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1. Introduction.

In the present paper we study C*-dynamical systems which are highly non-asymptotically abelian. More specifically we consider a unital C*-algebra A with an automorphism α such that there is a self-adjoint subset S of A which together with the identity spans a dense subset of A , and with the property that the anticommutators $[\alpha^n(w), w^*]_+$ converge to 0 for some properly chosen sequence of n 's depending on w for $w \in S$. It turns out that then there exists a unique α -invariant state ϕ , ϕ restricted to S is zero, and the entropy of α with respect to ϕ in the sense of [ST] is zero. Hence if A is nuclear the entropy in the sense of [CNT] vanishes, thus we have another example of a highly nonabelian C*-dynamical system with vanishing entropy.

Examples of systems as above can be found among the C*-algebras introduced by Powers [P], see also [Pr], in the study of binary shifts of the hyperfinite II_1 -factor. The set S will consist of finite products of self-adjoint unitary operators $\{s_i\}_{i \in \mathbb{Z}}$ with the property that $s_i s_j = \pm s_j s_i$, and α is the shift $\alpha(s_i) = s_{i+1}$. If $w_1, w_2 \in S$ then $w_1 w_2 = \pm w_2 w_1$, hence the C*-subalgebra C of $A \otimes A$ generated by $w \otimes w, w \in S$, is abelian. As pointed out in [AN] $\alpha \otimes \alpha$ restricted to C is the baker's transform so has entropy $\log 2$. It follows that $h_{\phi \otimes \phi}(\alpha \otimes \alpha) \geq h_{\phi \otimes \phi|C}(\alpha \otimes \alpha|_C) = \log 2 > 0 = h_\phi(\alpha) + h_\phi(\alpha)$, hence the tensor product formula " $h_{\phi \otimes \psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta)$ ", see [SV], is false in general.

2. General results.

Let A be a unital C*-algebra, $\alpha \in \text{Aut } A$, and ϕ an α -invariant state. We shall show that if the C*-dynamical system (A, α) is highly nonasymptotically abelian then the entropy $H_\phi(\alpha)$ in the sense of [ST] is zero. For our purposes it is unnecessary to repeat the definition, only the following. Let B be an abelian C*-algebra with an automorphism β , μ a β -invariant state on B , and λ an $\alpha \otimes \beta$ -invariant state on $A \otimes B$ such that $\lambda(a \otimes 1) = \phi(a)$, $\lambda(1 \otimes b) = \mu(b)$ for $a \in A, b \in B$. If $\lambda = \phi \otimes \mu$ then by [ST, 2.1] the "mutual information" $\varepsilon_\lambda(A, B) = 0$, hence the "conditional entropy" $H_\lambda(B|A) = H_\mu(B)$, so by [ST, 3.1] the entropy $h'(B, \lambda) = 0$. If $\lambda = \phi \otimes \mu$ is the only $\alpha \otimes \beta$ -invariant state as above then by [ST, Lemma 3.2] the entropy $H_\phi(\alpha) = 0$.

It was also shown in [ST, Prop. 4.1] that if A is nuclear then $H_\phi(\alpha) = h_\phi(\alpha)$, where $h_\phi(\alpha)$ is the entropy of α with respect to ϕ in the sense of [CNT].

Recall that a subset $S \subset A$ is said to be *self-adjoint* if $a \in S$ implies $a^* \in S$, and *total* if its linear span is norm dense in A . We shall use the notation $[a, b]_+$ for the anticommutator

$$[a, b]_+ = ab + ba, \quad a, b \in A.$$

Theorem 2.1. Let A be a unital C^* -algebra and $\alpha \in \text{Aut}A$. Suppose S is a self-adjoint subset of A such that $S \cup \{1\}$ is total in A and for which the following condition holds:

$$(*) \quad \forall w \in S, \forall \varepsilon > 0, \forall N \in \mathbf{N} \text{ there exist } n_1, \dots, n_N \in \mathbf{N}$$

such that if $i \neq j$ then

$$\|[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+\| < \varepsilon, \quad i, j = 1, \dots, N.$$

Then there exists a unique α -invariant state ϕ . ϕ satisfies $\phi|_S = 0$, and the entropy $H_\phi(\alpha) = 0$.

The theorem generalizes [NT] and is an easy consequence of the following lemma, where we use the notation $\|x\|_{2,\phi} = \phi(x^*x)^{\frac{1}{2}}$ if $x \in A$ and ϕ is a state.

Lemma 2.2. Let A be a unital C^* -algebra, $\alpha \in \text{Aut}A$, and ϕ an α -invariant state. Suppose S is a self-adjoint subset of A for which $S \cup \{1\}$ is total in A and such that the following condition holds:

$$(**) \quad \forall w \in S, \forall \varepsilon > 0, \forall N \in \mathbf{N} \text{ there exist } n_1, \dots, n_N \in \mathbf{N}$$

such that if $i \neq j$ then

$$\|[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+\|_{2,\phi} < \varepsilon, \quad i, j = 1, \dots, N.$$

Suppose B is an abelian C^* -algebra, $\beta \in \text{Aut}B$, and μ a β -invariant state on B . Let λ be an $\alpha \otimes \beta$ invariant state on $A \otimes B$ such that $\lambda(a \otimes 1) = \phi(a)$, $\lambda(1 \otimes b) = \mu(b)$, $a \in A, b \in B$. Then $\lambda|_{S \otimes B} = 0$ and $\lambda = \phi \otimes \mu$.

Proof. Note that if $a \in A, b \in B$ then by the Cauchy-Schwarz inequality

$$\begin{aligned} |\lambda(a \otimes b)| &= |\lambda((1 \otimes b)(a \otimes 1))| \leq \lambda((1 \otimes b)(1 \otimes b)^*)^{\frac{1}{2}} \lambda((a \otimes 1)^*(a \otimes 1))^{\frac{1}{2}} \\ &= \mu(bb^*)^{\frac{1}{2}} \phi(a^*a)^{\frac{1}{2}} = \|a\|_{2,\phi} \|b\|_{2,\mu}, \end{aligned}$$

Furthermore, again since B is abelian

$$[a^* \otimes b^*, a \otimes b]_+ = [a^*, a]_+ \otimes b^*b.$$

Let now $w \in S, b \in B$, where we for simplicity assume $\|b\| \leq 1$. Let $\varepsilon > 0$, and choose $N \in \mathbf{N}$ so large that

$$\frac{1}{N} \lambda([w^*, w]_+ \otimes b^*b) < \varepsilon.$$

Using the inequality $|\rho(x)|^2 \leq \frac{1}{2}\rho([x^*, x]_+)$ for a state ρ , we have for $n_1, \dots, n_N$ as given by (**)

$$\begin{aligned}
|\lambda(w \otimes b)|^2 &= \left| \frac{1}{N} \lambda \left(\sum_{i=1}^N \alpha^{n_i}(w) \otimes \beta^{n_i}(b) \right) \right|^2 \\
&\leq \frac{1}{2N^2} \lambda \left(\left[\sum_i \alpha^{n_i}(w^*) \otimes \beta^{n_i}(b^*), \sum_j \alpha^{n_j}(w) \otimes \beta^{n_j}(b) \right]_+ \right) \\
&= \frac{1}{2N^2} \sum_{i,j} \lambda \left([\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+ \otimes \beta^{n_i}(b^*) \beta^{n_j}(b) \right) \\
&= \frac{1}{2N^2} \sum_i \lambda \left([\alpha^{n_i}(w^*), \alpha^{n_i}(w)]_+ \otimes \beta^{n_i}(b^* b) \right) \\
&\quad + \frac{1}{2N^2} \sum_{i \neq j} \lambda \left([\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+ \otimes \beta^{n_i}(b^*) \beta^{n_j}(b) \right) \\
&\leq \frac{1}{2N} \lambda([w^*, w]_+ \otimes b^* b) + \frac{1}{2N^2} N(N-1) \|[\alpha^{n_i}(w^*), \alpha^{n_j}(w)]_+\|_{2,\phi} \\
&\quad \times \|\beta^{n_i}(b^*) \beta^{n_j}(b)\|_{2,\mu} \\
&< \varepsilon/2 + \frac{1}{2} \varepsilon \|b\| \mu(b^* b)^{\frac{1}{2}} \\
&\leq \varepsilon.
\end{aligned}$$

Since ε is arbitrary, $\lambda(w \otimes b) = 0$, hence $\lambda|_S \otimes B = 0$. In particular $\phi(w) = \lambda(w \otimes 1) = 0$. Thus if $c \in \mathbf{C}$ then

$$\begin{aligned}
\lambda((c1 + w) \otimes b) &= c\lambda(1 \otimes b) = c\mu(b) = \phi(c1 + w)\mu(b) \\
&= \phi \otimes \mu((c1 + w) \otimes b).
\end{aligned}$$

Since $(S \cup \{1\}) \otimes B$ is total in $A \otimes B$, $\lambda = \phi \otimes \mu$.

QED.

Proof of Theorem 2.1. Since the group $\{\alpha^n : n \in \mathbf{Z}\}$ is amenable there exists an α -invariant state ϕ on A . For all $x \in A$ $\|x\|_{2,\phi} \leq \|x\|$, so that condition (**) of Lemma 2.2 follows from (*). If we apply Lemma 2.2 to the case $B = \mathbf{C}$ we conclude that $\phi|_S = 0$, so ϕ is unique by the assumption that $S \cup \{1\}$ is total in A . The conclusion of Lemma 2.2 holds for all triples (B, β, μ) and all λ , hence from the discussion preceding the statement of the theorem, $H_\phi(\alpha) = 0$. QED.

Corollary 2.3. If in Theorem 2.1 A is nuclear then $h_\phi(\alpha) = 0$.

Proof. As remarked before $h_\phi(\alpha) = H_\phi(\alpha)$ if A is nuclear.

Remark 2.4. If we as in Lemma 2.2 assume the existence of the invariant state satisfying (**) then as in the proof of Theorem 2.1 we obtain $H_\phi(\alpha) = 0$.

Remark 2.5. If we in Theorem 2.1 assume S has the property that $z, w \in S$ implies $zw \in S \cup \mathbf{C}1$, and $zw \in \mathbf{C}1$ implies $zw = wz$, then ϕ is a trace. Indeed, if $a, b \in \mathbf{C}$ then

$$\phi((a1 + w)(b1 + z)) = ab + \phi(wz) = \begin{cases} ab & \text{if } wz \in S \\ ab + wz = ba + zw & \text{if } wz \in \mathbf{C}1 \end{cases}$$

hence the assertion follows from the totality of $S \cup \{1\}$.

Remark 2.6. If there exists a tracial state τ on A (e.g. if A is a unital AF-algebra) then the unique invariant state is tracial. Indeed, we can take an invariant mean over $\tau \circ \alpha^n$ to obtain the invariant state.

3. A number theoretic lemma.

In order to find sequences (n_i) as in Theorem 2.1 we need a result on the existence of certain sequences in $\mathbf{N}$. We use the notation

$$[k, m] = \{k, k+1, \dots, m\} \text{ when } k \leq m \text{ in } \mathbf{Z}$$

Lemma 3.1. For each $j \in \mathbf{N}$ there exists a sequence $(x_s^j)_{s \in \mathbf{N}}$ in $\mathbf{N}$ such that $x_1^j = j, j + (j+2)^s < x_s^j - x_{s-1}^j$, and if

$$W_j = \{x_s^j - x_t^j + i : i \in [-j, j], 1 \leq t < s, s, t \in \mathbf{N}\}$$

then the sets W_j are pairwise disjoint.

We shall need an estimate for the growth of the cardinality of unions of sets of the form $W_j \cap [1, m]$ as above.

Lemma 3.2. If $W_1, \dots, W_k$ are constructed as in Lemma 3.1 then

$$\text{card}\left(\bigcup_{j=1}^k W_j \cap [1, m]\right) \leq ((k+1) \log m)^2, \quad m \in \mathbf{N}.$$

Proof. Fix m and let $1 \leq j \leq k$. Let s be chosen as the minimal natural number such that $x_{s+1}^j - x_r^j - j > m$ for all $1 \leq r \leq s$. Then

$$\begin{aligned} \text{card}(W_j \cap [1, m]) &\leq \sum_{r=2}^s \text{card}\{x_r^j - x_t^j + i : i \in [-j, j], 1 \leq t < r\} \\ &= (2j+1) \sum_{r=2}^s (r-1) < (j+1)s^2. \end{aligned}$$

By choice of s

$$x_s^j - x_{s-1}^j - j \leq m$$

by minimality of this element among the numbers $x_s^i - x_{s-1}^i + i$, $i \in [-j, j]$. By assumption then,

$$(j+2)^s \leq x_s^j - x_{s-1}^j - j \leq m,$$

hence

$$s \leq \frac{\log m}{\log(j+2)} < \log m,$$

using that $\log(j+2) \geq \log 3 > 1$. Thus

$$\text{card}(W_j \cap [1, m]) < (j+1)(\log m)^2.$$

It follows that

$$\begin{aligned} \text{card}\left(\bigcup_{j=1}^k W_j \cap [1, m]\right) &\leq \sum_{j=1}^k \text{card}(W_j \cap [1, m]) \\ &< (\log m)^2 \sum_{j=1}^k (j+1) \\ &< ((k+1) \log m)^2. \end{aligned}$$

QED.

Proof of Lemma 3.1. We shall construct the sets W_j by induction on j . If $j = 1$ put $x_1^1 = 1$ and choose $x_s^1 \in \mathbf{N}$ such that $3^s + 1 < x_s^1 - x_{s-1}^1$, and put

$$W_1 = \{x_s^1 - x_t^1 + i : i \in [-1, 1], 1 \leq t < s, s, t \in \mathbf{N}\}$$

Suppose the sequences $(x_s^j)_{s \in \mathbf{N}}$, $j = 1, \dots, p-1$, are chosen such that $x_s^j - x_{s-1}^j > (j+2)^s + j$ and such that the sets $W_1, \dots, W_{p-1}$ are pairwise disjoint.

Put $x_1^p = p$. We first seek x_2^p such that

- (i) $[x_2^p - 2p, x_2^p] \cap \bigcup_{j=1}^{p-1} W_j = \emptyset$
- (ii) $x_2^p > x_1^p + (p+2)^2 + p = p^2 + 6p + 4$.

By Lemma 3.2 $\text{card}\left(\bigcup_{j=1}^{p-1} W_j \cap [1, m]\right) \leq (p \log m)^2$ for all m , hence for m sufficiently large

$$(2p+1) \text{card}\left(\bigcup_{j=1}^{p-1} W_j \cap [1, m]\right) \leq m - (p^2 + 6p + 4)$$

Thus there exists $x_2^p \in \mathbf{N}$, $x_2^p \leq m$ satisfying (ii) such that

$$[x_2^p - 2p, x_2^p] \cap \bigcup_{j=1}^{p-1} W_j = \emptyset,$$

from which (i) follows.

Let $r \geq 2$ and suppose $x_1^p, \dots, x_r^p$ are constructed such that $x_s^p - x_{s-1}^p > (p+2)^s + p$ and such that the sets

$$\{x_s^p - x_t^p + i : i \in [-p, p], 1 \leq t < s\}, \quad 2 \leq s \leq r$$

are disjoint from $\bigcup_{j=1}^{p-1} W_j$.

Let $m_0 = x_r^p + (p+2)^{r+1} + p + 1$. By Lemma 3.2

$$(1) \quad \text{card}([m_0, m] \setminus \bigcup_{j=1}^{p-1} W_j) \geq m - m_0 - (p \log m)^2.$$

Choose by Lemma 3.2 m so large that

$$(x_r^p + p + 1) \text{card}(\bigcup_{j=1}^{p-1} W_j \cap [1, m]) \leq m - m_0 - (p \log m)^2.$$

Then by (1) we can find $x_{r+1}^p \in [m_0, m]$ such that

$$(2) \quad \{x_{r+1}^p - n : n \in [0, x_r^p + p]\} \cap \bigcup_{j=1}^{p-1} W_j = \emptyset.$$

If $1 \leq t \leq r$ we find for $i \in [-p, p]$

$$x_{r+1}^p \geq x_{r+1}^p - x_t^p + i \geq x_{r+1}^p - x_r^p - p = x_{r+1}^p - (x_r^p + p).$$

Thus by (2)

$$\{x_{r+1}^p - x_t^p + i : i \in [-p, p], 1 \leq t \leq r\} \cap \bigcup_{j=1}^{p-1} W_j = \emptyset.$$

This completes the induction, since by choice of m_0 , $x_{r+1}^p - x_r^p > (p+2)^s + p$. QED.

4. The C*-algebras of Powers.

In [P] Powers introduced a class of C*-algebras obtained from binary shifts of the hyperfinite II_1 -factor, see also [Pr]. The definition is as follows. Let $X \subset \mathbf{N}$ be a subset considered as a subset of $\mathbf{Z}$, and let g be its characteristic function. Put

$$\sigma(n) = (-1)^{g(n)}.$$

Changing Powers' definition slightly we let $(s_i)_{i \in \mathbf{Z}}$ be a sequence of self-adjoint unitary operators satisfying the commutation relations

$$s_i s_j = \sigma(|i - j|) s_j s_i.$$

We denote by

$$I = \{i_1 < \cdots < i_r\}$$

the ordered subset $\{i_k : k \in [1, r], i_1 < i_2 < \cdots < i_r\}$, and we denote by

$$w_I = s_{i_1} s_{i_2} \cdots s_{i_r}, \quad w_\emptyset = 1.$$

If $J = \{j_1 < \cdots < j_s\}$ an easy calculation yields

$$w_I w_J = \prod_{k,l} \sigma(|i_k - j_l|) w_\tau w_I$$

Put

$$S = \{w_I : I = \{i_1 < \cdots < i_r\}, I \neq \emptyset\}.$$

Let $A(X)$ denote the C*-algebra generated by the set of s_i , $i \in \mathbf{Z}$. Then $S \cup \{1\}$ is total in $A(X)$. We define $\alpha \in \text{Aut } A(X)$ to be the shift $\alpha(s_i) = s_{i+1}$.

Theorem 4.1. With the above notation there exists $X \subset \mathbf{N}$ such that the C*-dynamical system $(A(X), \alpha)$ satisfies the assumptions of Theorem 2.1.

Proof. For each $j \in \mathbf{N}$ let $(x_s^j)_{s \in \mathbf{N}}$ be the sequence found in Lemma 3.1. Put

$$U_j = \{x_s^j - x_t^j + j : 1 \leq t < s, s, t \in \mathbf{N}\},$$

and put

$$X = \bigcup_{j=1}^{\infty} U_j.$$

Let $I = \{i_1 < \cdots < i_r\}$ and $N \in \mathbf{N}$. We shall find $n_1 < n_2 < \cdots < n_N \in \mathbf{N}$ such that

$$(3) \quad [\alpha^{n_s}(w_I), \alpha^{n_t}(w_I)]_+ = 0 \text{ if } s \neq t.$$

Note that this is sufficient since $w_I^* = \pm w_I$. Since (3) holds if we replace n_s and n_t by $n_s + n$ and $n_t + n$ for any $n \in \mathbf{Z}$, we may assume $1 \leq i_1 < \cdots < i_r$. Since also $[a, b]_+ = [b, a]_+$ for

all a, b , it suffices to show (3) for $n_s > n_t$. Put $j = i_r - i_1$, $n_s = x_s^j$, $n_t = x_t^j$. Then with W_j as in Lemma 3.1 we have

$$\begin{aligned} n_s - n_t + i_\ell - i_m &\in W_j \setminus U_j \text{ if } i_\ell - i_m < j, \\ n_s - n_t + i_r - i_1 &\in U_j. \end{aligned}$$

By Lemma 3.1 the sets W_k are pairwise disjoint, so the only contribution to σ applied to the numbers $n_s - n_t + i_\ell - i_m$ comes from $U_j \subset W_j$. Thus we have

$$\begin{aligned} \prod_{l,m} \sigma(|n_s - n_t + i_\ell - i_m|) &= \prod_{l,m} \sigma(n_s - n_t + i_\ell - i_m) \\ &= \sigma(n_s - n_t + i_r - i_1) = -1. \end{aligned}$$

Thus (3) holds whenever $n_s = x_s^j$, $n_t = x_t^j$. This completes the proof. QED.

By Remark 2.5 or by [P] the unique invariant state ϕ found in Theorem 2.1 is a trace. Also by [P] $A(X)$ is an AF-algebra, hence is nuclear, so the entropies $H_\phi(\alpha)$ and $h_\phi(\alpha)$ coincide. We shall now prove that with X and α as in Theorem 4.1 the tensor product formula for entropy fails for $\alpha \otimes \alpha$ and $\phi \otimes \phi$.

Theorem 4.2. Let $A(X)$ and α be as in Theorem 4.1, and let ϕ be the unique α -invariant trace. Then the tensor product formula fails for $\alpha \otimes \alpha$ and $\phi \otimes \phi$. More specifically we have

$$h_{\phi \otimes \phi}(\alpha \otimes \alpha) \geq \log 2, \quad h_\phi(\alpha) = 0.$$

Proof. Let A_0 denote the C^* -subalgebra of $A(X) \otimes A(X)$ generated by operators of the form $w_I \otimes w_J$. Since $w_I w_J = \pm w_J w_I$, A_0 is abelian. Since A_0 is generated by the self-adjoint unitaries $s_i \otimes s_i$, and $\alpha \otimes \alpha$ is the shift, and the invariant state $\phi \otimes \phi$ vanishes on each $s_i \otimes s_i$, the dynamical system $(A_0, \alpha \otimes \alpha, \phi \otimes \phi)$ is isomorphic to the two shift, or equivalently to the baker's transform, see [AN], and has entropy $\log 2$. Thus

$$h_{\phi \otimes \phi}(\alpha \otimes \alpha) \geq h_{\phi \otimes \phi}(\alpha \otimes \alpha|_{A_0}) = \log 2.$$

By Theorem 2.1 $h_\phi(\alpha) = 0$, so that

$$h_{\phi \otimes \phi}(\alpha \otimes \alpha) > h_\phi(\alpha) + h_\phi(\alpha),$$

proving the theorem. QED.

By [P, Theorem 3.9] the trace ϕ in the above theorem is a factor state. Since in the GNS-representation of ϕ the entropy of [CNT] equals that of [CS] we have

Corollary 4.3. There exists an automorphism α of the hyperfinite II_1 -factor such that the tensor product formula fails for $\alpha \otimes \alpha$.

Remark 4.4. In a recent paper [V] Voiculescu has introduced some alternative definitions of entropy in AF and hyperfinite von Neumann algebras based on approximation of given operators by operators in finite dimensional C^* -subalgebras. For all these entropies Voiculescu shows the inequality

$$“h'_{\tau \otimes \sigma}(\alpha \otimes \beta) \leq h'_\tau(\alpha) + h'_\sigma(\beta)”$$

hence by Theorem 4.2 and Corollary 4.3 his entropies are in general different from the entropies of [CS] and [CNT] considered in the present paper.

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