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by

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Stability properties of stochastic partial differential equations

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Abstract

Stochastic partial differential equations (SPDEs) often have solutions that are known to be pure Schwartz distributions i.e. not functions. To make sense of such equations one needs to introduce some kind of smoothing parameters. This paper is concerned with stability properties of the solutions as one lets the smoothing parameters approach some kind of delta function. The first part of the paper concentrates on linear functionals in connection with SPDEs. In the second part we address similar problems related to functionals of Hida distributions.

§1 Introduction

It is well known that noise in SPDEs leads to very singular problems. Even simple equations like the Poisson equation:

$$(1.1) \quad \Delta u = W \quad W \text{ White noise in } \mathbf{R}^d$$

have solutions which are pure Schwartz distributions (i.e. not functions) when the space dimension $d > 2$. See e.g. [W]. It would be natural to believe that the expression:

$$(1.2) \quad u(x) = \int G(x, y) dB(y)$$

where $G(x, y)$ is the Green function, gives the solution. This, however, fails when the dimension is larger than two. The Green function is not square integrable in this case and the point value is meaningless in the classical sense. To bypass this difficulty, it is necessary to smear out the values. The classical way of doing this is through the mapping:

$$(1.3) \quad u(\phi) = \int \int G(x, y) \phi(x) dx dB(y)$$

which turns out to be a Schwartz distribution, and solves the Poisson problem in the weak sense. In this way we get a well defined solution which fits well into the framework of functional analysis. If we want to use this framework on more general SPDEs, however, we are faced with serious technical problems. In light of the result above, we can not in general expect to find function solutions of SPDEs. We then have to deal with difficult problems related to multiplication of singular objects.

To be able to work with these kinds of singular structures, several spaces of generalized functionals have been introduced. The theory can be viewed from two different directions. One possibility is to use spaces of generalized random variables. Much work in this direction has concentrated on Hida distributions, see e.g. [HKPS], [HP] and [K]. The objects then act as functionals in the random variable. From our point of view this is really a question of smearing out random variables. The point values in (1.2) can be given an alternative meaning in the sense of Hida distributions, and we will discuss various aspects of this theory in the second part of the paper.

Another possibility is to smoothen the behaviour in the time/space variables. This angle of attack has its roots in the theory of Columbeau distributions, see [C]. Columbeau distributions makes it possible to handle products of pure distributions in a rigorous sense. The working environment also has pleasant features when we want to make connections with physics.

The basic idea with Columbeau distributions is to represent generalized functions by a family of smooth mappings equipped with a certain equivalence relation. A locally integrable function $f : D \rightarrow \mathbf{R}$ can be represented by the smooth mapping $f : \mathcal{D} \times D \rightarrow \mathbf{R}$:

$$(1.4) \quad f(\phi, x) = \int \phi_x(u) f(u) du$$

where $\mathcal{D}$ is the Schwartz test function space and $\phi_x(u) = \phi(u - x)$. For ϕ fixed this is a C^∞ mapping, and objects in this form we can differentiate and multiply freely. We look at the expression $f(\phi, x)$ as a measurement or observation of f . This idea has been used in [LØU 1], [LØU 2], [LØU 3], [HLØUZ 1] and [HLØUZ 2]. In these papers, however, limiting behaviour of the solutions have not been discussed.

To proceed further, any smooth mapping $F : \mathcal{D} \times D \rightarrow \mathbf{R}$ is considered to be the family of measurements of a physical quantity. The only problem with this kind of construction, is that the same function can be represented in several different ways. To get uniqueness in the multiplicative structure, we have to equip the space with an equivalence relation. In the Columbeau setting the difference between two different representations should approach zero very fast as we let ϕ approach a delta function (for a precise definition see [C]). In this paper we will usually consider weaker forms of equality, so we just keep the basic idea; one should try to control the behaviour as $\phi \rightarrow \delta$.

SPDEs with white noise coefficients, represent idealized physical models. In practical terms this means that physical measurements are performed on scales where the space/time

fluctuations of the noise are large. As we have seen, the solution of a model like this may very well be a pure distribution (or something even more singular!). A behaviour of such kind indicates that the physical phenomenon is very unstable. The mathematical solution also makes little sense in relation to the physics of the problem. The physical solution, as unstable as it might be, should still be a function. This means that the mathematical solution has to be smeared out before it is related to the physics in question.

In this sense a pure distribution solution functions only as an intermediate step. One may perhaps say that it acts like some kind of mirror. On the one side we have the actual physical phenomenon and on the other side we have smeared out approximations of the distribution. The smeared out versions can then be looked upon as images or shadows of the original behaviour. It is these approximations that make sense, and the mathematical idealization is only some kind of phantom object.

In light of the above, one may ask whether idealizations of the above type have any interest. If we smear out the coefficients before we solve the equations, the equations are usually relatively easy to handle. Doing this we can circumvent some difficult problems. The purpose of the idealized solutions is, however, to measure how stable these models are under approximations. Distribution solutions, whatever singular they may be, also represent stability in the approximations. Testing a distribution against two testfunctions which are close in the testfunction topology, will give nearly the same results. So even if SPDEs give rise to singular solutions, a well defined limit means that there are stable properties attached to the solution.

§2 Stability properties of the Poisson equation

In this section we want to study the stability of SPDEs under approximation. We let $L = \nabla[a \nabla(\cdot)]$ be a strictly elliptic operator on a bounded domain D in $\mathbf{R}^d$. We will assume that the coefficients $a = [a_{ij}]$ are bounded measurable functions, and want to consider the SPDE:

$$(2.1) \quad LV = W \quad V|_{\partial D} = 0$$

If we let $G(x, y)$ denote the Green function, then the expression:

$$(2.2) \quad V(\phi) = \int_D \int_D G(x, y) \phi(x) dx dB(y)$$

is the obvious candidate for the solution. It is not difficult to see that the expression above defines a distribution, i.e. that $V \in \mathcal{D}'(D)$. If $\psi \in L^2(D)$, then (see [GT] p 173) the problem:

$$(2.3) \quad Lu = \psi \quad V|_{\partial D} = 0$$

is weakly solvable with solution $u_\psi(x) = \int_D G(x, y)\psi(y)dy$, and we have:

$$(2.4) \quad \|u_\psi\|_{W^{1,2}(D)} \leq C\|\psi\|_{L^2(D)}$$

i.e. u_ψ is in the Sobolev space $W^{1,2}(D)$, and:

$$(2.5) \quad \int_D |u_\psi(x)|^2 + |\nabla u_\psi(x)|^2 dx \leq C^2 \|\psi\|_{L^2(D)}^2$$

By this estimate and the Ito isometry we see that:

$$(2.6) \quad \mathbb{E}[|V(\phi)|^2] = \int_D |u_\phi(x)|^2 dx \leq C^2 \|\phi\|_{L^2(D)}^2 \leq C' \sup_{x \in D} |\phi(x)|^2$$

This shows that the mapping $\phi \mapsto V(\phi)$ is well defined, almost linear and continuous in probability on $\mathcal{D}(D)$. By [W] p 332, it then has a version in $\mathcal{D}'(D)$. It is not clear, however, in what sense this distribution is a solution of the original equation. The operator involves a singular product, and it is necessary to examine the product of the non-smooth function a and the pure distribution ∇V . To pursue this further, we introduce an approximate identity $\{\delta_n\}_{n=1}^\infty$, i.e. we choose $\delta_1 \in C_c^\infty(\mathbf{R}^d)$ such that $\delta_1 \geq 0$, $\int \delta_1(x)dx = 1$ and put $\delta_n(x) = n^d \delta_1(nx)$. We extend V and a to $\mathbf{R}^d$ putting them equal to zero outside of D , and introduce the following error function (the $*$ denotes a convolution product):

$$(2.7) \quad \text{Error}_n(V)(x) = \nabla[a * \delta_n(x) \cdot \nabla[V * \delta_n(x)]] - W * \delta_n(x)$$

We can then prove the following stability result:

Theorem 2.1

The distribution:

$$(2.8) \quad V(\phi) = \int_D \int_D G(x, y)\phi(x)dx dB(y)$$

solves the problem:

$$(2.9) \quad LV = W \quad V|_{\partial D} = 0$$

in the sense that:

$$(2.10) \quad \text{Error}_n(V)(x) \rightarrow 0 \quad \text{in the sense of distributions.}$$

Remark

In the language of Columbeau distributions, the result says that we have an associate equality between LV and W . It is known, see [C], that two representations of Schwartz distributions are strictly equal if and only if they are associately equal. Since the objects

in theorem 2.1 just slightly extend beyond the class of distributions, it seems reasonable to conjecture that the equality above is strict.

Proof

We pick $\phi \in \mathcal{D}(D)$, and want to prove that $\text{Error}_n(V)(\phi) \rightarrow 0$. Since W is a distribution, we clearly have:

$$(2.11) \quad W * \delta_n(x) \rightarrow W(\phi) = \int_D \phi(x) dB(x)$$

We then consider the action of the term $\nabla[a * \delta_n(x) \cdot \nabla[V * \delta_n(x)]]$ on ϕ . Here we have:

$$(2.12) \quad \begin{aligned} & \nabla[a * \delta_n(x) \cdot \nabla[V * \delta_n(x)]](\phi) \\ &= - \int_D a * \delta_n(x) \cdot V * \nabla \delta_n(x) \cdot \nabla \phi(x) dx \\ &= - \int_D a * \delta_n(x) \int_D \int_D G(\xi, \eta) \nabla \delta_n(x - \xi) d\xi dB(\eta) \nabla \phi(x) dx \\ &= - \int_D \left[\int_D \int_D a * \delta_n(x) G(\xi, \eta) \nabla \delta_n(x - \xi) \nabla \phi(x) d\xi dx \right] dB(\eta) \end{aligned}$$

In the last equality we have used the stochastic Fubini theorem, see [W] p 297. To prove the theorem, we have to prove that the term:

$$(2.13) \quad \phi_n(\eta) = - \int_D \int_D a * \delta_n(x) G(\xi, \eta) \nabla \delta_n(x - \xi) \nabla \phi(x) d\xi dx$$

is close to $\phi(\eta)$ in $L^2(D)$, i.e. we want to prove that $\|\phi_n - \phi\|_{L^2(D)} \rightarrow 0$. To prove this we pick $\psi \in L^2(D)$ and estimate the integral:

$$(2.14) \quad \int_D (\phi_n(\eta) - \phi(\eta)) \psi(\eta) d\eta$$

It is then enough to prove that:

$$(2.15) \quad \int_D \phi_n(\eta) \psi(\eta) d\eta = \int_D \phi(\eta) \psi(\eta) d\eta + O_n(\psi)$$

together with the estimate:

$$(2.16) \quad |O_n(\psi)| \leq O(n) \|\psi\|_{L^2(D)}$$

where $O(n) \rightarrow 0$ independent of the choice of $\psi \in L^2(D)$. For this we have:

$$(2.17) \quad \begin{aligned} & - \int_D \phi_n(\eta) \psi(\eta) d\eta \\ &= \int_D \left[\int_D \int_D a * \delta_n(x) G(x - \xi, \eta) \nabla \delta_n(\xi) \nabla \phi(x) d\xi dx \right] \psi(\eta) d\eta \\ &= \int_D a * \delta_n(x) \left[\int_D \int_D G(x - \xi, \eta) \psi(\eta) d\eta \nabla \delta_n(\xi) d\xi \right] \nabla \phi(x) dx \end{aligned}$$

In the second equality we used the Fubini theorem again. We want to estimate the integral in the middle and put:

$$\begin{aligned}
u_{\psi,n}(x) &= \int_D \int_D G(x - \xi, \eta) \psi(\eta) d\eta \nabla \delta_n(\xi) d\xi \\
(2.18) \quad &= \int_D u_\psi(x - \xi) \nabla \delta_n(\xi) d\xi \\
&= - \int_D \nabla u_\psi(x - \xi) \delta_n(\xi) d\xi
\end{aligned}$$

Now observe that for any $f \in L^2(D)$ which is extended to $\mathbf{R}^d$ by zero, we have:

$$\begin{aligned}
&\| \int_D f(x - \xi) \delta_n(\xi) d\xi \|_{L^2(D)}^2 \\
&= \int_D \left[\int_D f(x - \xi) \delta_n(\xi) d\xi \cdot \int_D f(x - \eta) \delta_n(\eta) d\eta \right] dx \\
(2.19) \quad &= \int_D \int_D \left[\int_D f(x - \xi) f(x - \eta) dx \right] \delta_n(\xi) \delta_n(\eta) d\xi d\eta \\
&\leq \int_D \int_D \left[\int_D f^2(x - \xi) d\xi \right]^{\frac{1}{2}} \left[\int_D f^2(x - \eta) d\eta \right]^{\frac{1}{2}} \delta_n(\xi) \delta_n(\eta) d\xi d\eta \\
&\leq \int_D \int_D \|f\|_{L^2(D)}^2 \delta_n(\xi) \delta_n(\eta) d\xi d\eta = \|f\|_{L^2(D)}^2
\end{aligned}$$

If we apply the estimate above to $f = \nabla u_\psi$ and use the estimate (2.5), we get:

$$(2.20) \quad \|u_{\psi,n}\|_{L^2(D)} \leq \|\nabla u_\psi\|_{L^2(D)} \leq C \|\psi\|_{L^2(D)}$$

With the equality above in mind, we start to rewrite the term $\int_D \phi_n(\eta) \psi(\eta) d\eta$:

$$\begin{aligned}
- \int_D \phi_n(\eta) \psi(\eta) d\eta &= \int_D a(x) u_{\psi,n}(x) \nabla \phi(x) dx \\
(2.21) \quad &+ \int_D (a * \delta_n(x) - a(x)) u_{\psi,n}(x) \nabla \phi(x) dx \\
&= \int_D a(x) u_{\psi,n}(x) \nabla \phi(x) dx + O_n(\psi)
\end{aligned}$$

In the last equality we have used the estimate (2.20) and the fact that $\|a * \delta_n - a\|_{L^2(D)} \rightarrow 0$, see [H] p 22. We then proceed to rewrite the leading term:

$$\begin{aligned}
& \int_D a(x) u_{\psi,n}(x) \nabla \phi(x) dx \\
&= \int_D a(x) \left[\int_D \nabla u_{\psi}(x - \xi) \delta_n(\xi) d\xi \right] \nabla \phi(x) dx \\
&= \int_D \left[\int_D a(x) \nabla u_{\psi}(x - \xi) \nabla \phi(x) dx \right] \delta_n(\xi) d\xi \\
&= \int_D \left[\int_D (a(x) - a(x - \xi)) \nabla u_{\psi}(x - \xi) \nabla \phi(x) dx \right] \delta_n(\xi) d\xi \\
&\quad + \int_D \left[\int_D a(x - \xi) \nabla u_{\psi}(x - \xi) \nabla \phi(x) dx \right] \delta_n(\xi) d\xi \\
&= O_n(\psi) \\
&\quad + \int_D \left[\int_D a(x - \xi) \nabla u_{\psi}(x - \xi) \nabla \phi(x - \xi) dx \right] \delta_n(\xi) d\xi \\
&\quad + \int_D \left[\int_D a(x - \xi) \nabla u_{\psi}(x - \xi) (\nabla \phi(x) - \nabla \phi(x - \xi)) dx \right] \delta_n(\xi) d\xi \\
&= O_n(\psi) \\
&\quad + \int_D \left[\int_D a(x - \xi) \nabla u_{\psi}(x - \xi) \nabla \phi(x - \xi) dx \right] \delta_n(\xi) d\xi \\
&= O_n(\psi) \\
&\quad + \int_D \left[\int_D a(x) \nabla u_{\psi}(x) \nabla \phi(x) dx \right] \delta_n(\xi) d\xi \\
&= - \int_D \left[\int_D \psi(x) \phi(x) dx \right] \delta_n(\xi) d\xi + O_n(\psi) \\
&= - \int_D \psi(x) \phi(x) dx + O_n(\psi)
\end{aligned}
\tag{2.22}$$

In the sixth equality we choose n so large that the support of $x \mapsto \nabla \phi(x - \xi)$ is contained in D for all $\xi \in \text{supp}(\delta_n)$. In the seventh equality we have used that u_{ψ} is a weak solution of the ODE $Lu = \psi$. We have thus proved that:

$$\int_D \phi_n(\eta) \psi(\eta) d\eta = \int_D \phi(\eta) \psi(\eta) d\eta + O_n(\psi)
\tag{2.23}$$

with:

$$|O_n(\psi)| \leq O(n) \|\psi\|_{L^2(D)}
\tag{2.24}$$

and that concludes the proof of the theorem.

□

§3 Stability properties of the heat equation

Let $a(x) = [a_{ij}(x)] : \mathbf{R}^d \rightarrow \mathbf{R}^d \otimes \mathbf{R}^d$ be a measurable symmetric matrix valued function which satisfies the ellipticity condition:

$$(3.1) \quad \lambda I \leq a(\cdot) \leq \frac{1}{\lambda} I, \quad \lambda \in (0, 1]$$

Let L denote the divergence operator defined by:

$$(3.2) \quad L = \sum_{i=1}^d \sum_{j=1}^d \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right)$$

It is well known, see [S], that there exists a unique Feller continuous Markov semigroup P_t with the properties that:

$$(3.3) \quad t \in [0, \infty) \rightarrow P_t \phi \in W^{1,2}(\mathbf{R}^d) \text{ is a weakly continuous map for each } \phi \in C_0^\infty(\mathbf{R}^d).$$

$$(3.4) \quad \text{For all } \phi, \psi \in C_0^\infty(\mathbf{R}^d):$$

$$< P_t \phi, \psi > = < \phi, \psi > - \int_0^t < \nabla P_s \phi, a \nabla \psi > ds, \quad t \in [0, \infty)$$

Moreover there is a $p \in C((0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d)$ such that:

$$(3.5) \quad P_t \phi(x) = \int_{\mathbf{R}^d} p(t, x, y) \phi(y) dy$$

$$(3.6) \quad \|\nabla P_t \phi\|_{L^2} \leq \frac{1}{\lambda^{\frac{1}{2}} t^{\frac{1}{2}}} \|\phi\|_{L^2}, \quad t > 0$$

$$(3.7) \quad \frac{1}{M t^{\frac{d}{2}}} e^{-M \frac{|y-x|^2}{t}} \leq p(t, x, y) \leq \frac{M}{t^{\frac{d}{2}}} e^{-\frac{|y-x|^2}{M t}}$$

Now consider the heat equation:

$$(3.8) \quad \frac{\partial u}{\partial t} = Lu + W_{t,x} \quad u(0, x) = 0$$

When the space dimension $d > 1$, this equation has solutions that are pure Schwartz distributions, see [W]. The formal candidate for the solution is:

$$(3.9) \quad u(t, x) = \int_0^t \int_{\mathbf{R}^d} p(t-s, x, y) W(ds, dy)$$

We make sense of this through the distribution:

$$(3.10) \quad u_t(\phi) = \int_0^t \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p(t-s, x, y) \phi(x) dx \right) W(ds, dy)$$

In fact, from the Ito-isometry:

$$(3.11) \quad \begin{aligned} \mathbb{E}[|u_t(\phi)|^2] &= \int_0^t \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p(t-s, x, y) \phi(x) dx \right)^2 ds dy \\ &\leq \int_0^t \left(\int_{\mathbf{R}^d} \phi^2(x) dx \right) ds = t \|\phi\|_{L^2}^2 \end{aligned}$$

This shows that the mapping $\phi \mapsto u_t(\phi)$ is well defined, almost linear and continuous in probability on $\mathcal{S}(\mathbf{R}^d)$. Thus, by [W], u_t is well defined as an element of $\mathcal{S}'(\mathbf{R}^d)$.

We let $\{\delta_n\}_{n=1}^\infty$ denote an approximative identity, i.e:

$$(3.12) \quad \delta_1 \in C_c^\infty(\mathbf{R}^{d+1}), \delta_1 \geq 0, \int \delta_1(t, x) dx dt = 1, \delta_n(t, x) = n^{d+1} \delta_1(nt, nx)$$

and introduce the errorfunction:

$$(3.13) \quad \text{Error}_n(u) = \frac{\partial}{\partial t}(u * \delta_n(t, x)) - \nabla[a * \delta_n(t, x) \cdot \nabla[u * \delta_n(t, x)]] - W * \delta_n(t, x)$$

where the ∇ is acting on the x variable only. We then have the following:

Theorem 3.1

The distribution:

$$(3.14) \quad u_t(\phi) = \int_0^t \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} p(t-s, x, y) \phi(x) dx \right) W(ds, dy)$$

solves the problem:

$$(3.15) \quad \frac{\partial u}{\partial t} = Lu + W_{t,x} \quad u(0, x) = 0$$

in the sense that:

$$(3.16) \quad \text{Error}_n(u)(t, x) \rightarrow 0 \quad \text{in the sense of distributions.}$$

Proof

Take $\phi \in C_0^\infty((0, \infty) \times \mathbf{R}^d)$. We need to prove that $\text{Error}_n(u)(\phi) \rightarrow 0$ as $n \rightarrow \infty$. Since $W_{t,x}$ is a distribution, we clearly have:

$$(3.17) \quad W * \delta_n(\phi) \rightarrow \int_0^\infty \int_{\mathbf{R}^d} \phi(t, x) W(dt, dx)$$

We consider the first term in the errorfunction:

$$(3.18) \quad \begin{aligned} \frac{\partial}{\partial t}(u * \delta_n)(t, x) &= u * \frac{\partial \delta_n}{\partial t}(t, x) \\ &= \int_0^\infty \int_{\mathbf{R}^d} \left(\int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \frac{\partial \delta_n(t-s, x-y)}{\partial t} dy ds \right) W(du, dz) \end{aligned}$$

From this we can see that if n is large enough, then:

$$(3.19) \quad \begin{aligned} \frac{\partial}{\partial t}(u * \delta_n)(\phi) &= \int_0^\infty \int_{\mathbf{R}^d} \int_{\mathbf{R}^{d+1}} \phi(t, x) \cdot \\ &\left(\int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \frac{\partial \delta_n(t-s, x-y)}{\partial t} ds dy \right) dt dx W(du, dz) \end{aligned}$$

We can also express $\langle -\nabla[a * \delta_n(t, x) \cdot \nabla[u * \delta_n(t, x)]], \phi \rangle$ as:

$$(3.20) \quad \begin{aligned} &\int_{\mathbf{R}} \int_{\mathbf{R}^d} -\phi(t, x) \nabla[a * \delta_n(t, x) \cdot \nabla[u * \delta_n(t, x)]] dx dt \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^d} \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], \nabla_x[u * \delta_n(t, x)] \rangle dx dt \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^d} \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], u * \nabla_x \delta_n(t, x) \rangle dx dt \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^d} \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], \\ &\int_0^\infty \int_{\mathbf{R}^d} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \nabla_x \delta_n(t-s, x-y) \rangle ds dy W(du, dz) dx dt \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}^d} \int_0^\infty \int_{\mathbf{R}^d} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \cdot \\ &\quad \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], \nabla_x \delta_n(t-s, x-y) \rangle ds dy W(du, dz) dx dt \\ &= \int_0^\infty \int_{\mathbf{R}^d} \int_{\mathbf{R}} \int_{\mathbf{R}^d} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \cdot \\ &\quad \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], \nabla_x \delta_n(t-s, x-y) \rangle ds dy dx dt W(du, dz) \end{aligned}$$

From this we see that:

$$(3.21) \quad \begin{aligned} &\langle u * \frac{\partial \delta_n}{\partial t} - \nabla[a * \delta_n(t, x) \cdot \nabla[u * \delta_n(t, x)]], \phi \rangle \\ &= \int_0^\infty \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^{d+1}} \phi(t, x) \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \frac{\partial \delta_n(t-s, x-y)}{\partial t} ds dy dt dx \right. \\ &\quad + \int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \cdot \\ &\quad \left. \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], \nabla_x \delta_n(t-s, x-y) \rangle ds dy dx dt \right) W(du, dz) \end{aligned}$$

Thus, to prove the theorem, it suffices to prove that:

$$(3.22) \quad \begin{aligned} &\int_{\mathbf{R}^{d+1}} \phi(t, x) \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \frac{\partial \delta_n(t-s, x-y)}{\partial t} ds dy dt dx \\ &+ \int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \cdot \\ &\quad \langle a * \delta_n(t, x) \nabla_x[\phi(t, x)], \nabla_x \delta_n(t-s, x-y) \rangle ds dy dx dt \\ &\rightarrow \phi(u, z) \quad \text{in } L^2(\mathbf{R}_+ \times \mathbf{R}^d) \end{aligned}$$

The variable s is in fact integrated over a finite interval since ϕ has compact support. Note that:

$$\begin{aligned}
(3.23) \quad & \int_{\mathbf{R}^{d+1}} \phi(t, x) \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \frac{\partial \delta_n(t-s, x-y)}{\partial t} ds dy dt dx \\
&= \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \int_{\mathbf{R}^{d+1}} \phi(t, x) \frac{\partial \delta_n(t-s, x-y)}{\partial t} dt dx ds dy \\
&= - \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \int_{\mathbf{R}^{d+1}} \frac{\partial \phi(t, x)}{\partial t} \delta_n(t-s, x-y) dt dx ds dy \\
&\rightarrow - \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \int_{\mathbf{R}^{d+1}} \frac{\partial \phi(s, y)}{\partial s} ds dy \quad \text{in } L^2(\mathbf{R}_+ \times \mathbf{R}^d)
\end{aligned}$$

It is thus sufficient to show that:

i)

$$\begin{aligned}
(3.24) \quad & I_n(u, z) := \int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \cdot \\
& \quad < a(x) \nabla_x [\phi(t, x)], \nabla_x \delta_n(t-s, x-y) > ds dy dx dt \\
& \rightarrow I(u, z) := \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \frac{\partial \phi(s, y)}{\partial s} ds dy + \phi(u, z) \quad \text{in } L^2(\mathbf{R}_+ \times \mathbf{R}^d)
\end{aligned}$$

and

ii)

$$\begin{aligned}
(3.25) \quad & II_n(u, z) := \int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \cdot \\
& \quad < (a * \delta_n(t, x) - a(x)) \nabla_x [\phi(t, x)], \nabla_x \delta_n(t-s, x-y) > ds dy dx dt \\
& \rightarrow 0 \quad \text{in } L^2(\mathbf{R}_+ \times \mathbf{R}^d)
\end{aligned}$$

Let us first prove ii):

We will prove that for any $\psi(u, z) \in L^2(\mathbf{R}_+ \times \mathbf{R}^d)$:

$$(3.26) \quad | < II_n, \psi > | \leq O(n) \|\psi\|_{L^2(\mathbf{R}_+ \times \mathbf{R}^d)}$$

where $O(n)$ does not depend on ψ and $O(n) \rightarrow 0$ as $n \rightarrow \infty$.

We write the inner product $\langle II_n, \psi \rangle$ on the form:

$$\begin{aligned}
(3.27) \quad \langle II_n, \psi \rangle &= \int_{\mathbf{R}} \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} p(s-u, y, z) \langle (a * \delta_n(t, x) - a(x)) \cdot \right. \\
&\quad \left. \nabla_x[\phi(t, x)], \nabla_x \delta_n(t-s, x-y) \rangle dy ds dx dt \right) \psi(u, z) du dz \\
&= \int_{\mathbf{R}} \int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} P_{s-u}(\psi(u, \cdot))(y) \langle (a * \delta_n(t, x) - a(x)) \cdot \\
&\quad \nabla_x[\phi(t, x)], \nabla_x \delta_n(t-s, x-y) \rangle dy ds dx dt du \\
&= \int_{\mathbf{R}} \int_{\mathbf{R}^{d+1}} \int_u^\infty \int_{\mathbf{R}^d} P_{s-u}(\psi(u, \cdot))(y) \langle (a * \delta_n(t, x) - a(x)) \cdot \\
&\quad \nabla_x[\phi(t, x)], -\nabla_y \delta_n(t-s, x-y) \rangle dy ds dx dt du \\
&= \int_{\mathbf{R}} \int_{\mathbf{R}^{d+1}} \int_u^\infty \langle (a * \delta_n(t, x) - a(x)) \nabla_x[\phi(t, x)], \\
&\quad \int_{\mathbf{R}^d} -P_{s-u}(\psi(u, \cdot))(y) \nabla_y \delta_n(t-s, x-y) dy \rangle ds dx dt du \\
&= \int_{\mathbf{R}} \int_{\mathbf{R}^{d+1}} \int_u^\infty \langle (a * \delta_n(t, x) - a(x)) \nabla_x[\phi(t, x)], \\
&\quad \int_{\mathbf{R}^d} \nabla_y P_{s-u}(\psi(u, \cdot))(y) \delta_n(t-s, x-y) dy \rangle ds dx dt du
\end{aligned}$$

We now put $s-u=v$ and have (remember that the interval on which v varies is actually finite):

$$\begin{aligned}
(3.28) \quad |\langle II_n, \psi \rangle| &\leq \int_0^\infty \int_{\mathbf{R}^{d+1}} |a * \delta_x(t, x) - a(x)| |\nabla_x \phi(t, x)| \cdot \\
&\quad \int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(u, \cdot))(y)| \delta_n(t-(u+v), x-y) du dy dt dx dv \\
&\leq \int_0^\infty \left(\int_{\mathbf{R}^{d+1}} |a * \delta_x(t, x) - a(x)|^2 |\nabla_x \phi(t, x)|^2 dt dx \right)^{\frac{1}{2}} \\
&\quad \left(\int_{\mathbf{R}^{d+1}} \left(\int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(u, \cdot))(y)| \right. \right. \\
&\quad \left. \left. \delta_n(t-(u+v), x-y) du dy \right)^2 dt dx \right)^{\frac{1}{2}} dv
\end{aligned}$$

Putting $t - (u + v) = \bar{u}$, $x - y = z$, one has:

$$\begin{aligned}
& \int_{\mathbf{R}^{d+1}} \left(\int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(u, \cdot))(y)| \delta_n(t - (u + v), x - y) du dy \right)^2 dt dx \\
&= \int_{\mathbf{R}^{d+1}} \left(\int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(t - \bar{u} - v, \cdot))(x - z)| \delta_n(\bar{u}, z) d\bar{u} dz \right)^2 dt dx \\
&= \int_{\mathbf{R}^{d+1}} \int_{\mathbf{R}^{d+1}} \int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(t - \bar{u}_1 - v, \cdot))(x - z_1)| \delta_n(\bar{u}_1, z_1) \\
&\quad |\nabla_y P_v(\psi(t - \bar{u}_2 - v, \cdot))(x - z_2)| \delta_n(\bar{u}_2, z_2) d\bar{u}_1 dz_1 d\bar{u}_2 dz_2 dt dx \\
(3.29) \quad &\leq \int_{\mathbf{R}^{d+1}} \int_{\mathbf{R}^{d+1}} \delta_n(\bar{u}_1, z_1) \delta_n(\bar{u}_2, z_2) \\
&\quad \left(\int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(t - \bar{u}_1 - v, \cdot))(x - z_1)|^2 dt d\mathbf{x} \right)^{\frac{1}{2}} \\
&\quad \left(\int_{\mathbf{R}^{d+1}} |\nabla_y P_v(\psi(t - \bar{u}_2 - v, \cdot))(x - z_2)|^2 dt d\mathbf{x} \right)^{\frac{1}{2}} d\bar{u}_1 dz_1 d\bar{u}_2 dz_2 dt dx \\
&= \int_{\mathbf{R}^{d+1}} |\nabla_x P_v \psi(t, x)|^2 dt dx \leq \frac{C}{v} \|\psi\|_{L^2}^2
\end{aligned}$$

In the last estimate we used (3.6). Substituting this into (3.28), we get:

$$\begin{aligned}
| \langle II_n, \psi \rangle | &\leq \left(C^{\frac{1}{2}} \int_0^K v^{-\frac{1}{2}} \|\psi\|_{L^2} dv \right) \cdot \\
(3.30) \quad &\left(\int_{\mathbf{R}^{d+1}} |a * \delta_x(t, x) - a(x)|^2 |\nabla_x \phi(t, x)|^2 dt dx \right)^{\frac{1}{2}} \\
&\leq O(n) \|\psi\|_{L^2}
\end{aligned}$$

This completes the proof of ii). Before we prove i), we remark that the following formula can be deduced from (3.4).

$$(3.31) \quad - \langle \nabla P_t \phi, a \nabla \psi(t, \cdot) \rangle + \int P_t \phi(t, x) \frac{\partial \psi(t, x)}{\partial t} dx = \frac{\partial}{\partial t} \langle P_t \phi, \psi(t, \cdot) \rangle$$

To prove i), taking $\psi \in L^2(\mathbf{R}_+ \times \mathbf{R}^d)$, we need to show that:

$$(3.32) \quad |(I_n - I, \psi)| \leq O(n) \|\psi\|_{L^2}$$

First using (3.31), we have:

$$\begin{aligned}
(3.33) \quad & \langle I, \psi \rangle = \int_{\mathbf{R}^{d+1}} \phi(u, z) \psi(u, z) du dz \\
& + \int_{\mathbf{R}_+} \int_0^\infty \int_{\mathbf{R}^d} \frac{\partial \phi(u+v, y)}{\partial v} \int_{\mathbf{R}^d} p(v, y, z) \psi(u, z) dz dv du \\
& = \int_{\mathbf{R}^{d+1}} \phi(u, z) \psi(u, z) du dz + \int_{\mathbf{R}_+} \int_0^\infty \int_{\mathbf{R}^d} \frac{\partial \phi(u+v, y)}{\partial v} P_v(\psi(u, \cdot))(y) dy dv du \\
& = \int_{\mathbf{R}^{d+1}} \phi(u, z) \psi(u, z) du dz \\
& + \int_{\mathbf{R}_+} \int_0^\infty \left(\frac{\partial}{\partial v} \int_{\mathbf{R}^d} \phi(u+v, y) P_v(\psi(u, \cdot))(y) dy + \right. \\
& \quad \left. \int_{\mathbf{R}^d} \langle \nabla_y P_v(\psi(u, \cdot))(y), a \nabla_y \phi(u+v, y) \rangle dy \right) dv du \\
& = \int_{\mathbf{R}_+} \int_0^\infty \int_{\mathbf{R}^d} \langle \nabla_y P_v(\psi(u, \cdot))(y), a \nabla_y \phi(u+v, y) \rangle dy dv du
\end{aligned}$$

Now we look at:

$$\begin{aligned}
(3.34) \quad & \langle I_n, \psi \rangle = \int_{\mathbf{R}^d} \int_{\mathbf{R}_+} \psi(u, z) \int_{\mathbf{R}^{d+1}} \int_0^\infty \int_{\mathbf{R}^d} p(v, y, z) \cdot \\
& \quad \langle a(x) \nabla_x \phi(t, x), \nabla_x \delta_n(t - (u+v), x - y) \rangle dv dy dt dx du dz \\
& = \int_{\mathbf{R}_+} \int_{\mathbf{R}^{d+1}} \int_0^\infty \int_{\mathbf{R}^d} P_v(\psi(u, \cdot))(y) \cdot \\
& \quad \langle a(x) \nabla_x \phi(t, x), \nabla_x \delta_n(t - (u+v), x - y) \rangle dv dy dt dx du \\
& = \int_{\mathbf{R}_+} \int_{\mathbf{R}^{d+1}} \int_0^\infty \int_{\mathbf{R}^d} -P_v(\psi(u, \cdot))(y) \cdot \\
& \quad \langle a(x) \nabla_x \phi(t, x), \nabla_y \delta_n(t - (u+v), x - y) \rangle dv dy dt dx du \\
& = \int_{\mathbf{R}_+} \int_{\mathbf{R}^{d+1}} \int_0^\infty \int_{\mathbf{R}^d} \langle a(x) \nabla_x \phi(t, x), \\
& \quad \int_{\mathbf{R}^d} \nabla_y P_v(\psi(u, \cdot))(y) \delta_n(t - (u+v), x - y) \rangle dv dy dt dx du \\
& = \int_{\mathbf{R}_+} \int_0^\infty \int_{\mathbf{R}^d} \langle \nabla_y P_v(\psi(u, \cdot))(y), \\
& \quad \int_{\mathbf{R}^{d+1}} a(x) \nabla_x \phi(t, x) \delta_n(t - (u+v), x - y) dt dx \rangle dy dv du
\end{aligned}$$

From (3.33) and (3.34) we get:

$$\begin{aligned}
(3.35) \quad & \langle I - I_n, \psi \rangle = \int_{\mathbf{R}_+} \int_0^\infty \int_{\mathbf{R}^d} \langle \nabla_y P_v(\psi(u, \cdot))(y), a(y) \nabla_y \phi(u+v, y) \rangle \\
& \quad - \int_{\mathbf{R}^{d+1}} a(x) \nabla_x \phi(t, x) \delta_n(t - (u+v), x - y) dt dx \rangle dy dv du
\end{aligned}$$

From here, using exactly the same arguments as in the proof of ii), we obtain:

$$(3.36) \quad |(I - I_n, \psi)| \leq O(n) \|\psi\|_{L^2}$$

This completes the proof of Theorem 3.1. □

§4 Limits of Hida distributions

So far we have made sense out of singular objects like:

$$(4.1) \quad u(x, \omega) = \int G(x, y) dB(y)$$

by smearing them out in the x -variable. Another possibility is to smear them out in the direction of the random variable. That is to say we consider the action of these object as functionals in the ω -variable. In this connection it is natural to begin work in the space $(\mathcal{S})^*$ of Hida distributions, see e.g. [HKPS], [HP], [K] and [Z]. This space can be introduced briefly in the following way:

We consider the Schwartz space $\mathcal{S}(\mathbf{R}^d)$ together with its dual $\mathcal{S}'(\mathbf{R}^d)$ equipped with the white noise measure μ which is uniquely characterized by the characteristic function:

$$(4.2) \quad \int_{\mathcal{S}} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2} \|\phi\|_{L^2(\mathbf{R}^d)}^2}; \quad \text{for all } \phi \in \mathcal{S}$$

The mapping $W_\phi := \langle \omega, \phi \rangle$ is a white noise, and by a standard extension argument (see e.g. [HLØUZ 1] for more details) this can be used to show that: $B_t(\omega) := \langle \omega, \mathcal{X}_{[0,t]} \rangle$ is a brownian motion. With the coupling above, we have:

$$(4.3) \quad W_\phi = \int \phi dB$$

By the Wiener-Ito chaos theorem, every element $X \in (L^2) = L^2(\mathcal{S}'(\mathbf{R}^d), \mu)$ admits a chaos decomposition:

$$(4.4) \quad X = \sum_{n=0}^{\infty} I_n(f^{(n)})$$

Here I_n denotes a multiple Ito integral, and the integrands $f^{(n)} \in \hat{L}^2(\mathbf{R}^{nd})$, i.e. the symmetric L^2 functions, see e.g. [HKPS]. The (L^2) norm of this expansion can be calculated by:

$$(4.5) \quad \|X\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{L^2(\mathbf{R}^{nd})}^2$$

Now consider the (densely defined) selfadjoint operator $A : L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$:

$$(4.6) \quad Af(x) = -f''(x) + (1 + x^2)f(x)$$

Let $\mathcal{P} \subset (L^2)$ denote the algebra of smooth linear functionals on the form $\langle \cdot, \phi \rangle$; $\phi \in \mathcal{S}(\mathbf{R}^d)$, and for every integer p let $(S)_p$ denote the completion of $\mathcal{P}$ with respect to the norm:

$$(4.7) \quad |X|_{2,p}^2 := \sum_{n=0}^{\infty} n! |(A^{\otimes n d})^p f^{(n)}|_{L^2(\mathbf{R}^{nd})}^2$$

The Hida test function space (S) is then the projective limit of the spaces $(S)_p$. For $\phi \in \mathcal{S}(\mathbf{R}^d)$ the Wick exponential $Exp[W_\phi]$ is defined by the expression:

$$(4.8) \quad Exp[W_\phi] = \sum_{n=0}^{\infty} I_n(\phi^{\otimes n})$$

It is easy to see that $Exp[W_\phi] \in (S)$. The space $(S)^*$ of Hida distributions is the dual space of (S) . The S -transform $S : (S)^* \rightarrow \mathcal{S}(\mathbf{R}^d)$ can then be introduced as follows:

First we define it on (L^2) by:

$$(4.9) \quad SX(\phi) = \int X(\omega) Exp[W_\phi] d\mu(\omega) \quad \phi \in \mathcal{S}(\mathbf{R}^d)$$

For $\lambda \in \mathbb{C}$ and $\phi \in \mathcal{S}(\mathbf{R}^d)$ we may extend the definition above to $X \in (S)^*$ by the dual pairing:

$$(4.10) \quad SX(\lambda\phi) = \langle X, Exp[\lambda W_\phi] \rangle$$

It is well known that the elements $X \in (S)^*$ are uniquely characterized by their S -transforms; see [PS]. S -transforms always extend to ray entire functionals $\Psi = SX$ satisfying an estimate of the form:

$$(4.11) \quad |\Psi(\lambda\phi)| \leq C e^{K|\lambda|^2|\phi|_{2,p}^2}$$

where C and K are constants, p a non-negative integer and $|\phi|_{2,p} := |(A^{\otimes d})^p \phi|_{L^2(\mathbf{R}^d)}$. Conversely every ray entire functional Ψ satisfying an estimate as above is the S -transform of some element $X \in (S)^*$. The S -transform can also be used to prove the convergence in $(S)^*$. In this case we have the following result, see [PS]:

Proposition 4.1

Let $X_n, X \in (S)^*$. Then $X_n \rightarrow X$ if and only if the following conditions are satisfied:

$$(4.12) \quad SX_n(\phi) \rightarrow SX(\phi)$$

$$(4.13) \quad |SX_n(\lambda \phi)| \leq C e^{K|\lambda|^2|\phi|_{2,p}^2}$$

where the constants C, K and p do not depend on n .

Limit behaviour of SPDEs

In the following we will consider equations on the form:

$$(4.14) \quad Lu_\phi = \text{Noise}_\phi$$

where L is some differential operator and the subscript ϕ indicates that the noise is smeared by a smooth function. We then want to examine the limit:

$$(4.15) \quad \lim_{\phi \rightarrow \delta} u_\phi$$

as an $(\mathcal{S})^*$ valued object. We will consider the equations with two different kinds of noise; the white noise W_ϕ and the positive noise $\text{Exp}[W_\phi]$. Obviously, it is also possible to study other kinds of noise by the same machinery.

The results we summarize in the table below:

Equations with solutions in $(\mathcal{S})^*$		
Equation \ f	$f = W_x$	$f = \text{Exp}[W_x]$
$Lu = f$	Thm 4.1: Yes	Thm 4.2: Yes
$\frac{\partial u}{\partial t} = Lu + f$	Thm 4.3: Yes	Thm 4.4: Yes
$\frac{\partial u}{\partial t} = Lu + f \cdot u$	Thm 4.5: No	Thm 4.6: No
$\frac{\partial u}{\partial t} = Lu + f \diamond u$	Thm 4.7: Yes	Thm 4.8: No

Remark

Theorem 4.7 is the first example we have come across where a SPDE involving a nontrivial product of generalized functionals has a well defined limit solution. The $\diamond$ in this theorem indicates that a Wick product is performed, see e.g. [GHLØUZ].

The proofs are all easy applications of the S -transform, and we now proceed to fill in the details.

Theorem 4.1

If L is a strictly elliptic operator with bounded measurable coefficients on a bounded domain $D \subset \mathbf{R}^d$, then the family of equations:

$$(4.16) \quad Lu_\phi = W_\phi \quad u_\phi|_{\partial D} = 0$$

have a limit in $(\mathcal{S})^*$ which is given by the $(\mathcal{S})^*$ valued integral:

$$(4.17) \quad V(x) = \int_D G(x, y) W_y dy$$

Proof

We obviously have:

$$(4.18) \quad u_\phi = \int_D G(x, y) W_{\phi_y} dy$$

Taking S -transforms on this, we get:

$$(4.19) \quad Su_\phi(\lambda\xi + \eta) = \int_D G(x, y) \langle \lambda\xi + \eta, \phi_y \rangle_{L^2(\mathbf{R}^d)} dy$$

Passing to the limit, then:

$$(4.20) \quad Su_\delta(\lambda\xi + \eta) = \int_D G(x, y) (\lambda\xi(y) + \eta(y)) dy$$

Since $\int_D |G(x, y)| dy < \infty$, this obviously extends to an entire function and satisfies the requirements in (4.11) (with $p = 0$). From this we get $u_\delta \in (\mathcal{S})^*$. Using proposition 4.1, it is then easy to see that $u_\phi \rightarrow u_\delta$ in $(\mathcal{S})^*$. □

Theorem 4.2

If L is a strictly elliptic operator with bounded measurable coefficients on a bounded domain $D \subset \mathbf{R}^d$, then the family of equations:

$$(4.21) \quad Lu_\phi = \text{Exp}[W_\phi] \quad u_\phi|_{\partial D} = 0$$

have a limit in $(\mathcal{S})^*$ which is given by the $(\mathcal{S})^*$ valued integral:

$$(4.22) \quad V(x) = \int_D G(x, y) \text{Exp}[W_y] dy$$

Proof

We repeat the argument in the previous theorem, and get:

$$(4.23) \quad Su_\delta(\lambda\xi + \eta) = \int_D G(x, y) e^{\lambda\xi(y) + \eta(y)} dy$$

This too extends to an entire function and satisfies the requirements in (4.11) (with $p = 0$), so again we get $u_\phi \rightarrow u_\delta \in (\mathcal{S})^*$. □

Theorem 4.3

If L is a strictly elliptic operator with bounded measurable coefficients on $\mathbf{R}^d$, then the family of equations:

$$(4.24) \quad \frac{\partial u_\phi}{\partial t} = Lu_\phi + W_\phi \quad u(0, x) = 0$$

have a limit in $(S)^*$ which is given by the $(S)^*$ valued integral:

$$(4.25) \quad V(t, x) = \int_{\mathbf{R}^d \times [0, t)} G_{t-s}(x, y) W_y dy ds$$

Proof

In this case we have the solution formula:

$$(4.26) \quad u_\phi(t, x) = \int_{\mathbf{R}^d \times [0, t)} G_{t-s}(x, y) W_{\phi_y} dy ds$$

Taking S -transforms again, we end up with:

$$(4.27) \quad Su_\phi(\lambda\xi + \eta) = \int_{\mathbf{R}^d \times [0, t)} G_{t-s}(x, y) \langle \lambda\xi + \eta, \phi_y \rangle_{L^2(\mathbf{R}^d)} dy ds$$

and the rest of the proof is similar to the previous. □

Theorem 4.4

If L is a strictly elliptic operator with bounded measurable coefficients on $\mathbf{R}^d$, then the family of equations:

$$(4.28) \quad \frac{\partial u_\phi}{\partial t} = Lu_\phi + \text{Exp}[W_\phi] \quad u(0, x) = 0$$

have a limit in $(S)^*$ which is given by the $(S)^*$ valued integral:

$$(4.29) \quad V(t, x) = \int_{\mathbf{R}^d \times [0, t)} G_{t-s}(x, y) \text{Exp}[W_y] dy ds$$

Proof

By essentially the same arguments as in Theorems 4.3 and 4.2. □

Theorem 4.5

If L is a strictly elliptic operator with bounded measurable coefficients on $\mathbf{R}^d, d > 1$, then the family of equations:

$$(4.30) \quad \frac{\partial u_\phi}{\partial t} = Lu_\phi + W_\phi \cdot u_\phi \quad u(0, x) = f(x)$$

has no limit in $(S)^*$.

Proof

The solutions can in this case be given by the formulas:

$$(4.31) \quad u_\phi(t, x) = \mathbb{E}^x[f(X_t)e^{\int_0^t W_{\phi_{X_s}} ds}]$$

Taking S -transforms of this, we end up with:

$$(4.32) \quad Su_\phi(\xi) = \mathbb{E}^x[f(X_t)e^{\langle \xi, \int_0^t \phi_{X_s} ds \rangle_{L^2(\mathbb{R}^d)} e^{\frac{1}{2} \|\int_0^t \phi_{X_s} ds\|_{L^2(\mathbb{R}^d)}^2}]$$

If we let $\phi \rightarrow \delta$, then $e^{\langle \xi, \int_0^t \phi_{X_s} ds \rangle_{L^2(\mathbb{R}^d)}} \rightarrow e^{\int_0^t \xi(X_s) ds}$ and $\int_0^t \phi_{X_s} ds \rightarrow L(t, y)$ where $L(t, y)$ is the local time of the process X_t . This is well defined if the space dimension $d = 1$. If $d > 1$, however, it is well known that $L(t, y) = +\infty$, so the S -transform cannot satisfy the estimate in (4.11). □

Theorem 4.6

If L is a strictly elliptic operator with bounded measurable coefficients on $\mathbb{R}^d$, $d > 1$, then the family of equations:

$$(4.33) \quad \frac{\partial u_\phi}{\partial t} = Lu_\phi + \text{Exp}[W_\phi] \cdot u_\phi \quad u(0, x) = f(x)$$

has no limit in $(\mathcal{S})^*$.

Proof

The presence of the exponential term makes the S -transform grow even faster than in the previous theorem. □

Theorem 4.7

If L is a strictly elliptic operator with bounded measurable coefficients on $\mathbb{R}^d$, then the family of equations:

$$(4.34) \quad \frac{\partial u_\phi}{\partial t} = Lu_\phi + W_\phi \diamond u_\phi \quad u(0, x) = f(x)$$

has a limit in $(\mathcal{S})^*$ which is given by the $(\mathcal{S})^*$ valued integral:

$$(4.35) \quad V(t, x) = \mathbb{E}^x[f(X_t) \text{Exp}[\int_0^t W_{X_s} ds]]$$

Proof

The solutions can be expressed by the formulas (see [HLØUZ 2]):

$$(4.36) \quad u_\phi(t, x) = \mathbb{E}^x \left[f(X_t) \text{Exp} \left[\int_0^t W_{\phi_{X_s}} ds \right] \right]$$

Taking S -transforms again, we end up with:

$$(4.37) \quad Su_\phi(\xi) = \mathbb{E}^x[f(X_t)e^{<\xi, \int_0^t \phi_{X_s} ds>_{L^2(\mathbb{R}^d)}}] \rightarrow \mathbb{E}^x[f(X_t)e^{\int_0^t \xi(X_s) ds}]$$

From this we easily see that (4.11) and (4.13) can be satisfied. □

Theorem 4.8

If L is a strictly elliptic operator with bounded measurable coefficients on $\mathbb{R}^d$, then the family of equations:

$$(4.38) \quad \frac{\partial u_\phi}{\partial t} = Lu_\phi + \text{Exp}[W_\phi] \diamond u_\phi \quad u(0, x) = f(x)$$

has no limit in $(\mathcal{S})^*$.

Proof

See [HLØUZ 2], where it is proved that already $u_\phi \notin (\mathcal{S})^*$. □

§5 Some remarks on other stabilized properties

In [HLØUZ 1] and [HLØUZ 2], we presented L^1 solutions of the equations:

$$(5.1) \quad \Delta u_\phi^{(1)} = -\epsilon \text{Exp}[W_\phi] \diamond u_\phi^{(1)} - f \quad u_\phi^{(1)}|_{\partial D} = 0$$

and

$$(5.2) \quad \frac{\partial u_\phi^{(2)}}{\partial t} = \Delta u_\phi^{(2)} + \text{Exp}[W_\phi] \diamond u_\phi^{(2)} \quad u_\phi^{(2)}(0, x) = f(x)$$

As it turned out, these equations did not have any solutions in $(\mathcal{S})^*$. With this in mind, there is certainly no hope of finding any limiting object in $(\mathcal{S})^*$. There are, nevertheless, some stable properties attached to these solutions. Taking expected values of the solutions, we find that: $\overline{u} = \mathbb{E}[u_\phi^{(\cdot)}]$ is (regardless of the choice of ϕ) solutions of the equations:

$$(5.3) \quad \Delta \overline{u}_1 = -\epsilon \overline{u}_1 - f \quad \overline{u}_1|_{\partial D} = 0$$

and

$$(5.4) \quad \frac{\partial \overline{u}_2}{\partial t} = \Delta \overline{u}_2 + \overline{u}_2 \quad \overline{u}_2(0, x) = f(x)$$

Since these quantities does not depend on ϕ , certainly the limit $\lim_{\phi \rightarrow \delta} E[u_\phi]$ exists! We also think that it is possible to find other stable properties. One possibility is to consider the space of random variables with a chaos expansion of finite order. The dual space of this would consist of generalized variables which have a stable action in each chaos component. Note that we can look upon the expected value as the action on the chaos of order zero. There is some hope that we can extend this behaviour to other chaos components as well.

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