ISBN 82-553-0826-1
Pure Mathematics

No 6
February 1993

**The Rohlin property for quasi-free
automorphisms of the Fermion algebra**

by

O. Bratteli, D.E. Evans and A. Kishimoto

PREPRINT SERIES – Matematisk institutt, Universitetet i Oslo

**THE ROHLIN PROPERTY FOR QUASI-FREE AUTOMORPHISMS OF
THE FERMION ALGEBRA**

Ola Bratteli
Matematisk Institutt
Universitetet i Oslo
PB 1053-Blindern
N-0316 Oslo 3
Norway

David E. Evans
Department of Mathematics
University College of Swansea
Singleton Park
Swansea SA2 8PP
Wales
United Kingdom

Akitaka Kishimoto
Department of Mathematics
Hokkaido University
Sapporo
060 Japan

1. Introduction and main results

Let A be the CAR (or Fermion) algebra over a Hilbert space $\mathcal{H}$, i.e. A is the universal C^* -algebra generated by operators $a(f)$, $f \in \mathcal{H}$, satisfying

$$\begin{aligned} f \rightarrow a(f) & \text{ is antilinear} \\ a(f) a(g) + a(g) a(f) &= 0 \\ a(f) a^*(g) + a^*(g) a(f) &= (f, g) 1, \end{aligned}$$

[BR2]. It is well known that A is isomorphic to M_{2^∞} , the UHF algebra of Glimm type 2^∞ . If U is a unitary operator on $\mathcal{H}$, then U defines an automorphism α_U on A by the requirement

$$\alpha_U(a(f)) = a(U(f)).$$

The automorphism α_U is called the quasi-free automorphism (or Bogoliubov transformation) defined by U .

If M_m is the C^* -algebra of complex $m \times m$ matrices, the cyclic shift α_m on M_m is defined up to conjugacy as the automorphism determined by

$$\sigma_m(e_{ij}) = e_{i+1j+1}$$

for some complete set of matrix units for M_m .

If A is a unital C^* -algebra and α is an automorphism, we will use the terminology that (A, α) has the Rohlin property if for any $m_0 \in \mathbb{N}$, there exists $m \geq m_0$ such that for any $\varepsilon > 0$ and any finite subset $\{x_1, \dots, x_n\} \subseteq A$, there is a subalgebra $B \subseteq A$ such that $1 \in B$ and an automorphism σ of B such that $(B, \sigma) \cong (M_m, \sigma_m)$ and

$$\|\alpha(x) - \sigma(x)\| \leq \varepsilon \|x\|$$

for all $x \in B$, and

$$||[x_k, x]|| \leq \varepsilon ||x||$$

for all $x \in B$, $k = 1, \dots, n$.

This definition is seemingly slightly stronger than the definition in [HO], where B is replaced by the diagonal $\sum_i \mathbb{C}e_{ii}$ in the above definition. We will, however, prove in Theorem 1.6 that these definitions are equivalent even for general automorphisms.

The algebra $A = M_{2^\infty}$ has a unique trace state τ , which is then invariant under all automorphisms σ of A . We say that σ is weakly inner, respectively outer, in the trace representation if the extension of σ to the weak closure of A in this representation is inner, respectively outer.

Let $|||_2$ be the Hilbert-Schmidt norm on $B(\mathcal{H})$. Our first main result is then the following theorem. For completeness we have included some implications which are already known.

Theorem 1.1 Let $\alpha = \alpha_U$ be a quasi-free automorphism of the CAR algebra $A = M_{2^\infty}$. The following conditions are equivalent:

- (1) (A, α) has the Rohlin property.
- (2) α^n is weakly outer in the trace representation for all $n \in \mathbb{Z} \setminus \{0\}$.
- (3) $||U^n - 1||_2 = +\infty$ for all $n \in \mathbb{Z} \setminus \{0\}$.
- (4) $A \rtimes_\alpha \mathbb{Z}$ has a unique trace state.
- (5) $A \rtimes_\alpha \mathbb{Z}$ has real rank zero.
- (6) $A \rtimes_\alpha \mathbb{Z}$ is the inductive limit of an increasing sequence B_n of subalgebras containing the unit of $A \rtimes_\alpha \mathbb{Z}$, such that

$$B_n \cong M_{4^n} \otimes C(\mathbb{T})$$

and the embedding $B_n \hookrightarrow B_{n+1}$ is 2 copies of the twice around embedding, [Bla2].

Furthermore, the implications $(1) \Rightarrow (2) \Leftrightarrow (4)$, $(1) \Rightarrow (6) \Rightarrow (5)$ and $(6) \Rightarrow (4)$ are true for general (not necessarily quasi-free) automorphisms of A .

Theorem 1.1 says among other things that there is only one crossed product $A \rtimes_{\alpha} \mathbb{Z}$ up to isomorphism when $\|U^n - 1\|_2 = \infty$ for $n = 1, 2, \dots$. We can in fact classify all the crossed products $A \rtimes_{\alpha} \mathbb{Z}$ when $\alpha = \alpha_U$ is quasifree. Recall that α_U is outer on the C^* -algebra if and only if $\|U - 1\|_1 = \infty$, where $\|\cdot\|_1$ is the trace class norm, [Lan].

Theorem 1.2 Let $\alpha = \alpha_U$ be a quasifree automorphism of the CAR algebra $A = M_{2^{\infty}}$. Assume that

$$\|U^n - 1\|_1 = +\infty$$

for all $n \in \mathbb{N}$, but

$$\|U^n - 1\|_2 < +\infty$$

for some $n \in \mathbb{N}$. Then $A \rtimes_{\alpha} \mathbb{Z}$ is a simple unital inductive limit of subalgebras of the form

$$B_n \cong M_{2^n} \otimes C(\mathbb{T})$$

where the embedding of B_n into B_{n+1} is given by the sum of the identity embedding and the same translated by a rotation (depending on n) of the circle. The K -theory of $A \rtimes_{\alpha} \mathbb{Z}$ is given by $K_0(A) = K_1(A) = \mathbb{Z}[\frac{1}{2}]$, the positive cone in $K_0(A)$ consists of the nonnegative dyadic rationals and the order unit is 1. Furthermore, the trace state space T of $A \rtimes_{\alpha} \mathbb{Z}$ is isomorphic to the space of probability measures on $\mathbb{T}$, and the canonical pairing $T \times K_0(A) \rightarrow \mathbb{R}$ is given by $\tau \times g \rightarrow g \in \mathbb{R}$, where $g \in K_0(A)$ is identified with its value as a dyadic rational.

Remark 1.3 George A. Elliott has informed us that his classification of simple inductive limits of interval algebras given in [Ell] extends to inductive limits of

circle algebras (of course K_1 is now part of the invariant). Hence all the crossed products described in Theorem 1.2 are $*$ -isomorphic.

Remark 1.4 Alex Kumjian has informed us that he independently has proved Theorem 1.2.

Remark 1.5 When $\|U^n - 1\|_1 = +\infty$ for all $n \in \mathbb{N}$, Theorems 1.1, 1.2 and Remark 1.3 imply that $M_{2^\infty} \rtimes_\alpha \mathbb{Z}$ belongs to one of two isomorphism classes according to whether $\|U^m - 1\|_2 < +\infty$ for some $m \in \mathbb{N}$ or not. If $\|U^n - 1\|_1 < +\infty$ for some $n \in \mathbb{N}$, let m be the smallest such n . We may then perturb U by a unitary having a trace class difference with 1 in such a way that the perturbed U has the property $U^m = 1$. This amounts to perturbing α_U by an inner perturbation, so it does not change the crossed product $M_{2^\infty} \rtimes_\alpha \mathbb{Z}$, [Tak]. Since $\alpha^m = 1$, a little computation shows that

$$M_{2^\infty} \rtimes_\alpha \mathbb{Z} \cong \{f \in C[0, 1] \otimes (M_{2^\infty} \rtimes_\alpha \mathbb{Z}_m) \mid f(0) = \hat{\alpha}_1(f(1))\}$$

where $\hat{\alpha}$ is the dual action of the action α of $\mathbb{Z}_m$ on M_{2^∞} , see [Bla1, Proposition 10.3.3]. Since $U^m = 1$, but $\|U^n - 1\|_1 = +\infty$ for $n = 1, 2, \dots, m-1$, it follows that the spectrum of U is contained in the m m^{th} roots of 1 , and that the greatest common divisor of the $n = 1, 2, \dots, m-1$ such that the eigenvalue $\exp(2\pi i n/m)$ occur with infinite multiplicity must be 1 . Hence α_U is a product type action on M_{2^∞} , and $M_{2^\infty} \rtimes_\alpha \mathbb{Z}_m$ is an AF algebra which is an inductive limit of subalgebras

$$B_n = \bigoplus_{k=1}^m M_{2^k}$$

with embedding matrix for $B_n \hookrightarrow B_{n+1}$ given by

$$\begin{pmatrix} 1 & 0 & & & 0 \\ 0 & 1 & & & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}^{k_n}$$

where $k_n = 0, 1, \dots, m-1$, and each k_n is determined by the requirement that $\exp(2\pi i k_n/m)$ is the n^{th} eigenvalue of U .

Thus, when $k_n = 0$ the embedding diagram is given by

$$|| \quad || \quad \dots \quad ||$$

and when $k_n = 1$

The dual action $\hat{\alpha}_1$ is given by a cyclic shift on the m vertices of each level of the diagram. Thus the classification of the crossed products in this case boils down to classifying the pairs $(M_{2^\infty} \rtimes_{\alpha} \mathbb{Z}_m, \hat{\alpha}_1)$ described above up to isomorphism of $M_{2^\infty} \rtimes_{\alpha} \mathbb{Z}_m$ and homotopy of $\hat{\alpha}_1$ when the crossed products by $\mathbb{Z}_m$ are isomorphic. This task can presumably be accomplished using [HR1], [HR2]. (When $m = 2$, there is only one isomorphism class of crossed products, when m is a prime > 2 there are two, but in addition $\hat{\alpha}_1$ has to be classified up to homotopy).

We now turn to the alternative characterizations of the Rohlin property alluded to earlier. This extends results from [Con] and [HO].

Theorem 1.6 Let σ be an automorphism of a UHF algebra A . Let

$$P = \{p \mid p \in \mathbb{N} \text{ is prime and } A \cong A \otimes M_{p^\infty}\}$$

and assume that $P \neq \emptyset$. The following three conditions are equivalent:

(1.6.1) There is a strictly increasing sequence $(n_k)_k$ of positive integers such that for each $m = n_k$ there is a sequence $(e_i^{(j)} \mid i = 0, 1, \dots, m-1)$ of families of projections with

$$\sum_{i=0}^{m-1} e_i^{(j)} = 1$$

$$\lim_{j \rightarrow \infty} \|\sigma(e_i^{(j)}) - e_{i+1}^{(j)}\| = 0$$

for $i = 0, 1, \dots, m-1$, where $e_m^{(j)} = e_0^{(j)}$.

(1.6.2) There is a strictly increasing sequence $(n_k)_k$ of positive integers such that for each $m = n_k$ there is a sequence $(e_{i_1 i_2}^{(j)} \mid i_1, i_2 = 0, 1, \dots, m-1)$ of matrix units with

$$\sum_{i=0}^{m-1} e_{ii}^{(j)} = 1$$

$$\lim_{j \rightarrow \infty} \|\sigma(e_{i_1 i_2}^{(j)}) - e_{i_1+1, i_2+1}^{(j)}\| = 0$$

for $i_1, i_2 = 0, 1, \dots, m-1$, where $e_{mi}^{(j)} = e_{0i}^{(j)}$ and $e_{im}^{(j)} = e_{i0}^{(j)}$.

(1.6.3) For each central sequence $(v_k)_k$ of unitaries there is a sequence $(w_k)_k$ of unitaries such that

$$\lim_{k \rightarrow \infty} \|v_k - w_k \sigma(w_k^*)\| = 0.$$

Furthermore, the following three conditions are equivalent.

(1.6.4) Condition (1.6.1) with the $e_i^{(j)}$ satisfying the additional property

$$\lim_{j \rightarrow \infty} \|[x, e_i^{(j)}]\| = 0$$

for all $x \in A$, $i = 0, 1, \dots, m-1$.

(1.6.5) Condition (1.6.2) with the $e_{i_1 i_2}^{(j)}$ satisfying the additional property

$$\lim_{j \rightarrow \infty} \|[x, e_{i_1 i_2}^{(j)}]\| = 0$$

for all $x \in A$, $i_1, i_2 = 0, 1, \dots, m-1$.

(1.6.6) For each sequence $(v_k)_k$ of unitaries there is a sequence $(w_k)_k$ of unitaries such that

$$\lim_{k \rightarrow \infty} \|v_k - w_k \sigma(w_k^*)\| = 0.$$

If $(v_k)_k$ is central, $(w_k)_k$ can be taken to be central.

Furthermore, if any of the conditions (1.6.1) - (1.6.6) are satisfied we may take $\{p^\ell \mid p \in P, \ell = 1, 2, \dots\}$ as the set $\{n_k \mid k = 1, 2, \dots\}$.

It will be clear from the proof that in conditions (1.6.2) and (1.6.5) we may replace the requirement that σ asymptotically approximates the cyclic shift on M_m by the requirement that σ asymptotically approximates any given automorphism on M_m .

In the following, when we refer to the Rohlin condition without any more qualifications, we mean the condition in the form (1.6.4) - (1.6.6).

As stated in [Voi], and expressed in more detail in [BKRS], if an automorphism σ of a UHF algebra has the Rohlin property in the form 1.6.4, then for any $\mathcal{D} \in \mathcal{F}(A)$ (= the finite-dimensional *-subalgebras of A with the same unit as A) and any $\varepsilon > 0$ there exists a $\mathcal{E} \in \mathcal{F}(A)$ such that $\mathcal{D} \subseteq^\varepsilon \mathcal{E}$ and $\text{dist}(\mathcal{E}, \sigma(\mathcal{E})) < \varepsilon$. In this construction, the centre of $\mathcal{E}$ has dimension at least the order of the Rohlin tower employed. But in fact we can find an $\mathcal{E}$ which is a full matrix algebra. If A is a unital C*-algebra, let $\mathcal{F}(A)$ denote the class of finite dimensional *-subalgebras of A containing the unit of A .

Theorem 1.7 Let A be a UHF algebra containing a tensor factor M_{p^∞} for some prime p , and let σ be an automorphism of A satisfying any of the equivalent versions (1.6.4) - (1.6.6) of the Rohlin condition. Then for any finite dimensional subalgebra $\mathcal{D} \in \mathcal{F}(A)$ and $\varepsilon > 0$, there exists a $\mathcal{B} \in \mathcal{F}(A)$ isomorphic to a full

matrix algebra and a unitary $U \in A$ such that $D \subseteq B$, $\text{Ad}U \circ \sigma(B) = B$, and $\|U - 1\| < \varepsilon$. Furthermore, there exists an increasing sequence $B_n \in \mathcal{F}(A)$ all isomorphic to full matrix algebras and a unitary $V \in A$ such that $\text{Ad}V \circ \sigma(B_n) = B_n$ for all n , $\bigcup_n B_n$ is dense in A , and $\|V - 1\| < \varepsilon$.

The plan of the rest of the paper is as follows: In Section 2 we prove Theorem 1.6, and in Section 3 we prove Theorem 1.7. In Section 4 we collect some auxilliary facts on equi-distribution of points on the circle which is needed in the proof of Theorem 1.1, which then follows in Section 5. Theorem 1.2 is proved in Section 6.

2. The Rohlin property

In this section we prove Theorem 1.6. So adopt the hypotheses of that theorem. We will prove

$$(1.6.1) \Rightarrow (1.6.3) \Rightarrow (1.6.2) \Rightarrow (1.6.1)$$

and

$$(1.6.4) \Rightarrow (1.6.6) \Rightarrow (1.6.5) \Rightarrow (1.6.4).$$

(1.6.1) $\Rightarrow$ (1.6.3): We follow the proof of Theorem 1 of [HO], which again is based on the proof of Theorem 2.1.3 in [Con]: Let $(v_k)_k$ be a central sequence of unitaries, let m be an integer for which there exists a Rohlin partition $e_0, e_1, \dots, e_{m-1}$ with

$$\sum_{i=0}^{m-1} e_i = 1$$

$$\sigma(e_i) \approx e_{i+1}$$

and go so far out in the sequence (v_k) that $v = v_k$ approximately commute with $e_0, e_1, \dots, e_{m-1}$. For this fixed v put

$$v_n = v \sigma(v) \sigma^2(v) \dots \sigma^{n-1}(v)$$

for $n = 1, \dots, m$. (Don't confuse v_n 's with v_k 's). Since A is a UHF algebra we may by choosing the original $v = v_k$ even further out obtain that all the v_n , $n = 1, \dots, m$ approximately commute with $e_0, e_1, \dots, e_{m-1}$, and, furthermore, we may find a string of unitaries

$$U_m = 1, \sigma(U_{m-1}), \sigma^2(U_{m-2}), \dots, \sigma^{m-1}(U_1) = \sigma^m v_m$$

such that the U_n 's still approximately commute with $e_0, \dots, e_{m-1}$, and such that the distance between two consecutive unitaries in the above string is less than $\frac{\pi}{m-2}$:

$$\|U_{n+1} - \sigma(U_n)\| = \|\sigma^{m-n-1}(U_{n+1}) - \sigma^{m-n}(U_n)\| < \frac{\pi}{m-2}.$$

Now put

$$w = \sum_{n=1}^m v_n U_n e_n.$$

Since e_n approximately commute with U_n, v_n , w is approximately a unitary.

Furthermore, using this and $\sigma(e_n) \approx e_{n+1}$ and $v_{n+1} = v \sigma(v_n)$ one computes

$$w^* v \sigma(w) \simeq e_1 + \sum_{n=1}^{m-1} U_{n+1}^* \sigma(U_n) e_{n+1}$$

Now, using $\|U_{n+1}^* \sigma(U_n) - 1\| < \frac{\pi}{m-2}$ one deduces

$$\|w^* v \sigma(w) - 1\| < \frac{\pi}{m-2} + \text{small contribution}$$

and replacing w by an approximating unitary we deduce (1.6.3).

(1.6.3) $\Rightarrow$ (1.6.2): Let $m = p^\ell$ with $p \in P$, $\ell = 1, 2, \dots$. Since A is UHF, there exists a central sequence

$$(f_{i_1 i_2}^{(j)} \mid i_1, i_2 = 0, 1, \dots, m)_j$$

of matrix units with $\sum_i f_{ii}^{(j)} = 1$.

If $\lambda_i \in \mathbb{T}$, $i = 0, 1, \dots, m-1$ is a finite sequence of phases, then

$$\sigma(f_{i_1 i_2}^{(j)}) \rightarrow \lambda_{i_1} \overline{\lambda_{i_2}} f_{i_1 i_2}^{(j)}, i_1, i_2 = 0, 1, \dots, m-1,$$

is a *-automorphism of full matrix algebras, and since $(\sigma(f_{i_1 i_2}^{(j)}))_j$ also is central, there exists a central sequence $(v_j)_j$ of unitaries such that

$$\text{Ad } v_j \circ \sigma(f_{i_1 i_2}^{(j)}) = \lambda_{i_1} \overline{\lambda_{i_2}} f_{i_1 i_2}^{(j)}.$$

By (1.6.3) there is a sequence w_j of unitaries such that

$$\lim_{j \rightarrow \infty} \|v_j - w_j^* \sigma(w_j)\| = 0.$$

Then

$$\begin{aligned} & \sigma \circ \text{Ad } w_j(f_{i_1 i_2}^{(j)}) \\ &= \text{Ad}(\sigma(w_j)) \sigma(f_{i_1 i_2}^{(j)}) \\ &\approx \text{Ad } w_j \circ \text{Ad } v_j \circ \sigma(f_{i_1 i_2}^{(j)}) \\ &= \lambda_{i_1} \overline{\lambda_{i_2}} \text{Ad } w_j(f_{i_1 i_2}^{(j)}) \end{aligned}$$

Thus, if

$$g_{i_1 i_2}^{(j)} = \text{Ad } w_j(f_{i_1 i_2}^{(j)})$$

then

$$\lim_{j \rightarrow \infty} \|\sigma(g_{i_1 i_2}^{(j)}) - \lambda_{i_1} \overline{\lambda_{i_2}} g_{i_1 i_2}^{(j)}\| = 0.$$

Specialize now to the case

$$\lambda_k = \exp(2\pi i k/m) = \rho^k, k = 0, 1, \dots, m-1$$

and define

$$e_{i_1 i_2}^{(j)} = \frac{1}{m} \sum_{k_1, k_2=0}^{m-1} \rho^{i_1 k_1 - i_2 k_2} g_{k_1 k_2}^{(j)}.$$

Since $g_{k_1 k_2}^{(j)}$ are matrix units satisfying

$$\lim_{j \rightarrow \infty} \|\sigma(g_{k_1 k_2}^{(j)}) - \rho^{k_1 - k_2} g_{k_1 k_2}^{(j)}\| = 0$$

one now verifies that $e_{i_1 i_2}^{(j)}$ are matrix units satisfying

$$\lim_{j \rightarrow \infty} \|\sigma(e_{i_1 i_2}^{(j)}) - e_{i_1+1, i_2+1}^{(j)}\| = 0.$$

Thus (1.6.2) holds.

(1.6.2) $\Rightarrow$ (1.6.1) is trivial.

(1.6.4) $\Rightarrow$ (1.6.6) is again proved by following [H0] as in the proof of (1.6.1) $\Rightarrow$ (1.6.3), but now one exploits that $(e_i^{(j)})_j$ is central to make e_n to approximately commute with the constructed v_n and U_n . Furthermore, if v_j also is central it follows from the construction of w_j (by choosing the U_n 's carefully) that (w_j) can be made central.

(1.6.6) $\Rightarrow$ (1.6.5): this is proved as (1.6.3) $\Rightarrow$ (1.6.2), using only the last statement of (1.6.6) (so one could replace (1.6.6) by this statement in the theorem). Since the w_j in that proof can be taken to form a central sequence, it follows that $g_{i_1 i_2}^{(j)} = \text{Ad} w_j (f_{i_1 i_2}^{(j)})$ becomes central, as does

$$e_{i_1 i_2}^{(j)} = \frac{1}{m} \sum_{k_1, k_2=0}^{m-1} \rho^{i_1 k_1 - i_2 k_2} g_{k_1 k_2}^{(j)}.$$

Hence (1.6.5) follows.

(1.6.5) $\Rightarrow$ (1.6.4) is trivial.

3. Approximately invariant matrix algebras.

In this section we prove Theorem 1.7. Adopt the hypotheses of that theorem.

We follow the arguments of Voiculescu [Voi]. Choose m sufficiently large with $m = p^\ell$, $p \in P$, $\ell = 1, 2, \dots$. There exists by [Voi, Lemma 3.1] a family $\{B_i \mid i = 0, 1, \dots, m-1\}$ in $\mathcal{F}(A)$ such that

$$d(\sigma(B_i), B_{i+1}) < \frac{5\pi}{m}$$

$$B_i \supset D$$

for $i = 0, \dots, m-1$, where $B_m = B_0$. Furthermore, it follows from the proof in [Voi] that the B_i 's can be taken to be full matrix algebras. We may also assume that there is a full matrix subalgebra $N \in \mathcal{F}(A)$ such that $N \supset B_i$, $i = 0, \dots, m-1$. By the Rohlin property there exists a type I_m subfactor C of $A \cap N'$ with matrix units e_{ij} such that $\sigma(e_{ij}) \approx e_{i+1,j+1}$. More precisely we assume that there is a unitary U such that $\|U - 1\| = \delta$ where δ is very small, and $\text{Ad}U \circ \sigma(e_{ij}) = e_{i+1,j+1}$, $i, j = 0, \dots, m-1$. Choosing δ sufficiently small and replacing σ by $\text{Ad}U \circ \sigma$, we assume that

$$D \subseteq B_i \subseteq N, \quad i = 0, \dots, m-1$$

$$C \subseteq A \cap N'$$

$$d(\sigma(B_i), B_{i+1}) < \frac{5\pi}{m}$$

$$\sigma(e_{ij}) = e_{i+1,j+1}.$$

Suppose that $\frac{5\pi}{m} < 10^{-4}$. By [Chr] there is a unitary $v_i \in C'$ such that

$$\text{Adv}_i \circ \sigma(B_i) = B_{i+1}$$

$$\|v_i - 1\| < 64 \sqrt{\frac{5\pi}{m}}.$$

Let

$$v = \sum_{i=0}^{m-1} v_i e_{i+1}$$

$$M = \sum_{i=0}^{m-1} B_i e_i$$

where $e_i = e_{ii}$. As the v_i 's and e_j 's commute, v is a unitary element such that $\|v - 1\| < 64 \sqrt{\frac{5\pi}{m}}$ and $\text{Adv} \circ \sigma(B_i e_i) = \text{Adv}(\sigma(B_i) e_{i+1}) = B_{i+1} e_{i+1}$. Hence $\text{Adv} \circ \sigma(M) = M$. Since the B_i 's are factors, $M \cap M'$ is spanned by e_i , $i = 0, \dots, m-1$ and as $(\text{Adv} \circ \sigma)^m(e_i) = e_i$, $(\text{Adv} \circ \sigma)^m|_M$ is inner. Let $\sigma_1 = \text{Adv} \circ \sigma$ and let U be a unitary of M such that $\sigma_1^m|_M = \text{Ad} U$. Since σ_1 maps M onto M we have $\text{Ad} U|_M = \text{Ad} \sigma_1(U)|_M$, and it follows that $\sigma_1(U) = ZU$ where Z is a unitary in $M \cap M'$. Since $M \cap M' \simeq \mathbb{C}^m$, and $\sigma_1(e_i) = e_{i+1}$, $\sigma_1|_{M \cap M'}$ is the cyclic shift, and since $\sigma_1^m(U) = U$ we have $Z \sigma_1(Z) \cdots \sigma_1^{m-1}(Z) = 1$. It follows that there is a unitary $\zeta \in M \cap M'$ such that $\sigma_1(\zeta) = Z\zeta$. Thus by taking $\bar{\zeta}U$ instead of U we may assume that $\sigma_1(U) = U$. Then by the spectral calculus there is a unitary u of M such that

$$u^m = U$$

$$\|u - 1\| \leq \frac{\pi}{m}$$

$$\sigma_1(u) = u.$$

Then $(\text{Ad} u^* \circ \sigma_1)^m = \text{Ad} u^{*m} \circ \sigma_1^m = \text{Ad} U^* \circ \sigma_1^m$ is the identity on M . Let $w = u^*v$.

Then, as u, v both commute with $e_0, \dots, e_{m-1}$, $w \in \{e_0, \dots, e_{m-1}\}'$ and

$$\text{Ad} w \circ \sigma(M) = M$$

$$(\text{Ad } w \circ \sigma)^m | M = \text{id}$$

$$||w - 1|| < \frac{\pi}{m} + 64 \sqrt{\frac{5\pi}{m}}.$$

Let $\sigma_2 = \text{Ad } w \circ \sigma$. Then $\sigma_2(M) = M$ and $\sigma_2(e_i) = e_{i+1}$. As $B_0 e_0$ is a full matrix algebra there exists a partial isometry $b \in A$ such that

$$\text{Ad } b | B_0 e_0 = \sigma_2 | B_0 e_0$$

(in particular $b^*b = e_0$, $bb^* = e_1$). Note that $\sigma_2^k(b)$ is a partial isometry between e_k and e_{k+1} and using $B_k e_k = \sigma_2^k(B_0 e_0)$ one deduces $\text{Ad } \sigma_2^k(b) | B_k e_k = \sigma_2 | B_k e_k$.

Let

$$\xi = \sigma_2^{m-1}(b) \sigma_2^{m-2}(b) \dots \sigma_2(b) b.$$

Then ξ is a unitary in $e_0(A \cap B_0')e_0$ since for $x \in B_0 e_0$

$$\xi x \xi^* = \prod_{k=0}^{m-1} \text{Ad } \sigma_2^k(b)(x) = \sigma_2^m(x) = x.$$

Since σ_2^m leaves $e_0(A \cap B_0')e_0$ invariant, and $\sigma_2^m | e_0(A \cap B_0')e_0$ has the Rohlin property in the sense of (1.6.4) - (1.6.6) by the following Lemmas 3.1 - 3.3, it follows from Theorem 1.6, (1.6.6), applied to the constant sequence $v_k = \xi$ that there is a unitary $\eta \in e_0(A \cap B_0')e_0$ such that

$$||\xi - \sigma_2^m(\eta)\eta^*|| = \varepsilon_1,$$

where ε_1 is very small. If

$$c = \sigma_2(\eta^*)b\eta,$$

then

$$\sigma_2^{m-1}(c) \sigma_2^{m-2}(c) \dots c = \sigma_2^m(\eta^*) \xi \eta$$

which is close to e_0 up to ε_1 and

$$\begin{aligned} \text{Ad } c | B_0 e_0 &= \text{Ad } \sigma_2(\eta^*) \circ \text{Ad } b \circ \text{Ad } \eta | B_0 e_0 \\ &= \text{Ad } \sigma_2(\eta^*) \circ \sigma_2 | B_0 e_0 \\ &= \sigma_2 \circ \text{Ad } \eta^* | B_0 e_0 \\ &= \sigma_2 | B_0 e_0 \end{aligned}$$

Hence the C^* -algebra generated by M and the partial isometries $\sigma_2^k(c)$ between e_k and e_{k+1} , $k = 0, 1, \dots, m-1$, is a full matrix algebra. Since $\|\sigma_2^m(c) \sigma_2^{m-1}(c) \dots c - c\| = \varepsilon_1$, B is nearly σ_2 -invariant up to $2m\varepsilon_1$. This is because any $x \in B$ can be represented as a linear combination of approximate matrix units

$$x = \sum_{i,j=0}^{m-1} \sigma_2^{i-1}(c) \dots c x_{ij} c^* \dots \sigma_2^{j-1}(c^*)$$

with $x_{ij} \in B_0 e_0$, and then

$$\begin{aligned} \|\sigma_2(x) - \sum_{i,j=0}^{m-2} \sigma_2^i(c) \dots c x_{ij} c^* \dots \sigma_2^j(c^*) - \sum_{i=0}^{m-2} x_{m-1,i} c^* \dots \sigma_2^i(c^*) \\ - \sum_{j=0}^{m-2} \sigma_2^j(c) \dots c x_{j,m-1} - x_{m-1,m-1}\| \\ \leq 2m\varepsilon_1 \|x\|. \end{aligned}$$

Since $\|\sigma_2 - \sigma\| \leq \frac{\pi}{m} + 64 \sqrt{\frac{5\pi}{m}}$, B is nearly σ -invariant.

The last statement of Theorem 1.7 can now for example be established as follows. Let (x_n) be a dense sequence in A . By induction, we will construct an increasing sequence $B_n \in \mathcal{F}(A)$ of full matrix algebras, and a sequence U_n of unitaries in B such that

$$U_n \in B'_{n-1} \quad (B_0 = \mathbb{C}1)$$

$$\text{Ad}(U_n U_{n-1} U_{n-2} \cdots U_1) \circ \sigma(B_n) = B_n$$

$$\|U_n - 1\| < \varepsilon 2^{-n}$$

$$\text{dist}(x_i, B_n) < 2^{-n} \text{ for } i = 1, \dots, n$$

B_1, U_1 are constructed using the first part of the theorem, and when $B_1, \dots, B_n$ and $U_1, \dots, U_n$ have been constructed, one notices that $\sigma_n = \text{Ad} U_n U_{n-1} \cdots U_1 \circ \sigma$ has the Rohlin property by Lemma 3.2 and as $B'_n \cap A$ is left invariant by σ_n and is a UHF algebra one deduces that $\sigma_n|_{B'_n \cap A}$ has the Rohlin property. Take a finite-dimensional subalgebra $D \in \mathcal{F}(B'_n \cap A)$ such that $C^*(B_n, D) \cong B_n \otimes D$ approximately contains $x_1, \dots, x_{n+1}$. One now applies the first part of the theorem with $A \rightarrow A \cap B'_n, D \rightarrow D, \sigma \rightarrow \text{Ad}(U_n U_{n-1} \cdots U_1) \circ \sigma|_{A \cap B'_n}$ to find a full matrix subalgebra $C \subseteq \mathcal{F}(A \cap B'_n)$ and a unitary $U_{n+1} \in A \cap B'_n$ such that

$$\text{Ad}(U_{n+1} U_n \cdots U_1) \circ \sigma(C) = C$$

$$\|U_{n+1} - 1\| < \varepsilon 2^{-n-1}$$

and then puts $B_{n+1} = C^*(B_n, C) \cong B_n \otimes C$.

If $V = \lim_{n \rightarrow \infty} U_n U_{n-1} \cdots U_1$, then B_n, V satisfies the requirements in the last statement of Theorem 1.7.

Lemma 3.1 If σ has the Rohlin property then σ^m has Rohlin property for any integer $m \neq 0$.

Proof Let $n \gg |m|$ and let $\{e_i\}$ be an orthogonal family of projections such that

$$\sum_{i=0}^{n-1} e_i = 1$$

$$\|\sigma(e_i) - e_{i+1}\| < \varepsilon.$$

Let k be the greatest common divisor of $|m|$ and n and let

$$E_i = \sum_{s=0}^{k-1} e_{mi+s}$$

where all the indices are calculated mod n . Then

$$\sum_{i=0}^{\frac{n}{k}-1} E_i = \sum_{i=0}^{\frac{n}{k}-1} \sum_{s=0}^{k-1} e_{mi+s} = 1.$$

Now $m\mathbb{Z} + n\mathbb{Z} = k\mathbb{Z}$ (or $m\mathbb{Z}/n\mathbb{Z} = k\mathbb{Z}/n\mathbb{Z}$), and

$$\|\sigma^m(E_i) - E_{i+1}\| \leq \sum_{s=0}^{k-1} \|\sigma^m(e_{mi+s}) - e_{mi+m+s}\| \leq k m \varepsilon \leq m^2 \varepsilon.$$

Since $\frac{n}{k} \geq \frac{n}{m}$, this concludes the proof.

Lemma 3.2 If σ has the Rohlin property and $U \in A$ is unitary, then $\text{Ad} U \circ \sigma$ has the Rohlin property.

Proof The very same central sequences $(e_i^{(j)})_j$ in condition (1.6.4) which works for σ has exactly the same property with respect to $\text{Ad} U \circ \sigma$ because $(\sigma(e_i^{(j)}))_j$ asymptotically commute with U .

Lemma 3.3 If σ has the Rohlin property and $\alpha(e) = e$ for some nonzero projection e , then $\sigma|_{eAe}$ has the Rohlin property.

Proof First replace the $e_i^{(j)}$ in condition (1.6.4) by $f_i^{(j)} = e e_i^{(j)} e$. Then $f_i^{(j)}$ are positive elements such that $\sum_i f_i^{(j)} = e$, $\lim_{n \rightarrow \infty} \|\sigma(f_i^{(j)}) - f_{i+1}^{(j)}\| = 0$ and $(f_i^{(j)})_j$ are central sequences in $e A e$. But as $e_i^{(j)}$ asymptotically commute with e as $j \rightarrow \infty$, the $f_i^{(j)}$ are approximately orthogonal approximate projections for large j . Modifying the $f_i^{(j)}$ by spectral theory, we derive the Rohlin property for $\sigma|_{e A e}$.

4. Uniformly distributed sets on the circle

The following lemma is a small modification of Lemma 7.1.1 in [CFS], and is proved in the same manner.

Lemma 4.1 Let $(x_n)_{n=1}^{\infty}$ be a sequence in the unit interval $[0, 1]$. The following conditions are equivalent

(i) $(x_n)_{n=1}^{\infty}$ is uniformly distributed on $[0, 1]$ in the sense that

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k=1}^{2^n} f(x_k) = \int_0^1 f(x) dx$$

for each $f \in C([0, 1])$

(ii) If $v_n([b_1, b_2]) = \#\{k \mid 1 \leq k \leq 2^n \text{ such that } v_1 \leq x_k < v_2\}$,

where $\#F$ denotes the cardinality of the set F , then

$$\lim_{n \rightarrow \infty} 2^{-n} v_n([b_1, b_2]) = b_2 - b_1$$

for all $0 \leq b_1 < b_2 \leq 1$.

(iii)
$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k=1}^{2^n} \exp(2\pi_i s x_k) = 0$$

for any integer $s \neq 0$.

We will apply this lemma on certain sequences of points on the circle $\mathbb{T}$:

Lemma 4.2 Let $(\rho_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{T} \subseteq \mathbb{C}$, and define a new sequence $(\xi_n)_{n=1}^{\infty}$ in $\mathbb{T}$ inductively as follows

$$\xi_1 = 1, \xi_2 = \rho_1.$$

When $\xi_1, \dots, \xi_{2^n}$ are defined, define

$$\xi_{2^{n+k}} = \xi_k \rho^{n+1}, k = 1, \dots, 2^n.$$

Thus the subsequence $(\xi_k)_{k=1}^{2^n}$ consists of all products $\rho_{i_1} \rho_{i_2} \dots \rho_{i_k}$, $1 \leq i_1 < i_2 < \dots < i_k \leq n$, where each product occurs exactly once.

The following conditions are equivalent

(i) $(\xi_n)_{n=1}^{\infty}$ is uniformly distributed on $\mathbb{T}$ in the sense that

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k=1}^{2^n} f(\xi_k) = \int_{\mathbb{T}} f(t) dt$$

where dt is normalized Haar measure on the circle group $\mathbb{T}$.

(ii) If

$$v_n([b_1, b_2]) = \# \{k \mid 1 \leq k \leq 2^n \text{ such that } b_1 \leq \arg \xi_k < b_2\},$$

then

$$\lim_{n \rightarrow \infty} 2^{-n} v_n([v_1, v_2]) = \frac{1}{2\pi} (v_2 - v_1)$$

for all $0 \leq v_1 < v_2 \leq 2\pi$.

(iii) $\sum_{k=1}^{\infty} |\rho_k^s - 1|^2 = +\infty$ or $\rho_k^s = -1$ for some k

for $s = 1, 2, \dots$

(iv) Define

$$\mathcal{F}_n = (\xi_k)_{k=1}^{2^n} \text{ and } \mathcal{G}_n = \left(\exp \frac{2\pi i k}{2^n} \right)_{k=1}^{2^n}.$$

Then for any $\varepsilon > 0$ there exists an n_0 and for any $n > n_0$ a bijection $\varphi_n: \mathcal{F}_n \rightarrow \mathcal{G}_n$ such that

$$|\varphi_n(\xi_k) - \xi_k| < \varepsilon$$

for $1 \leq k \leq 2^n$.

Proof The equivalence of (i) and (ii) is immediate from the equivalence of (i) and (ii) in Lemma 2.1. But

$$\begin{aligned} 2^{-n} \sum_{k=1}^{2^n} \xi_k^s &= 2^{-n} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \rho_{i_1}^s \rho_{i_2}^s \dots \rho_{i_k}^s \\ &= 2^{-n} \prod_{k=1}^n (1 + \rho_k^s) \\ &= \prod_{k=1}^n \left(\frac{1 + \rho_k^s}{2} \right). \end{aligned}$$

Hence we see that the condition

$$\lim_{n \rightarrow \infty} 2^{-n} \sum_{k=1}^{2^n} \xi_k^s = 0, \quad s = 1, 2, \dots$$

which is condition (iii) of Lemma 2.1, is equivalent to the condition

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \left| \frac{1 + \rho_k^s}{2} \right| = 0$$

in the present context, which again is equivalent to

$$\lim_{n \rightarrow \infty} \prod_{k=1}^n \left| \frac{1 + \rho_k^s}{2} \right|^2 = 0.$$

But as $|\rho_k^s| = 1$,

$$\left| \frac{1 + \rho_k^s}{2} \right|^2 = \left| 1 - \frac{1 - \rho_k^s}{2} \right|^2 = 1 - \left(\frac{|1 - \rho_k^s|}{2} \right)^2$$

so the last condition is again equivalent to

$$\sum_{k=1}^{\infty} |1 - \rho_k^s|^2 = +\infty \text{ or } \rho_k^s = -1 \text{ for some } k$$

which is (iii) of the present lemma. Hence the conditions (i), (ii) and (iii) are all equivalent

The proof that (iv) implies (ii) is left to the reader, and we will finally prove (ii) $\Rightarrow$ (iv) by using the matching theorem, (Hall's theorem), from elementary combinatorics:

Let $\mathcal{F}$, $\mathcal{G}$ be finite sets, and assume that to each $x \in \mathcal{F}$ is associated a nonempty set $S_x \subseteq \mathcal{G}$ such that

$$\# \left(\bigcup_{x \in F} S_x \right) \geq \#(F)$$

for each $F \subseteq \mathcal{F}$. Then there exists an injection $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ such that $\varphi(x) \in S_x$ for all x , see [Bol, Theorem 7, page 54]. In fact we will essentially apply the form of the matching theorem which is stated as Lemma 2.1 in [Su].

To prove (ii) $\Rightarrow$ (iv), choose first m so large that

$$2^{-m} < \epsilon/2.$$

By condition (ii), we may then choose $n_0 \geq m$ so large that if $n \geq n_0$, then

$$\left| \frac{\#(\mathcal{F}_n \cap I)}{2^n} - |I| \right| < 2^{-m-1}$$

for each subset $I \subseteq \mathbb{T} = [0, 1]$ which is a union of a subset of the intervals

$$[r2^{-m}, (r+1)2^{-m}), \quad r = 0, 1, \dots, 2^m - 1$$

(so there are only a finite number, 2^{2^m} , of such I 's).

Now, to each $\xi \in \mathcal{F}_n$, associate the subset

$$S_\xi \subseteq \mathcal{G}_n$$

defined by

$$S_\xi = \{w \in \mathcal{G}_n \mid |w - \xi| < \varepsilon\}$$

and for subsets $F \subseteq \mathcal{F}_n$, define

$$S_F = \bigcup_{\xi \in F} S_\xi = \{w \in \mathcal{G}_n \mid \text{dist}(w, F) < \varepsilon\}.$$

To apply the matching theorem, we have to show that

$$\#(S_F) \geq \#(F)$$

for any F . But for given F with $S_F \neq \mathcal{G}_n$, let I be the union of the $k(F)$ intervals of the form $[r2^{-m}, (r+1)2^{-m})$ which intersects F . Then, as $2^{-m} < \varepsilon/2$, S_F contains those points in $\mathcal{G}_n$ which are contained in these intervals as well as those points in $\mathcal{G}_n$ which are contained in the $\ell(F)$ adjacent intervals. Now

$$\begin{aligned} \#(F) &= \#(F \cap I) \\ &\leq \#(\mathcal{F}_n \cap I) \\ &< 2^n |I| + 2^n 2^{-m-1} \end{aligned}$$

$$= 2^n \frac{k(F)}{2^m} + 2^n 2^{-m-1}.$$

If J is the union of I by the adjacent intervals, then

$$\begin{aligned} \#(S_F) &\geq \#(\mathcal{G}_n \cap J) \\ &= 2^{n-m} (k(F) + l(F)) \\ &\geq 2^{n-m} (k(F) + 1) \\ &> 2^{n-m} (k(F) + \frac{1}{2}) > \#(F), \end{aligned}$$

where $l(F) \geq 1$, since $S_F \neq \mathcal{G}_n$.

Thus the matching theorem applies to show (iv).

We will also need a slightly stronger version of Lemma 4.2:

Lemma 4.3 Let $(\rho_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{T} \subseteq \mathbb{C}$, and for each n , let $(\xi_k^e)_{k=1}^{2^{n-1}}$ consist of those points in $(\xi_k)_{k=1}^{2^n}$ which are products $\rho_{i_1} \rho_{i_2} \dots \rho_{i_{2m}}$ with $1 \leq i_1 < i_2 < \dots < i_{2m} \leq n$, with an even number of indices $i_1, \dots, i_{2m}$, and let $(\xi_k^\sigma)_{k=1}^{2^{n-1}}$ consist of those points in $(\xi_k)_{k=1}^{2^n}$ which are products of an odd number of ρ_i 's.

The following conditions are equivalent:

(i) $(\xi_n^e)_{n=1}^{\infty}$ and $(\xi_n^\sigma)_{n=1}^{\infty}$ are both uniformly distributed on $\mathbb{T}$ in the sense of Lemma 4.2.

(ii) $\left(\sum_{k=1}^{\infty} |\rho_k^s - 1|^2 = +\infty \text{ or } \rho_k^s = -1 \text{ for some } k \right)$

and $\left(\sum_{k=1}^{\infty} |\rho_k^s + 1|^2 = +\infty \text{ or } \rho_k^s = 1 \text{ for some } k \right)$

for $s = 1, 2, \dots$

(iii) Define

$$\mathcal{F}_n^e = (\xi_k^e)_{k=1}^{2^n}$$

$$\mathcal{F}_n^\sigma = (\xi_k^\sigma)_{k=1}^{2^n} \text{ and } \mathcal{G}_n = (\exp \frac{2\pi i k}{2^n})_{k=1}^{2^n}.$$

Then for any $\varepsilon < 0$ there exists an n_0 and for any $n > n_0$ bijections

$$\varphi_n^e: \mathcal{F}_n^e \rightarrow \mathcal{G}_n, \quad \varphi_n^\sigma: \mathcal{F}_n^\sigma \rightarrow \mathcal{G}_n$$

such that

$$|\varphi_n^e(\xi_k^e) - \xi_k^e| < \varepsilon, \quad |\varphi_n^\sigma(\xi_k^\sigma) - \xi_k^\sigma| < \varepsilon$$

for $1 \leq k \leq 2^n$.

Finally, these conditions are implied by

$$(iv) \quad \sum_{k=1}^{\infty} |\rho_k^s - 1|^2 = +\infty$$

for $s = 1, 2, \dots$.

Proof The equivalence of (i) and (iii) is established as (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) in Lemma 4.2.

We now show (ii) $\Leftrightarrow$ (i). By Lemma 4.1, (i) is equivalent to

$$\lim_{n \rightarrow \infty} E_n(s) = \lim_{n \rightarrow \infty} O_n(s) = 0$$

for $s \in \mathbb{Z} \setminus \{0\}$, where

$$E_n(s) = 2^{-n+1} \sum_{k=1}^{2^{n-1}} (\xi_k^e)^s,$$

$$O_n(s) = 2^{-n+1} \sum_{k=1}^{2^{n-1}} (\xi_k^\sigma)^s.$$

We consider this condition for $s = 1$, the other cases are analogous. Using the inductive definition of $(\xi_k)_k$ in Lemma 4.2, and the notation $E_n = E_n(1)$, $O_n = O_n(1)$, we see that

$$E_{n+1} = \frac{1}{2} (E_n + O_n \rho_{n+1}),$$

$$O_{n+1} = \frac{1}{2} (E_n \rho_{n+1} + O_n),$$

with the initial condition

$$E_1 = 1, O_1 = \rho_1.$$

Now, note that the 2×2 matrix

$$\frac{1}{2} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

has the eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with eigenvalues $\frac{1}{2}(1 + \rho)$, $\frac{1}{2}(1 - \rho)$, respectively. Thus, if we do the change of basis

$$\begin{pmatrix} E_n \\ O_n \end{pmatrix} = F_n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + G_n \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

then F_n , G_n satisfy the recursion relation

$$F_{n+1} = \frac{1}{2} (1 + \rho_{n+1}) F_n$$

$$G_{n+1} = \frac{1}{2} (1 - \rho_{n+1}) G_n$$

with the initial condition

$$F_1 = \frac{1}{2}(E_1 + O_1) = \frac{1}{2}(1 + \rho_1)$$

$$G_1 = \frac{1}{2}(E_1 - O_1) = \frac{1}{2}(1 - \rho_1)$$

Thus

$$F_n = \prod_{k=1}^n \left(\frac{1 + \rho_k}{2} \right)$$

$$G_n = \prod_{k=1}^n \left(\frac{1 - \rho_k}{2} \right)$$

But the pair E_n, O_n converges to zero if and only if the pair F_n, G_n converges to zero. By the same argument as in the proof of Lemma 4.2, F_n converges to zero if and only if

$$\sum_{k=1}^{\infty} |1 - \rho_k|^2 = +\infty \text{ or } \rho_k = -1 \text{ for some } k,$$

and G_n converges to zero if and only if

$$\sum_{k=1}^{\infty} |1 + \rho_k|^2 = +\infty \text{ or } \rho_k = +1 \text{ for some } k.$$

Thus (ii) $\Leftrightarrow$ (i) follows from (iii) $\Leftrightarrow$ (i) in Lemma 4.1.

Finally, if (iv) is valid, i.e.

$$\sum_{k=1}^{\infty} |\rho_k^s - 1|^2 = +\infty$$

for $s = 1, 2$, then the estimate

$$|\rho_k^{2s} - 1|^2 = |\rho_k^s + 1|^2 |\rho_k^s - 1|^2 \leq 4 |\rho_k^s + 1|^2$$

implies that

$$\sum_{k=1}^{\infty} |\rho_k^s + 1|^2 \geq \frac{1}{4} \sum_{k=1}^{\infty} |\rho_k^{2s} - 1|^2 = +\infty$$

so (ii) holds

5. Actions which are outer in the trace representation

In this section we will prove Theorem 1.1. The following implications are already known:

(2) $\Leftrightarrow$ (3): This is [Blat, Theorem 1]. (See also [SS, Theorem 2.5] and [Ara].)

(2) $\Rightarrow$ (4): This is [Bed, Theorem 2.1].

(1) $\Rightarrow$ (6): This follows from the arguments of [Voi] just as in [BKRS, Lemma 2.1] and Remark 1.4.

(6) $\Rightarrow$ (5): This follows from [BBEK, Theorem 1.3]. (One easily checks that the inductive limit has small eigenvalue variation.)

(4) $\Rightarrow$ (3) and (5) $\Rightarrow$ (3): If $\|U^n - 1\|_2 < +\infty$ for some $n \in \mathbb{N}$, one checks from Theorem 1.2 and Remark 1.5 that the projections in $A \rtimes_{\alpha} \mathbb{Z}$ do not separate the trace states of A , so $A \rtimes_{\alpha} \mathbb{Z}$ can neither be of real rank zero nor have a unique trace state. We will also see during the proof of Theorem 1.2 that (4) $\Rightarrow$ (2) is independent of α being a quasifree automorphism. (If α^n is weakly inner for some n , the representation of $A \rtimes_{\alpha} \mathbb{Z}$ canonically induced by the trace of A has a unitary with full spectrum in its centre).

It remains to prove (3) $\Rightarrow$ (1). It was proved in [Con, Theorem 1.2.5] that (2) implies a weaker version of (1) for general automorphisms, but our proof is independent: If $\|U^n - 1\|_2 = +\infty$ for all $n \in \mathbb{Z} \setminus \{0\}$, then there exist elements

$\rho_1, \rho_2, \dots$ in the spectrum of U , with ρ_k occurring with multiplicity at most n if ρ_k is an eigenvalue of degeneracy n , such that

$$\sum_{k=1}^{\infty} |\rho_k^n - 1|^2 = +\infty$$

for all $n \in \mathbb{Z} \setminus \{0\}$. If U has pure point spectrum one just let ρ_k be the set of eigenvalues of U , with each ρ_k occurring as many times as its degeneracy dictates, and if U does not have pure point spectrum take $(\rho_k)_k$ to consist of the single point $\exp(2\pi i\theta)$ which is contained in the spectrum but not the point spectrum, and where θ is irrational. Call these cases case 1 and case 2.

In case 1, choose an orthonormal basis $(\eta_k)_{k=1}^{\infty}$ for the one-particle Hilbert space $\mathcal{H}$ such that

$$U\eta_k = \rho_k \eta_k$$

for $k = 1, 2, \dots$, and define $\mathcal{H}_n$ as the linear span of $\eta_1, \eta_2, \dots, \eta_n$.

In case 2, choose by spectral theory an orthonormal set $(\eta_k)_{k=1}^{\infty}$ in $\mathcal{H}$ such that

$$\|U\eta_k - \rho\eta_k\| < 2^{-k}$$

for $k = 1, 2, \dots$, where $\rho = \rho_k = \rho_1$, and define $\mathcal{H}_n$ as the orthogonal complement of $\eta_{n+1}, \eta_{n+2}, \dots$.

In both cases, $(\mathcal{H}_n)_n$ is an increasing sequence of closed subspaces with dense union in $\mathcal{H}$.

We now verify the Rohlin property. If $\{x_1, \dots, x_n\} \subseteq A$, $\varepsilon > 0$ and $m_0 \in \mathbb{N}$, one first chooses a n_0 such that $x_1, \dots, x_n$ are approximately contained in the CAR algebra $A(\mathcal{H}_{n_0})$ over $\mathcal{H}_{n_0}$. Next one applies Lemma 4.3 on the sequence $(\rho_{n+n_0})_{n=1}^{\infty}$ in lieu of (ρ_n) . By condition (iii) there exists for any $\delta > 0$ an n_1

such that the sets $\mathcal{F}_{n_1}^e$ and $\mathcal{F}_{n_1}^\sigma$ can be put in one-one correspondence with $\mathcal{G}_{n_1} = (\exp \frac{2\pi i k}{2^{n-1}})_{k=1}^{2^{n-1}}$ such that the distance between corresponding elements is at most δ . Consider the $n_1 + 1$ - dimensional subspace $\mathcal{H}'$ spanned by $\eta_{n_0+1}, \eta_{n_0+2}, \dots, \eta_{n_0+n_1+1}$ in case 1, or by $\eta_{n_0+n_2}, \eta_{n_0+n_2+1}, \dots, \eta_{n_0+n_2+n_1}$ for a sufficiently large n_2 in case 2. Then $\alpha_U|_{A(\mathcal{H}')}$ identifies with the adjoint of

$$\begin{pmatrix} 1 & 0 \\ 0 & \rho_{n_0+1} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 & 0 \\ 0 & \rho_{n_0+n_1+1} \end{pmatrix}$$

in case 1. The eigenvalues of this matrix is the set of all products

$$\rho_{i_1} \rho_{i_2} \dots \rho_{i_k}$$

with

$$n_0 + 1 \leq i_1 < i_2 < \dots < i_k \leq n_0 + n_1 + 1,$$

i.e. the eigenvalues are the elements ξ_k 's, $1 \leq k \leq 2^{n_1+1}$ in the notation of Lemmas 4.2 and 4.3. The even automorphism σ , defined by $a(f) \rightarrow -a(f)$, identifies with the adjoint of

$$V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the even subalgebra $A^e(\mathcal{H}')$ of $A(\mathcal{H}')$ is isomorphic to

$$M_{2^{n_1}} \oplus M_{2^{n_1}}$$

where the central projection corresponding to each of these two factors are the eigen-projections of V corresponding to the eigenvalues ± 1 . Thus α_U restricted to the first $M_{2^{n_1}}$, corresponding to eigenvalue $+1$, is the adjoint of the unitary matrix with the 2^{n_1} eigenvalues equal the set of all products

$$\rho_{i_1} \rho_{i_2} \dots \rho_{i_k}$$

where $n_0+1 \leq i_1 < i_2 < \dots < i_k \leq n_0 + n_1 + 1$ and k is even, while α_U restricted to the last M_{2n} , corresponding to the eigenvalue -1 of V , is the adjoint of the unitary matrix whose set of eigenvalues is equal to the set of products as above with k odd. Thus the two sets of eigenvalues are equal to $\mathcal{F}_n^e$ and $\mathcal{F}_n^\sigma$, respectively. But $A^e(\mathcal{H}')$ is contained in the commutant of $A(\mathcal{H}_{n_0})$, and by choosing δ small and then n_1 large for a fixed m_0 , the restriction of α to $A^e(\mathcal{H}')$ become approximately equal to

$$(\beta \otimes \iota) \oplus (\beta \otimes \iota)$$

where β is the cyclic shift of order 2^{m_0} and ι is the identity automorphism on $M_{2n_1-m_0}$. The approximation is up to $2^{-m_0} + \delta$, where δ can be made arbitrary small. But

$$\beta \otimes \iota \oplus \beta \otimes \iota \quad \text{on} \quad A^e(\mathcal{H}') \cong M_{2n_1} \oplus M_{2n_1}$$

contains

$$\beta \quad \text{on} \quad M_{2m_0}$$

as a dynamical subsystem. This proves the Rohlin property in case 1. In case 2 one has to do an extra approximation since the η_k are only approximate eigenvectors for U , but since $\rho_k = \rho$ is independent of k in this case the sets $\mathcal{F}_n^e, \mathcal{F}_n^\sigma$ depend only on n , not on the subsequence of (ρ_k) used, and hence one may choose n_2 after having chosen m_0, n_0, n , and δ and the additional approximation can be made as good as one likes.

Thus the Rohlin property is valid in this case too.

6. Actions which are inner in the trace representation

In this section we prove Theorem 1.2. First note that the condition $\|U^n - 1\| = +\infty$ for all $n \in \mathbb{Z} \setminus \{0\}$ means that all nonzero powers of α_U are

outer, [Lan]. Thus $A \rtimes_{\alpha} \mathbb{Z}$ is simple, [Kis]. Since $\|U^n - 1\|_2 < +\infty$ for some $n > 0$, U has purely discrete spectrum, say $(\rho_i)_{i=1}^n$. By the Jordan-Wigner transformation, [BR2], (A, α) is then isomorphic to the product type dynamical system:

$$\left(\bigotimes_{k=1}^{\infty} M_2, \bigotimes_{k=1}^{\infty} \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \rho_k \end{pmatrix} \right).$$

By [Bra], it follows $A \rtimes_{\alpha} \mathbb{Z}$ is the inductive limit of

$$B_n = \bigotimes_{k=1}^n M_2 \rtimes \bigotimes_{k=1}^n \text{Ad} \begin{pmatrix} 1 & 0 \\ 0 & \rho_k \end{pmatrix} \quad \mathbb{Z} \cong M_{2^n} \otimes C(\hat{\mathbb{Z}}) \cong M_{2^n} \otimes C(\mathbb{T})$$

where the embedding of B_n into B_{n+1} is given by the sum of the identity embedding and the same translated by rotation by ρ_{n+1} . Thus the K-theory is given as in Theorem 1.2. Finally, if τ_0 is the trace state on A , and U is the canonical unitary in the crossed product $A \rtimes_{\alpha} \mathbb{Z}$, then $A \rtimes_{\alpha} \mathbb{Z}$ has a canonical trace state given by

$$\tau(aU^n) = \begin{cases} \tau_0(a) & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

for $a \in A, n \in \mathbb{Z}$. Thus, if φ is a general trace state on $A \rtimes_{\alpha} \mathbb{Z}$, then

$$\tau = \int_{\mathbb{T}} dt \varphi \circ \hat{\alpha}_t$$

where $\hat{\alpha}$ is the dual action of $\hat{\mathbb{Z}} = \mathbb{T}$ on $A \rtimes_{\alpha} \mathbb{Z}$. Thus, if $f \in C(\mathbb{T})$ is a positive function, then

$$\int_{\mathbb{T}} f_n(t) dt \varphi \circ \hat{\alpha}_t \leq \|f\|_{\infty} \tau.$$

Letting f_n converge to the δ_0 -measure, we see that φ is a weak *-limit of trace states which are normal in the τ -representation of $A \rtimes_{\alpha} \mathbb{Z}$. Let now τ_n be the

smallest positive n such that α^n is inner in the trace representation, and let $v \in (A \rtimes_\alpha \mathbb{Z})''$ be a unitary in the weak closure implementing U^n . Then $v^* U^n$ generate the centre of $(A \rtimes_\alpha \mathbb{Z})''$ in this representation, and hence this centre is isomorphic to $L^\infty(\mathbb{T})$, and the trace states which are normal in this representation are given by positive functions on $L^\infty(\mathbb{T})$ of integral 1. It follows from decomposition theory that any trace state on $A \rtimes_\alpha \mathbb{Z}$ is defined by a probability measure on $\mathbb{T}$, and conversely any such measure defines a trace state, the correspondence being given by

$$\varphi = \int_{\mathbb{T}} d\mu(t) \psi \circ \hat{\alpha}_t$$

where the measure μ on $\mathbb{T}$ is periodic with period $1/n$, and ψ is an extremal trace state on $A \rtimes_\alpha \mathbb{T}$ so that

$$\tau = \int_{\mathbb{T}} dt \psi \circ \hat{\alpha}_t,$$

[BR1]. Thus the convex compact set of trace states on $A \rtimes_\alpha \mathbb{Z}$ is affinely isomorphic to the set of probability measures on $\mathbb{T}$.

Finally, as $A \rtimes_\alpha \mathbb{Z}$ is an inductive limit of the particular form described, and the restriction of any trace φ on $A \rtimes_\alpha \mathbb{Z}$ to

$$B_n \equiv M_{2n} \otimes C(\mathbb{T})$$

has the form $\text{tr} \otimes \mu$, where tr is normalized trace on M_{2n} and μ is a Radon measure on $C(\mathbb{T})$, it follows that any two trace states on $A \rtimes_\alpha \mathbb{Z}$ assume the same value on any projection, namely the K_0 -representative of the projection in $\mathbb{Z}[\frac{1}{2}]$.

This ends the proof of Theorem 1.2.

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