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Cogitations over Berezin's integral

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(1,0) and (0,1). (the latter does not exist for even manifolds). We prove that Berezin's integral over a s-domain of this kind is well-defined and state a Stokes type theorem. This theorem is extended for the class of a s-domains whose boundary has normal crossing singularity.

Our analysis of the Stokes' theorem implies a whimsical odd part topology on a supermanifold. The (0,1)-part of boundary looks unlike the (1,0)-part. In particular the origin in an odd line $\mathbb{R}^{0,1}$ is a s-domain whose (0,1)-boundary is the line.

1. IMPLICATION OF ODD TOPOLOGY ON AN ODD SPACE

We discuss the conception of the integral starting with the modelling cases.

Even case. Take an even line $\mathbb{R}$. The Riemann integral over the negative halfline $\mathbb{R}_- \subset \mathbb{R}$ of a 1-form w can be calculated by means of the Newton-Leibniz formula

$$(1.1) \quad \int_{\mathbb{R}_-} da = a(0) \equiv \int_{\partial \mathbb{R}_-} a$$

where a is a primitive that vanishes at $-\infty$.

Odd case. Denote Ξ an odd line. Its support is one point space denoted $\odot$ and the corresponding structure algebra is $\mathcal{O} = \mathbb{R}[\xi]/(\xi^2)$. The integration table (0.2) looks differently from that of the even case. Comparing it with (1.1) we rearrange this formula as follows

$$\int_{\Xi} aD(\xi) = \tau(a)(\odot).$$

where $\tau = d/d\xi$. To interpret this equation like (1.1) we assume that the value of the function $\tau(a)$ at $\odot$ is the integral over this point and $\tau(a) = d(\tau \otimes a)$, where $\otimes$ means the tensor product over the field $\mathbb{R}$. Then we get the equation

$$(1.2) \quad \int_{\odot} d(\tau \otimes a) = \int_{\Xi} aD(\xi)$$

which looks similar to (1.1) if we assume that

- *the odd line Ξ is the boundary of the point $\odot$*
- *the restriction of the tensor $\tau \otimes a$ on this boundary is equal $aD(\xi)$.*

Assenting that the boundary of point is an odd line we get an unusual kind of topology. Nevertheless this assumption agrees with the basic algebraic formula

$$\text{str}(A) = \text{tr}(A_{00}) - \text{tr}(A_{11})$$

for supertrace of endomorphism A of a $\mathbb{Z}_2$ -graded vector space V . In particular $\text{str}(E) = n - \nu$ for the identity endomorphism E where $\dim V = (n, \nu)$. This means that an effective dimension of V is equal $n - \nu$! Therefore the boundary operator we have introduced for an odd space has the effective dimension -1 as well as the standard boundary operator on even manifold.

2. INTEGRAL FORMS

We call s-manifold any real supermanifold $(X, \mathcal{O}(X))$ of the class C^∞ . For basic definitions and notations see [B-L, Le, B3]. In particular we denote by $\mathbb{R}^{n,\nu} = (\mathbb{R}^n, \mathcal{O}^{n,\nu})$ a coordinate s-space of dimension (n, ν) with even and odd coordinate functions x, ξ . A chart in a s-manifold X of dimension (n, ν) is an arbitrary embedding of s-manifolds $U \rightarrow X$, where U is an open part of $\mathbb{R}^{n,\nu}$. The functions (x, ξ) are used as local coordinates on X . The sheaf Vol of volume forms is a locally free left $\mathcal{O}$ -sheaf of rank 1. A mapping $\mathcal{O}|U \cong Vol|U, a \mapsto aD(x, \xi)$ is an isomorphism for any chart (x, ξ) . The sections $D(x, \xi)$ are connected by the transformations (0.3).

Definition 2.1 [Bn-L] The sheaf of integral forms of a s-manifold $(X, \mathcal{O})$ is defined as follows

$$\Upsilon = \Upsilon_* = \oplus \Upsilon_i, \quad \Upsilon_{n-\nu-i} = (\Omega^i)' \otimes_{\mathcal{O}} Vol,$$

where Ω^* denotes the $\mathcal{O}$ -sheaf of differential forms on X ; $(\Omega^*)'$ means its left $\mathcal{O}$ -dual and the tensor product taken for the structures of left $\mathcal{O}$ -modules. (We write this product in the opposite order to that of [Bn-L]).

The differential of an integral form is defined in a chart (x, ξ) in the following way: $\delta \otimes aD(x, \xi) \mapsto \delta \cdot da D(x, \xi)$ where $\delta \in (\Omega^*)'$ and the multiplication means the natural pairing $(\Omega^*)' \times \Omega^* \rightarrow \mathcal{O}$.

We display this formula in a coordinate independent form. For this we consider the sheaf $\mathcal{T} \equiv \text{Der } \mathcal{O}(X)$ of germs of tangent fields on X . Recall that a homogeneous element $v \in \mathcal{T}_p, p \in X$ of parity $k \in \mathbb{Z}_2$ is a $\mathbb{R}$ -endomorphism of $\mathcal{O}(X)_p$ satisfying the Leibniz equation

$$v(ab) = v(a)b + (-1)^{kp(a)}av(b), \quad a, b \in \mathcal{O}(X)_p$$

where a is homogeneous of parity $p(a)$. This is a $\mathbb{Z}_2$ -Lie algebra sheaf acting on $\mathcal{O}$. At the other hand the sheaf $\mathcal{T}$ possesses a natural structure of left $\mathcal{O}$ -module that agrees with this representation. The latter means that

$$(2.1) \quad [u, av] = u(a)v + (-1)^{kp(a)}a[u, v], \text{ where } u, v \in \mathcal{T}, a \in \mathcal{O}.$$

We say that $\mathcal{T}$ is a $\mathcal{O}$ -Lie algebra referring to this combination of structures.

The sheaf $\mathcal{T}$ is a locally free $\mathcal{O}$ -module of rank $n + \nu$. For any chart $\theta : U \rightarrow X$ it is generated on $\theta(U)$ by the derivations $\partial/\partial x_i, i = 1, \dots, n; \partial/\partial \xi_j, j = 1, \dots, \nu$; they are called coordinates fields.

By definition we have $\Omega^1 \cong (\mathcal{T})' \equiv \text{Hom}_{\mathcal{O}}(\mathcal{T}, \mathcal{O})$. Therefore $(\Omega^1)' \cong \mathcal{T}$ since $\mathcal{O}$ is a sheaf of regular $\mathbb{R}$ -algebras. Whence $(\Omega^i)^* \cong \Lambda^i \mathcal{T}$ where the exterior product Λ^* is taken over the structure sheaf $\mathcal{O}$ and

$$\Upsilon_{n-\nu-i} \cong \Lambda^i \mathcal{T} \otimes_{\mathcal{O}} Vol$$

Since the $\mathcal{O}$ -sheaves $Vol, \mathcal{T}$ are locally free and soft any integral form $\sigma \in \Upsilon_{n-\nu-i}$ can be written as a finite sum

$$\sigma = \sum v_1 \wedge \dots \wedge v_i \otimes \rho_v$$

where ρ_v is a volume form, $v := (v_1, \dots, v_i)$ are tangent fields on X and the usual commutation rule

$$(2.2) \quad u \wedge v = -(-1)^{p(u)p(v)}v \wedge u$$

is assumed.

Proposition 2.1. ([Le]) *For any s -manifold X there exists an action $L(\cdot)$ of the $\mathbb{Z}_2$ -Lie algebra sheaf $\mathcal{T}$ on Υ with the following properties:*

$$(2.3) \quad L(v)(a\rho) = v(a)\rho + (-1)^{p(a)p(v)}aL(v)\rho,$$

$$(2.4) \quad L(av)\rho = (-1)^{p(a)p(v)}L(v)(a\rho),$$

where $\rho \in \Upsilon$, $a \in \mathcal{O}$ and

$$(2.5) \quad L(v)D(x, \xi) = 0,$$

for any chart (x, ξ) on X and any coordinate field v on the chart.

The equation (2.5) means that $D(x, \xi)$ is always a flat section of the sheaf Vol with respect to this action of $\mathcal{T}$ (called Lie derivative). This property together with (2.3) and (2.4) defines this action uniquely for any chart (x, ξ) . To check that these actions coincide for adjoint charts Leites used an infinitesimal version of the transitivity equation for the Berezinian in (0.3).

Now we write down a formula for the differential.

Proposition 2.2. *For any volume form ρ and tangent fields $v_1, \dots, v_i$ on X we set*

$$(2.6) \quad d(v_1 \wedge \dots \wedge v_i \otimes \rho) = \\ = \sum_{k=1}^i (-1)^{k-1} \left(\sum_{j=k+1}^i (-1)^{\varepsilon(k, j-1)} v_1 \wedge \dots \wedge \hat{v}_k \wedge \dots \wedge v_{j-1} \wedge [v_k, v_j] \wedge v_{j+1} \wedge \dots \wedge v_i \otimes \rho + \right. \\ \left. + (-1)^{\varepsilon(k, i)} v_1 \wedge \dots \wedge \hat{v}_k \wedge \dots \wedge v_i \otimes L(v_k)\rho \right).$$

where $\varepsilon(k, j) = p(v_k)[(p(v_{k+1}) + \dots + p(v_j))]$ is a well-defined differential in the graded sheaf $\Upsilon(X)_*$. For any chart on X it agrees with the differential given by Definition 2.1.

The choice of numbers $\sigma(k, j)$ in this formula agrees with the standard sign convention for any homogeneous elements of $\mathbb{Z}_2$ -graded modules. Note that the action of a field v_k on a density is consistent with the tensor product in virtue of (2.1), (2.3), (2.4). A proof of (2.6) can be done by a routine computation.

It follows that there is defined a sheaf complex of integral forms

$$\Upsilon_* : \dots \xrightarrow{d} \Upsilon_i \xrightarrow{d} \Upsilon_{i+1} \dots \rightarrow \Upsilon_{n-\nu} \rightarrow 0$$

Proposition 2.2 implies that this is a variant of Chevalley-Eilenberg chain complex ([C-E]) for the $\mathcal{O}$ -Lie algebra $\mathcal{T}$ and for the $\mathcal{T}$ -module structure in Vol defined in Proposition 2.1.

The natural pairing

$$\mathcal{T} \times \Omega^1 \rightarrow \mathcal{O}, \quad (v \times w) \mapsto v \triangleright w$$

is a morphism of left $\mathcal{O}$ -modules called *interior product*. For any $i \geq j > 0$ it generates a morphism of $\mathcal{O}$ -modules

$$\Lambda^i \mathcal{T} \times \text{Hom}(\Lambda^j \mathcal{T}, \mathcal{O}) \rightarrow \Lambda^{i-j} \mathcal{T}$$

by the rule

$$(v_1 \wedge \dots \wedge v_i) \times \omega \mapsto \sum_{\sigma} \varepsilon(\sigma) \varepsilon(v, \omega) (v_1 \wedge \dots \wedge v_{i-j}) \omega(v_{i-j+1}, \dots, v_i)$$

where $\varepsilon(v, \omega) = (-1)^{p(\omega)[p(v_{i-j+1}) + \dots + p(v_i)]}$, the sum is taken over the set of all transpositions σ of elements $1, \dots, i$ and the number $\varepsilon(\sigma) = \pm 1$ is defined according to (2.2). The interior product equal zero for $i < j$.

Any tangent field v on X generates Lie derivatives $L(v)$ acting on the sheaf Ω^* :

$$L(v)\omega = d(v \lrcorner \omega) + v \lrcorner d\omega.$$

It commutes with the differential in Ω^* .

Remark. Compare this construction with the de Rham complex $\Omega^*(X)$. The latter looks like Chevalley-Eilenberg cochain complex for the $\mathcal{T}$ -module $\mathcal{O}$:

$$\Omega^*(X) : \mathcal{O} \xrightarrow{\delta} \text{Hom}(\mathcal{T}, \mathcal{O}) \xrightarrow{\delta} \dots \rightarrow \text{Hom}_{\mathcal{O}}(\Lambda^i \mathcal{T}, \mathcal{O}) \xrightarrow{\delta} \text{Hom}_{\mathcal{O}}(\Lambda^{i+1} \mathcal{T}, \mathcal{O}) \rightarrow \dots$$

where

$$\begin{aligned} \delta\omega(v_0 \wedge \dots \wedge v_i) = \\ = \sum_{k=0}^i (-1)^k \left(\sum_{j=0}^{k-1} (-1)^{\pi(k, j+1)} \omega(v_0 \wedge \dots \wedge v_{j-1} \wedge [v_j, v_k] \wedge v_{j+1} \wedge \dots \wedge \hat{v}_k \wedge \dots \wedge v_i) + \right. \\ \left. + (-1)^{\pi(k)} L(v_k)(\omega(v_0 \wedge \dots \wedge \hat{v}_k \wedge \dots \wedge v_i)) \right). \end{aligned}$$

where $\pi(k, j) = p(v_k)[p(v_j) + \dots + p(v_{k-1})]$, $\pi(k) = \pi(k, 0) + p(v_k)p(\omega)$. Therefore there is an $\mathcal{O}$ -bilinear pairing

$$\Upsilon_* \times \Omega^* \rightarrow \Upsilon_*, \quad (v_1 \wedge \dots \wedge v_i \otimes \rho) \times \omega \mapsto (-1)^{p(\rho)p(\omega)} (v_1 \wedge \dots \wedge v_i) \lrcorner \omega \otimes \rho$$

that is consistent with the differentials. As far as I know this complex is not as useful for the integration theory as it is for even manifolds.

3. INTEGRAL

Recall some properties of Berezin's integral. A s -manifold X is called orientated if so is the underlying manifold X^e .

Proposition 3.1. *Let X be an orientated s -manifold. For any volume form α on X with compact support there is defined integral*

$$\int_X \rho$$

that coincides with (0.1) for an arbitrary chart θ if $\text{supp } \rho \subset \theta(U)$ and U^e is endowed with the induced orientation. This integral satisfies the equation

$$(3.1) \quad \int_X d\sigma = 0$$

for any integral form $\sigma \in \Gamma_c(X, \Upsilon_{n-\nu-1})$.

Corollary 3.2. *For any $\rho \in \Gamma_c(X, Vol)$ and any tangent field v on X we have*

$$\int_X L(v)\rho = 0.$$

This equation follows from (3.1) if we set $\sigma = v \otimes \rho$.

Proof of Proposition 3.1. Take an arbitrary volume form ρ on X with compact support and choose a finite set $\{\theta_i : U_i \rightarrow X, i \in I\}$ of charts such that the open sets $\theta_i(U_i)^e$ cover the set $\text{supp } \rho$. Then we choose a decomposition $\rho = \sum \rho_i$ such that $\text{supp } \rho_i$ is a compact subset of $\theta_i(U_i)^e$ for any $i \in I$. Set

$$\int \rho := \sum_i \int_{U_i} \theta_i^*(\rho_i).$$

where each term in the righthand side is defined by means of (0.1). The sum does not depend on the choice of charts and of the decomposition of the volume form in virtue of (0.3).

Check (3.1) for a form $\sigma = v \otimes \rho$ where ρ is a density with compact support. Take a covering $\{\theta_i\}$ of $\text{supp } \rho$ and a decomposition $\rho = \sum \rho_i$ as above. Find $d\sigma = \sum L(v)\rho_i$ whence

$$\int d\sigma = \sum \int_{U_i} \theta_i^*(L(v)\rho_i) = \sum \int_{U_i} L(\theta_i^*(v))\theta_i^*(\rho) = 0.$$

The integrals in the righthand side vanish since of [Le, Lemma 2.4.8] and (3.1) follows.

4. "STOKES" THEOREM

Recall more basics ([B3, ch.IV]). Let $(X, \mathcal{O}(X))$ be a s-manifold. For any point $p \in X$ the stalk $\mathcal{O}_p$ of the structure sheaf $\mathcal{O} = \mathcal{O}(X)$ is a $\mathbb{Z}_2$ -commutative local $\mathbb{R}$ -algebra with $\mathbb{Z}_2$ -grading $\mathcal{O}_p = \mathcal{O}_p^0 \oplus \mathcal{O}_p^1$ equipped with the *residue* morphism $\text{res}_p : \mathcal{O}_p \rightarrow \mathbb{R}$ that sends a function germ a to $a(p)$ (hence vanishes on $\mathcal{O}_p^1$).

Let $\mathcal{I}$ be a sheaf ideal in $\mathcal{O}(X)$, i.e. a $\mathcal{O}$ -subsheaf of $\mathcal{O}$. The $\mathbb{R}$ -subspace S of X determined by $\mathcal{I}$ is the subspace $S^e \subset X^e$ of common zeros of functions $a \in \mathcal{I}$ endowed with the algebra sheaf $\mathcal{O}(S) := \mathcal{O}(X)/\mathcal{I}$. This is a $\mathbb{Z}_2$ -graded ringed space if the sheaf $\mathcal{I}$ is generated locally by homogeneous elements of $\mathcal{O}$. This subspace is called closed if so is S^e . The space $(S, \mathcal{O}(S))$ is a s-submanifold if the sheaf ideal $\mathcal{I}$ is generated locally by homogeneous elements $g_1, \dots, g_m \in \mathcal{O}^0(X); \gamma_1, \dots, \gamma_\mu \in \mathcal{O}^1(X)$ such that the differentials $dg_1, \dots, dg_m; d\gamma_1, \dots, d\gamma_\mu$ are linearly independent. The couple (m, μ) is called codimension of S .

In particular the underlying manifold X^e for X is the closed s-submanifold of X determined by the sheaf ideal $\mathcal{I}$ generated by $\mathcal{O}^1(X)$.

Definition 4.1 A morphism of s-manifolds (or s-morphism) $g : (Y, \mathcal{O}(Y)) \rightarrow (X, \mathcal{O}(X))$ is a mapping $g^e : Y^e \rightarrow X^e$ of underlying s-manifolds together with a morphism of $\mathbb{Z}_2$ -graded sheaves of $\mathbb{R}$ -algebras $G : g^{e*}(\mathcal{O}(X)) \rightarrow \mathcal{O}(Y)$ on Y such that for any $q \in Y$ the relation $\text{res}_p = \text{res}_q G_q p = g(q)$ holds.

Domains A s-domain in a s-manifold X is circumscribed by means of fences like a domain in even geometry. The difference is that the boundary of the former consists of two parts: one of codimension $(1, 0)$ another of codimension $(0, 1)$. The construction of the first one is close to the conventional one.

Definition 4.2. We call *even fence* on a s-manifold X any even function f on X such that $df \neq 0$ on the subset $f^e = 0$ (where f^e means the restriction of f on X^e). We say that even fences f, f' are equivalent if $f' = af$ for some function a such that $a^e > 0$. We call a fence system on X any system F of fences defined on open submanifolds $U_i \subset X$ such that

(i) $\cup U_i = X$; (ii) for any adjoin sets U_i, U_j the corresponding fences are equivalent on $U_i \cap U_j$. We say that an even fence system F on X *fences a s-domain* $Z = X_F$; any equivalent fence system fences the same s-domain. This s-domain is underlaid by the domain $Z^e := \{f^0(p) \leq 0, p \in X^e\}$. In a more formal way we mean s-domain to be the ringed $\mathbb{Z}_2$ -space $(Z^e, \mathcal{O}(X)|_{Z^e})$ endowed with the class of equivalent fence systems.

Take an even fence system F and consider the system of equations $f = 0, f \in F$. The solution is a smooth closed s-submanifold of X of codimension $(1, 0)$ underlaid by the manifold $\partial Z^e = \{f^0(p) = 0\}$. For any equivalent fence system we get the same s-submanifold. We call it *even boundary* of Z and denote $\partial^0 Z$. There is a natural immersion of $\mathbb{Z}_2$ -ringed spaces $b_0 : \partial^0 Z \rightarrow Z$.

If X is orientated we use for Z^e the induced orientation. However we endow the boundary $\partial^0 Z$ with a nonstandard orientation that is equal to the standard one times the factor $(-1)^\nu$, i.e. the orientation of $\partial^0 Z$ is given by a frame $(-1)^\nu w_1 \wedge \dots \wedge w_{n-1}$ if $df^e \wedge w_1 \wedge \dots \wedge w_{n-1}$ is an orientation frame for X .

An odd counterpart of this construction is the following. We call *odd fence* on a s-manifold X any odd function φ on X that satisfies the same condition as above $d\varphi \neq 0, \varphi = 0$. Two odd barriers φ, φ' are called equivalent if $\varphi' = \alpha\varphi$ for some function $\alpha \neq 0$. Here we need not to assume that the function α^e is positive since the integral (1.2) does not need orientation of the odd line. An odd fence system Φ on X is defined by the above conditions (i),(ii). In fact any odd fence system fences a submanifold $Y = X_\Phi$ in X of codimension $(0, 1)$ and vice versa a $(0, 1)$ -submanifold is fenced by a class of equivalent odd fence systems. The manifold X is considered as *odd boundary* of Y and denoted $\partial^1 Y$. To keep a symmetry with the even case we assume that there exists a morphism $b_1 : \partial^1 Y \rightarrow Y$ such that $b_1|_Y = \text{id}(Y)$. Now we combine both constructions:

Definition 4.3 Let even and odd fence systems F, Φ be given on a s-manifold X . We call Z *s-domain fenced* by these systems the s-domain $Z = X_{F, \Phi}$ fenced by F in the $(0, 1)$ -submanifold X_Φ that is fenced by Φ . The boundary of Z consists of two parts

$$\partial Z = \partial^0 Z \cup \partial^1 Z$$

where $\partial^0 Z := \{f = 0, f \in F; \varphi = 0, \varphi \in \Phi\}$ is the even boundary of Z in s-manifold X_Φ and $\partial^1 Z = X_F$ is the s-domain in X fenced by the system F . There is a natural immersion $b_0 : \partial^0 Z \rightarrow Z$ and we assume that there exists a morphism $b_1 : \partial^1 Z \rightarrow Z$ such that $b_1|_Z = \text{id}(Z)$. We call b_0, b_1 *boundary mappings*.

Note that the double boundaries $\partial^1 \partial^0 Z, \partial^0 \partial^1 Z$ coincide and we have the follow-

ing commutative diagram

$$\begin{array}{ccc} \partial^0 \partial^1 Z = \partial^1 \partial^0 Z & \xrightarrow{b_{01}} & \partial^1 Z \\ \downarrow b_{10} & & \downarrow b_1 \\ \partial^0 Z & \xrightarrow{b_0} & Z \end{array}$$

where b_{01} , b_{10} are corresponding boundary morphisms. Note that $\partial^0 Z$ has no even boundary and $\partial^1 Z$ has no odd boundary.

Suppose that X is orientated. Then the domain $X_{F,\Phi}$ and parts of its boundary are equipped with induced orientations as above. Therefore the double boundaries $\partial^1 \partial^0 Z$, $\partial^0 \partial^1 Z$ are equipped with opposite orientations.

Integral forms on s-domain. Let $Z = X_{F,\Phi}$ be a s-domain in s-manifold X of dimension (n, ν) as above. We have the commutative diagram of immersions

$$\begin{array}{ccc} Z = X_{F,\Phi} & \xrightarrow{(0,0)} & X_\Phi \\ (0,1) \downarrow & & (0,1) \downarrow \\ X_F & \xrightarrow{(0,0)} & X \end{array}$$

where the codimensions of the immersions are shown. Define

$$(4.1) \quad \Upsilon_{n-\nu-1-i}(Z) \cong \Lambda^i \mathcal{T}(X_F) \otimes_{\mathcal{O}(Z)} \text{Vol}(X_\Phi).$$

where the sheaf $\Lambda^* \mathcal{T}(X_F) := \Lambda^* \mathcal{T}(X)|_{X_F}$ is considered as $\mathcal{O}(Z)$ -sheaf via the algebra morphism $b_1^* : \mathcal{O}(Z) \rightarrow \mathcal{O}(X_F)$. We call $\Upsilon(Z) = \oplus \Upsilon_j(Z)$ the sheaf of volume forms on the domain Z . It possesses a structure of left $\mathcal{O}(X)$ -module.

Proposition 4.1. *There exists an action $L(\cdot)$ of the Lie algebra sheaf $\mathcal{T}(X)$ on $\mathcal{O}(X)$ -module $\text{Vol}(X_\Phi)$ that satisfies the conditions (2.3), (2.4) and (2.5) for any chart (x, ξ) on X_Φ and any coordinate field v on this chart.*

It follows from Proposition 2.1. We can define a differential d on that sheaf by the formula (2.6) where L means the action of the sheaf $\mathcal{T}(X)$.

Pullbacks We call a mapping $\theta : U \rightarrow X$ a *even (odd) fence chart* for a s-domain Z if it is a chart and the coordinate function $x_1(\xi_1)$ is a local fence for Z . For any integral form σ on Z we construct its pullbacks with regard to the boundary morphisms. First take a volume form $\sigma = v_1 \wedge \dots \wedge v_i \otimes D(X_\Phi)$ on Z and set

$$b_0^*(\sigma) = (-1)^{\nu-1} (v_1 \wedge \dots \wedge v_i) \triangleright dx_1 \otimes D(\partial^0 Z)$$

in any even fence chart (x, ξ) . The result is well-defined as an element of $\Upsilon(\partial^0 Z)$. We set analogously

$$b_1^*(\sigma) = (-1)^{\nu-1} (v_1 \wedge \dots \wedge v_i) \triangleright d\xi_1 \otimes D(X)$$

for any odd fence chart and the result is in $\Upsilon(X_F)$. We denote here by $D(X)$, $D(X_\Phi)$ and $D(\partial^0 Z)$ the generator of the corresponding volume sheaf related to the corresponding chart θ ; $\theta|_{\xi_1} = 0$ and $\theta|_{x_1} = 0$, $\xi_1 = 0$.

Integral on s-domain Suppose that X is an orientated s-manifold and F is an even fence system on it. Define for arbitrary form $\rho \in \Gamma_c(X, Vol(X))$ an integral

$$(4.2) \quad \int_{X_F} \rho = \sum \int_{U_{i-}} \theta_i^*(\rho_i)$$

where $\{U_i\}$ is a covering of $\text{supp } \rho$ by even fence charts and $\rho = \sum \rho_i$ a corresponding decomposition of the form (cf. Proposition 3.1).

Proposition 4.2. *The integral (4.2) does not depend on the choice of fence charts and of decomposition of the volume form.*

A proof follows from Corollary 6.2.

Summarize the aforesaid. Let Z be a s-domain in X given by even and odd fence systems F, Φ . Taking a form $\sigma \in \Gamma_c(Z, \Upsilon_{n-\nu-1}(Z))$ we can define an integral of $b_1^*(\sigma)$ over $\partial^1 Z$, an integral of $b_0^*(\sigma)$ over the manifold $\partial^0 Z$ and an integral of $d\sigma$ over Z . Now we formulate

Stokes Theorem for smooth boundary. *Let Z be a s-domain in an orientated s-manifold X of dimension (n, ν) . Then for any $\sigma \in \Gamma_c(Z, \Upsilon_{n-\nu-1}(Z))$ the following equation holds*

$$(4.3) \quad \int_Z d\sigma = \int_{\partial^0 Z} b_0^*(\sigma) + \int_{\partial^1 Z} b_1^*(\sigma)$$

We omit a proof which is a routine.

5. GENERALIZATION

The conventional analysis includes Stokes' theorem for domain with singular boundary. In fact the class of domains with normal crossing singular points is big enough for many applications. We say that a domain Z in an even manifold X has normal crossing points (nc-domain) if it is locally diffeomorphic to a modelling singular domain $\mathbb{R}_-^k \times \mathbb{R}^n$. The boundary ∂Z is again a domain with normal crossing boundary points.

An odd counterpart of above construction can be given in the following way. Instead of $\mathbb{R}_-$ we consider a couple $\mathcal{O} = (\odot, \Xi)$ where $\odot$ is the origin in the odd line Ξ . We call it *a point with boundary in odd line*. Next we exchange the direct product by a *bouquet* operation applied to $\mathcal{O}$. This is a closed singular $\mathbb{Z}_2$ -subspace

$$B^\kappa(\mathcal{O}) \subset \mathbb{R}^{0,\kappa} \quad \mathcal{O}(B^\kappa(\mathcal{O})) := \mathcal{O}(\mathbb{R}^{0,\kappa})/I,$$

where I is the ideal generated by the monomials $\xi_i \xi_j$, $i \leq j$. Taking a geometrical point of view we image the bouquet as an union of coordinate lines in an odd space $\mathbb{R}^\kappa$. Really these lines are sub-s-manifolds of this space. Combining even and odd singularities we take a modelling singular s-space in the form

$$(5.1) \quad \mathbb{R}_-^k \times B^\kappa(\mathcal{O}) \times \mathbb{R}^{n,\nu}$$

with some integers k, κ, n, ν .

Definition 5.1 We call Z s-domain with *normal crossing boundary* (ncs-domain) any $\mathbb{Z}_2$ -graded $\mathbb{R}$ -ringed space $(S, \mathcal{O}(S))$ that is locally isomorphic to a modelling space M of the form (5.1).

We call a ncs-domain W *semiregular* if it is locally isomorphic to a modelling space (5.1) with $\kappa \leq 1$. Any ncs-domain Z is locally a bouquet of semiregular ncs-domains $Z(j)$ over an underlying nc-domain. Therefore we can define an integral $\int_Z \sigma$ for an orientated ncs-domain Z to be a sum of integrals $\int_{Z(ij)} \sigma_i$ over its local semiregular parts $Z(ij)$. The latter is calculated by a reduction to an integral over the set (5.1) with $\kappa = 0$ which is a s-domain according to Sect.4. To check that the result does not depend on charts and on decomposition of the form $\sigma = \sum \sigma_i$ we can use a corresponding generalization of Theorem 6.1.

The boundary ∂Z of an orientated ncs-domain Z is an union of ncs-domains with the induced orientations. Note that it can be decomposed on the whole in union two parts $\partial Z = \partial^0 Z \cup \partial^1 Z$ of codimension (1,0) and (0,1) correspondingly.

Stokes Theorem for singular boundary. *Let Z be an orientated ncs-domain. Then an arbitrary integral form $\alpha \in \Gamma_c(Z, \Upsilon(Z))$ the equation (4.3) still holds.*

We omit details which are routine.

6. CHANGE OF VARIABLES IN BEREZIN'S INTEGRAL

Here we prove that the Berezin's formula is valid also for change of coordinates in s-domain.

Fix an open subspace $U \subset \mathbb{R}^{n,\nu}$ with coordinates x, ξ and consider s-domain U_- with the fence function x_1 . This domain is underlaid by the set $U_-^e = \{x \in U^e, x_1 \leq 0\}$. Take an arbitrary differential form w on U with compact support and define an integral over this domain as follows

$$\int_{U_-} w := \int_{U_-^e} (\varepsilon)^* [L(\tau_\nu) \dots L(\tau_1)(w)],$$

where $\tau_i := \partial/\partial \xi_i$ $i = 1, \dots, \nu$ and $\varepsilon : U^e \rightarrow U$ is the canonical embedding. In particular for a form $w = a\Lambda(dx) := a dx_1 \wedge \dots \wedge dx_n$ we get

$$(6.1) \quad \int_{U_-} a\Lambda(dx) := \int_{U_-^e} (\varepsilon)^* [L(\tau_\nu) \dots L(\tau_1)(a)]\Lambda(dx) \equiv \int_{U_-} aD(x, \xi)$$

since the form $\Lambda(dx)$ vanishes under action of the coordinate fields.

Theorem 6.1. *Let $V_- \subset \mathbb{R}^{n,\nu}$ be an open subspace with coordinates (y, η) , V_- be the s-domain with fence function y_1 and $\lambda : U \rightarrow V$ be another fence chart for U_- . Then for any function $a \in \Gamma_c(V, \mathcal{O}^{n,\nu})$ we have*

$$\int_{V_-} a\Lambda(dy) = \int_{U_-} a \det[(\partial y/\partial x) - (\partial y/\partial \xi)(\partial \eta/\partial \xi)^{-1}(\partial \eta/\partial x)] \det^{-1}(\partial \eta/\partial \xi) \Lambda(dx)$$

Here and later we write $a, dy, \dots$ instead of $\lambda^*(a), \lambda^*(dy), \dots$.

Corollary 6.2. *The equation holds*

$$\int_{V_-} aD(y, \eta) = \int_{U_-} a\text{Ber}(y, \eta|x, \xi)D(x, \xi)$$

for any couple of equivalent fence charts.

This follows from (6.1) and Theorem 6.1 .

Remark. The similar assertion is valid for any nc-domain of the form $U \cap (\mathbb{R}_-^k \times \mathbb{R}^{n, \nu})$.

Lemma 6.3. *The equation*

$$\int_{V_-^e} L(v)w = 0$$

holds for any form $w \in \Gamma_c(V, \Omega)$ and any tangent field v on V that satisfies the equations

$$(6.2) \quad v(\eta_1) = \dots = v(\eta_\nu) = 0, \quad v(y_1) = ey_1, \quad e \in \Gamma(V, \mathcal{O})$$

Lemma 6.4. *The equation*

$$\int_{V_-} y_1 w + dy_1 \wedge w' = 0$$

holds for any $w, w' \in \Theta \wedge \Gamma_c(V, \Omega^*)$, where Θ is the linear span of forms $d\eta_1, \dots, d\eta_\nu$.

Proof of Theorem. First we note that the equivalence of the fences implies the following equation

$$(6.3) \quad y_1 = cx_1, \quad c \in \Gamma(V, \mathcal{O}), \quad c|V^e > 0.$$

Changing variables we get equations

$$dy = (\partial y / \partial x)dx + (\partial y / \partial \xi)d\xi, \quad d\eta = (\partial \eta / \partial x)dx + (\partial \eta / \partial \xi)d\xi.$$

where $dx, dy, \dots$ are thought columns and find

$$dy - (\partial y / \partial \xi)(\partial \eta / \partial \xi)^{-1}d\eta = B_0 dx,$$

where

$$B_0 := (\partial y / \partial x) - (\partial y / \partial \xi)(\partial \eta / \partial \xi)^{-1}(\partial \eta / \partial x).$$

Consider the product

$$(6.4) \quad \Lambda(dy - (\partial y / \partial \xi)(\partial \eta / \partial \xi)^{-1}d\eta) = \det B_0 \Lambda(dx)$$

and show that it can be substituted for $\Lambda(dy)$ in the integral over V_- . For this we note that the first factor is equal

$$(6.5) \quad dy_1 - \sum_{i,j} \partial y_1 / \partial \xi_i \delta_i^j d\eta_j$$

where δ_i^j are the entries of the matrix $(\partial\eta/\partial\xi)^{-1}$. We derive from (6.3) that

$$(6.6) \quad \partial y_1 / \partial \xi_i = x_1 \partial c / \partial \xi_i = y_1 c_1, \quad c_1 \in \Gamma_c(V, \mathcal{O})$$

Therefore the second term in (6.5) belongs to $y_1 \Theta \wedge \Gamma_c(V, \Omega^*)$ and we conclude that

$$\Lambda(dy - (\partial y / \partial \xi)(\partial\eta/\partial\xi)^{-1}d\eta) - \Lambda(dy) \in y_1 \Theta \wedge \Gamma_c(V, \Omega^*) + dy_1 \wedge \Theta \wedge \Gamma_c(V, \Omega^*).$$

According to Lemma 6.4 the integral of this form over V_- vanishes. Thus

$$(6.7) \quad \begin{aligned} \int_{V_-} a \Lambda(dy) &= \int_{V_-} a \det B_0 \Lambda(dx) \\ &\equiv \int_{V_-^e} L(\partial/\partial\eta_\nu) \dots L(\partial/\partial\eta_1) a B_0 \Lambda(dx). \end{aligned}$$

We can change the domain V_-^e to U_-^e in the righthand side in virtue of (6.3). Then we find $\partial/\partial\xi = (\partial\eta/\partial\xi)^t \partial/\partial\eta + (\partial y/\partial\xi)^t \partial/\partial y$, where the $(\cdot)^t$ means the conjugated matrix. Whence

$$\partial/\partial\eta_j = \sum_i \delta_j^i [\partial/\partial\xi_i - \sum_k \partial y_k / \partial \xi_i \partial/\partial y_k], \quad j = 1, \dots, \nu.$$

We can use this equation in (6.7) and omit each term $v_k := \partial y_k / \partial \xi_i \partial/\partial y_k$ since of Lemma 6.3. Really this field vanishes identically on y_1 for any $k > 1$. The field v_1 satisfies the condition (6.2) because of (6.6). Therefore

$$L(\partial/\partial\eta_\nu) \dots L(\partial/\partial\eta_1) a \Lambda(dx) = \delta_\nu \partial/\partial\xi \dots \delta_1 \partial/\partial\xi (a) \Lambda(dx)$$

since $L(\partial/\partial\xi)dx = 0$. Then we use the following identity

$$\delta_1 \partial/\partial\xi \dots \delta_\nu \partial/\partial\xi = \partial/\partial\xi_1 \dots \partial/\partial\xi_\nu B_1 \quad B_1 := \det(\partial\eta/\partial\xi)^{-1}.$$

This identity was proved in an equivalent form in [B2]. Therefore the form to be integrated in (6.7) is equal to

$$L(\partial/\partial\xi_1) \dots L(\partial/\partial\xi_\nu) a B_1 B_0 \Lambda(dx),$$

since B_1 is an even function. Taking in account that $B_0 B_1 = \text{Ber}(y, \eta|x, \xi)$ we complete the proof of Theorem 6.1

Proof of Lemma 6.3 We have $L(v)w = d(v \triangleright w) + v \triangleright dw$. No term of $v \triangleright dw$ contains the product $dx_1 \wedge \dots \wedge dx_n$, hence $\varepsilon^*(v \triangleright dw) = 0$. The form $v \triangleright w$ vanishes on the boundary of $\partial V_- = \{y_1 = 0\}$ because of (6.2). The integral of $d(v \triangleright w)$ over V_-^e equals zero since of the Gauss-Ostrogradski theorem.

Proof of Lemma 6.4. We start with the formula

$$\begin{aligned} L(s)(u \wedge u') &= L(s)u \wedge u' + (-1)^{kp(u)} u \wedge L(s)u' + \\ &+ (-1)^{\deg(u)} s \triangleright u \wedge du' + (-1)^{\deg(u) + kp(u)} du \wedge s \triangleright u', \end{aligned}$$

where u, u' are arbitrary differential forms and s is any tangent field of parity k . This implies an equation for any odd coordinate tangent field τ and any element $d\zeta \in \Theta$:

$$L(\tau)(u \wedge d\zeta) = L(\tau)u \wedge d\zeta + (-1)^{\deg(u)+p(u)}\tau(\zeta)du,$$

where $\tau \triangleright d\zeta = \tau(\zeta)$ is a constant. Therefore the second term of the righthand side vanishes. Applying this formula once for another odd coordinate field τ' we get

$$L(\tau')L(\tau)(u \wedge d\zeta) = L(\tau')L(\tau)u \wedge d\zeta \pm \tau'(\zeta)L(\tau)du \pm \tau(\zeta)L(\tau')du$$

and so on. Then we apply the restriction morphism ε^* . It vanishes on the factor $d\zeta$. Hence the integral over V_-^e of any form that contains this factor vanishes. Other terms are of the form

$$(6.8) \quad L(\tau)L(\tau') \dots du$$

for some odd coordinate fields $\tau, \tau', \dots$. Now we assume that

$$(6.9) \quad u = y_1 w + dy_1 \wedge w', \quad w, w' \in \Gamma_c(V, \Omega^*).$$

and show that the integral of the form (6.8) over V_-^e vanishes. This will imply Lemma 6.4. We have

$$(6.10) \quad du = d(y_1 w + dy_1 \wedge w').$$

For any odd coordinate field τ the form $\tau \triangleright du$ admits the representation (6.9) with some w, w' , hence $L(\tau)du \equiv d(\tau \triangleright du)$ is of the form (6.10) again and so on. Arguing inductively we conclude that any term (6.8) belongs to the class (6.10). The equality

$$\int_{V_-^e} d(y_1 w + dy_1 \wedge w) = 0$$

is once more a corollary of the Gauss-Ostrogradski Theorem since the form (6.9) vanishes on the boundary of the domain.

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