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by

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FLUID FLOW IN A MEDIUM DISTORTED BY A QUASICONFORMAL MAP CAN PRODUCE FRACTAL BOUNDARIES

Olli Martio¹⁾ and Bernt Øksendal²⁾

Abstract

Physical experiments indicate that when an expanding fluid flows through a porous rock then the boundary between the wet and the dry region can be very irregular (see e.g. [OMBAFJ] and the references there). In fact, it has been conjectured that this boundary is a fractal with Hausdorff dimension about 2.5.

The (one-phase) fluid flow in a porous medium can be modelled mathematically by a system of partial differential equations, which - under some simplifying assumptions - can be reduced to a family of semi-elliptic boundary value problems involving the (unknown) pressure $p_t(x)$ of the fluid (at the point x and at time t) and the (unknown) wet region U_t at time t . (See (1.5)–(1.7) below). This set of equations, called *the moving boundary problem*, involves the *permeability* matrix $K(x)$ of the medium at x .

A question which has been debated is whether this relatively simple mathematical model can explain such a complicated fractal nature of ∂U_t . More precisely, does there exist a symmetric non-negative definite matrix $K(x)$ such that the solution U_t of the corresponding (expanding) moving boundary problem has a fractal boundary? The purpose of this paper is to prove that this is indeed the case. More precisely, we show that a porous medium which produce fractal wet boundaries can be obtained by distorting a completely homogeneous medium by means of a quasiconformal map.

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FLUID FLOW IN A MEDIUM DISTORTED BY A QUASICONFORMAL MAP CAN PRODUCE FRACTAL BOUNDARIES

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§1. The moving boundary problem of fluid flow in a porous medium.

The basic equations for the flow of an incompressible fluid in a heterogeneous, anisotropic porous medium are the following, (1.1) and (1.2):

$$(1.1) \quad \text{Darcy's law: } \vec{q}_t = -\frac{1}{\mu} K \nabla p_t$$

where $\vec{q}_t(x)$ is the (seepage) velocity of the fluid at time t and at the point $x \in \mathbf{R}^n$, $p_t(x)$ is the pressure of the fluid (the gradient is taken with respect to x), μ is the viscosity and $K = K(x)$ is the *permeability matrix* of the medium. $K(x)$ is a symmetric, non-negative definite matrix of order n , for each x .

$$(1.2) \quad \text{The continuity equation: } \frac{\partial \theta}{\partial t} = -\operatorname{div}(\rho \vec{q}_t) + \xi_t$$

where $\theta = \theta(t, x)$ is the saturation of the fluid, $\xi_t = \xi_t(x)$ is the source/sink rate of the fluid ($\xi \geq 0$ corresponds to source, $\xi \leq 0$ corresponds to sink) and ρ is the density of the fluid. The divergence is taken with respect to x .

If we put $\rho = \mu = 1$ and combine (1.1) and (1.2) we get

$$(1.3) \quad \frac{\partial \theta}{\partial t} = \operatorname{div}(K \nabla p_t) + \xi_t$$

Now let us assume that at any point x and for any time t we have either no fluid at x , i.e. $\theta(t, x) = 0$, or we have complete saturation $\theta_0(x)$ at x , i.e. $\theta(t, x) = \theta_0(x) > 0$. Define

$$(1.4) \quad U_t = \{x; \theta(t, x) = \theta_0(x)\} \quad (\text{the wet region})$$

Then (1.3) gets the following interpretation (see §2):

$$(1.5) \quad \operatorname{div}(K \nabla p_t)(x) = -\xi_t(x); \quad x \in U_t$$

$$(1.6) \quad p_t(x) = 0; \quad x \in \partial U_t$$

$$(1.7) \quad \theta_0 \cdot \frac{d}{dt}(\partial U_t) = -N^T K \nabla p_t \quad \text{on } \partial U_t; \quad U_0 \text{ given}$$

where $N = (N_1(x), \dots, N_n(x))^T$ is the outer unit normal of U_t at $x \in \partial U_t$ and $(\cdot)^T$ denotes transposed. The exact meaning of (1.7) will be explained in §2.

We assume that $\operatorname{supp} \xi_t(\cdot) \subset U_t$ for all $t \geq 0$.

The problem with the system (1.5)–(1.7) is that in many important applications, for example in porous rocks, the permeability $K(x)$ varies rapidly from point to point in an irregular manner. Such functions are of course difficult to handle analytically. Therefore one might be tempted to replace $K(x)$ by its (constant) average $\bar{K}$. However, if U_0 is a ball centered at the origin, if $\xi_t \equiv \xi > 0$ is a radially symmetric source supported in U_0 and K is constant, then it is clear from symmetry that U_t must be a ball for all $t \geq 0$. This is far from what is observed in physical experiments (see e.g. the example in [LØU]). In fact, it has been conjectured that for expanding fluid flow in porous rocks (in $\mathbf{R}^3$) the Hausdorff dimension of ∂U_t is 2.5. This conjecture is based on physical experiments which indicate that such flow can be modelled by DLA processes (see [MFJ], [OMBAFJ]), combined with dimension estimates of such processes based on computer simulations [WS], [M]. In $\mathbf{R}^2$ the conjectured Hausdorff dimension is about 1.7.

In view of this it is natural to ask if it is possible to explain such a complicated fractal nature of ∂U_t by the relatively simple mathematical model (1.5)–(1.7), if we allow the permeability matrix $K(x)$ to be sufficiently irregular.

The purpose of this work is to prove that the answer to this question is yes. More precisely, we want to show that we can construct (mathematically) a medium which produce fractal wet boundaries, by distorting a completely homogeneous (isotropic) medium by means of a quasiconformal map g .

Figure 1

§2. Weak formulation of the (expanding) moving boundary problem

If we want to consider irregular permeability functions $K(x)$ in (1.5)–(1.7) we must find the corresponding weak formulation. To do this we need to formulate some basic assumptions and terminology:

In the following $m(x) \geq 0$ will denote a *p-admissible weight* for some $p, 1 < p < \infty$, in the sense of [HKM]. This means that $m(x)$ satisfies the following conditions (I) - (IV):

- (I) The doubling condition
- (II) The gradient of a Sobolev function is well defined in $L^p(mdx)$
- (III) The weighted Sobolev inequality in $L^p(mdx)$
- (IV) The weighted Poincaré inequality in $L^p(mdx)$

Examples of *p*-admissible weights include the Muckenhoupt A_p weights (with the same p). In our application we will use the weight

$$(2.1) \quad m(x) = J_f(x)^{1-\frac{2}{n}}$$

where J_f is the Jacobian determinant of a quasiconformal map $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$. We emphasize that f is defined (and quasiconformal) everywhere on $\mathbf{R}^n$. This weight is in fact 2-admissible.

For proofs of these statements, exact definitions and more information, we refer to [HKM, Chapters 1 and 15].

Assume from now on that $K(x) \in \mathbf{R}^{n \times n}$ is a symmetric non-negative definite matrix for each $x \in \mathbf{R}^n$ such that

$$(2.2) \quad \begin{aligned} &\text{there exists a } p\text{-admissible weight} \\ &m(x) \geq 0 \quad \text{and constants } c_1, c_2 > 0 \quad \text{such that} \\ &c_1 m(x) |y|^2 \leq y^T K(x) y \leq c_2 m(x) |y|^2 \quad \text{for all } y \in \mathbf{R}^n \end{aligned}$$

If $U \subset \mathbf{R}^n$ is open, we let $H_0^1 = H_0^1(U, m)$ denote the weighted Sobolev space consisting of the closure of $C_0^\infty(U)$ (the smooth functions with compact support in U) with respect to the norm

$$(2.3) \quad \|\psi\|_{H_0^1} = \left[\int_U [\psi^2(x) + |\nabla \psi(x)|^2] m(x) dx \right]^{1/2}; \quad \psi \in C_0^\infty(U)$$

where dx denotes Lebesgue measure on $\mathbf{R}^n$.

We will assume that all sets U considered are subsets of a fixed bounded open set Ω . All functions $\psi \in H_0^1(U, m)$ are regarded as functions on the whole set Ω by the definition $\psi(x) = 0$ for $x \notin U$. The set Ω has been chosen as a general reference set only to avoid working on unbounded domains. It has no mechanical significance.

In the following we will assume that the domains are **expanding**, i.e.

$$(2.4) \quad U_s \subseteq U_t \quad \text{if } 0 \leq s \leq t.$$

This will be the case if e.g. $\xi_t(x) \geq 0$ for all t, x (i.e. $\xi_t(x)$ represents a source rate) and under additional conditions on $K(x)$. We will also assume that $\xi_t(x)$ is bounded with respect to t and x :

$$(2.5) \quad \exists C < \infty \quad \text{s.t.} \quad |\xi_t(x)| \leq C \quad \text{for all } t, x.$$

We now proceed to formulate the weak formulation of the moving boundary problem:

The natural variational interpretation of (1.3) is that $p_t(\cdot) \in H_0^1(U_t, m)$ for each t and

$$(2.6) \quad \begin{aligned} & - \iint \theta(t, x) \psi(x) \varphi'(t) dx dt \\ & = - \iint [\nabla \psi(x)]^T K(x) \nabla p_t(x) \varphi(t) dx dt + \iint \xi_t(x) \psi(x) \varphi(t) dx dt \end{aligned}$$

for all $\varphi \in C_0^\infty(0, \infty)$ and $\psi \in C_0^\infty(\Omega)$.

From the definition (1.4) of U_t we get

$$(2.7) \quad - \iint \theta(t, x) \psi(x) \varphi'(t) dx dt = \int \frac{\partial}{\partial t} \left(\int_{U_t} \theta_0(x) \psi(x) dx \right) \varphi(t) dt$$

Substituting (2.7) in (2.6) and choosing $\varphi(s) = \chi_{[0,t]}(s)$ (which is allowed by a standard approximation procedure) we get

$$(2.8) \quad \int_{U_t \setminus U_0} \theta_0(x) \psi(x) dx = - \int [\nabla \psi(x)]^T K(x) \nabla u_t(x) dx + \int \psi(x) \left(\int_0^t \xi_s(x) ds \right) dx$$

for all $\psi \in C_0^\infty(\Omega)$, where

$$(2.9) \quad u_t(x) = \int_0^t p_s(x) ds \in H_0^1(U_t, m),$$

(u_t is called the Baiocchi transformation of p_t . See e.g. [G], [BG 1–2] and the references therein.)

We call (2.8), (2.9) *the weak formulation of the expanding moving boundary problem* and we call the pair (u_t, U_t) satisfying (2.8) *the weak solution*.

Conversely, suppose we start with a weak solution (w_t, W_t) of (2.8). We will show how to obtain a solution of the variational version of problem (1.5)–(1.7) under additional assumptions:

First assume that $t \rightarrow w_t$ is differentiable, with

$$(2.10) \quad p_t := \frac{dw_t}{dt} \in H_0^1(W_t, m) \quad \text{for all } t.$$

Then from (2.8) we get

$$(2.11) \quad \frac{d}{dt} \left(\int_{W_t \setminus W_0} \theta_0(x) \psi(x) dx \right) = - \int [\nabla \psi(x)]^T K(x) \nabla p_t(x) dx + \int \psi(x) \xi_t(x) dx$$

Next assume that

$$(2.12) \quad t \rightarrow W_t \quad \text{is left continuous,}$$

in the sense that for all compact sets $M \subset W_t$ there exists $\varepsilon > 0$ such that $M \subset W_s$ for all $s \in (t - \varepsilon, t]$. Then choosing $\psi \in C_0^\infty(W_t)$ we get from (2.11)

$$(2.13) \quad \int [\nabla \psi(x)]^T K(x) \nabla p_t(x) dx = \int \psi(x) \xi_t(x) dx$$

which is the variational interpretation of (1.5).

Next assume that $K(x)$, $\nabla p_s(x)$ and ∂W_s are smooth, for all s . Then by the divergence theorem (Green's theorem) we get

$$(2.14) \quad \int_{W_s} \nabla \psi^T K \nabla p_s dx = \int_{\partial W_s} \psi N^T K \nabla p_s d\sigma - \int_{W_s} \psi \operatorname{div}(K \nabla p_s) dx$$

where $d\sigma$ denotes surface measure on ∂W_s . Since

$$\operatorname{div}(K \nabla p_s) = -\xi_s \quad \text{in } W_s \quad \text{by (2.13),}$$

we get by combining (2.11) and (2.14)

$$(2.15) \quad \frac{d}{dt} \left(\int_{W_t} \theta_0(x) \psi(x) dx \right) = - \int_{\partial W_t} \psi(x) [N(x)]^T K(x) \nabla p_t(x) d\sigma(x),$$

which is the precise formulation of (1.7). Note that since $W_0 \subset W_t$ the integral on the left of (2.15) can be taken over the whole of W_t , because of the differentiation with respect to t .

The calculations above justify the following definition:

DEFINITION 2.1 Suppose that conditions (2.2), (2.4) and (2.5) hold. We say that (u_t, U_t) is a *weak solution* of the moving boundary problem (1.5)–(1.7) (in the expanding case) if $u_0(\cdot) = 0$, U_0 is given and

$$(2.16) \quad u_t(\cdot) \in H_0^1(U_t, m) \quad \text{and } u_t \geq 0 \quad \text{for all } t \geq 0$$

and

$$(2.17) \quad \int_{U_t \setminus U_0} \theta_0(x) \psi(x) dx = - \int \nabla \psi^T K \nabla u_t dx + \int \psi(x) \left(\int_0^t \xi_s(x) ds \right) dx$$

for all $\psi \in C_0^\infty(\Omega)$.

This weak solution concept is closely related to one of the definitions given by Y. Reichelt [R]. It may be regarded as an extension to the *heterogeneous, anisotropic* case of the weak solution definition for homogeneous media formulated by Elliott and Janovsky [EJ], Gustafsson [G] and others.

We mention the following result, due to Y. Reichelt:

THEOREM ([R]). There exists a unique weak solution (u_t, U_t) of the expanding moving boundary problem (2.16), (2.17).

Note that the connection between the weak solution u_t of (2.16), (2.17) and the solution p_t of (1.5)–(1.7) is given by the Baiocchi transformation

$$(2.18) \quad u_t(x) = \int_0^t p_s(x) ds$$

§3. Transformation of the weak solution under a quasiconformal map.

Recall that a homeomorphism $g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is called a *quasiconformal mapping* if the coordinate functions g_i of g belong to the classical Sobolev space $H^{1,n}(G)$ for each bounded open set $G \in \mathbf{R}^n$ and if there exists a constant $Q \geq 1$ such that

$$(3.1) \quad \|g'(x)\|^n \leq Q J_g(x) \quad \text{a.e. in } \mathbf{R}^n.$$

Here $H^{1,n}(G)$ denotes the class of functions $f : G \rightarrow \mathbf{R}$ with

$$\int_G (|f|^n + |\nabla f|^n) dx < \infty,$$

where ∇f is the distributional gradient of f ,

$$(3.2) \quad g'(x) = \left[\frac{\partial g_i}{\partial x_j} \right]_{i,j=1}^n$$

is the derivative matrix of g ,

$$(3.3) \quad J_g(x) = \det(g'(x))$$

is the Jacobian determinant of g and, in general,

$$(3.4) \quad \|A\| = \sup_{|h|=1} |Ah|$$

is the usual operator norm of the $n \times n$ matrix A .

Recall that if g is quasiconformal then the inverse mapping $f = g^{-1}$ is quasiconformal as well. For more information see e.g. [HKM] and the references there.

We now pose the following question:

Suppose $(v_t(x) = \int_0^t q_s(x) ds, V_t)$ solves in the weak sense the classical moving boundary problem

$$(3.5) \quad \begin{cases} \Delta q_t(x) = \tilde{\xi}_t(x); & x \in V_t \subset \mathbf{R}^n \\ q_t(x) = 0; & x \in \partial V_t \\ \tilde{\theta}_0 \cdot \frac{d}{dt}(\partial V_t) = -\nabla q_t & \text{on } \partial V_t; V_0 \text{ given} \end{cases}$$

where Δ denotes the Laplacian operator and $\tilde{\xi}_t(x), \tilde{\theta}_0(x)$ are given functions.

Let Ω be a bounded domain containing V_0 and let $g : \Omega \rightarrow \mathbf{R}^n$ be a non-constant quasiconformal map.

Put $f = g^{-1}$ and define

$$(3.6) \quad U_t = g(V_t), \quad u_t(x) = v_t(f(x)) \quad \text{for } x \in U_t. \quad (\text{See figure 1})$$

What moving boundary problem (if any) will $(u_t(x), U_t)$ solve?

Define

$$(3.7) \quad S = J_f \cdot (f')^{-1}((f')^{-1})^T$$

Then we have

LEMMA 3.1. Let $\varphi = \psi \circ f$ where either $\psi \in C_0^\infty(\Omega)$ or $\varphi \in C_0^\infty(g(\Omega))$. Then

$$(3.8) \quad \int_{V_t} [\nabla \psi(y)]^T \nabla v_t(y) dy = \int_{U_t} [\nabla \varphi(x)]^T S(x) \nabla u_t(x) dx$$

Proof. By the integral transformation formula we have, substituting $y = f(x)$,

$$\begin{aligned} \int_{V_t} [\nabla \psi(y)]^T \nabla v_t(y) dy &= \int_{U_t} [\nabla \psi(f(x))]^T \nabla v_t(f(x)) \cdot J_f(x) dx \\ &= \int_{U_t} [\nabla(\varphi \circ g)(f(x))]^T \cdot \nabla(u_t \circ g)(f(x)) \cdot J_f(x) dx \\ &= \int_{U_t} [\nabla \varphi(x)]^T g'(f(x)) g'(f(x))^T \nabla u_t(x) J_f(x) dx \\ &= \int_{U_t} [\nabla \varphi(x)]^T S(x) \nabla u_t(x) dx, \quad \text{as claimed.} \end{aligned}$$

Since (v_t, V_t) solves (3.5) in the weak sense we have by Definition 2.1

$$(3.9) \quad \int_{V_t \setminus V_0} \tilde{\theta}_0(y) \psi(y) dy = - \int [\nabla \psi(y)]^T \nabla v_t(y) dy + \int \psi(y) \left(\int_0^t \tilde{\xi}_s(y) ds \right) dy$$

for all $\psi \in C_0^\infty(\Omega)$. By approximation one sees that (3.9) holds as well for all ψ of the form $\psi = \varphi \circ g$; $\varphi \in C_0^\infty(f(\Omega))$.

Substituting $y = f(x)$ and using Lemma 3.1 we get

$$(3.10) \quad \int_{U_t \setminus U_0} \tilde{\theta}_0(f(x)) \varphi(x) J_f(x) dx = - \int [\nabla \varphi(x)]^T S(x) \nabla u_t(x) dx + \int \varphi(x) J_f(x) \left(\int_0^t \tilde{\xi}_s(f(x)) ds \right) dx$$

Comparing (3.10) with Definition 2.1 we conclude that

THEOREM 3.2 If (v_t, V_t) solves the homogeneous moving boundary problem (3.5) then

$$u_t := v_t \circ f, \quad U_t := g(V_t)$$

solves in the weak sense moving boundary problem (2.16)–(2.17) for an anisotropic, heterogeneous medium with

$$(3.11) \quad \text{permeability matrix} \quad K(x) := S(x) = J_f(x) \cdot f'^{-1}(x) \left(f'^{-1}(x) \right)^T,$$

$$(3.12) \quad \text{source rate} \quad \xi_t(x) := J_f(x) \cdot \tilde{\xi}_t(f(x))$$

and

$$(3.13) \quad \text{maximal saturation} \quad \theta_0(x) := J_f(x) \tilde{\theta}_0(f(x))$$

REMARK 3.3 For the physical interpretation it is necessary to assume that

$$(3.14) \quad 0 \leq \tilde{\theta}_0(y) \leq 1 \quad \text{and} \quad 0 \leq \theta_0(x) \leq 1$$

for all x and y . This will be satisfied if

$$(3.15) \quad 0 \leq \tilde{\theta}_0(y) \leq J_f^{-1}(g(y)) \wedge 1$$

In particular, if $J_f \leq 1$ we can choose $\tilde{\theta}_0 \equiv 1$.

REMARK 3.4 Note that from (3.12) we have

$$(3.16) \quad \int_{V_t} \tilde{\xi}_t(y) dy = \int_{U_t} \xi_t(x) dx \quad \text{for all } t,$$

which means that the amount of incoming fluid in V_t is the same as that in U_t , for all t .

§4. Existence of fractal boundaries of the wet region in a porous medium flow.

We now apply the general transformation formula obtained in §3 to quasiconformal mappings with certain properties:

According to Gehring and Väisälä [GV] there exists a quasiconformal map

$$g : \mathbf{R}^n \rightarrow \mathbf{R}^n$$

such that, if in general $B(x, r) = \{y \in \mathbf{R}^n; |x - y| < r\}$,

$$(4.1) \quad g(B(0, 1)) = B(0, 1)$$

and

$$(4.2) \quad U := g(B(0, r))$$

has a fractal boundary for some $r > 1$.

In fact, given $\varepsilon > 0$ one can choose g such that

$$(4.3) \quad \dim_{\mathcal{H}}(\partial U) \geq n - \varepsilon,$$

where $\dim_{\mathcal{H}}$ denotes Hausdorff dimension.

This construction has been done in [GV] but since we also want the property $J_f \leq 1$ (see Remark 3.3), we explain the construction of g more carefully. Let Q be the unit cube $\{x : 0 \leq x_i \leq 1\}$ of $\mathbf{R}^n$ and let C_s^n , $s = (4^n + 1)^{-1}$, be a Cantor set as in [GV, p. 505] constructed from 2^n disjoint closed cubes Q_i of side length s in $\text{int } Q$ with centers at $Q \cap T$ where T is the line $\{x : x_2 = x_3 = \dots = x_n = \frac{1}{2}\}$, see [GV, p. 510]. Fix a bilipschitz map $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $F(\partial B(0, r)) \supset Q \cap T$ for some $r > 1$ and $F|_{B(0, 1)}$ is a similarity $F(x) = \lambda x + b$ for some $\lambda > 0$ and $b \in \mathbf{R}^n$ with $F(B(0, 1)) \cap Q = \emptyset$. Next given $t, s \leq t < 1/2$, we construct a Cantor map $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}^n$ with $f_0(C_s^n) = C_t^n$ and $f_0 = \text{id}$ outside Q . Also from the proof of [GV, Theorem 5] it follows that $J_{f_0} \geq c = c(t, n) > 0$ a.e. in $\mathbf{R}^n$. This is due to the selfsimilar construction of f_0 and to the fact $s \leq t$. Thus the mapping f_0 does not contract too much in this case.

Let $g = F^{-1} \circ f_0 \circ F$. Since F^{-1} and F preserve the Hausdorff dimension of any set in $\mathbf{R}^n$ and $f_0(C_s^n) = C_t^n$ with

$$\dim_{\mathcal{H}}(C_s^n) = n \frac{\log \frac{1}{2}}{\log s},$$

we conclude that, choosing $t < 1/2$ such that

$$\dim_{\mathcal{H}}(C_t^n) = n \frac{\log \frac{1}{2}}{\log t} > \min(n - \varepsilon, n - 1),$$

the map g satisfies all the required conditions except possibly $J_{g^{-1}} \leq 1$. In fact, we now have $\dim_{\mathcal{H}}(g(\partial B(0, r))) = \dim_{\mathcal{H}}(C_t^n)$. Let $L \geq 1$ be the Lipschitz constant of F . Since

$$J_f = J_{g^{-1}} = 1/J_g \leq L^{2n}/J_{f_0} \leq L^{2n}/c < \infty,$$

the mapping f can be composed with an auxiliary bilipschitz mapping F_1 such that $(F_1 \circ f)^{-1} = g$ has all the desired properties.

Therefore, if we let (v_t, V_t) be the solution of the homogeneous medium moving boundary problem (3.1) with $V_0 = B(0, 1)$ and we choose $\tilde{\xi}_s(x) = \tilde{\xi}(|x|)$ with support in $B(0, 1)$, then for all $t > 0$

$$(4.4) \quad V_t = B(0, r(t))$$

with $t \rightarrow r(t)$ continuous, increasing and $r(0) = 1$. Hence there must exist $t > 0$ such that

$$(4.5) \quad U_t := g(V_t)$$

has a fractal boundary.

FINAL REMARKS

In this paper we have *constructed* a mathematical porous medium (i.e. a permeability matrix $K(x)$ and a complete saturation function $\theta_0(x)$) such that in a (one phase) fluid flow through this medium the boundary of the corresponding wet region is fractal, at least for *some* time $t > 0$ (even though the initial wet region is the unit ball). This indicates that in principle this presented generalized moving boundary model for fluid flow in general heterogeneous porous media (allowing very irregular $K(x)$) may be able to explain the observed fractal nature of the moving wet boundary in experiments.

However, our approach leaves many important questions unanswered:

Q1): It is natural to ask to what extent a given permeability matrix $K(x)$ can be written on the form (3.11) for some suitable quasiconformal f . In other words, given a symmetric, non-negative $n \times n$ matrix $K(x)$ does there exist a quasiconformal map $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that

$$(4.6) \quad K(x) = J_f(x) \cdot f'(x)^{-1} (f'(x)^{-1})^T?$$

For $n \geq 3$ this is not possible in general, because the system of partial differential equations needed for the construction of f is overdetermined.

However, for $n = 2$ the answer to this question is yes, if we add the (necessary) assumption on K that

$$(4.7) \quad \det K(x) = 1.$$

(This is a consequence of (4.6) if $n = 2$).

The argument for this is the following: Since $\det K(x) = 1$ we see that K can be described by just 2 parameters and these are connected to the *dilatation* $\mu(x)$ of f . And from $\mu(x)$ we can construct f . (See [LV]).

Q2): For “how many” t can U_t (given by (4.5)) have a fractal boundary?

If $n = 2$ one can show that U_t must have a *rectifiable* (and hence non-fractal) boundary for *a.a.* $t > 0$ with respect to Lebesgue measure. This follows from the modulus inequality

$$(4.8) \quad M_2(\Gamma) \leq Q M_2(g(\Gamma))$$

(see [LV] and [V]), combined with the fact that the modulus of non-rectifiable curves is equal to zero.

For general $n > 2$ the situation seems to be more complicated.

- Q3): As mentioned in the introduction it is conjectured that for $n = 3$ the Hausdorff dimension of the boundary of the wet region is approximately 2.5. Can the construction of this paper explain this?
- Q4): For $n = 2$ the conjectured Hausdorff dimension of the boundary of the wet region is approximately 1.7. Can the construction of this paper explain this? Since quasiconformal maps in the plane are better known than in the space, this seems to offer an easier problem than Q3.

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