ISBN 82-553-0928-4
Pure Mathematics

No. 25
October 1994

**A Hierarchy of Domains with Totality,
but without Density**

by
Dag Normann

PREPRINT SERIES – Matematisk institutt, Universitetet i Oslo

A Hierarchy of Domains with Totality, but without Density

Dag Normann

October 17, 1994

1 Introduction

This paper is a continuation of the papers [6, 7, 9] where we give a pure semantical analysis of some of the constructors of pure type theory.

The constructors we have focused on so far are the dependent sums $\Sigma(x \in A)B_x$ and dependent products $\Pi(x \in A)B_x$. It is important that we discuss possible interpretations of the constructors without having particular type theories in mind, because this is the only way to construct semantics that are robust with respect to extensions or alterations of theories.

The naive interpretation will be to let

$$\Sigma(X \in A)B_x = \{(x, y) \mid x \in A \wedge y \in B_x\}$$

and

$$\Pi(x \in A)B_x = \{f \mid \text{dom}(f) = A \wedge \forall x \in A (f(x) \in B_x)\}$$

and then to construct a full hierarchy over some base types where we include sums and products of all possible parameterisations of types.

There are two problems with this naive approach.

One is that we cannot construct this hierarchy using standard set theory ZFC without a minimum of large cardinal axioms.

The other, and more serious problem, is that the types in a hierarchy like this will have little internal structure beyond what can be obtained from general set theory, and that there is thus no real information to be obtained from the naive approach.

We have to find a ballance between the need for generality, i.e. the construction lends itself to the implementation of the relevant fragment of any type theory, and the need for restrictions in order to get structured interpretations.

To us it is also important that the interpretation is as close to the set-theoretic intuition as possible. This means that in the end a sum-type should be interpreted as a set of ordered pair, while a product-type should be interpreted as a set of choice functions.

There are two principles for restriction that suggest themselves, we require that all functions or parameterisations are continuous (locally finitely based), or we may even require that they are effective. Here, as in previous papers, we focus on continuity. In [6] we also investigated a third alternative, stability in qualitative domains. This part of [6] is based on Kristiansen [5].

In the previous papers we have worked with coherence-spaces, see Girard [2], or with Scott-Ershov domains, see Stoltenberg-Hansen, Lindström and Griffor [15] as a reference book, and constructed hierarchies based on interpretations for the set of natural numbers and the set of boolean values. The elements of the hierarchy will in each case be a two-level structure, because we have the underlying domain together with a canonical set of total objects. Consistency will be an equivalence relation on the set of total elements, and the equivalence classes will form structures that share many properties with the Kleene-Kreisel continuous functionals. The hierarchy of equivalence classes will be isomorphic to a hierarchy of sets of ordered pairs and choice functions.

In this paper we will construct a transfinite hierarchy of domains where we will use the canonical domains for the empty set, the set of boolean values and the set of natural numbers as the base of the hierarchy.

The construction of the hierarchy will be in two steps. First we construct one domain S of *syntactic forms* together with an interpretation $I(s)$ as a domain for each $s \in S$. Similar constructions can be found in Palmgren [12] and Palmgren and Stoltenberg-Hansen [13, 14], and this part of our construction is mainly covered by the methods of those papers.

The second step will be to define the well founded syntactic forms S_{wf} and the total elements $I(s)_{TOT}$ for $s \in S_{wf}$. It is important that this definition is accepted as canonical, since we do want to investigate the total elements of natural interpretations of typed expressions.

We will work in the context of domains. We will assume knowledge to

the basic theory of domains, see e.g. Stoltenberg-Hansen, Lindström and Griffor [15]. Following [12, 14] we will represent a domain as the set of ideals over a preordering with an explicit least upper bound construction. We will strengthen the concept from [14] slightly. This will be discussed in section 2.

Though we will work with total elements in a domain, our work will not fit into the axiomatisation of totality suggested by Berger [1]. In section 3 we will discuss attempts to formalise totality and discuss the connections with the construction in this paper.

Since the total elements are in general not dense in the underlying domains, it will not always be the case that a continuous function on the total objects can be extended to a continuous function on the underlying domain. Since we want to be able to represent all continuous functions defined on the total objects, we introduce continuous multivalued functions. The total functions will be singlevalued when restricted to total input. In order to give a decent treatment of multivalued functions, we introduce a domain-constructor mD , the domain of multivalued elements of D , and we characterize the compact elements of mD . This will be done in section 4.

In section 5, we give an explicit construction of the domains D and S together with the interpretation of $s \in S$ as a subdomain of D .

In section 6, we define the well founded hierarchy of domains with totality, and we prove that all total elements will respect extensional equality. The fact that this hierarchy can be constructed, is the main result of the paper.

In section 7 we first show that a hierarchy of domains of singlevalued functions, where we systematically avoid the empty type, can be seen as a subhierarchy of the one constructed in this paper. A consequence will be that the height of the hierarchy will be the first ordinal not recursive in 3E and any real. We then give a simpler proof of this fact, using the presence of the empty type.

2 Implicit and explicit information

One advantage of a theory like domain theory is that the constructions and proofs will have a flavour of generality instead of ad hoc'nes. The danger is, however, that when working within a general framework one loses the opportunity to use the extra information that follows with an ad hoc construction. Let us give one example.

In Kristiansen and Normann [6] we give an explicit construction of a hierarchy of domains with totality based on the domains for the boolean values and the natural numbers, and in a natural way, closed under the Σ - and Π -types of continuously parameterized families of types. (Normann [11] contains a construction of the same hierarchy restricted to Π -types.) We prove the density theorem, i.e. the total objects is a dense subset of each domain. However, attempts to give a general description of a parameterised family of domains with totality permitting the transfer of the density theorem has failed.

One important aspect of the ad hoc construction in [6, 11] is that we first construct a partial preordering representing the compact objects of the domain, and we replace the general *least upper bound*-operator on bounded pairs of compacts with an explicit sup-construction on the expressions representing the compacts. Technically in the proof of the density theorem for the hierarchy, it is used that the set of sup-predecessors for any object is finite. Thus we have made use of an extra structure on the underlying set of compacts in proving the density theorem.

In Palmgren and Stoltenberg-Hansen [14] it is observed that using pre-orderings of compacts makes it easier to interpret equality between types as true equality, and not just as being isomorphic. They introduce the notion of a *conditional upper semilattice with least element, c usl*. Below we define what we call an *iei-structure*. An iei-structure will be a *c usl* with the one extra property used in the proof of the density theorems in [6, 11].

The intuition behind the notion of an iei-structure is that a description of a compact is finite, and though there may be infinitely many compacts with less implicit information, there are only finitely many compacts with less explicit information. The least upper bound construction *sup* is seen as the process of taking the union of the explicit information of the expressions, and it is thus monotone in the explicit information ordering. The choice of axioms for the *sup*-operator is not canonical, and it may be practical to exclude some of these axioms.

Definition 1 An *implicit-explicit-information-structure, iei-structure* for short, is a partial preordering $(X, \leq)$ together with a partial, binary operator *sup* such that:

- i) *sup*(x, y) is defined if and only if $\{x, y\}$ is bounded in the preordering, and then, for all z , $\text{sup}(x, y) \leq z \Leftrightarrow x \leq z \wedge y \leq z$.

- ii) $\sup(x, x) = x$ for all x .
- iii) $\sup(x, y) = \sup(y, x)$ whenever $\{x, y\}$ is bounded.
- iv) $\sup(\sup(x, y), z) = \sup(x, \sup(y, z))$ whenever $\{x, y, z\}$ is bounded.
- v) $\sup(x, \sup(y, z)) = \sup(\sup(x, y), \sup(x, z))$ whenever $\{x, y, z\}$ is bounded.
- vi) $E_x = \{y \mid \sup(y, x) = x\}$ is finite for all x .
The elements of E_x are called the *subelements* of x .

When appropriate, we use the notation E_x^X for the set of subelements.

If $x \leq y$ we say that

‘ x has less *implicit* information than y ’

while if $x \in E_y$ we say that

‘ x has less *explicit* information than y ’.

Definition 2 Let $(X, \leq), (Y, \leq')$ be two iei-structures, $f : X \rightarrow Y$ be a function.

f is an *embedding* if

- i) $x \leq y$ if and only if $f(x) \leq' f(y)$ for all x and y .
- ii) $f(\sup(x, y)) = \sup(f(x), f(y))$ whenever $\{x, y\}$ is bounded in X
- iii) f maps E_x^X onto $E_{f(x)}^Y$ for all $x \in X$

Lemma 1 Let $(X, \leq)$ be an iei-structure, $x \in X$.

Then $y \in E_x$ if and only if for some z , $\sup(y, z) = x$.

Proof:

The proof is by simple calculation.

If $(X, \leq)$ is a preordering, we let the *quotient* be the canonical ordering of the equivalence classes of the relation

$$x \leq y \text{ and } y \leq x.$$

We then have

Theorem 1 *Any countable ordering where each finite, bounded set has a least upper bound, is isomorphic to the quotient of an iei-structure.*

Proof:

Let $(C, \sqsubseteq)$ be a countable partial ordering, and let $\sqcup$ be a least-upper-bound-operator for finite, bounded sets.

Let $\{(c_i, n_i)\}_{i \in \mathbb{N}}$ be an enumeration of $C \times \mathbb{N}$.

We let $i \leq j$ if $c_i \sqsubseteq c_j$.

By recursion on n we will construct a finite subset A_n of $\mathbb{N}$ and define $\sup$ on A_n such that it forms an iei-structure. We start with A_0 as a singleton $\{i\}$ such that $c_i = \perp$ in C .

Assume that A_n is defined. Let j be the least natural number not in A_n . We define A_{n+1} and the $\sup$ -operator in 4 steps:

Step 1 Let $j \in A_{n+1}$ with $\sup(j, j) = j$

Step 2 Let $i \in A_n$. Let k be the least number not in A_n such that $c_k = c_i \sqcup c_j$.

Let $k \in A_{n+1}$ and let $\sup(i, j) = \sup(j, i) = k$.

Step 3 Let i_1 and i_2 be in A_n with $k = \sup(i_2, j)$.

We then let $\sup(i_1, k) = \sup(k, i_1) = \sup(\sup(i_1, i_2), j)$.

Step 4 Let $k_1 = \sup(i_1, j)$ and $k_2 = \sup(i_2, j)$

Then $\sup(k_1, k_2) = \sup(\sup(i_1, i_2), j)$.

If $k \in A_{n+1}$ then $E_k \subseteq A_{n+1}$ and will thus be finite.

It is also clear from the construction that the given ordering is the quotient of the constructed preordering, and that $\sup$ is a least-upper-bound operator. The algebraic properties of A_{n+1} follows from the algebraic properties of A_n . The proof is trivial.

If $(X, \leq)$ is a finitely complete preordering, an *ideal* in X will be a nonempty subset α of X such that if $x \in \alpha$ and $y \leq x$ then $y \in \alpha$ and such that all finite subsets of α are bounded in α .

It is well known that the set of ideals of a partial preordering ordered by inclusion will form a domain, and that all domains can be constructed this way.

We call a domain *separable* if the set of compact elements is countable. We then have

Corollary 1 *Any separable domain is isomorphic to the domain of ideals in a countable iei-structure.*

Definition 3 Let $(X, \leq)$ be an iei-structure.

X is *saturated* if for all pairs A, B of finite iei-structures where A is a sub-structure of B , any embedding of A into X can be extended to an embedding of B into X

Lemma 2 *The category of iei-structures permits amalgamations (push-outs) and direct limits.*

Proof:

The direct-limit construction is trivial. The amalgamation is obtained by a direct construction in the style of the proof of Theorem 1, and is left for the reader.

Standard modeltheoretic methods then give us

Theorem 2 a) *There is a countable, saturated iei-structure.*

b) *Two countable, saturated iei-structures are isomorphic.*

c) *If $(X, \leq)$ is a saturated iei-structure, then all countable iei-structures can be embedded into X .*

We use the standard notion of embedding of one domain into another used e.g. by Palmgren and Stoltenberg-Hansen in [14]: An embedding from D to E is a *projection pair* (f^+, f^-) where f^+ is a continuous map from D to E , f^- is a continuous map from E to D , $f^-(f^+(p)) = p$ for all $p \in D$, and $f^+(f^-(\beta)) < \beta$ for all $\beta \in E$.

We then have

Corollary 2 *There is a separable domain D such that all other separable domains E can be embedded into D .*

Remark:

There is no need to use iei-structures in this construction, a saturated finitely complete partial ordering will do. Neither is this result new, see [15].

3 The concept of totality

If we want to give an analysis of the concept of totality in a domain, there are two main approaches.

The one taken in Normann [9] is that the elements of a set of objects are total because they represent procedures that give defined outputs to a given set of inputs. This reflects the standard case where the total objects are total because the domain in question is the interpretation of some type term or of a similar object, and then the total elements are canonically given.

The other approach is that totality is an intrinsic property of a domain. This view is taken in Berger [1], where he axiomatizes the notion of a set of total objects in terms of boolean valued tests.

Berger's axioms are used to prove that density of the total objects are preserved when we construct function spaces. In the domains with totality that we will construct, density is not satisfied because we introduce the empty type as one of the base types. Thus our sets of total objects will not satisfy Berger's axioms.

In this section we will translate the axioms of [9] to the setting of iei-structures, and later we will show to what extent our sets of total objects will satisfy these axioms.

In [9], totality is defined via a system of chains. The idea is that the elements of a domain X represent processes to which we can give various sorts of input information, and possibly obtain well defined, atomic outputs.

Some pieces of input information are essential. These pieces are called *questions*. An object is total if it gives a defined, atomic output to each input question.

The compact elements will actually represent finite sets of finitary information about how an input is transformed to an output. In [9] we follow this intuitive idea and identify a question with its *chain*, the set of minimal (in the explicit sense) pieces of information about how to answer that particular question. Formally we define

Definition 4 Let X be an iei-structure, D the domain of X -ideals.

- a) A *chain* in X is a set $C \subseteq X$ of elements that are pairwise incomparable in the explicit information ordering.
- b) If C is a chain, and $x \in D$ we say that x *meets* C if $x \cap C \neq \emptyset$.

c) If Q is a set and C_q is a chain in X for each $q \in Q$, then

$$(X, Q, \{C_q\}_{q \in Q})$$

is a *totality space* and

$$D_{\text{TOT}} = \{x \in D \mid x \text{ meets } C_q \text{ for all } q \in Q\}.$$

Berger's [1] axioms for sets of total objects are in reality based on the same idea, but in principle he defines sets of total objects as a property of the domain, and not of a richer structure. We will not discuss Berger's axioms in detail, only one first approach, the axiom for one object to be total. Here we call it premaximal in order to distinguish Berger's definition from the definition above.

Definition 5 Let D be a domain.

An element $x \in D$ is called *premaximal* if for all compacts A and B in D , if both A and B are consistent with x , then A and B are consistent with each other.

It is easily seen that consistency between premaximal objects will be an equivalence relation, and that x is premaximal if and only if x has a unique maximal extension. Thus every equivalence class will have a largest element. Now assume that X is a totality-space as defined above with a total x that is not premaximal. This means that there are inconsistent compacts A and B that both are consistent with x , but that are inconsistent with each other. Consistency with x will imply that both A and B will answer all questions $q \in Q$ in the same way as q is answered by x . Inconsistency between A and B should mean that there is some finitary input material for which A and B produces different, well defined outputs. Thus the relevant questions cannot be dense in the set of available finitary input material.

This discussion has been at a rather intuitive level. The conclusion is that Berger's axioms essentially reflects the situation where a procedure is total because it works on a dense set of input material.

Now let us assume that the total elements of X are total because they answer a set Q of questions q that are not necessarily dense in the underlying set of possible input material. Define the relation E by

xEy if $x \sqcap y$ is total.

We claim that it is reasonable to assume that E is an equivalence relation by the following informal argument:

E is obviously reflexive and symmetric.

Let xEy and yEz . We explain why $x \sqcap z$ should be total.

Let q be a question. Then $x \sqcap y$ will use some finite information q_0 from q in order to produce the answer a , and $y \sqcap z$ will use some finite information q_1 from q to produce the answer b .

Because y then give both answers a and b to q , we must have that $a = b$.

Let q_2 be the union of q_0 and q_1 . Then, assuming that x and z are ideals of pieces of information coded in the standard way, the information

'we obtain the answer a from q_2 '

will be in both x and z , i.e. in $x \sqcap z$. Thus $x \sqcap z$ will answer q .

We will prove that our total objects will satisfy that the relation E is an equivalence relation, and that it in a natural sense corresponds to hereditarily extensional equality between total elements. This property of the hierarchy will be the main technical result of the paper. We will further prove that each equivalence-class will have a unique maximal element.

4 Multiobjects

As mentioned in the introduction, we want to consider multivalued functions, and in order to give a clean account for this, we introduce the domains of multiobjects.

Definition 6 Let D be a domain. A *multiobject* in D is a nonempty set $X \subseteq D$ such that

- i) If $\alpha \in X$ and $\beta \sqsubseteq \alpha$, then $\beta \in X$.
- ii) If $Y \subseteq X$ is a bounded subset of D , then $\sqcup Y \in X$.

We let mD be the set of multiobjects in D ordered by inclusion.

Lemma 3 mD is a domain.

Proof:

If $\{X_i\}_{i \in I}$ is a directed set from ${}^m D$, we let

$Y = \sqcup \{X_i\}_{i \in I} = \{\alpha \mid \alpha = \sqcup \{\alpha_i \mid i \in I\} \text{ where } \{\alpha_i\}_{i \in I} \text{ is a directed set from } D \text{ with } \alpha_i \in X_i \text{ for each } i \in I\}$.

It is easy to see that Y will be the least upper bound of the directed set, but it requires some more work to show that it actually is an element of ${}^m D$. We still leave this for the reader.

This will show that ${}^m D$ is a complete partial ordering.

The compact elements will be represented by finite sets $\{C_1, \dots, C_n\}$ of compact elements from D such that if C_i and C_j are consistent, then for some k , $C_i \sqcup C_j = C_k$. (This is one possible representation of the compact sets, which makes the preordering simple.) The corresponding compact set will be

$$X_{C_1, \dots, C_n} = \{\alpha \mid \alpha \sqsubseteq C_i \text{ for some } i \leq n\}$$

It is not hard to show that this will be the compact sets, and that they will generate all elements in ${}^m D$.

The preordering on the representation of compact sets is determined from the following fact:

$X_{C_1, \dots, C_n} \sqsubseteq X_{D_1, \dots, D_m}$ if and only if for each $i \leq n$ there is a $j \leq m$ with $C_i \sqsubseteq D_j$.

There will be multiobjects that are pairwise bounded. These will be exactly the multiobjects of the form

$$\{\beta \mid \beta \sqsubseteq \alpha\}$$

for some $\alpha \in D$.

We will call an object of this form *single*, and we will identify it with α .

When X is a multiobject in D , and we write $X \in D$, or $X \in D'$ for some subset D' of D , we mean that X is single, and without further mentioning, we will identify X with its corresponding element in D .

We will of course represent the domain ${}^m D$ of multiobjects using ideals of compacts. Thus, if D is the set of ideals in the iei-structure $(|D|, \leq, lub)$, the elements of ${}^m D$ will be sets of sets $\{C_1, \dots, C_n\}$ where $C_1, \dots, C_n$ are elements of $|D|$ closed under sup, and $sup(X, Y) = X \cup Y$ for $X, Y \in |{}^m D|$.

If $\alpha \in {}^m D$ we may identify α with

$$\{C \mid \exists X \in \alpha (C \in X)\}.$$

With this identification, we see D as a subset of ${}^m D$. Since we want ${}^m D$ to be based on an iei-structure, we will not make this identification explicitly.

5 The domains S and D

We will now define two domains D and S .

D will be a domain of all possible objects that we might be interested in, and we might have used a universal domain based on a saturated iei-structure instead.

The domain S will represent possible typeterms, we call the elements "syntactic forms" for types. As an integrated part of the definition of S , we have an interpretation $I(s)$ of $s \in S$ as a subdomain of D .

We find an analogue construction in the domain of types in [14], and we have used the same main idea in [6, 7, 11].

For technical reasons we will use another representation of the compact elements of a cartesian product than what is standard. We will explain later exactly where we need this seemingly unnecessarily complex representation. In order to give some intuition we can say that since our aim is to use pairs as elements in Σ -types, it will be an advantage to have a representation of the objects where the dependence of the second component from the first component is taken into account.

Definition 7 Let X and Y be two domains generated from iei-structures $|X|$ and $|Y|$ resp.

We represent the compact elements in $X \times Y$ by sets $\{(p_1, q_1), \dots, (p_n, q_n)\}$ where $\{p_1, \dots, p_n\}$ is a bounded set of compacts in $|X|$ and $\{q_1, \dots, q_n\}$ is a bounded set of compacts in $|Y|$.

We let $\{(p_1, q_1), \dots, (p_n, q_n)\} \leq \{(u_1, v_1), \dots, (u_m, v_m)\}$ if $\sup\{p_1, \dots, p_n\} \leq \sup\{u_1, \dots, u_m\}$ and $\sup\{q_1, \dots, q_n\} \leq \sup\{v_1, \dots, v_m\}$

We may then use union as the sup-operator on the representations for compact sets in a cartesian product.

If $\gamma \in |X \times Y|$ we let

$$\pi_0(\gamma) = \{p \mid (\exists q \in |D|)(\exists C \in \gamma)(p, q) \in C\}$$

$$\pi_1(\gamma) = \{q \mid (\exists p \in |D|)(\exists C \in \gamma)(p, q) \in C\}$$

Definition 8 By recursion we define the set $|D|$ with the preordering $\leq$, and then the induced domain D , as follows:

$\perp \in |D|$ and $\perp$ will be the minimal object in $|D|$.

The Boolean values tt and ff will be atomic elements of $|D|$.

Each natural number n will be an atomic element of $|D|$.

If $r = \{(p_1, q_1), \dots, (p_n, q_n)\}$ represents a compact set in $D \times D$, then

$(P, r) \in |D|$. P is just a syntactic entity, meaning "pair".

We let $(P, r_1) \leq (P, r_2)$ if $r_1 \leq r_2$ in the preordering on the representations of compacts in cartesian products.

If C represents a compact in $D \rightarrow {}^m D$, we let $(\lambda, C) \in |D|$. λ is just a syntactic entity.

We define the preordering between the (λ, C) 's in the obvious way.

$(|D|, \leq)$ is extended to an iei-structure, using the obvious interpretation of *sup*.

Essentially, D will be a solution to the domain equation

$$D = \text{Boolean} \oplus \text{Nat} \oplus (D \times D) \oplus (D \rightarrow {}^m D)$$

The domain S will contain atomic objects B and N which will be interpreted as the subdomains of D representing Boolean and Nat. It will also contain the atomic element O representing the empty set.

If $s \in S$ with interpretation $I(s)$ and $F : I(s) \rightarrow {}^m S$ is continuous, then the pair s, F represents the multivalued parameterization $\{I(F(x))\}_{x \in I(s)}$, and S will essentially contain the elements $\sum(s, F)$ and $\prod(s, F)$.

We define the set of compacts $|S|$ with a preorder, and the interpretation $|I(\sigma)|$ as a subset of $|D|$ for each $\sigma \in |S|$ in a simultaneous definition:

Definition 9 Let $\perp_S \in |S|$ with $|I(\perp_S)| = \{\perp\}$.

Let O be an atomic element of $|S|$ with $|I(O)| = \{\perp\}$.

Let B be an atomic element of $|S|$ with $|I(B)| = \{\perp, tt, ff\}$.

Let N be an atomic element of $|S|$ with $|I(N)| = \{\perp, 0, 1, \dots\}$.

If $\sigma \in |S|$, $p_1, \dots, p_n \in |I(\sigma)|$ and $C_1, \dots, C_n \in {}^m S$, then

$$(\prod, \sigma, \{(p_1, C_1), \dots, (p_n, C_n)\})$$

and

$$(\sum, \sigma, \{(p_1, C_1), \dots, (p_n, C_n)\})$$

are in $|S|$.

We let $(\Pi, \sigma, X) \leq (\Pi, \tau, Y)$ and $(\Sigma, \sigma, X) \leq (\Sigma, \tau, Y)$ if $\sigma \leq \tau$ as elements of $|S|$, and $X \leq Y$ as elements of $|D \rightarrow {}^m S|$.

Now let $C_i = \{\tau_{i,1}, \dots, \tau_{i,n_i}\}$ for $i = 1, \dots, n$.

$|I((\Pi, \sigma, \{(p_1, C_1), \dots, (p_n, C_n)\}))|$ contains $\perp$ together with all objects of the form

$$(\lambda, \{(q_1, Y_1), \dots, (q_m, Y_m)\})$$

such that for each j we have

- i) $q_j \in |I(\sigma)|$.
- ii) Let Z_j be the subdomain of D with compacts $|Z_j|$ where $|Z_j|$ is the closure of

$$\cup \{|I(\tau_{i,k})| \mid p_i \leq q_j \wedge 1 \leq k \leq n_i\}$$

under sup.

We then in addition demand that $Y_j \in |{}^d(Z_j)|$.

Let $|I((\Sigma, \sigma, \{(p_1, C_1), \dots, (p_n, C_n)\}))| = \{(P, \{(q_1, v_1), \dots, (q_m, v_m)\}) \in |D| \mid q_j \in |I(\sigma)| \wedge v_j \in |I(Z_j)| \text{ for } j = 1, \dots, m\}$

where Z_j is as constructed above.

If $s \in S$, we let $|I(s)| = \cup \{|I(\sigma)| \mid \sigma \in s\}$.

$(|S|, \leq)$ is extended to an iei-structure in the obvious way.

Lemma 4 a) *The function $\sigma \rightarrow |I(\sigma)|$ is monotone*

- b) *If $p \in |I(s)|$ for some $s \in S$, there is a unique minimal $\sigma(p) \in |S|$ with $p \in |I(\sigma(p))|$ such that $\sup(\sigma(p), \sigma(q)) = \sigma(\sup(p, q))$ whenever the latter is defined.*

Proof

a) is trivial.

b) is proved by induction on the rank of p .

If $p = \perp$, then $\sigma(p) = \perp_S$.

If $p = tt$ or ff , then $\sigma(p) = B$.

If $p = n$ for some natural number n , then $\sigma(p) = N$.

Let $p = (\lambda, \{(q_1, Y_1), \dots, (q_m, Y_m)\})$. Then

$\sigma(p) = (\prod, \sup\{\sigma(q_1), \dots, \sigma(q_m)\}, \{(q_1, E_1), \dots, (q_m, E_m)\})$ whenever this is defined, where $E_j = \{\sigma(q) \mid q \in Y_j\}$.

It is easy to see that $p \in |I(\sigma(p))|$ whenever $\sigma(p)$ is defined, and that the construction of $\sigma(p)$ in this case commutes with least upper bounds.

It remains to show that if

$$p \in |I(\tau)| \text{ with } \tau = (\prod, \sigma, \{(p_1, C_1), \dots, (p_n, C_n)\})$$

then $\sigma(p)$ is defined and $\sigma(p) \leq \tau$.

First we have that $q_j \in |I(\sigma)|$ for $1 \leq j \leq m$ so $\sigma(q_j) \leq \sigma$.

Thus $\sup\{\sigma(q_1), \dots, \sigma(q_m)\}$ exist by the induction hypothesis, and is bounded by σ .

Then we show that E_j is defined and that each element of E_j is bounded by a consistent subset of $\cup\{C_i \mid p_i \leq q_j\}$.

Let $q \in Y_j$. Then $\sigma(q)$ is a typical element of E_j .

Using the notation from the definition, we have $Y_j \in |^d(Z_j)|$ so $q \in |Z_j|$.

By the induction hypothesis, $\sigma(q)$ is defined, and by the definition of Z_j we have that $\sigma(q)$ is bounded by the sup of a consistent set of $\tau_{i,k}$ where $p_i \leq q_j$ and $1 \leq k \leq n_i$.

But this is exactly what we aimed to prove.

If $p = (P, \{(q_1, v_1), \dots, (q_m, v_m)\})$ we let

$$\sigma(p) = (\sum, \sup\{\sigma(q_1), \dots, \sigma(q_m)\}, \{(q_1, E_1), \dots, (q_m, E_m)\})$$

where E_j is as above.

Here we use our special representation of compacts for pairs to see that the definition of $\sigma(p)$ commutes with sup.

The rest of the argument is like the $\prod$ -case.

As a consequence we get

Corollary 3 a) If $s, t \in S$ then $|I(s \cap t)| = |I(s)| \cap |I(t)|$
b) If $s, t \in S$, $\alpha \in I(s)$ and $\beta \in I(t)$, then $\alpha \cap \beta \in I(s \cap t)$.

The proof is trivial.

If $s \in S$, either $s = \{\perp_S\}$ or s will be one of five sorts; s represents the empty type, the type of boolean values, the type of natural numbers, or it represents a $\sum$ - or $\prod$ -type.

If s contains an element of the form $(\prod, \sigma, X)$, we let

$$s_1 = \{\tau \mid (\Pi, \tau, Y) \in s \text{ for some } Y\},$$

and we let

$$F = \{Y \mid (\Pi, \tau, Y) \in s \text{ for some } \tau\}.$$

s_1 will be an element of S , and F will be the domain interpretation of a multivalued function from $I(s_1)$ to S . s_1 and F will determine s completely. In this case we write $s = \Pi(s_1, F_s)$

Likewise, if s contains an element of the form (Σ, σ, X) , we write $s = \Sigma(s_1, F_s)$.

If $s, t \in S$ with $s \subseteq t$, we will use the following notation:

If $\alpha \in I(s)$, we let $[\alpha]_t$ be the closure of α to an ideal in $I(t)$.

Lemma 5 a) *If $s = \Pi(s_1, F_s)$ and $t = \Pi(t_1, F_t)$ with $r = s \cap t$, then*

$$r \in S, r_1 = s_1 \cap t_1 \text{ and } F_r(\alpha) = F_s([\alpha]_{s_1}) \cap F_t([\alpha]_{t_1}) \\ \text{for any } \alpha \in I(s_1 \cap t_1).$$

b) *If s, t and r are as in a),*

$$\alpha_1 \in I(s), \alpha_2 \in I(t) \text{ and } \beta \in I(r_1), \\ \text{then } (\alpha_1 \cap \alpha_2)(\beta) = \alpha_1([\beta]_{s_1}) \cap \alpha_2([\beta]_{t_1}).$$

c) *If $s = \Sigma(s_1, F_s)$ and $t = \Sigma(t_1, F_t)$ with $r = s \cap t$, then*

$$r \in S, r_1 = s_1 \cap t_1 \text{ and } F_r(\alpha) = F_s([\alpha]_{s_1}) \cap F_t([\alpha]_{t_1}) \\ \text{for any } \alpha \in I(s_1 \cap t_1).$$

d) *If s, t and r are as in c), $\alpha \in I(s)$, $\beta \in I(t)$ and $\gamma = \alpha \cap \beta$ then*

$$\pi_0(\gamma) = \pi_0(\alpha) \cap \pi_0(\beta) \\ \pi_1(\gamma) = \pi_1(\alpha) \cap \pi_1(\beta) \cap |I(F_s(\pi_0(\alpha)) \cap F_t(\pi_1(\beta)))|.$$

We prove a) and leave the rest for the reader. The proof of c) is almost identical, the proof of b) follows the same pattern while the proof of d) follows the same idea, but is slightly more complicated.

Proof of a):

Clearly $r_1 \subseteq t_1$.

If $\sigma \in s_1 \cap t_1$, then $(\Pi, \sigma, \emptyset) \in s \cap t = r$, so $\sigma \in r_1$.

We also clearly have

$$F_r(\alpha) \subseteq F_s([\alpha]_{s_1}) \cap F_t([\alpha]_{t_1}).$$

Let $X \in {}^m S$ be in $F_s([\alpha]_{s_1}) \cap F_t([\alpha]_{t_1})$.

Then for some $p_0 \in [\alpha]_{s_1}$ and $p_1 \in [\alpha]_{t_1}$ we have that

$$(\lambda, \{p_0, X\}) \in F_s \text{ and } (\lambda, \{p_1, X\}) \in F_t.$$

Then there is a $p_2 \in \alpha$ with $p_0 \leq p_2$ and $p_1 \leq p_2$.

It follows that $(\lambda, \{p_2, X\}) \in F_s \cap F_t$, so $X \in (F_s \cap F_t)(\alpha) = F_r(\alpha)$.

This complete the proof in case a).

6 Wellfounded types and total objects

We have introduced a system of multivalued functions and multivalued parameterisations. We are however mainly interested in a hierarchy of hereditarily singlevalued total functions and singlevalued, wellfounded parameterisations.

Definition 10 By simultaneous recursion we define the set S_{wf} and $I(s)_{\text{TOT}}$ for $s \in S_{\text{wf}}$ as follows:

$$\begin{aligned} s &= \{\perp_S, O\} \in S_{\text{wf}} \text{ with } I(s)_{\text{TOT}} = \emptyset. \\ s &= \{\perp_S, B\} \in S_{\text{wf}} \text{ with } I(s)_{\text{TOT}} = \{\{\perp, tt\}, \{\perp, ff\}\} \\ s &= \{\perp_S, N\} \in S_{\text{wf}} \text{ with } I(s)_{\text{TOT}} = \{\{\perp, n\} \mid n \text{ is a natural number}\}. \end{aligned}$$

If $s = \prod(s_1, F)$ we let $s \in S_{\text{wf}}$ if $s_1 \in S_{\text{wf}}$ and if $F(\alpha) \in S_{\text{wf}}$ for all $\alpha \in I(s_1)_{\text{TOT}}$.

See the conventions on single objects in section 4 for the precise interpretation of this definition.

If $\gamma \in I(s)$ we let $\gamma \in I(s)_{\text{TOT}}$ if $\gamma(\alpha) \in I(F(\alpha))_{\text{TOT}}$ for all $\alpha \in I(s_1)_{\text{TOT}}$.

If $s = \sum(s_1, F_s)$ we let $s \in S_{\text{wf}}$ if $s_1 \in S_{\text{wf}}$ and if $F(\alpha) \in S_{\text{wf}}$ for all $\alpha \in I(s_1)_{\text{TOT}}$.

Let $\gamma \in I(s)$. Then $\gamma \in I(s)_{\text{TOT}}$ if $\pi_0(\gamma) \in I(s_1)_{\text{TOT}}$ and $\pi_1(\gamma) \in I(F_s(\pi_0(\gamma)))_{\text{TOT}}$.

Theorem 3 Let $s \subseteq t$, $s, t \in S_{\text{wf}}$

a) Let $\alpha \in I(s)_{\text{TOT}}$. Then $[\alpha]_t \in I(t)_{\text{TOT}}$.

b) Let $\beta \in I(t)_{\text{TOT}}$. Then $\beta \cap |I(s)| \in I(s)_{\text{TOT}}$.

Proof:

We use induction on s in S_{wf} .

The cases $s = [O]_S$, $s = [B]_S$ and $s = [N]_S$ are trivial.

Case $s = \prod(s_1, F_s)$, $t = \prod(t_1, F_t)$.

a): Let $\alpha \in I(s)_{\text{TOT}}$. Let $\gamma \in I(t_1)_{\text{TOT}}$.

We will show that $[\alpha]_t(\gamma) \in I(F_t(\gamma))_{\text{TOT}}$.

Using the induction hypothesis we have

$$[\alpha(\gamma \cap |I(s_1)|)]_{F_t(\gamma)} \in I(F_t(\gamma))_{\text{TOT}}$$

so it is sufficient to prove

$$[\alpha]_t(\gamma) = [\alpha(\gamma \cap |I(s_1)|)]_{F_t(\gamma)}.$$

$\supseteq$ is trivial, so let $q \in [\alpha]_t$.

Then there is a $p \in \gamma$ with

$$(\lambda, \{(p, \{q\})\}) \in [\alpha]_t.$$

Further there is a $(\lambda, \{(p_1, X_1), \dots, (p_m, X_m)\}) \in \alpha$ with

$$(\lambda, \{(p, \{q\})\}) \leq (\lambda, \{(p_1, X_1), \dots, (p_m, X_m)\}).$$

Let $K = \{i \mid p_i \leq p\}$. Then there is a bounded set $Y \subseteq \cup\{X_i \mid i \in K\}$ with $q \leq \sup Y$.

Let $p' = \sup\{p_i \mid i \in K\}$. Then $p' \in I(s_1)$ so $p' \in \gamma \cup |I(s_1)|$.

Moreover $q' = \sup Y \in \alpha(\gamma \cap |I(s_1)|)$ and $q \leq q'$ so $q \in [\alpha(\gamma \cap |I(s_1)|)]_{F_t(\gamma)}$.

b): let $\gamma \in I(s_1)_{\text{TOT}}$. Then

$$[\gamma]_{t_1} \in I(t_1)_{\text{TOT}}$$

and

$$\beta([\gamma]_{t_1}) \in I(F_t([\gamma]_{t_1}))_{\text{TOT}}$$

so

$$\beta([\gamma])_{t_1} \cap |I(F_s(\gamma))| \in I(F_s(\gamma))_{\text{TOT}}$$

We will show that

$$\beta([\gamma]_{t_1}) \cap |I(F_s(\gamma))| = (\beta \cap |I(s)|)(\gamma).$$

$\supseteq$ is simple, and is left for the reader.

Let $q \in \beta([\gamma]_{t_1}) \cap |I(F_s(\gamma))|$.

Since $q \in \beta([\gamma]_{t_1})$ there is a $p \in [\gamma]_{t_1}$ with $(\Pi, \{(p, \{q\})\}) \in \beta$, and since $p \in [\gamma]_{t_1}$ there is a $p_1 \in \gamma$ with $p \leq p_1$.

Then $(\Pi, \{(p_1, \{q\})\}) \in \beta$.

Since $q \in |I(F_s(\gamma))|$ there is a $p_2 \in \gamma$ and a finite set X such that $(p_2, X) \in F_s$ and $q \in |I(X)|$.

(where $|I(X)|$ is the closure of $\cup\{|I(r)| \mid r \in X\}$ under sup.)

X has to be singlevalued since $p_2 \in \gamma$, γ is total and $F_s(\gamma) \in S$.

Then $(\lambda, \{(p_2, \{q\})\}) \in |I(s)|$.

Let $p_3 = \sup\{p_1, p_2\}$.

Then $(\lambda, \{(p_3, \{q\})\}) \in |I(s)| \cap \beta$ and $p_3 \in \gamma$ so $q \in (\beta \cap |I(s)|)(\gamma)$.

Case $s = \sum(s_1, F_s)$, $t = \sum(t_1, F_t)$.

a): Let $\alpha \in I(s)_{\text{TOT}}$

Then $[\pi_0(\alpha)]_{t_1} = \pi_0([\alpha]_t)$ so $\pi_0([\alpha]_t) \in I(t_1)_{\text{TOT}}$.

It is sufficient to show

$$\pi_1([\alpha]_t) = [\pi_1(\alpha)]_{F_t([\pi_0(\alpha)]_{t_1})}.$$

$\supseteq$ is easy because $\pi_1([\alpha]_t) \in I(F_t(\pi_0(\alpha)))_{t_1}$.

Let $q \in \pi_1([\alpha]_t)$. Then for some p we have that $(P, \{(p, q)\}) \in [\alpha]_t$ and for some p_1 and q_1 we have that $p \leq p_1$, $q \leq q_1$ and $(P, \{(p_1, q_1)\}) \in \alpha$.

Then $q_1 \in \pi_1(\alpha)$.

It remains to show that $q \in I([F_t(\pi_0(\alpha))]_{t_1})$.

But this is trivial, since $p \in \pi_0([\alpha]_t)$ and $(P, \{(p, q)\}) \in [\alpha]_t$ and a requirement for this last fact is that $q \in |I(F_t(p))|$.

Let $\beta \in I(t)_{\text{TOT}}$.

Then

$$\pi_0(\beta \cap |I(s)|) = \pi_0(\beta) \cap |I(s_1)|$$

and

$$\pi_1(\beta \cap |I(s)|) = \pi_1(\beta) \cap |I(F_s(\pi_0(\beta) \cap |I(s_1)|))|.$$

The second fact requires a small proof following the same line of argument as the proofs above. We leave this proof for the reader.

Corollary 4 a) If $s \subseteq t \subseteq r$, s and r are in S_{wf} and $t \in S$, then $t \in S_{\text{wf}}$.

b) If $s \subseteq t \subseteq r$, $\alpha \subseteq \beta \subseteq \gamma$, $s, r \in S_{\text{wf}}$, $t \in S$, $\alpha \in I(s)_{\text{TOT}}$, $\gamma \in I(r)_{\text{TOT}}$ and $\beta \in I(t)$, then $\beta \in I(t)_{\text{TOT}}$.

Proof:

This is proved by induction on $s \in S_{\text{wf}}$.

Using the above theorem, the steps are trivial.

Definition 11 By a simultaneous recursion we define the binary relation E on S_{wf} and the relation R on

$$\{(s, \alpha) \mid s \in S_{\text{wf}} \wedge \alpha \in I(s)_{\text{TOT}}\}$$

The relations are meant to represent extentional equality between types and extentional equality between objects in extentionally equal types.

We write $[O]$ instead of $\{\perp_S, O\}$ or $[O]_S$ etc. in this definition.

1. $[O] E [O]$
2. $[B] E [B]$, $([B], [tt]) R ([B], [tt])$ and $([B], [ff]) R ([B], [ff])$
3. $[N] E [N]$ and $([N], [n]) R ([N], [n])$ for all numbers n .
4. If $s = \prod(s_1, F_s)$ and $t = \prod(t_1, F_t)$, let
 $s E t$ if $s_1 E t_1$ and for all $\alpha \in I(s_1)_{\text{TOT}}$ and $\beta \in I(t_1)_{\text{TOT}}$ we have that if
 $(s_1, \alpha) R (t_1, \beta)$ then $F_s(\alpha) E F_t(\beta)$.
 If $s E t$ in this case, $\gamma_1 \in I(s)_{\text{TOT}}$ and $\gamma_2 \in I(t)_{\text{TOT}}$ then
 $(s, \gamma_1) R (t, \gamma_2)$ if for all $\alpha \in I(s_1)_{\text{TOT}}$ and $\beta \in I(t_1)_{\text{TOT}}$ we have that if
 $(s_1, \alpha) R (t_1, \beta)$ then $(F_s(\alpha), \gamma_1(\alpha)) R (F_t(\beta), \gamma_2(\beta))$.
5. If $s = \sum(s_1, F_s)$ and $t = \sum(t_1, F_t)$, let
 $s E t$ if $s_1 E t_1$ and for all $\alpha \in I(s_1)_{\text{TOT}}$ and $\beta \in I(t_1)_{\text{TOT}}$ we have that if
 $(s_1, \alpha) R (t_1, \beta)$ then $F_s(\alpha) E F_t(\beta)$.
 If $s E t$ in this case, $\gamma_1 \in I(s)_{\text{TOT}}$ and $\gamma_2 \in I(t)_{\text{TOT}}$ then
 $(s, \gamma_1) R (t, \gamma_2)$ if $(s_1, \pi_0(\gamma_1)) R (t_1, \pi_0(\gamma_2))$ and
 $(F_s(\pi_0(\gamma_1)), \pi_1(\gamma_1)) R (F_t(\pi_0(\gamma_2)), \pi_1(\gamma_2))$.

We will characterise the relations and show that they are equivalence relations. The only obvious property the relations E and R have is symmetry.

Theorem 4 *Let E and R be the relations defined above.*

- a) *Let s and t be elements of S_{wf} .
Then $s E t$ if and only if $s \cap t \in S_{\text{wf}}$*
- b) *Let s and t be in S_{wf} with $s E t$.
Let $\alpha \in I(s)_{\text{TOT}}$ and let $\beta \in I(t)_{\text{TOT}}$.
Then $(s, \alpha) R (t, \beta)$ if and only if $\alpha \cap \beta \in I(s \cap t)_{\text{TOT}}$.*

Proof:

We will prove the theorem by simultaneous induction on the rank of s in S_{wf} . The induction start, when s represents the empty type, the type of boolean values or the type of natural numbers, is trivial, since E and R are the identity-relations in these cases.

Case II:

a): Let $s = \prod(s_1, F_s)$ and $t = \prod(t_1, F_t)$.

By Lemma 5

$$s \cap t = \prod(s_1 \cap t_1, F_{s \cap t})$$

where

$$F_{s \cap t}(\alpha) = F_s([\alpha]_{s_1}) \cap F_t([\alpha]_{t_1})$$

for $\alpha \in I(s_1 \cap t_1)$.

First assume that $s E t$.

By the induction hypothesis, $s_1 \cap t_1 \in S_{\text{wf}}$ and by Theorem 3

$$[\alpha]_{s_1} \in I(s_1)_{\text{TOT}} \text{ and } [\alpha]_{t_1} \in I(t_1)_{\text{TOT}}$$

We further have that $\alpha \subseteq [\alpha]_{s_1} \cap [\alpha]_{t_1}$ so by Corollary 4

$$[\alpha]_{s_1} \cap [\alpha]_{t_1} \in I(s_1 \cap t_1)_{\text{TOT}}.$$

By the induction hypothesis

$$(s_1, [\alpha]_{s_1}) R (t_1, [\alpha]_{t_1})$$

so by assumption

$$F_s([\alpha]_{s_1}) E F_t([\alpha]_{t_1}).$$

Then by the induction hypothesis

$$F_s([\alpha]_{s_1}) \cap F_t([\alpha]_{t_1}) \in S_{\text{wf}}.$$

This shows that $s \cap t \in S_{\text{wf}}$.

Now assume that $s \cap t \in S_{\text{wf}}$.

Then $s_1 \cap t_1 \in S_{\text{wf}}$ so $s_1 E t_1$ by the induction hypothesis.

Let $\alpha \in I(s_1)_{\text{TOT}}$ and $\beta \in I(t_1)_{\text{TOT}}$ such that $(s_1, \alpha) R (t_1, \beta)$.

Then $\alpha \cap \beta \in I(s_1 \cap t_1)_{\text{TOT}}$.

Clearly

$$F_{s \cap t}(\alpha \cap \beta) \subseteq F_s(\alpha) \cap F_t(\beta)$$

Since $s \cap t \in S_{\text{wf}}$ we have that $F_{s \cap t}(\alpha \cap \beta) \in S_{\text{wf}}$.

It follows from Corollary 4 that

$$F_s(\alpha) \cap F_t(\beta) \in S_{\text{wf}}$$

and then $F_s(\alpha) E F_t(\beta)$ by the induction hypothesis. This shows that $s E t$.

The proof of b) follows the same pattern, and is left for the reader.

Case Σ :

The proof of a) is exactly like the proof of a) in case Π .

b): Let $s = \Sigma(s_1, F_s)$ and $t = \Sigma(t_1, F_t)$

Let $\alpha \in I(s)_{\text{TOT}}$ and let $\beta \in I(t)_{\text{TOT}}$.

Assume first that $(s, \alpha) R (t, \beta)$.

By Lemma 5 we have

$$\begin{aligned} \pi_0(\alpha \cap \beta) &= \pi_0(\alpha) \cap \pi_0(\beta) \\ \pi_1(\alpha \cap \beta) &= \pi_1(\alpha) \cap \pi_1(\beta) \cap |I(F_{s \cap t}(\pi_0(\alpha \cap \beta)))| \end{aligned}$$

By the induction hypothesis

$$\pi_0(\alpha \cap \beta) \in I(s_1 \cap t_1)_{\text{TOT}}$$

and

$$\pi_1(\alpha) \cap \pi_1(\beta) \in I(F_s(\alpha) \cap F_t(\beta))_{\text{TOT}}.$$

Then by Theorem 3

$$\pi_1(\alpha) \cap \pi_1(\beta) \cap |I(F_{s \cap t}(\pi_0(\alpha \cap \beta)))| \in I(F_{s \cap t}(\pi_0(\alpha \cap \beta)))_{\text{TOT}}.$$

This shows that $\alpha \cap \beta \in I(s \cap t)_{\text{TOT}}$.

Conversely, assume that $\alpha \cap \beta \in I(s \cap t)_{\text{TOT}}$.

Then

$$\pi_0(\alpha) \cap \pi_0(\beta) \in I(s_1 \cap t_1)_{\text{TOT}}$$

so by the induction hypothesis

$$(s_1, \pi_0(\alpha)) R (t_1, \pi_0(\beta)).$$

Moreover

$$\pi_1(\alpha \cap \beta) \in I(F_{s \cap t}(\alpha \cap \beta))_{\text{TOT}}$$

and

$$\pi_1(\alpha \cap \beta) \subseteq \pi_1(\alpha) \cap \pi_1(\beta) \subseteq \pi_1(\alpha) \in I(F_s(\pi_0(\alpha)))_{\text{TOT}}$$

so

$$\pi_1(\alpha) \cap \pi_1(\beta) \in I(F_s(\pi_0(\alpha)) \cap F_t(\pi_0(\beta)))_{\text{TOT}}.$$

This shows

$$(F_s(\pi_0(\alpha)), \pi_1(\alpha)) R (F_t(\pi_0(\beta)), \pi_1(\beta))$$

which in turn shows $(s, \alpha) R (t, \beta)$. This ends the proof of the theorem.

Lemma 6 *Let s and t be in S_{wf} .*

If $s E t$ and $\alpha \in I(s)_{\text{TOT}}$, then there is a $\beta \in I(t)_{\text{TOT}}$ with $(s, \alpha) R (t, \beta)$

Proof:

Let $\beta = [\alpha \cap |I(s \cap t)|]_t \in I(t)_{\text{TOT}}$.

Then

$$\beta \cap \alpha = \alpha \cap |I(s \cap t)| \in I(s \cap t)_{\text{TOT}}$$

so

$$(s, \alpha) R (t, \beta)$$

by Theorem 4.

Lemma 7 *If s and t are in S_{wf} and $s E t$, then s and t have the same rank in S_{wf} .*

Proof:

We use induction on s . Using Lemma 6, the proof is trivial.

Theorem 5 *The relations E and R are equivalence relations.*

Proof:

Symmetry is obvious from the definition.

Reflexivity follows from Theorem 4.

By Lemma 7, The relation E respect the rank in S_{wf} . Transitivity is proved by induction on this rank.

The induction start, when s represents a base type, is trivial because both relations are the identity relation on the domains in this case.

We prove transitivity of E in the Π -case. Transitivity of R in the Π -case and transitivity of E in the Σ -case will follow by the same argument. Transitivity of R in the Σ -case is trivial, since R is defined by coordinatewise satisfaction in this case.

Let $s = \Pi(s_1, F_s)$, $t = \Pi(t_1, F_t)$ and $r = \Pi(r_1, F_r)$ be in S_{wf} , with $s E t$ and $t E r$.

Let $\alpha \in I(s_1)_{\text{TOT}}$ and $\gamma \in I(r_1)_{\text{TOT}}$ such that $(s_1, \alpha) R (r_1, \gamma)$.

By Lemma 6, let $\beta \in I(t_1)_{\text{TOT}}$ such that $(s_1, \alpha) R (t_1, \beta)$.

Then $(t_1, \beta) R (r_1, \gamma)$ by the induction hypothesis.

Then $F_s(\alpha) E F_t(\beta)$ and $F_t(\beta) E F_r(\gamma)$,

so $F_s(\alpha) E F_r(\gamma)$ by the induction hypothesis.

Then $s E t$ by definition. This ends the proof.

Remark

The relation E will represent extensional equality between types (or rather between the expressions for the types), and R will represent extensional equality between total elements of extensionally equal types. The relations are defined the way they are in order to make this explicit. In [8], Longo and Moggi showed that the continuous functionals of finite types will respect extensional equality without proving first that the total objects of a type is dense in the underlying domain. Their observation was that the intersection $F \cap G$ of two total objects F and G that are extensionally equivalent will be total, and thus for any Φ at the next level,

$$\Phi(F) = \Phi(F \cap G) = \Phi(G)$$

The proof of Theorem 4 is an elaboration of the idea from [8].

Reflexivity of the relations E and R simply means that all wellfounded parameterisations and all total functions will respect extensional equality, and Theorem 4 is essential in proving reflexivity.

Going back to the discussion in Section 3, we see that one of our reasonable axioms of totality is satisfied via Theorem 5. Another reasonable axiom, that a superobject of a total object is total, will not be satisfied, because a superobject might introduce multivaluedness on total input, and we do not want that.

We will see that each equivalence class will contain a unique, maximal element. Thus one important aspect of the notion of *premaximal* is kept.

Theorem 6 a) If $s \in S_{\text{wf}}$, then $\sqcup\{t \in S_{\text{wf}} \mid s E t\} \in S_{\text{wf}}$.

b) If $s \in S_{\text{wf}}$, $t \in S_{\text{wf}}$, $s E t$ and $\alpha \in I(s)_{\text{TOT}}$, then

$$\sqcup\{\beta \in I(t)_{\text{TOT}} \mid (s, \alpha) R (t, \beta)\} \in I(t)_{\text{TOT}}.$$

Proof:

We use induction on the rank of s . Since E and R are relations on S_{wf} and the total objects, the restriction in the statement of the theorem is unnecessary. The base cases are trivial. We prove a) for the Π -case, and leave the rest of the proof for the reader.

The intuitive idea is that we cannot break out of singlevaluedness by taking

unions of equivalent objects. Corollary 4 makes it possible to prove that an object is well founded or total using estimates from both sides, and the estimates from above will be given by the induction hypothesis.

Let $s = \prod(s_1, F_s)$ and let $r = \sqcup\{t \mid s E t\}$.
Then

$$r = \prod(r_1, F_r).$$

and

$$s_1 \sqsubseteq r_1 = \sqcup\{t_1 \mid s E t\} \sqsubseteq \sqcup\{t_1 \mid s_1 E t_1\} \in S_{\text{wf}}$$

so $r_1 \in S_{\text{wf}}$ by Corollary 4.

If $\alpha \in I(r_1)_{\text{TOT}}$, then

$$\begin{aligned} F_s(\alpha \cap |I(s_1)|) &\sqsubseteq F_r(\alpha) \sqsubseteq \sqcup\{F_t(\alpha) \mid s E t\} \sqsubseteq \\ &\sqcup\{t' \mid F_s(\alpha \cap |I(s_1)|) E t'\} \in S_{\text{wf}}, \end{aligned}$$

so $F_r(\alpha) \in S_{\text{wf}}$.

Then $r \in S_{\text{wf}}$.

This completes the proof of the theorem.

Corollary 5 *Each equivalence-class of E , R and of the relations*

$$\alpha R_s \beta \text{ if and only if } (s, \alpha) R (s, \beta)$$

will be lattices under $\sqcup$ and $\sqcap$ that are upwards complete.

The proof is trivial and is left for the reader.

7 Further discussion and results

In this section we will relate the construction of the hierarchy and its properties to previous constructions.

In section 3 we introduced the notion of a *chain* first discussed in Normann [9]. The definition of a chain is based on a conceptual analysis of what totality may be, but in [9], and also in Kristiansen [5], chains are used to obtain technical results, e.g. about strictly positive inductive definitions and

(in the case of [5]) to obtain a density-theorem for a transfinite hierarchy of qualitative domains. This latter application of the concept is also worked out in Kristiansen and Normann [6].

In the present hierarchy it is clear that the total objects of one type cannot be defined simply as the objects meeting a given set of chains, simply because we may have that α is total, $\alpha \subseteq \beta$ but β is not total because β is too much multivalued. Meeting a chain is a positive property. If we were to develop the concept of chains to this situation, we will have not only to represent each question by the chain of working finite bits of procedures , but also with a set of overflow reactions to the question that will be used negatively in order to ensure singlevaluedness.

There does not seem to be any obstacles to a program of developing these concepts, even on the basis of a conceptual understanding like that of [9], but as we have no applications in mind, we will not carry out that program here. The only motivation would be to understand what totality means in the Σ -case, but we think that this is sufficiently illustrated and discussed in [6].

We will now see that the hierarchies of domains from Kristiansen and Normann [6] and Normann [11] can be seen as subhierarchies of the one constructed here. This is not obvious since these hierarchies are based on singlevalued functions, and the hierarchy here is based on multivalued objects, and thus the underlying domains of even the lowest types in the hierarchies, like $\mathbb{N}^{\mathbb{N}}$, are not the same. The underlying domains from [6] will in general not even be subdomains of the corresponding domains here. It will, however be the case that the spaces of equivalence classes essentially will be the same, and without going into too many details we will indicate why.

Let D' and S' be the domains from [6] with $J(s)$ as the interpretation of $s \in S'$.

Lemma 8 *There is a map $\rho : |D'| \rightarrow |D|$ with the following three properties:*

$p \leq' q \Leftrightarrow \rho(p) \leq \rho(q)$.

$\{p, q\}$ is bounded in $|D'| \Rightarrow \{\rho(p), \rho(q)\}$ is bounded in D .

If $q \leq \rho(p)$ then q is equivalent to some $\rho(p')$.

Proof:

Any object in $|D'|$ can be seen as a hereditarily singlevalued object only operating on hereditarily single valued objects. ρ can thus be defined by

recursion adding set-builder parentheses whenever it is appropriate to transform an object to a one-point set. We leave the details for the reader.

ρ cannot be used to form a projection pair, because $\{p, q\}$ will not necessarily be bounded in $|D'|$ when $\{\rho(p), \rho(q)\}$ is bounded in $|D|$.

We still can define the function $f^+ : D' \rightarrow D$ as the ρ -image, and the partial function $f^- : D \rightarrow D'$ as the inverse ρ -image.

Since we are working with domains that consist of sets of objects in a pre-ordering, $f^-(\alpha)$ will be defined as a subset of $|D'|$ for all $\alpha \in D$, and this we will make use of below.

We will call a pair like f^+ and f^- a *weak projection pair*. We will show that there is a system of weak projection pairs that will connect the hierarchy from [6] and the hierarchy of this paper.

Lemma 9 *There is a continuous function*

$$g : S' \rightarrow S$$

and for each $s \in S'$, a weak projection pair (f_s^+, f_s^-) from $J(s)$ to $I(g(s))$.

Proof:

We define g , f_s^+ and f_s^- as the joint fixpoint of a set of equations.

g is defined in the canonical way on the representatives for the types of boolean values and natural numbers.

$g(\Pi(s, F_s)) = \Pi(g(s), G_s)$ where $G_s(\alpha) = g(F_s(f_{s_1}^-(\alpha)))$ for $\alpha \in I(g(s_1))$.

Here we use that $f_{s_1}^-(\alpha)$ actually is a subset X of $|D'|$, that $F_s(X)$ is a subset of $|S'|$ and finally that $g(F_s(X))$ is a subset of $|S|$.

Since S accepts multivaluedness, it is easily seen that this object will be an element of S . Such considerations are lying under the other parts of the definition as well.

In this case

$f_s^+(\gamma)(\alpha) = f_{F_s(f_{s_1}^+(\alpha))}^+(\gamma(f_{s_1}^-(\alpha)))$ for $\alpha \in I(g(s_1))$.

$f_s^-(\gamma)(\alpha) = f_{F_s(\alpha)}^-(\gamma(f_{s_1}^+(\alpha)))$ for $\alpha \in J(s_1)$.

$g(\Sigma(s_1, F_s))$ is defined entirely analogous.

f_s^+ and f_s^- are defined coordinatewise with the obvious typing.

It is tedious, but simple to check that $f_s^-(f_s^+(\gamma)) = \gamma$ for all $\gamma \in J(s)$.

In order to obtain a complete proof, we must verify the equation for $J(\sigma)$ by induction on the rank of $\sigma \in |S'|$.

Theorem 7 *We use the notation from the above lemma.*

- a) $s \in S'_{\text{wf}} \Rightarrow g(s) \in S_{\text{wf}}$.
- b) If $\alpha \in J(s)_{\text{TOT}}$ then $f_s^+(\alpha) \in I(g(s))_{\text{TOT}}$.
- c) If $\beta \in I(g(s))_{\text{TOT}}$ then $f_s^-(\beta) \in J(s)_{\text{TOT}}$.

Proof:

The proof is by induction on the rank of $s \in S'_{\text{wf}}$.

The only problem will be to prove that $f_s^-(\beta)$ is defined when β is total in a product-type. In that case we will use the induction hypothesis and that the total objects are dense in the parameter-type $J(s_1)$ to ensure that no unwanted multivaluedness is present.

Corollary 6 *The hierarchy of topological spaces from [6] (when we have divided out by the consistency relation on S'_{wf}) is a subhierarchy of the corresponding hierarchy of this paper.*

Proof:

The weak projection pairs will respect the relevant equivalence relations in both hierarchies, and they will then induce a typed isomorphism between the equivalence-classes of total objects.

Kleene [4] introduced recursion in higher types, and he also introduced the functional 3E of quantification over Baire-space $\mathbf{N}^{\mathbf{N}}$. Recursion in 3E is a natural transfinite extension of the *analytic* or *2. order definable* sets.

Corollary 7 *The rank of the hierarchy is the first ordinal not recursive in 3E and any real.*

Proof:

We have just showed that the hierarchy from [11] is a subhierarchy of the one developed here. The corollary is thus a consequence of the main theorem from [11], since it is trivial to show that the whole construction can be coded semirecursively in 3E .

The first estimate of the complexity of a hierarchy like this can be found in the unpublished Normann [10]. We will give the construction from [10] here, because the construction is much more natural from a type-theorists point of view than the one in [11]. This proof makes use of the empty type. We will assume that the reader is familiar with the basic definition of Kleene recursion and of 3E . [11] contains sufficient background. For a deeper introduction, see the original paper [4], or Sacks [16].

When we write $\{e\}(\mathbf{f}) = n$, we actually mean that $\{e\}(\mathbf{f}_1, {}^3E, \mathbf{f}_2) = n$ where $\mathbf{f}$ is the concatenation of $\mathbf{f}_1$ and $\mathbf{f}_2$, and the position of 3E is determined by e .

When we write $\{e\}(\mathbf{f}) \downarrow$, we mean that for some n , $\{e\}(\mathbf{f}) = n$.

For the construction below, it will be an advantage to change and extend the notation a bit.

We introduce cartesian products $\times$ and disjoint sums $\oplus$ using products and sums of parameterisations over the type of boolean values.

We also introduce the function space constructor, $\rightarrow$, via constant parameterisations.

In formulating infinite products and sums, it will be an advantage to include the variable of the parameterisation in the notation. Thus, instead of $\prod(s, F)$, we may write $\prod(\alpha \in s)F(\alpha)$, ignoring in the notation the distinction between an $s \in S$ and $I(s)$.

When p is a compact, we let $[p]$ be the closure of p to an ideal.

Theorem 8 *Let e be a natural number, $\mathbf{f}$ a finite sequence of functions on the natural numbers.*

Uniformly in e and $\mathbf{f}$ there is an element

$$T(e, \mathbf{f}) \in S$$

and a continuous map

$$v(e, \mathbf{f}) : D \rightarrow \mathbf{N}$$

such that $\{e\}(\mathbf{f}) \downarrow$ if and only if $T(e, \mathbf{f}) \in S_{\text{wf}}$, and in this case

- $I(T(e, \mathbf{f}))_{\text{TOT}} \neq \emptyset$
- $v(e, \mathbf{f})$ is constant $\{e\}(\mathbf{f})$ on $I(T(e, \mathbf{f}))_{\text{TOT}}$.

Proof:

T and v are defined as the least fixpoints of the following set of equations:

$$\begin{aligned} 1. \quad & \{e\}(x, \mathbf{f}) = x + 1 \\ & T(e, x, \mathbf{f}) = [N] \\ & v(e, x, \mathbf{f})([n]) = x + 1 \end{aligned}$$

2.,3.and 7. All other initial computations are treated the same way.

$$\begin{aligned} 4. \quad & \{e\}(\mathbf{f}) = \{e_1\}(\{e_2\}(\mathbf{f}), \mathbf{f}) \\ & T(e, \mathbf{f}) = \sum(\alpha \in T(e_2, \mathbf{f}))T(e_1, v(e_2, \mathbf{f})(\alpha), \mathbf{f}) \\ & v(e, \mathbf{f})(\gamma) = v(e_1, v(e_2, \mathbf{f})(\pi_0(\gamma)), \mathbf{f})(\pi_1(\gamma)). \end{aligned}$$

$$\begin{aligned} 6. \quad & \{e\}(\mathbf{f}) = \{e_1\}(\sigma(\mathbf{f})) \text{ where } \sigma \text{ is a permutation.} \\ & \text{Let } T(e, \mathbf{f}) = [B] \times T(e_1, \sigma(\mathbf{f})). \\ & v(e, \mathbf{f})(y) = v(e_1, \sigma(\mathbf{f}))(y([tt])). \\ & \text{For technical reasons we want to increase the rank in this case.} \end{aligned}$$

$$\begin{aligned} 8. \quad & \{e\}(\mathbf{f}) = {}^3E(\lambda f \{e_1\}(f, \mathbf{f})). \\ & \text{Let } T_1(f, y) = [N] \text{ and } T_2(f, y) = [O] \text{ if } v(e_1, f, \mathbf{f}) > 0. \\ & \text{Let } T_1(f, y) = [O] \text{ and } T_2(f, y) = [N] \text{ if } v(e_1, f, \mathbf{f}) = 0. \\ & \text{Let} \\ & T(e, \mathbf{f}) = \prod(f \in \mathbf{N}^{\mathbf{N}})((T(e_1, f, \mathbf{f}) \rightarrow [O]) \rightarrow [O]) \\ & \quad \times [\sum(f \in \mathbf{N}^{\mathbf{N}}) \sum(y \in T(e_1, f, \mathbf{f}))T_1(f, y) \\ & \quad \oplus \prod(f \in \mathbf{N}^{\mathbf{N}}) \prod(y \in T(e_1, f, \mathbf{f}))T_2(f, y)]. \\ & \text{Let } v(e, \mathbf{f})(x, r(y)) = 1. \\ & \text{Let } v(e, \mathbf{f})(x, l(y)) = 0. \end{aligned}$$

$$9. \quad \{e\}(e_1, \mathbf{f}) = \{e_1\}(\mathbf{f}) \text{ is treated like 6.}$$

Claim 1

If $\{e\}(\mathbf{f}) \downarrow$ then $T(e, \mathbf{f}) \in S_{\text{wf}}$, $I(T(e, \mathbf{f}))_{\text{TOT}} \neq \emptyset$ and $v(e, \mathbf{f})(\alpha) = \{e\}(\mathbf{f})$ for $\alpha \in I(T(e, \mathbf{f}))_{\text{TOT}}$.

Proof:

We will use induction on the rank of the computation $\{e\}(\mathbf{f})$.

All cases but case 8 are left for the reader.

Let

$$\{e\}(\mathbf{f}) = {}^3E(\lambda f \{e_1\}(f, \mathbf{f})) \downarrow.$$

Then for all $f \in \mathbf{N}^{\mathbf{N}}$ we have that $\{e_1\}(f, \mathbf{f}) \downarrow$, so $T(e_1, f, \mathbf{f}) \in S_{\text{wf}}$ with a nonempty set of total elements.

This shows that

$$\prod(f \in \mathbf{N}^{\mathbf{N}})((T(e_1, f, \mathbf{f}) \rightarrow [O]) \rightarrow [O]) \in S_{\text{wf}}$$

with all its elements total.

Moreover, if $y \in I(T_1(e_1, f, \mathbf{f}))_{\text{TOT}}$, then $T_1(f, y) \in S_{\text{wf}}$ and $T_2(f, y) \in S_{\text{wf}}$.

All together this shows that $T(e, \mathbf{f}) \in S_{\text{wf}}$.

By the induction hypothesis there will be total elements in

$$I(\prod(f \in \mathbf{N}^{\mathbf{N}}) \prod(y \in T(e_1, f, \mathbf{f})) T_2(f, y))$$

if and only if

$$\forall f \in \mathbf{N}^{\mathbf{N}}(\{e_1\}(f, \mathbf{f}) = 0),$$

and there will be total elements in

$$\sum(f \in \mathbf{N}^{\mathbf{N}}) \sum(y \in T(e_1, f, \mathbf{f})) T_1(f, y)$$

if and only if

$$\exists f \in \mathbf{N}^{\mathbf{N}}(\{e_1\}(f, \mathbf{f}) > 0).$$

This shows that v acts as it shall, and claim 1 is proved.

Claim 2

If $T(e, \mathbf{f}) \in S_{\text{wf}}$ then $\{e\}(\mathbf{f}) \downarrow$.

Proof:

We use induction on the rank of $T(e, \mathbf{f})$. In cases S4 and S8 we will need Claim 1 as well.

It is in order to be able to prove this claim that we increase the rank in cases S6 and S9.

In case S8, observe that

$$\prod(f \in \mathbf{N}^{\mathbf{N}})((T(e_1, f, \mathbf{f}) \rightarrow [O]) \rightarrow [O]) \in S_{\text{wf}}$$

will imply that $T(e_1, f, \mathbf{f}) \in S_{\text{wf}}$ for all $f \in \mathbf{N}^{\mathbf{N}}$, so $\{e_1\}(f, \mathbf{f}) \downarrow$ for all $f \in \mathbf{N}^{\mathbf{N}}$.

The rest of the argument is trivial, and is left for the reader.

This ends the proof of the claim and of the theorem.

8 Concluding remarks

The main construction in this paper is based on two principles, the fact that equivalence between two total objects is the same as the intersection being total, and the use of multivaluedness to get all continuous functions defined on the total sets represented in the hierarchy even when the total sets are not dense. The idea to use the first principle evolved during a visit to Munich in spring 1993, when I discussed the possibilities of constructing semantics for type theory with Ulrich Berger and other members of the Munich group. It then seemed natural to rework the material from Normann [10] in this setting. In [10] types and total elements were represented by elements and subsets of $\mathbb{N}^{\mathbb{N}}$ in a Kleene's associates kind of way. The representation based on domains seemed more natural. In addition, equivalence between representatives for types or total object was not an integrated part of the hierarchies defined in [10].

One important concept in [10] is the concept of *typestream*. In the setting of this paper, a typestream will be a certain nonwellfounded element of S for which we still can define the total objects. Typestreams will generalise the class of types defined by strictly positive inductive definitions. We conjecture that most of the properties of S_{wf} can be extended to a hierarchy of typestreams.

In [10] we used recursion in 3E and what we called E-structures to model a second order constructor. Continued work along these lines has indicated that this idea was not very fruitful, we do not obtain models of type theory with second order elements that are better, i.e. contains more information about the underlying theories, than the already known models.

Finally, in [10] we looked at an effective version of the hierarchy constructed there, i.e. a hierarchy where we only used recursive functions to code types and total elements. The result was that the hierarchy then corresponding to S_{wf} will have height ω_1^{CK} , the first non-recursive ordinal, while the corresponding hierarchy with type-streams will have the first recursively inaccessible ordinal as its height. If we define an appropriate effective version of the hierarchy in this paper, we can without difficulty prove that the height will be ω_1^{CK} , and if our constructions can be extended to type-streams at all, there is every reason to believe that the proof from [10] can be transferred to this setting, and that the height of the hierarchy will be the first recursively inaccessible ordinal. This result is a semantical version of the proof-theoretic

estimates of Martin-Löf type theory with induction, see Griffor and Rathjen [3] and Setzer [17].

References

- [1] Berger, U.: *Total sets and objects in domain theory*, Annals of pure and applied logic 60 (1993) 91-117
- [2] Girard, J.-Y.: *The system F of variable types fifteen years later*, Theor. Comp. Science 45 (1986) 159-192
- [3] Griffor, E. and Rathjen, M.: *The Strength of Some Martin-Löf Type Theories*, in manuscript.
- [4] Kleene, S. C.: *Recursive Functionals and Quantifiers of Finite Types I*, T.A.M.S 91 (1959), 1-52
- [5] Kristiansen, L.: *Totality in Qualitative Domains*, Thesis, University of Oslo (1993)
- [6] Kristiansen, L. and Normann, D.: *Semantics for some constructors of type theory*, To appear in the proceedings of the Gauss symposium, der Ludwig-Maximilians-Universität München 1993
- [7] Kristiansen, L. and Normann, D.: *Interpreting higher computations as types with totality*, to appear in Archives for Mathematical Logic
- [8] Longo, G. and Moggi, E.: *The hereditarily partial effective functionals and recursion theory in higher types* JSL 49 (1984) 1319-1332
- [9] Normann, D.: *Formalizing the notion of total information*, In P. P. Petkov (ed), Mathematical Logic, Plenum Press (1990) 67-94
- [10] Normann, D.: *Wellfounded and non-wellfounded types of continuous functionals*, Oslo Preprint Series in Mathematics No 6 (1992)
- [11] Normann, D.: *Closing the gap between the continuous functionals and recursion in 3E* , Presented at the Sacks-symposium, MIT 1993

- [12] Palmgren, E.: *An Information System Interpretation of Martin-Löf's Partial Type Theory with Universes*, Information and Computation 106 (1993) 26-60
- [13] Palmgren, E. and Stoltenberg-Hansen, V.: *Domain interpretations of Martin-Löf's partial type theory*, Annals of pure and applied Logic 48 (1990) 135-196
- [14] Palmgren, E. and Stoltenberg-Hansen, V.: *Remarks on Martin-Löf's partial type theory*, BIT 32 (1992) 70-83
- [15] Stoltenberg-Hansen, V., Lindström, I. and Griffor, E.: *Mathematical Theory of Domains*, Cambridge Tracts in Theor. Comp. Science 22 (1994)
- [16] Sacks, G. E.: *Higher recursion theory*, Springer-Verlag (1990)
- [17] Setzer, A.: *Proof theoretical strength of Martin-Löf Type Theory with W-type and one universe*, Thesis, der Ludwig-Maximilians-Universität München (1993)