim, we introduce the concept of tracially properly outeress of actions.

Introduction.

Let $(\mathcal{A}, G, \alpha, u)$ denote a twisted C^* -dynamical system, where $\mathcal{A}$ is a unital C^* -algebra, G is a discrete group and (α, u) is a cocycle-crossed action of G on $\mathcal{A}$, the 2-cocycle u taking values in the unitary group of $\mathcal{A}$. In this note, we discuss the problem of determining when the reduced twisted C^* -crossed product $\mathcal{B} = C_r^*(\mathcal{A}, G, \alpha, u)$ belongs to the class $\mathcal{S}_{u.t.}$ of simple C^* -algebras with a unique trace. By a trace, we always mean a tracial state in this paper.

For completeness, we give a review of most of the known results in this area in the first section, where we also fix some notation. Let us point out at once that the same problem concerning the full twisted C^* -crossed product may be seen as a subproblem to the one above, at least in the case when u is trivial (cf. Theorem 0).

In [Bed 3; Theorem 1], we proved that if $\mathcal{A}$ itself belongs to $\mathcal{S}_{u.t.}$ and the action is tracially outer (i.e. each $\alpha_g, g \neq e$, is outer when extended to the weak closure of $\mathcal{A}$ in its GNS-representation with respect to the trace), then $\mathcal{B}$ belongs to $\mathcal{S}_{u.t.}$ too. Our interest in this result was mainly motivated by the applications we could give in [Bed 3]. It has also been useful in two recent papers ([BEK],[BKRS]). However, one drawback of this result is that it doesn't provide a direct access to the class $\mathcal{S}_{u.t.}$. Moreover, when $\mathcal{A}$ is a finite C^* -algebra which doesn't belong to $\mathcal{S}_{u.t.}$, it would be nice to have a way to embed $\mathcal{A}$ into a C^* -algebra belonging to $\mathcal{S}_{u.t.}$. This point of view is of some importance in the recent work of Arveson ([Arv 1],[Arv 2]) about C^* -algebras and numerical linear algebra. It is also clear that the crossed product construction is a natural candidate for such a procedure.

It seems reasonably safe to propose the following *conjecture*: Suppose $\mathcal{A}$ has a unique α -invariant trace ϕ and that the action is minimal and tracially properly outer (i.e. each $\alpha_g, g \neq e$, is properly outer when extended to $\pi_\phi(\mathcal{A})''$). Then $\mathcal{B}$ belongs to $\mathcal{S}_{u.t.}$.

Note that the first two conditions are necessary (and sufficient for some highly non-commutative groups, cf. Theorem 1). In the second section (Theorem 10), we verify that this conjecture is true in the following cases:

- i) $\mathcal{A}$ is simple,
- ii) $\mathcal{A}$ is abelian,
- iii) $\mathcal{A}$ is separable and G is torsion-free,
- iv) $\mathcal{A}$ is separable and u takes values in the unitary group of the center of $\mathcal{A}$.
- v) G is abelian and u is trivial.

In the general case, we have to assume further that the action is known to be properly outer, hence the conjecture would be shown if one could prove that this assumption is always redundant.

We conclude the second section with the following result (Theorem 12): Suppose G is abelian, u is trivial and $\mathcal{A}$ has a faithful unique trace. Then $\mathcal{B}$ belongs to $\mathcal{S}_{u.t.}$ if and only if the action is minimal and tracially outer.

The third and final section is devoted to some examples.

1. Notation and known results.

1.1. Throughout this paper, $\mathcal{A}$ will denote a unital C^* -algebra with unit I , G a discrete group with identity e and (α, u) a cocycle crossed action of G on $\mathcal{A}$, i.e. α is a map from G into the group $\text{Aut}(\mathcal{A})$ of $*$ -automorphisms of $\mathcal{A}$ and u is a map from $G \times G$ into the unitary group $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$ satisfying

$$\begin{aligned}\alpha_g \alpha_h &= ad(u(g, h)) \alpha_{gh} \\ u(g, h) u(gh, k) &= \alpha_g(u(h, k)) u(g, hk) \\ u(g, e) &= u(e, g) = I\end{aligned}$$

for all $g, h, k \in G$.

As usual, $ad(v)$ denotes the inner automorphism of $\mathcal{A}$ implemented by a unitary v in $\mathcal{A}$. The quadruple $\Sigma = (\mathcal{A}, G, \alpha, u)$ is then called a *twisted (discrete) C^* -dynamical system* and we denote by $C_r^*(\Sigma)$ the associated reduced twisted C^* -crossed product.

Of course, when u is trivial (i.e. $u(g, h) = I$ for all $g, h \in G$), $C_r^*(\Sigma)$ is nothing but the usual reduced C^* -crossed product associated to the system $(\mathcal{A}, G, \alpha)$ (cf. [Ped], [Tom]). In the case when u takes values in the unitary group of the center of $\mathcal{A}$, the reader may consult [Z-M]. For the general case, we refer to [Bed 2] and [PR] where some background material and other references may be found.

1.2. We are interested in determining when $C_r^*(\Sigma)$ is simple with a unique trace, i.e. $C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$. Now, one can also associate to Σ the full twisted C^* -crossed product $C^*(\Sigma)$ and raise the same problem about $C^*(\Sigma)$. But we have:

Theorem 0: Suppose u is trivial. Then

- a) G is amenable if and only if $C^*(\Sigma) = C_r^*(\Sigma)$ and $\mathcal{A}$ has an α -invariant state.
- b) $C^*(\Sigma) \in \mathcal{S}_{u.t.}$ if and only if G is amenable and $C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$

Assertion b) is an immediate consequence of a), which is due to Zeller-Meier (and Arveson) ([Z-M; Proposition 5.2]). When u is not trivial, the if part of b) still holds true (cf. [PR; Theorem 3.11]). In view of this, we shall concentrate our attention on $C_r^*(\Sigma)$.

1.3. If $\mathcal{A}$ is faithfully represented as a C^* -subalgebra of the bounded linear operators $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$, then we may identify $\mathcal{A}$ with its canonical copy in $C_r^*(\Sigma)$ and $C_r^*(\Sigma)$ is then the C^* -subalgebra of $\mathcal{B}(l^2(G, \mathcal{H}))$ generated by $\mathcal{A}$ and $\lambda(G) = \{\lambda(g) | g \in G\}$ where λ is the (left) u -regular representation of G on $l^2(G, \mathcal{H})$ satisfying

$$\begin{aligned}\lambda(g) \lambda(h) &= u(g, h) \lambda(gh) \\ \alpha_g(a) &= ad(\lambda(g))(a) = \lambda(g) a \lambda(g)^*\end{aligned}$$

for all $g, h \in G, a \in \mathcal{A}$.

An important feature of $C_r^*(\Sigma)$ is that there exists a faithful conditional expectation E from $C_r^*(\Sigma)$ onto $\mathcal{A}$ satisfying

$$\begin{aligned}E(\lambda(g)) &= 0 \quad , \quad g \neq e, \text{ and} \\ E(\lambda(g) x \lambda(g)^*) &= \alpha_g(E(x)) \quad , \text{ for all } g \in G, x \in C_r^*(\Sigma).\end{aligned}$$

If $\mathcal{A}$ is actually a von Neumann algebra, we may also form the regular extension $\mathcal{A} \times_{(\alpha, u)} G$, which in the above setting is just the von Neumann subalgebra of $\mathcal{B}(l^2(G, \mathcal{H}))$ generated by $\mathcal{A}$ and $\lambda(G)$. Then there exists a faithful normal conditional expectation $\tilde{E}$ from $\mathcal{A} \times_{(\alpha, u)} G$ onto $\mathcal{A}$ satisfying the same properties as E .

We refer to [Bed 2] for these facts.

1.4. We shall say that the action (α, u) is *minimal* if there are no proper closed (two-sided) ideals in $\mathcal{A}$ which are invariant under each $\alpha_g, g \in G$, and that the action (α, u) is *uniquely trace-ergodic* (with respect to ϕ) if $\mathcal{A}$ possess a unique α -invariant trace ϕ . If (α, u) is both minimal and uniquely trace-ergodic, we say that (α, u) is *strictly trace-ergodic*. This terminology is of course directly inspired from the terminology in topological dynamics, cf. 1.8.

From the properties of the expectation E , one deduces easily that if $C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$, then (α, u) is strictly trace-ergodic.

For some highly non-abelian groups, this necessary condition is also sufficient:

Theorem 1: If G is a non-abelian free group, or more generally a Powers group or even an extension of weak Powers groups, then $C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$ if and only if (α, u) is strictly trace-ergodic.

Inspired by Powers' paper ([Pow]), this theorem was proved by de la Harpe and Skandalis ([HS; Corollary 7 and Proposition 10]) in the case of a Powers group and a trivial cocycle u . Their result was successively enlarged in [BN; Corollary 2.4 and Proposition 2.6], and [Bed 2; Corollary 4.8] to yield the above theorem. If G is an extension of ultraweak Powers groups, a slightly weaker result is available ([Bed 3; Corollary 5]). If one only considers the reduced group C^* -algebra $C_r^*(G)$, there are other known conditions on G which ensure that $C_r^*(G) \in \mathcal{S}_{u.t.}$. For a recent overview, see [Har].

Concerning the reduced twisted group C^* -algebra $C_r^*(G, u)$, let us also mention the result of Packer ([Pac; Theorem 1.7]) which says that when G is countable and nilpotent, then $C_r^*(G, u) \in \mathcal{S}_{u.t.}$ if and only if each u -regular conjugacy class of G is infinite.

1.5. We recall that a $*$ -automorphism β of a von Neumann algebra $\mathcal{M}$ is called *properly outer* (or *freely acting*) if 0 is the only element $a \in \mathcal{M}$ satisfying

$$ax = \beta(x)a \quad \text{for all } x \in \mathcal{M}.$$

(cf. [Str; §17], [Kal]). By Connes' characterization ([Str; Theorem 17.9]), this is equivalent to:

For every non-zero projection $p \in \mathcal{M}$, we have

$$\inf\{\|q\beta(q)\| \mid q \in \mathcal{M}, q \text{ non-zero projection}, q \leq p\} = 0.$$

1.6. We shall say that $\theta \in \text{Aut}(\mathcal{A})$ is *properly outer* if:

For every hereditary C^* -subalgebra $\mathcal{B}$ of $\mathcal{A}$, we have

$$\inf\{\|x\theta(x)\| \mid x \in \mathcal{B}, x \geq 0, \|x\| = 1\} = 0.$$

When $\mathcal{A}$ is separable, this definition is equivalent to ten other conditions ([OP; Theorem 6.6]), including the original one in [Ell], which says that for any non-zero θ -invariant closed two-sided ideal $\mathcal{J}$ in $\mathcal{A}$ and every unitary multiplier v of $\mathcal{J}$, we have

$$\|\theta|_{\mathcal{J}} - \text{ad}(v)|_{\mathcal{J}}\| = 2.$$

In the general case, it is only known that if θ is properly outer in the sense of Elliott, then θ is properly outer (cf. [OP; proof of ii) $\Rightarrow$ iv) in Theorem 6.6, which doesn't require that $\mathcal{A}$ is separable]).

We say further that the action (α, u) is *properly outer* if each α_g is properly outer, $g \in G \setminus \{e\}$.

1.7. Let $\theta \in \text{Aut}(\mathcal{A})$ and ρ denote a faithful θ -invariant state on $\mathcal{A}$. Then we may identify $\mathcal{A}$ with $\pi_\rho(\mathcal{A}) \subseteq \mathcal{B}(\mathcal{H}_\rho)$ and extend θ to a $*$ -automorphism $\tilde{\theta}$ on $\mathcal{A}''$ by a well known procedure. We say then that θ is ρ -outer (resp. ρ -properly outer) whenever $\tilde{\theta}$ is outer (resp. properly outer) on $\mathcal{A}''$.

Suppose now ϕ is a faithful α -invariant trace on $\mathcal{A}$.

We say that (α, u) is ϕ -outer (resp. ϕ -property outer) whenever each $\alpha_g, g \neq e$, is ϕ -outer (resp. ϕ -properly outer). Furthermore, if ϕ is the only α -invariant trace on $\mathcal{A}$, we just say that (α, u) is *tracially outer* (resp. *tracially properly outer*) when (α, u) is ϕ -outer (resp. ϕ -properly outer).

If $\mathcal{A}$ has a faithful unique trace ϕ , then ϕ is α -invariant and $\pi_\phi(\mathcal{A})''$ is a finite factor ([KR; 12.12]), so tracially outerness is equivalent to tracially properly outerness in this case by ([Kal], [Str; 17.4]). This fact was used in the proof of [Bed 3; Theorem 1] which we restate here:

Theorem 2: If $\mathcal{A} \in \mathcal{S}_{u.t.}$ and (α, u) is tracially outer, then $C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$

1.8. In this subsection, we restrict ourselves to the case when $\mathcal{A}$ is abelian and u is trivial, i.e. to topological dynamics ([Tom]). By Gelfand's theory, we may then suppose that α is the action of G on $\mathcal{A}$ associated to an action of G by homeomorphisms on X , where X is the character space of $\mathcal{A}$. We recall some terminology:

$$\text{For } g \in G, \text{ let } X^g = \{x \in X \mid g \cdot x = x\}.$$

We say that the action of G on X is *free* (resp. *topologically free*) whenever

$$X^g = \emptyset \quad (\text{resp. } \text{Int}(X^g) = \emptyset) \text{ for all } g \in G \setminus \{e\}.$$

Further, if μ is a G -invariant Radon probability measure on X , we say that the action of G on X is μ -free whenever

$$\mu(X^g) = 0 \quad \text{for all } g \in G \setminus \{e\},$$

while we say that it is *strictly ergodic* (w.r.t. μ) if G acts minimally on X and μ is the only G -invariant probability Radon measure on X .

Of course, the strict ergodicity of the action of G on X is equivalent to the strict trace-ergodicity of α . Further, it is well known that G acts μ -freely on X if and only if the action α on $\mathcal{A}$ is ϕ -properly outer, where ϕ is the trace on $\mathcal{A}$ associated to μ . On the other hand, if G acts topologically freely on X , then α is properly outer, while the converse is true when $\mathcal{A}$ is separable (i.e. X is metrizable), cf. 1.6 and [AS] or [ET].

Theorem 3: Suppose the action of G on X is strictly ergodic (w.r.t. μ) and μ -free. Then $C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$

Proof: The uniqueness of the trace on $C_r^*(\Sigma)$ is shown in [HS; digression following Corollary 8]. Since G acts minimally on X , the support of μ must be X . Hence, the μ -freeness of the action implies its topological freeness and the simplicity of $C_r^*(\Sigma)$ follows then from [Ell; 3.7] or [AS; Corollary to Theorem 1]. ■

When X is metrizable, G is amenable and the action is free, Theorem 3 has usually been deduced from [Z-M; Théorème 4.20 and Corollaire 9.6], a cocycle-crossed action being then allowed.

Theorem 4: $C^*(\Sigma) \in \mathcal{S}_{u.t.}$ if and only if G is amenable and the action of G on X is strictly ergodic (w.r.t. μ) and μ -free.

Proof: By [KTT; Corollary 2.8], $C^*(\Sigma)$ has a unique trace (if and) only if X has a unique G -invariant Radon probability measure μ and G acts μ -freely on X . The theorem follows then by combining this result with Theorem 0 and Theorem 3. Instead of Theorem 3, one may invoke here [KT; Theorem 4.1] or [AS; Corollary to Theorem 2]. ■

In order to apply Theorem 4, the following proposition is often useful (and surely well known):

Proposition 5: Suppose G is amenable and X has a unique G -invariant probability measure μ . Then G acts minimally on X (so the action of G on X is strictly ergodic) if and only if the support of μ is X .

Proof: The only if part is easy (and always true). So suppose $\text{supp}(\mu) = X$ and that Y is a closed G -invariant subset of X , $\emptyset \neq Y \neq X$. Then we have $\mu(X \setminus Y) > 0$, since $X \setminus Y$ is open and non-empty. Further, G acts on Y and there exists a G -invariant probability Radon measure ν on Y since G is amenable (we use here Theorem 0 with $\mathcal{A} = C(Y)$). But we may now define a G -invariant probability Radon measure $\tilde{\nu}$ on X by $\tilde{\nu}(A) = \nu(A \cap Y)$, A Borel set in X . Then $\tilde{\nu}(X \setminus Y) = \nu(\emptyset) = 0$. Hence $\tilde{\nu} \neq \mu$, which contradicts the uniqueness of μ . So there exists no such Y , i.e. G acts minimally on X as desired. ■

Corollary 6: Suppose G is abelian and acts effectively on X (i.e. for all $g \in G \setminus \{e\}$, $X^g \neq X$). Then $C^*(\Sigma) \in \mathcal{S}_{u.t.}$ if and only if X has a unique G -invariant Radon measure μ and the support of μ is X .

Proof: When G is abelian, then X^g is G -invariant for all $g \in G$. Hence, if the action is effective and minimal, we get $X^g = \emptyset$ for all $g \in G \setminus \{e\}$, i.e. the action is free. Now, as an abelian group is amenable, the corollary follows from Theorem 4 and Proposition 5. ■

Corollary 6 sharpens [KTT; Corollary 2.10] (where effectiveness of the action should be added).

2. Tracially properly outer actions.

Throughout this section, $\Sigma = (\mathcal{A}, G, \alpha, u)$ denotes a twisted C^* -dynamical system and we use freely the notation introduced in section 1.

We suppose that $\mathcal{A}$ has a faithful α -invariant trace ϕ and let $\sigma = \phi \circ E$ be the associated trace on $\mathcal{B} = C_r^*(\Sigma)$. We identify $\mathcal{A}$ with $\pi_\phi(\mathcal{A})$ and write $\mathcal{H}$ for $\mathcal{H}_\phi$ and ξ_0 for ξ_ϕ . By the α -invariance of ϕ , the action (α, u) extends to an action $(\tilde{\alpha}, u)$ of G on $\mathcal{A}'' \subseteq \mathcal{B}(\mathcal{H})$. Then $\tilde{\phi} = \omega_{\xi_0}|_{\mathcal{A}''}$ is a faithful normal trace on $\mathcal{A}''$ (cf. [Tak; Proposition 3.19]), which is $\tilde{\alpha}$ -invariant since $\tilde{\phi}|_{\mathcal{A}} = \phi$.

We now form the von Neumann algebra $\mathcal{M} = \mathcal{A}'' \times_{(\tilde{\alpha}, u)} G$ which acts on $l^2(G, \mathcal{H})$ and clearly satisfies $\mathcal{B}'' = \mathcal{M}$. Further, $\mathcal{M}$ has a faithful normal trace $\tilde{\sigma} = \tilde{\phi} \circ \tilde{E}$, where $\tilde{E}$ denotes the canonical conditional expectation from $\mathcal{M}$ onto $\mathcal{A}''$, and we have $\tilde{\sigma}|_{\mathcal{B}} = \sigma$.

If we assume that σ is the only trace on $\mathcal{B}$, then we obviously get that $\tilde{\sigma}$ is the only normal trace on $\mathcal{M}$, hence $\mathcal{M}$ must be a factor. We record this observation (which is essentially due to Longo ([Lon])) as a lemma.

Lemma 7: If $\mathcal{B}$ has a unique trace, then $\mathcal{M}$ is a (finite) factor.

The converse assertion is not generally true. Indeed, if one chooses $\mathcal{A} = \mathbb{C}$, $\alpha = id$, $u = I$ and let G be any non-trivial amenable ICC group, then $\mathcal{M} = vN(G)$ is a finite factor, while $\mathcal{B} = C_r^*(G)$ has more than one trace as is well known. However, it is clear that when searching for sufficient conditions in aim that $\mathcal{B} \in \mathcal{S}_{u.t.}$, one should have in mind the known conditions which ensure that $\mathcal{M}$ is a factor. By essentially the same proofs as the ones for usual crossed products ([Str; 22.6]), one shows that this will be the case if $(\tilde{\alpha}, u)$ acts ergodically on the center of $\mathcal{A}''$ and either G is ICC or (α, u) is ϕ -properly outer.

Since weak Powers groups are ICC, Theorem 1 illustrates the possibilities of the ICC-alternative. Before we look in the other direction, let us also record the following lemma.

Lemma 8: Suppose that (α, u) is uniquely trace-ergodic (w.r.t. ϕ). Then $(\tilde{\alpha}, u)$ acts ergodically on the center Z of $\mathcal{A}''$. Therefore, if we assume further that (α, u) is ϕ -properly outer, then $\mathcal{M}$ is a factor.

Proof: Suppose p is a non-zero $\tilde{\alpha}$ -invariant projection in Z . Then define $\tilde{\tau} : \mathcal{A}'' \rightarrow \mathbb{C}$ by

$$\tilde{\tau}(x) = \tilde{\phi}(p)^{-1} \tilde{\phi}(xp), \quad x \in \mathcal{A}''.$$

Then $\tilde{\tau}$ is clearly a normal trace on $\mathcal{A}''$. Let $\tau = \tilde{\tau}|_{\mathcal{A}}$. Then τ is an α -invariant trace on $\mathcal{A}$, so by the uniqueness assumption, we have $\tau = \phi$. Since $\tilde{\tau}$ and $\tilde{\phi}$ are normal, we get $\tilde{\tau} = \tilde{\phi}$, which implies that $\tilde{\phi}(p) = \tilde{\tau}(p) = 1$. But $\tilde{\phi}$ is faithful, so this gives $p = I$ and the first assertion is proved. The second follows then from our above comment. ■

With this lemma at hand, we can adapt the proof of [Bed 3; Theorem 1] to obtain:

Proposition 9: Suppose that (α, u) is uniquely trace-ergodic and tracially properly outer. Then $\mathcal{B} = C_r^*(\Sigma)$ has a unique trace.

Proof: We merely indicate where changes in the proof of [Bed 3; Theorem 1] have to be effected. Let ρ be a trace on $\mathcal{B}$. It is enough to show the existence of a $*$ -isomorphism π from $\mathcal{M} = \mathcal{B}''$ onto $\pi_\rho(\mathcal{B})''$ (π corresponds to Φ^{-1} in [Bed 3]) which extends $\pi_\rho : \mathcal{B} \rightarrow \pi_\rho(\mathcal{B})$. Indeed, as $\mathcal{M}$ is a (finite) factor by lemma 8, $\tilde{\sigma}$ is then the unique trace of $\mathcal{M}$. Hence we get

$$\tilde{\sigma}(x) = (\pi(x)\xi_\rho, \xi_\rho) \quad \text{for all } x \text{ in } \mathcal{M},$$

which gives

$$\sigma(b) = (\pi(b)\xi_\rho, \xi_\rho) = (\pi_\rho(b)\xi_\rho, \xi_\rho) = \rho(b) \quad \text{for all } b \text{ in } \mathcal{B},$$

i.e. $\rho = \sigma$ as desired.

Now, as $\mathcal{M} = \mathcal{A}'' \times_{(\tilde{\alpha}, u)} G$, the existence of π may be deduced as in [Bed 3] from the analog of Connes' characterization of discrete crossed products. Since $\rho|_{\mathcal{A}} = \phi$ (by the uniqueness of ϕ) and ϕ is faithful, we see that $\pi_\rho|_{\mathcal{A}}$ is a $*$ -isomorphism from $\mathcal{A}$ onto $\pi_\rho(\mathcal{A})$. Further, w_{ξ_ρ} restricts to a faithful normal trace on $\pi_\rho(\mathcal{B})''$ ([Tak; Proposition 3.19]) and $\tilde{\phi}|_{\mathcal{A}} = w_{\xi_\rho} \circ \pi_\rho|_{\mathcal{A}}$, so we may invoke [K.R; 7.6.7] and extend $\pi_\rho|_{\mathcal{A}}$ to a $*$ -isomorphism θ from $\mathcal{A}''$ onto $\pi_\rho(\mathcal{A})''$.

Then we may transport $(\tilde{\alpha}, u)$ via θ over to a twisted action (β, v) of G on $\pi_\rho(\mathcal{A})''$ which is properly outer, since $(\tilde{\alpha}, u)$ is properly outer by assumption. The rest of the proof proceeds just as in [Bed 3; Theorem 1]. ■

We can now prove the theorem mentioned in the introduction.

Theorem 10:

- a) Suppose that (α, u) is properly outer, strictly trace-ergodic and tracially properly outer. Then $\mathcal{B} = C_r^*(\Sigma) \in \mathcal{S}_{u.t.}$
- b) The assumption in a) that (α, u) is properly outer is redundant whenever one of the following conditions is satisfied:
 - i) $\mathcal{A}$ is simple
 - ii) $\mathcal{A}$ is abelian
 - iii) $\mathcal{A}$ is separable and G is torsion free
 - iv) $\mathcal{A}$ is separable and u takes values in the unitary group of the center of $\mathcal{A}$.
 - v) G is abelian and u is trivial.

Proof:

- a) By proposition 9, we only have to show that $\mathcal{B}$ is simple. When u is trivial, since the action is supposed to be minimal and properly outer, this follows from [OP;

Theorem 7.2] (We can overlook the separability condition in this theorem because our definition of properly outerness is the one that makes their proof work). An inspection of their proof shows that it also goes through in the case of a non-trivial cocycle. Alternatively, one may repeat the proof of [Bed 2; Theorem 3.2] combined with [OP; Lemma 7.1] instead of [Bed 2; Lemma 3.1] (which is a restatement of Kishimoto's key lemma [Kis; lemma 3.2]).

b) We suppose that (α, u) is strictly trace-ergodic and tracially properly outer. We shall show that if i) or v) is satisfied, then $\mathcal{B}$ is simple, while if ii), iii) or iv) is satisfied, then (α, u) is properly outer. In view of Proposition 9 and a), this will prove b).

- i) Suppose $\mathcal{A}$ is simple. For each $g \in G \setminus \{e\}$, $\tilde{\alpha}_g$ is properly outer on $\mathcal{A}''$, so α_g must be outer on $\mathcal{A}$. The simplicity of $\mathcal{B}$ then follows from [Bed 2; Theorem 3.2].
- ii) Suppose $\mathcal{A}$ is abelian, so $\mathcal{A} = C(X)$ and ϕ is given by integration w.r.t. μ . For each $g \in G \setminus \{e\}$, we have then $\mu(X^g) = 0$, hence $\text{Int}(X^g) = \emptyset$ since $\text{supp}(\mu) = X$, so α_g is properly outer (cf. 1.8).
- iii) Suppose $\mathcal{A}$ is separable and G is torsion-free. Let $g \in G \setminus \{e\}$. For each $n \in \mathbf{N}, n \neq 0$, we have then by assumption that $\tilde{\alpha}_{g^n}$ is properly outer. An easy computation gives that

$$(\tilde{\alpha}_g)^n = \text{ad}(u(g, g)u(g^2, g) \cdots u(g^{n-1}, g))\tilde{\alpha}_{g^n}, n \geq 2.$$

Further, if $\beta \in \text{Aut}(\mathcal{A}'')$ and $v \in \mathcal{U}(\mathcal{A}'')$, one checks that $\text{ad}(v)\beta$ and β^{-1} are properly outer whenever β is properly outer. Hence, we obtain that $(\tilde{\alpha}_g)^n$ is properly outer for each $n \in \mathbf{Z} \setminus \{0\}$. By [Str; 17.12], this implies that the Connes spectrum $\Gamma(\tilde{\alpha}_g) = \mathbf{T} = \{\lambda \in \mathbf{C}; |\lambda| = 1\}$. Now, as $\Gamma(\tilde{\alpha}_g) \subseteq \Gamma(\alpha_g) \subseteq \mathbf{T}$ by [Ped; Lemma 8.8.9] we get that the Connes spectrum $\Gamma(\alpha_g) = \mathbf{T}$. Since $\mathcal{A}$ is separable, we may use [OP; Theorem 10.4] and conclude that α_g is properly outer as desired.

- iv) We suppose that $\mathcal{A}$ is separable and u takes values in the unitary group of the center of $\mathcal{A}$. Especially, α (resp. $\tilde{\alpha}$) is then a group homomorphism from G into $\text{Aut}(\mathcal{A})$ (resp. $\text{Aut}(\mathcal{A}'')$).

Let $g \in G \setminus \{e\}$. If g has infinite period, then proceeding as in iii), we get that α_g is properly outer. Suppose so that the period p of g is finite, $2 \leq p < \infty$, and write $\theta = \alpha_g$ (resp. $\tilde{\theta} = \tilde{\alpha}_g$). We may then define an action β (resp. $\tilde{\beta}$) of $\mathbf{Z}_p$ on $\mathcal{A}$ (resp. $\mathcal{A}''$) by

$$\beta_j = \theta^j \text{ (resp. } \tilde{\beta}_j = \tilde{\theta}^j) \quad , \quad j \in \mathbf{Z}_p.$$

Now $\tilde{\beta}$ is properly outer by assumption. Proceeding exactly as in [Str; 17.12], we obtain that the Connes spectrum of β is full, i.e. $\Gamma(\beta) = \hat{\mathbf{Z}}_p (= \mathbf{Z}_p)$. Since $\Gamma(\tilde{\beta}) \subseteq \Gamma(\beta) \subseteq \hat{\mathbf{Z}}_p$ by [Ped; lemma 8.8.9], we get $\Gamma(\beta) = \hat{\mathbf{Z}}_p$, i.e. $\Gamma(\beta)^\perp = \{0\}$. By [OP; Proposition 4.2], this implies especially that there is no non-zero β -invariant hereditary C^* -subalgebra $\mathcal{C}$ of $\mathcal{A}$ such that $\beta_1|_{\mathcal{C}} = \exp \delta$ for some β -invariant $*$ -derivation δ on $\mathcal{C}$. As $\mathcal{A}$ is assumed to be separable and β -invariance

means the same as θ -invariance, we may conclude from [OP; Theorem 6.6 vii) $\Rightarrow$ iv)] that $\alpha_g = \theta$ is properly outer.

- v) Suppose G is abelian and u is trivial. Since $\mathcal{M} = \mathcal{A}'' \times_{\tilde{\alpha}} G$ is a factor (cf. Lemma 8), it follows from [Ped; Theorem 8.11.15] that the Connes spectrum $\Gamma(\tilde{\alpha})$ of $\tilde{\alpha}$ is full, i.e. $\Gamma(\tilde{\alpha}) = \hat{G}$ (the dual group of G). Again, as $\Gamma(\tilde{\alpha}) \subseteq \Gamma(\alpha) \subseteq \hat{G}$ by [Ped; Lemma 8.8.9], this means that $\Gamma(\alpha) = \hat{G}$; hence $\mathcal{B}$ is simple by [Ped; Theorem 8.11.12] since the action is minimal by assumption. ■

Cases i) and ii) in Theorem 10 are (slight) generalizations of Theorems 2 and 3, respectively.

We conclude this section by a partial converse to case v). Its proof requires the next proposition, which has its own interest.

Proposition 11: Suppose $\mathcal{N}$ is a finite factor and let $\beta : G \rightarrow \text{Aut}(\mathcal{N})$ denote an action of an abelian group G on $\mathcal{N}$. Then the following conditions are equivalent:

- a) $\mathcal{N} \times_{\beta} G$ is a (finite) factor
- b) The Connes spectrum $\Gamma(\beta)$ is full, i.e. $\Gamma(\beta) = \hat{G}$
- c) β is outer

Proof: a) is equivalent to b) by [Ped; Theorem 8.11.15], while c) implies a) by [Str; 22.6, Corollary 1]. To show that a) implies c) contrapositively, we suppose that there exists $h_1 \in G \setminus \{e\}$ such that β_{h_1} is inner, and let H denote the cyclic subgroup of G generated by h_1 .

We also denote by β the restriction of β to H and consider the von Neumann crossed product $\mathcal{N} \times_{\beta} H$. We may suppose that $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, so that $\mathcal{N} \times_{\beta} H$ is generated by $\pi_{\beta}(\mathcal{N})$ and $\lambda(H)$, π_{β} denoting as usual the embedding of $\mathcal{N}$ into $\mathcal{N} \times_{\beta} H$ and λ the (left) regular representation of H on $l^2(H, \mathcal{H})$.

Set $K = G/H$ and choose a section $n : K \rightarrow G$ for the canonical map from G onto the factor group K satisfying $n_e = e$. By the decomposition theorem [Bed 1; Theorem 1], there exists a cocycle-crossed action (γ, v) of K on $\mathcal{N} \times_{\beta} H$ such that

$$\begin{aligned} \gamma_k(\pi_{\beta}(x)) &= \pi_{\beta}(\beta_{n_k}(x)), & x \in \mathcal{N}, & \quad k \in K \\ \gamma_k(\lambda(h)) &= \lambda(h), & h \in H, & \quad k \in K \\ v(k, \ell) &= \lambda(n_k n_{\ell} n_{k\ell}^{-1}), & k, \ell \in K \end{aligned}$$

and $\mathcal{N} \times_{\beta} G \simeq (\mathcal{N} \times_{\beta} H) \times_{(\gamma, v)} K$.

We are going to show that $\mathcal{N} \times_{\beta} G$ is not a finite factor by showing that $\mathcal{N} \times_{\beta} G$ has more than one normal trace. In view of the above decomposition, it is enough to show that $\mathcal{N} \times_{\beta} H$ has more than one normal γ -invariant trace.

We use that there exists a unitary representation $h \rightarrow b(h)$ of H into $\mathcal{U}(\mathcal{N})$ satisfying $\beta_h = \text{ad}(b(h))$, $h \in H$. Indeed, we may choose by assumption $b \in \mathcal{U}(\mathcal{N})$ such that $\beta_{h_1} = \text{ad}(b)$. If $H \simeq \mathbf{Z}$, then we just define $b(h_1^m) = b^m$, $m \in \mathbf{Z}$. Otherwise, let p denote the period of

$H, 2 \leq p < \infty$. Then we have $ad(b^p) = \beta_{n_1}^p = \beta_e = id$, hence b^p belongs to the center of $\mathcal{N}$, so $b^p = \rho I$ for some $\rho \in \mathbf{T}$ since $\mathcal{N}$ is a factor. We now define $b(h_1^m) = (\bar{\rho}^{1/p} b)^m, m \in \mathbf{Z}_p$, where $\bar{\rho}^{1/p}$ denotes a p -th root of $\bar{\rho}$.

Let so $i : H \rightarrow \text{Aut}(\mathcal{N})$ denote the trivial action of H on $\mathcal{N}$, and consider the crossed product $\mathcal{N} \times_i H$, which is generated by $\pi_i(\mathcal{N})$ and $\lambda(H)$ on $l^2(H, \mathcal{H})$. Define the operator W on $l^2(H, \mathcal{H})$ by

$$(W\xi)(h) = b(h)\xi(h), \quad \xi \in l^2(H, \mathcal{H}), h \in H.$$

Then, by direct computation, one checks that W is a unitary operator satisfying

$$\begin{aligned} W\pi_\beta(x)W^* &= \pi_i(x), \quad x \in \mathcal{N}, \\ W\lambda(h)W^* &= \pi_i(b(h))\lambda(h), \quad h \in H, \end{aligned}$$

from which it follows that

$$W(\mathcal{N} \times_\beta H)W^* = \mathcal{N} \times_i H.$$

We denote by ψ the $*$ -isomorphism of $\mathcal{N} \times_\beta H$ onto $\mathcal{N} \times_i H$ implemented by W .

Further, as is well-known, there exists a $*$ -isomorphism ϕ from $\mathcal{N} \times_i H$ onto $\mathcal{N} \bar{\otimes} vN(H)$ satisfying

$$\begin{aligned} \phi(\pi_i(x)) &= x \bar{\otimes} 1, \quad x \in \mathcal{N}, \\ \phi(\lambda(h)) &= 1 \bar{\otimes} \rho(h), \quad h \in H, \end{aligned}$$

where $vN(H)$ denotes the (left) group von Neumann algebra of H generated by the (left) regular representation $h \rightarrow \rho(h)$ of H on $l^2(H)$.

If τ denotes the unique (normal) trace of $\mathcal{N}$ and μ any normal state of $vN(H)$, we clearly get a normal trace $\tilde{\mu}$ on $\mathcal{N} \times_\beta H$ by letting $\tilde{\mu} = (\tau \bar{\otimes} \mu) \circ \phi \circ \psi$, which is γ -invariant since

$$\begin{aligned} \tilde{\mu}(\gamma_k(\pi_\beta(x))) &= \tilde{\mu}(\pi_\beta(\beta_{n_k}(x))) = (\tau \bar{\otimes} \mu)(\phi(\pi_i(\beta_{n_k}(x)))) \\ &= (\tau \bar{\otimes} \mu)(\beta_{n_k}(x) \bar{\otimes} 1) = \tau(\beta_{n_k}(x)) \\ &= \tau(x) = (\tau \bar{\otimes} \mu)(x \bar{\otimes} 1) = \tilde{\mu}(\pi_\beta(x)), \end{aligned}$$

and $\tilde{\mu}(\gamma_k(\lambda(h))) = \tilde{\mu}(\lambda(h))$, for all $k \in K, x \in \mathcal{N}, h \in H$.

Finally, the map $\mu \rightarrow \tilde{\mu}$ is clearly injective. As $vN(H) \simeq L^\infty(\mathbf{T})$ or $\mathbf{C}^p, 2 \leq p < \infty$, it follows that $\mathcal{N} \times_\beta H$ has more than one γ -invariant normal trace which finishes the proof. $\blacksquare$

It would be interesting to know if this proposition may be enlarged to the setting of a finite von Neumann algebra $\mathcal{N}$ and a centrally ergodic action β if one replaces the outerness of β by the properly outerness of β in condition c). The problem is then to show that a) (or b)) $\Rightarrow$ c), since a) $\Leftrightarrow$ b) and c) $\Rightarrow$ a) are true by the same references as in the proof above.

A positive answer to this question would imply that the converse of Theorem 10 case v) is true. (Indeed, if $\mathcal{B} \in \mathcal{S}_{u.t.}$, then $\tilde{\alpha}$ is centrally ergodic by Lemma 8 and $\mathcal{B}'' = \mathcal{A}'' \times_{\tilde{\alpha}} G$ is a finite factor; hence we would obtain that $\tilde{\alpha}$ is properly outer, i.e. α is ϕ -properly outer).

We have to content ourselves with the following:

Theorem 12: Suppose ϕ is the only trace of $\mathcal{A}$, G is abelian and u is trivial. Then we have:

- a) $\mathcal{B}$ has a unique trace if and only if α is ϕ -outer.
- b) $\mathcal{B} \in \mathcal{S}_{u.t.}$ if and only if α is minimal and ϕ -outer.

Proof: Since ϕ is unique, α is uniquely trace-ergodic and $\mathcal{N} = \mathcal{A}''$ is a finite factor, so ϕ -properly outerness is equivalent to ϕ -outerness (cf. 1.7). Hence the if part of a) follows from Proposition 9. On the other hand, if α is not ϕ -outer, i.e. $\tilde{\alpha}$ is not outer on $\mathcal{N}$, we get from Proposition 11 that $\mathcal{M} = \mathcal{B}'' = \mathcal{N} \times_{\tilde{\alpha}} G$ is not a finite factor, so $\mathcal{B}$ has not a unique trace; this shows the only if part of a). Assertion b) follows then easily from a) and Theorem 10 (case v). ■

3. Some examples.

3.1. We shall mainly give examples involving usual actions, i.e. where the cocycle is trivial. Examples of cocycle crossed actions may then be obtained by perturbing the action by a multiplier of the group. Indeed, if $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ is an action and $u : G \times G \rightarrow \mathbf{T}$ is a map satisfying

$$\begin{aligned} u(g, h)u(gh, k) &= u(g, hk)u(h, k) \text{ and} \\ u(g, e)u(e, h) &= 1 \quad \text{for all } g, h, k \text{ in } G, \end{aligned}$$

then, identifying $\mathbf{T}$ with $\mathbf{T} \cdot I$, we get clearly a cocycle crossed action (α, u) of G on $\mathcal{A}$. If u is a coboundary (i.e. $[u] = 0$ in $H^2(G, \mathbf{T})$), then it is well known that $C_r^*(\mathcal{A}, G, \alpha, u) \simeq C_r^*(\mathcal{A}, G, \alpha)$; hence potentially new cases will arise only when $H^2(G, \mathbf{T}) \neq \{0\}$.

3.2. For examples of C^* -dynamical systems satisfying the assumptions of Theorem 10, case i), we refer to [Bed 3], [BKRS] and [BEK].

Examples illustrating Theorem 10, case ii), are well known, the most prominent being given by the action of $\mathbf{Z}$ on $C(\mathbf{T})$ induced from an irrational rotation of $\mathbf{T}$. A quite general class is provided by the following procedure.

Let D denote a topological group and $\mathcal{AP}(D)$ the abelian unital C^* -algebra of all continuous almost periodic functions on D . We refer to [Dix; §16], [HR; §18] and [Loo; §41] for the basic facts concerning $\mathcal{AP}(D)$ used below. If $\mathcal{D}$ denote the character space of $\mathcal{AP}(D)$, so that $\mathcal{AP}(D) \simeq C(\mathcal{D})$, then $\mathcal{D}$ may be organized as a compact group and we have a canonical continuous homomorphism $i : D \rightarrow \mathcal{D}$ given by $[i(d)](f) = f(d)$ ($d \in D, f \in \mathcal{AP}(D)$), such that $i(D)$ is dense in $\mathcal{D}$.

Let now $\tau : D_{\text{dis}} \rightarrow \text{Aut}(\mathcal{AP}(D))$ denote the action of D_{dis} (D considered as a discrete group) on $\mathcal{AP}(D)$ induced by left translation on D . Then $\mathcal{AP}(D)$ has a faithful unique τ -invariant state. Further, if τ is transported to an action $\bar{\tau}$ of D_{dis} on $C(\mathcal{D})$, one checks easily that

$$[\bar{\tau}_d(F)](\gamma) = F(i(d)^{-1}\gamma) \quad (d \in D, F \in C(\mathcal{D}), \gamma \in \mathcal{D}),$$

so τ acts minimally on $\mathcal{AP}(D)$. Thus, τ is strictly (tracially) ergodic.

Moreover, if D is maximally almost periodic (i.e. the canonical map i is injective), then we see from the above that D_{dis} acts freely on $\mathcal{D}$. Hence, Theorem 10, case ii), (or Theorem 3) applies and gives:

Proposition 13: $C_r^*(\mathcal{AP}(D), D_{\text{dis}}, \tau) \in \mathcal{S}_{u.t.}$ whenever D is maximally almost periodic.

Examples of maximally almost periodic topological groups are furnished by compact groups, locally compact abelian groups and any direct product of these. Of course, when D is compact, $\mathcal{D}$ identifies with D and Proposition 13 may be known.

For an application of this proposition when D is a countable amenable maximally almost periodic discrete group, related to [Arv 1], [Arv 2], we refer to [Bed 4].

3.3. To exemplify the situation of Theorem 10, cases iii), iv) or v), it is quite natural to consider product-type actions.

Suppose we are given a compact Hausdorff space X , an action $\alpha^{(1)}$ of a (discrete) group G on $\mathcal{A}_1 = C(X)$ which is strictly ergodic with respect to a state ϕ_1 on $\mathcal{A}_1$, and an action $\alpha^{(2)}$ of G on a simple unital C^* -algebra $\mathcal{A}_2$ with a unique trace ϕ_2 .

We denote then by $\mathcal{A}$ the C^* -tensor product $\mathcal{A}_1 \otimes_* \mathcal{A}_2$ of $\mathcal{A}_1$ and $\mathcal{A}_2$. Since $\mathcal{A}_1$ is nuclear by Takesaki's theorem ([Mur; Theorem 6.4.15]), we don't have to specify which C^* -norm we use on the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ to define $\mathcal{A}$. Further, we let $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ denote the product action given by $\alpha_g = \alpha_g^{(1)} \otimes \alpha_g^{(2)}$ ($g \in G$), and ϕ the product trace $\phi_1 \otimes \phi_2$ on $\mathcal{A}$. We have then:

Proposition 14:

- i) The product action α is strictly trace-ergodic.
- ii) Suppose that either $\alpha^{(1)}$ is ϕ_1 -properly outer or $\alpha^{(2)}$ is ϕ_2 -outer. Then α is ϕ -properly outer. Hence, if we assume further that G is abelian or that $\mathcal{A}$ is separable, then $C_r^*(\mathcal{A}, G, \alpha) \in \mathcal{S}_{u.t.}$

Proof:

- i) Suppose ψ is a α -invariant trace on $\mathcal{A}$. Then for each $a_1 \in \mathcal{A}_1, a_1 \geq 0$, the map $a_2 \rightarrow \psi(a_1 \otimes a_2), a_2 \in \mathcal{A}_2$, is clearly a tracial positive linear functional on $\mathcal{A}_2$. Since ϕ_2 is the unique trace on $\mathcal{A}_2$, one deduces easily that $\psi(a_1 \otimes a_2) = \psi(a_1 \otimes I)\phi_2(a_2)$ for all $a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2$. Now, $a_1 \rightarrow \psi(a_1 \otimes I), a_1 \in \mathcal{A}_1$, is an $\alpha^{(1)}$ -invariant trace on $\mathcal{A}_1$, so we get $\psi(a_1 \otimes I) = \phi_1(a_1), a_1 \in \mathcal{A}_1$, by the uniqueness assumption on ϕ_1 . By density, we conclude that $\psi = \phi$. Hence ϕ is the only α -invariant trace on $\mathcal{A}$.

To see that α is minimal, we use that $\mathcal{A}$ may be identified with the C^* -algebra $C(X, \mathcal{A}_2)$ of all $\mathcal{A}_2$ -valued continuous functions on X ([Mur; Theorem 6.4.17]), the action α being then given by

$$[\alpha_g(f)](x) = \alpha_g^{(2)}(f(g^{-1} \cdot x)),$$

$g \in G, f \in C(X, \mathcal{A}_2), x \in X$, where $x \rightarrow g \cdot x$ denotes the action of G on X associated to the action $\alpha^{(1)}$ of G on $\mathcal{A}_1$. Since $\mathcal{A}_2$ is simple, it follows from [Dix; Lemma 10.4.2] that all the (closed two-sided) ideals of $C(X, \mathcal{A}_2)$ are of the form

$$\mathcal{J}(E) = \{f \in C(X, \mathcal{A}_2) | f = 0 \text{ on } E\}, \text{ where } E \text{ is a closed subset of } X.$$

Now, if $\mathcal{J} = \mathcal{J}(E)$ is an α -invariant ideal of $\mathcal{A} = C(X, \mathcal{A}_2)$, we get easily that E is G -invariant, which gives $E = \emptyset$ or $E = X$ since G is assumed to act minimally on X . Thus, $\mathcal{J} = \mathcal{A}$ or $\mathcal{J} = \{0\}$ as desired.

- ii) We suppose that either $\alpha^{(1)}$ is ϕ_1 -properly outer or $\alpha^{(2)}$ is ϕ_2 -outer, which means that either $\tilde{\alpha}^{(1)}$ is properly outer on $\pi_{\phi_1}(\mathcal{A}_1)''$ or $\tilde{\alpha}^{(2)}$ is outer on $\pi_{\phi_2}(\mathcal{A}_2)''$. Since $\pi_{\phi_2}(\mathcal{A}_2)''$ is a factor, outerness is equivalent to properly outerness on $\pi_{\phi_2}(\mathcal{A}_2)''$. Hence we get from [Str; Proposition 17.6] that $\tilde{\alpha}^{(1)} \overline{\otimes} \tilde{\alpha}^{(2)}$ is properly outer on $\pi_{\phi_1}(\mathcal{A}_1)'' \overline{\otimes} \pi_{\phi_2}(\mathcal{A}_2)''$. Further, we have $[(\pi_{\phi_1} \otimes \pi_{\phi_2})(\mathcal{A})]'' = \pi_{\phi_1}(\mathcal{A}_1)'' \overline{\otimes} \pi_{\phi_2}(\mathcal{A}_2)''$ by [Tak; Proposition 4.13], while $(\pi_{\phi}, \mathcal{H}_{\phi}, \xi_{\phi})$ may be identified with $(\pi_{\phi_1} \otimes \pi_{\phi_2}, \mathcal{H}_{\phi_1} \otimes \mathcal{H}_{\phi_2}, \xi_{\phi_1} \otimes \xi_{\phi_2})$ by [Tak; Theorem 4.9].

Under this identification, we have $\tilde{\alpha} = \tilde{\alpha}^{(1)} \overline{\otimes} \tilde{\alpha}^{(2)}$, so it follows that $\tilde{\alpha}$ is properly outer on $\pi_{\phi}(\mathcal{A})''$, i.e. α is ϕ -properly outer.

If we assume further that G is abelian or that $\mathcal{A}$ is separable, we may then appeal to Theorem 10, cases iv) or v), and conclude that $C_r^*(\mathcal{A}, G, \alpha) \in \mathcal{S}_{u.t.}$. ■

As a special case of this construction, we may take $\alpha^{(1)}$ to be action of $\mathbf{Z}$ on $C(\mathbf{T})$ induced by an irrational rotation, and let $\alpha^{(2)}$ be any action of $\mathbf{Z}$ on a unital C^* -algebra $\mathcal{B}$ belonging to $\mathcal{S}_{u.t.}$. Then the resulting C^* -crossed product of $C(\mathbf{T}) \otimes_* \mathcal{B}$ by $\mathbf{Z}$ under the product action belongs to $\mathcal{S}_{u.t.}$ and may be viewed as a generalized non-commutative torus.

3.4. Recall that a (discrete) group G is said to be C^* -simple (resp. have a unique trace) if $C_r^*(G)$ is simple (resp. has a unique trace), cf. [Bed 2]. A wide class of C^* -simple groups with a unique trace consists of the so-called ultraweak Powers groups ([Bed 2], [Bed 3]). In all known examples, then either G is C^* -simple and has a unique trace or G doesn't have any of these properties, and it is an open question whether this is generally true (cf. [Har]).

We proved in [Bed 2; Corollaries 3.6 and 4.12] that the class of C^* -simple groups has the following extension property:

If H is a normal C^* -simple subgroup of G such that either the centralizer of H in G is trivial or G/H is a (repeated) extension of ultraweak Powers groups, then G is C^* -simple.

Further, in [Bed 3; Corollaries 4 and 6] we proved that the class of C^* -simple groups with a unique trace has the same property. As another indication that the answer to the above question should be positive, let us point out that the class of groups with a unique trace also has the same extension property.

Indeed, a group with a unique trace is necessarily ICC. Therefore, as in [Bed 3], this assertion may be deduced as an easy corollary to the following result (playing then the role of [Bed 3; Theorem 2 and Corollary 5]).

Proposition 15: Let $\Sigma = (\mathcal{A}, G, \alpha, u)$ denote a twisted C^* -dynamical system, and suppose $\mathcal{A}$ has a faithful α -invariant trace. Then

- i) If H is a normal subgroup of G such that G/H acts freely on H (i.e. the set $\{hgh^{-1}; h \in H\}$ is infinite for all $g \in G \setminus H$) and $C_r^*(\mathcal{A}, H, \alpha, u)$ has a unique trace,

then $C_r^*(\mathcal{A}, G, \alpha, u)$ has a unique trace.

- ii) If G is an ultraweak Powers group and $\mathcal{A}$ has a unique trace, then $C_r^*(\Sigma)$ has a unique trace.

Proof:

- i) Besides that one now invokes Proposition 9 instead of [Bed 3; Theorem 1], the proof proceeds exactly as the proof of [Bed 3; Theorem 2].
- ii) This follows from i) and [Bed 2; Theorem 4.2], cf. the proof of [Bed 3; Corollary 5].

■

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