ISBN 82-553-0907-1
Pure Mathematics

No.12
May 1994

**An equation modelling transport of a
substance in a stochastic medium**

by

J. Gjerde, H. Holden, B. Øksendal,
J. Ubøe and T. Zhang

AN EQUATION MODELLING TRANSPORT OF A SUBSTANCE IN A STOCHASTIC MEDIUM

Jon Gjerde⁽¹⁾, Helge Holden⁽²⁾, Bernt Øksendal⁽¹⁾
 Jan Ubøe⁽³⁾ and Tusheng Zhang⁽³⁾

- (1) Department of Mathematics, University of Oslo
 Box 1053, Blindern, N-0316 Oslo 3, Norway
- (2) Department of Mathematical Sciences, Norwegian Institute of Technology,
 N-7034 Trondheim, Norway
- (3) Department of Mathematics, National College of Safety Engineering
 Skåregaten 103, N-5500 Haugesund, Norway

Abstract

We find an explicit expression for the (unique) solution $u = u(t, x, \omega)$ of the stochastic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\nu^2\Delta u + \vec{W}_x \diamond \nabla u; \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^d$$

$$u(0, \cdot) = f(\cdot)$$

where Δ and ∇ are the Laplacian and gradient operators respectively, with respect to $x = (x_1, \dots, x_d)$ and $\vec{W}_x$ is d -dimensional white noise in the d parameters $(x_1, x_2, \dots, x_d)$. The symbol $\diamond$ denotes the (vector) Wick product, the use of which corresponds to an Ito/Skorohod interpretation of the equation.

This equation occurs in many situations. For example, it models the transport of a substance in a turbulent (stochastic) medium.

§1 INTRODUCTION

A substance dissolved in a moving fluid/medium in $\mathbf{R}^d$ is exposed to both a molecular diffusion and to a drift coming from the movement of the fluid. If the fluid is turbulent, a natural model for its velocity at the time t and the point x is d -dimensional white noise $\vec{W}_x$ in the d parameters $(x_1, \dots, x_d)$, $\omega \in \Omega$ is a random parameter. The concentration $u(t, x, \omega)$ of the substance at (t, x) will then satisfy the stochastic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\nu^2\Delta u + \vec{W}_x \cdot \nabla u; \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}^d \quad (1.1)$$

where $\nu > 0$ is the molecular viscosity of the medium. We assume that the concentration at time $t = 0$ is a known, random function $f(x, \omega)$:

$$u(0, x, \omega) = f(x, \omega); x \in \mathbf{R}^d \quad (1.2)$$

Equations of the type (1.1)-(1.2) have been studied by several authors. See e.g. [Ch],[NZ] and [Po]. However, there are several differences from these papers and ours (see §2 for the precise definitions):

a) We will adopt the functional process interpretation of the equation (see e.g. [HLØUZ 1] and [HLØUZ 2]). This means that we smoothen the singular white noise $\vec{W}_x$ by an x -shifted test function ϕ and consider the solution $u = u^\phi(t, x, \omega)$ of the corresponding equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\nu^2 \Delta u + \vec{W}_{\phi_x} \cdot \nabla u \quad (1.3)$$

where $\phi_x(y) = \phi(y-x)$ and $\vec{W}_{\phi_x}$ is the white noise smoothed by ϕ . This smoothing is not just a technical convenience; physically it represents taking a macroscopical average so as to obtain a more realistic model. For example, the support of ϕ will determine the maximal distance within which there is a correlation between the fluid velocities.

b) We interpret the product $\cdot$ in (1.3) as a **Wick product** $\diamond$. This corresponds to considering the equation (1.3) in the Ito/Skorohod sense. Previous authors have all adopted the Stratonovich interpretation, corresponding to using the usual, pointwise product in (1.3). (See e.g. [HLØUZ 1, p.398] for further explanation). Our interpretation leads to the equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\nu^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u \quad (1.4)$$

$$u(0, x, \omega) = f(x, \omega) \quad (1.5)$$

c) We allow the initial value $f(x, \omega)$ to be stochastic and anticipating.

The purpose of this paper is to give a rather explicit formula for the solution $u(t, x, \omega)$ of the equation (1.4)-(1.5). Moreover, we show that the solution is unique as an element of the space $(\mathcal{S})^*$ of generalized white noise distributions (Hida distributions).

§2 SOME PRELIMINARIES ON WHITE NOISE

Here we briefly recall some of the basic definitions and results from white noise calculus. For more information the reader is referred to [HKPS]. In the following we fix the parameter dimension d and let $\mathcal{S} = \mathcal{S}(\mathbf{R}^d)$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbf{R}^d$. The dual space $\mathcal{S}^* = \mathcal{S}^*(\mathbf{R}^d)$ is

the space of tempered distributions. By the Bochner-Minlos theorem there exists a probability measure μ on the Borel subsets $\mathcal{B}$ of $\mathcal{S}^*$ with the property that:

$$\int_{\mathcal{S}^*} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\phi\|^2}, \phi \in \mathcal{S}(\mathbf{R}^d) \quad (2.1)$$

where $\langle \omega, \phi \rangle$ denotes the action of $\omega \in \mathcal{S}^*$ on $\phi \in \mathcal{S}$ and $\|\phi\|^2 = \int_{\mathbf{R}^d} |\phi(x)|^2 dx$. The triple $(\mathcal{S}^*, \mathcal{B}, \mu)$ is called the **white noise space**. The (1-dimensional) **white noise process** is the map $W : \mathcal{S} \times \mathcal{S}^* \rightarrow \mathbf{R}$ defined by:

$$W(\phi, \omega) = W_\phi(\omega) = \langle \omega, \phi \rangle; \omega \in \mathcal{S}^*, \phi \in \mathcal{S} \quad (2.2)$$

Expressed in terms of Ito integrals with respect to d -parameter Brownian motion we have:

$$W_\phi(\omega) = \int_{\mathbf{R}^d} \phi(x) dB_x(\omega); \phi \in \mathcal{S} \quad (2.3)$$

By the Wiener-Ito chaos theorem, every element $X \in (L^2) = L^2(\mathcal{S}^*(\mathbf{R}^d), \mu)$ admits a chaos decomposition:

$$X = \sum_{n=0}^{\infty} I_n(f^{(n)}) \quad (2.4)$$

Here I_n denotes a multiple Ito integral of order n and the integrands $f^{(n)}$ belong to $\hat{L}^2((\mathbf{R}^d)^{\otimes n})$, i.e. the symmetric L^2 -space, see e.g. [HKPS]. The (L^2) norm of X with this expansion is given by:

$$\|X\|_{(L^2)}^2 = \sum_{n=0}^{\infty} n! \|f^{(n)}\|_{L^2((\mathbf{R}^d)^{\otimes n})}^2 \quad (2.5)$$

Consider the (densely defined) selfadjoint operator $A: L^2(\mathbf{R}) \rightarrow L^2(\mathbf{R})$:

$$Af(x) = -f''(x) + (1+x^2)f(x) \quad (2.6)$$

Let $\mathcal{P} \subset (L^2)$ denote the algebra generated by functionals on the form $\langle \cdot, \phi \rangle$; $\phi \in \mathcal{S}(\mathbf{R}^d)$ and for every integer p let $(\mathcal{S})_p$ denote the completion of $\mathcal{P}$ with respect to the norm:

$$\|X\|_{2,p}^2 := \sum_{n=0}^{\infty} n! \|(A^{\otimes nd})^p f^{(n)}\|_{L^2(\mathbf{R}^{nd})}^2 \quad (2.7)$$

The Hida test function space $(\mathcal{S})$ is then the projective limit of the spaces $(\mathcal{S})_p$. For $\xi \in \mathcal{S}(\mathbf{R}^d)$ the Wick exponential $Exp[W_\xi]$ is defined by the expression:

$$Exp[W_\xi] = \sum_{n=0}^{\infty} I_n(\xi^{\otimes n}) \quad (2.8)$$

It is easy to see that $Exp[W_\xi] \in (\mathcal{S})$ and it turns out that the following holds:

$$Exp[W_\xi] = exp(W_\xi - \frac{1}{2}\|\xi\|^2) \quad (2.9)$$

The space $(\mathcal{S})^*$ of Hida distributions is the dual space of $(\mathcal{S})$. The $\mathcal{S}$ -transform of a element $X \in (\mathcal{S})^*$ is a functional on $\mathcal{S}(\mathbf{R}^d)$ defined by:

$$SX(\xi) = \langle X, Exp[W_\xi] \rangle; \xi \in \mathcal{S}(\mathbf{R}^d) \quad (2.10)$$

where $\langle, \rangle$ denotes the dual pairing between $(\mathcal{S})^*$ and $(\mathcal{S})$. The $\mathcal{S}$ -transform plays an important role in the study of white noise analysis. The generalized functionals are completely determined by their $\mathcal{S}$ -transforms. For our purpose we recall the following definition and two propositions from [PS]:

Definition 2.1

A complex valued function F on $\mathcal{S}(\mathbf{R}^d)$ is called a U -functional if for every $\xi, \eta \in \mathcal{S}(\mathbf{R}^d)$, the mapping $\lambda \rightarrow F(\eta + \lambda\xi)$, $\lambda \in \mathbf{R}$, has an entire analytic extension, denoted by $F(\eta + z\xi)$, $z \in \mathbf{C}$, and moreover there exist constants K and $p \geq 0$ such that:

$$|F(z\xi)| \leq Ke^{K|z|^2\|\xi\|_{2,p}^2}, z \in \mathbf{C} \quad (2.11)$$

where $\|\xi\|_{2,p} := \|(A^{\otimes d})^p \xi\|_{L^2(\mathbf{R}^d)}$.

Proposition 2.1

A complex valued function F on $\mathcal{S}(\mathbf{R}^d)$ is the $\mathcal{S}$ -transform of a element in $(\mathcal{S})^*$ if and only if F is a U -functional.

Proposition 2.2

Let $X_n, X \in (\mathcal{S})^*$. Then $X_n \rightarrow X$ in $(\mathcal{S})^*$ if and only if the following conditions are satisfied:

$$SX_n(\xi) \rightarrow SX(\xi) \quad (2.12)$$

$$|SX_n(z\xi)| \leq Ke^{K|z|^2\|\xi\|_{2,p}^2} \quad (2.13)$$

where the constants K and p do not depend on n .

Before we close this section, let us recall the definition of the **Wick product** [HKPS], [GHLØUZ]. Let F, G be generalized functionals with the $\mathcal{S}$ -transforms $SF(\xi)$ and $SG(\xi)$. The Wick product of F and G , denoted by $F \diamond G$, is defined to be the unique element in $(\mathcal{S})^*$ with the $\mathcal{S}$ -transform $SF(\xi)SG(\xi)$. Interpreting the products in a stochastic differential equation as Wick products corresponds

to interpreting the equation in the sense of Ito/Skorohod: One can express the Ito/Skorohod integral in a striking way by using the Wick product. See e.g. [B], [LØU 1], [LØU 2].

Multidimensional white noise.

Finally we mention that to get the m -dimensional white noise, we need to work on the product probability space:

$$(\Omega = \prod_{i=1}^m \mathcal{S}^*(\mathbf{R}^d), \mathcal{B} = \otimes_{i=1}^m \mathcal{B}(\mathcal{S}^*(\mathbf{R}^d)), \mu_m = \prod_{i=1}^m \mu)$$

If $\omega = (\omega_1, \omega_2, \dots, \omega_m) \in \Omega$ and $\phi = \phi_1 \otimes \phi_2 \otimes \dots \otimes \phi_m$ with $\phi_i \in \mathcal{S}(\mathbf{R}^d)$ we define **m-dimensional white noise** $\vec{W}_\phi = (W_{\phi_1}^1, W_{\phi_2}^2, \dots, W_{\phi_m}^m)$ by

$$\vec{W}_\phi(\omega) = (\langle \omega_1, \phi_1 \rangle, \langle \omega_2, \phi_2 \rangle, \dots, \langle \omega_m, \phi_m \rangle) \in \mathbf{R}^m \quad (2.14)$$

All the machinery we used on white noise space $(\mathcal{S}^*(\mathbf{R}^d), \mathcal{B}(\mathcal{S}^*(\mathbf{R}^d)), \mu)$ carries over to the above product space $(\Omega, \mathcal{B}, \mu_m)$. The details can be found in [Gj].

§3 AN EQUATION FROM TURBULENT TRANSPORT

We now return to the stochastic partial differential equation which arises in modelling the transport of a substance in a turbulent medium:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2} \nu^2 \Delta u + \vec{W}_{\phi_x} \diamond \nabla u \\ u(0, x, \omega) &= f(x, \omega) \end{aligned} \quad (3.1)$$

where $u = u(t, x, \omega)$, $(t, x, \omega) \in \mathbf{R}_+ \times \mathbf{R}^d \times \Omega$ and

$$\vec{W}_{\phi_x} = (W_{\phi_{1,x_1}}^1, W_{\phi_{2,x_2}}^2, \dots, W_{\phi_{d,x_d}}^d)$$

is the d -dimensional white noise. Here $\phi \in \mathcal{S} := \bigotimes_{i=1}^d \mathcal{S}(\mathbf{R}^d)$ and $\phi_{j,x_j}(\cdot)$ is the x -shift of ϕ defined by:

$$\phi_{j,x_j}(y_j) = \phi_j(y_j - x_j) \quad ; \quad 1 \leq j \leq d \quad (3.2)$$

Note the use of the vector Wick product

$$\vec{W}_{\phi_x} \diamond \nabla u := W_{\phi_{1,x_1}}^1 \diamond \frac{\partial u}{\partial x_1} + \dots + W_{\phi_{d,x_d}}^d \diamond \frac{\partial u}{\partial x_d}$$

in (3.1). As mentioned above, this corresponds to interpreting the equation in the Ito/Skorohod sense.

Let $(b_t(\hat{\omega}), P)$ be a standard Brownian motion in $\mathbf{R}^d$ which is independent of the white noise $\vec{W}_\phi$ and let E denote the expectation with respect to P . Let $C_b^2(\mathbf{R}^d \rightarrow (L^2))$ denote the space of functions $f : \mathbf{R}^d \rightarrow (L^2)$ such that f is twice Fréchet differentiable and bounded. Our main result can be stated as follows:

Theorem 3.1

Assume the constant $\nu > 0$ and $f \in C_b^2(\mathbf{R}^d \rightarrow (L^2))$. Then for all $\phi \in \mathcal{S}$ there is a unique solution $u(t, x, \cdot) \in (\mathcal{S})^*$ of the equation (3.1) given by:

$$u(t, x, \omega) = E \left[f(x + \nu b_t) \diamond \exp \left[\sum_{i=1}^d \nu^{-1} \int_0^t [W_{\phi_y}(\omega)]_{y=x+\nu b_s} db_s^i - \frac{1}{2} \sum_{i=1}^d \nu^{-2} \int_0^t [W_{\phi_y}^{\diamond 2}(\omega)]_{y=x+\nu b_s} ds \right] \right] \quad (3.3)$$

Proof. Existence.

We will split the proof of this theorem into several lemmas. For simplicity, we assume that the dimension $d = 1$ and that f is deterministic. It is easily seen that the following proof carries over to the general case.

Lemma 3.1

The functional $u(t, x, \cdot)$ defined by (3.3) is indeed in $(\mathcal{S})^*$.

Proof. Taking the $\mathcal{S}$ -transform inside the expectation in (3.3) we obtain the following functional on $\mathcal{S}(\mathbf{R}^d)$:

$$F(t, x, \xi) = E[f(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\}] \quad (3.4)$$

According to Proposition 2.1 we need to show that $F(t, x, \xi)$ is a U -functional. For $\eta, \xi \in \mathcal{S}(\mathbf{R}^d)$ and $z \in \mathbf{C}$, define

$$\begin{aligned} & F(t, x, \eta + z\xi) \\ &= E[f(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s}) db_s - \frac{1}{2} \nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})^2 ds\}] \end{aligned} \quad (3.5)$$

We want to show that $z \rightarrow F(t, x, \eta + z\xi)$ is analytic on the complex plane. Note that $f(x)$ is bounded and $|(\xi, \phi_x)| \leq \|\xi\|_{2,0} \|\phi\|_{2,0}$. For any $M > 0$, it holds that

$$\begin{aligned} & \sup_{|z| \leq M} E[|f(x + \nu b_t)|^2 \times \\ & |\exp\{\nu^{-1} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s}) db_s - \frac{1}{2} \nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})^2 ds\}|^2] \\ & \leq K \exp(KM^2) E[\exp\{2\nu^{-1} \int_0^t (\eta + \operatorname{Re}(z)\xi, \phi_{x+\nu b_s}) db_s\}] \end{aligned} \quad (3.6)$$

where K is an appropriate constant depending on f, ξ, η, ϕ and t . This is further equal to

$$\begin{aligned}
& K \exp(KM^2) E[\exp\{2\nu^{-1} \int_0^t (\eta + \operatorname{Re}(z)\xi, \phi_{x+\nu b_s}) db_s \\
& - 2\nu^{-2} \int_0^t (\eta + \operatorname{Re}(z)\xi, \phi_{x+\nu b_s})^2 ds \\
& \times \exp\{2\nu^{-2} \int_0^t (\eta + \operatorname{Re}(z)\xi, \phi_{x+\nu b_s})^2 ds\}] \quad (3.7) \\
& \leq K_1 \exp(K_1 M^2) E[\exp\{2\nu^{-1} \int_0^t (\eta + \operatorname{Re} z \xi, \phi_{x+\nu b_s}) db_s \\
& - 2\nu^{-2} \int_0^t (\eta + \operatorname{Re} z \xi, \phi_{x+\nu b_s})^2 ds\}] = K_1 \exp(K_1 M^2)
\end{aligned}$$

Since the integrand in the right hand side of (3.5) is an analytic function, (3.7) implies that $F(t, x, \eta + z\xi)$ is continuous in z . Moreover, for any closed curve D in the complex plane we have

$$\begin{aligned}
& \int_D F(t, x, \eta + z\xi) dz \\
& = E\left[\int_D f(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s}) db_s \right. \\
& \quad \left. - \frac{1}{2}\nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})^2 ds\} dz\right] = 0 \quad (3.8)
\end{aligned}$$

Thus it follows from Morera's Theorem that $F(t, x, \eta + z\xi)$ is analytic. For $\xi \in \mathcal{S}(\mathbf{R}^d)$, $z \in \mathbf{C}$, we have

$$|F(t, x, z\xi)| \leq K \exp(K|z|^2 \|\xi\|_{2,0}^2) E[\exp\{\nu^{-1} \operatorname{Re}(z) \int_0^t (\xi, \phi_{x+\nu b_s}) db_s\}] \quad (3.9)$$

where K is a constant. The above is equal to

$$\begin{aligned}
& K \exp(K|z|^2 \|\xi\|_{2,0}^2) E[\exp\{\nu^{-1} \operatorname{Re}(z) \int_0^t (\xi, \phi_{x+\nu b_s}) db_s \\
& - \frac{1}{2}\nu^{-2} (\operatorname{Re}(z))^2 \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\} \\
& \times \exp\{\frac{1}{2}\nu^{-2} (\operatorname{Re}(z))^2 \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\}] \leq \exp(K_1 |z|^2 \|\xi\|_{2,0}^2) \quad (3.10)
\end{aligned}$$

This together with the analytic property of F shows that $F(t, x, \xi)$ is indeed a U -functional, which ends the proof of Lemma 3.1.

Lemma 3.2

$\frac{\partial u(t, x)}{\partial x}$ and $\frac{\partial^2 u(t, x)}{\partial x^2}$ exist in $(\mathcal{S})^*$ and

$$\mathcal{S} \frac{\partial u(t, x)}{\partial x}(\xi) = \frac{\partial \mathcal{S} u(t, x)(\xi)}{\partial x} \quad (3.11)$$

$$\mathcal{S} \frac{\partial^2 u(t, x)}{\partial x^2}(\xi) = \frac{\partial^2 \mathcal{S} u(t, x)(\xi)}{\partial x^2} \quad (3.12)$$

Proof. As in Lemma 3.1, we continue to denote the $\mathcal{S}$ -transform of $u(t, x)$ by $F(t, x, \xi)$. By saying that $\frac{\partial u(t, x)}{\partial x}$ exists in $(\mathcal{S})^*$ and is equal to $g(t, x) \in (\mathcal{S})^*$, we mean that the following limit holds in $(\mathcal{S})^*$:

$$\lim_{\Delta x \rightarrow 0} \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} = g(t, x) \quad (3.13)$$

Since $(\xi, \phi_{x+\nu b_s})$ is bounded and smooth in x , it is not difficult to show that

$$\begin{aligned} & \frac{\partial F(t, x, \xi)}{\partial x} \\ &= E[f'(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s \\ & \quad - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\}] \\ & \quad + E[f(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\} \\ & \quad \times [-\nu^{-1} \int_0^t (\xi, \phi'_{x+\nu b_s}) db_s + \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})(\xi, \phi'_{x+\nu b_s}) ds]] \\ &= I_1(t, x, \xi) + I_2(t, x, \xi) \end{aligned} \quad (3.14)$$

From the proof of Lemma 3.1, one can see that $I_1(t, x, \xi)$ is a U -functional. As for $I_2(t, x, \xi)$, we note that for $\eta, \xi \in \mathcal{S}(\mathbf{R}^d)$ and $M > 0$:

$$\begin{aligned}
& \sup_{|z| \leq M} E[|f(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s}) db_s \\
& - \frac{1}{2} \nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})^2 ds|^2 \cdot | - \nu^{-1} \int_0^t (\eta + z\xi, \phi'_{x+\nu b_s}) db_s \\
& + \nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})(\eta + z\xi, \phi'_{x+\nu b_s}) ds|^2] \\
& \leq \sup_{|z| \leq M} \{E[|f(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s}) db_s \\
& - \frac{1}{2} \nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})^2 ds|^4]^{\frac{1}{2}} \cdot E[| - \nu^{-1} \int_0^t (\eta + z\xi, \phi'_{x+\nu b_s}) db_s \\
& + \nu^{-2} \int_0^t (\eta + z\xi, \phi_{x+\nu b_s})(\eta + z\xi, \phi'_{x+\nu b_s}) ds|^4]^{\frac{1}{2}}\} \\
& \leq K \exp(K^2 M^2) [M + M^2]
\end{aligned} \tag{3.15}$$

where K is an appropriate constant. Using this inequality and the method in Lemma 3.1 we conclude that $I_2(t, x, \eta + z\xi)$ is analytic on the complex plane. Moreover, employing similar estimates as in the proof of Lemma 3.1 and the Schwartz inequality, one can show that:

$$\left| \frac{\partial F(t, x, z\xi)}{\partial x} \right| \leq K_1 \exp(K_1 |z|^2 |\xi|_{2,0}^2) \tag{3.16}$$

where K_1 is a constant which is independent of x . Thus according to Proposition 2.1, we have proved that $\frac{\partial F(t, x, \xi)}{\partial x}$ is the $\mathcal{S}$ -transform of an element in $(\mathcal{S})^*$, which is denoted by $g(t, x)$.

Next we prove that $\frac{\partial u(t, x)}{\partial x} = g(t, x)$:

Set

$$\bar{u}(t, x, \Delta x) = \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} \tag{3.17}$$

We need to show that

$$\bar{u}(t, x, \Delta x) \rightarrow g(t, x), \text{ as } \Delta x \rightarrow 0 \tag{3.18}$$

The above limit is taken in $(\mathcal{S})^*$. From the definition it follows that:

$$\mathcal{S}\bar{u}(t, x, \Delta x)(\xi) \rightarrow \mathcal{S}g(t, x)(\xi), \text{ as } \Delta x \rightarrow 0 \tag{3.19}$$

On the other hand, by (3.16) we have:

$$\begin{aligned}
|\mathcal{S}\bar{u}(t, x, \Delta x)(z\xi)| &= \left| \frac{1}{\Delta x} \int_x^{x+\Delta x} \frac{\partial F(t, y, z\xi)}{\partial y} dy \right| \\
&\leq K_1 \exp(K_1 |z|^2 |\xi|_{2,0}^2)
\end{aligned} \tag{3.20}$$

Here K_1 does not depend on Δx . By proposition 2.2, (3.19) and (3.20) imply that (3.18) indeed holds. The proof for $\frac{\partial u(t,x)}{\partial x}$ is completed. The remaining claim (3.12) can be proved similarly. We omit the details.

Lemma 3.3

$\frac{\partial u(t,x)}{\partial t}$ exists in $(\mathcal{S})^*$ and

$$\mathcal{S} \frac{\partial u(t,x)}{\partial t}(\xi) = \frac{\partial \mathcal{S}u(t,x)(\xi)}{\partial t} \tag{3.21}$$

Proof. Since

$$f(x + \nu b_t) = f(x) + \nu \int_0^t f'(x + \nu b_s) db_s + \frac{1}{2} \nu^2 \int_0^t f''(x + \nu b_s) ds \tag{3.22}$$

we can, using the properties of exponential martingales, rewrite the $\mathcal{S}$ -transform of $u(t, x)$ as:

$$\begin{aligned}
&F(t, x, \xi) \\
&= f(x) + \nu E \left[\left(\int_0^t f'(x + \nu b_s) db_s \right) \exp \left\{ \nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds \right\} \right] \\
&+ \frac{1}{2} \nu^2 E \left[\int_0^t f''(x + \nu b_s) ds \exp \left\{ \nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds \right\} \right] \\
&= f(x) + E \left[\int_0^t f'(x + \nu b_s) (\xi, \phi_{x+\nu b_s}) \exp \left\{ \nu^{-1} \int_0^s (\xi, \phi_{x+\nu b_v}) db_v \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \nu^{-2} \int_0^s (\xi, \phi_{x+\nu b_v})^2 dv \right\} ds \right] \\
&+ \frac{1}{2} \nu^2 E \left[\int_0^t f''(x + \nu b_s) \exp \left\{ \nu^{-1} \int_0^s (\xi, \phi_{x+\nu b_v}) db_v \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \nu^{-2} \int_0^s (\xi, \phi_{x+\nu b_v})^2 dv \right\} ds \right]
\end{aligned} \tag{3.23}$$

This gives:

$$\begin{aligned}
& \frac{\partial F(t, x, \xi)}{\partial t} \\
&= E[f'(x + \nu b_t)(\xi, \phi_{x+\nu b_t}) \exp\{\nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\}] \\
&+ \frac{1}{2} \nu^2 E[f''(x + \nu b_t) \exp\{\nu^{-1} \int_0^t (\xi, \phi_{x+\nu b_s}) db_s - \frac{1}{2} \nu^{-2} \int_0^t (\xi, \phi_{x+\nu b_s})^2 ds\}]
\end{aligned} \tag{3.24}$$

Using this expression, as in the proof of lemma 3.1, we can show that $\frac{\partial F(t, x, \xi)}{\partial t}$ is an $\mathcal{S}$ -transform and for any $T > 0$, there exists a constant K such that the following holds uniformly on $[0, T] \times \mathbf{R}^d$:

$$\left| \frac{\partial F(t, z, \xi)}{\partial t} \right| \leq K \exp(K|z|^2 \|\xi\|_{2,0}^2) \tag{3.25}$$

This is enough to conclude that $\frac{\partial u(t, x)}{\partial t}$ exists in $(S)^*$ and (3.21) holds. See the proof of Lemma 3.2.

Now we are ready to complete the proof of the main theorem. From the expression of the $\mathcal{S}$ -transform $F(t, x, \xi)$ of $u(t, x)$ one knows that it solves the deterministic equation

$$\frac{\partial F(t, x, \xi)}{\partial t} = \frac{1}{2} \nu^2 \Delta F(t, x, \xi) + (\xi, \phi_x) \cdot \nabla F(t, x, \xi) \tag{3.26}$$

$$F(0, x, \xi) = f(x)$$

This together with the above three lemmas shows that $u(t, x, \omega)$ is indeed a solution.

Uniqueness

If a solution $u(t, x)$ exists, the $\mathcal{S}$ -transform of u must satisfy the equation (3.26). From the standard probabilistic representation of the solution of the parabolic equation, we know that the $\mathcal{S}$ -transform $Su(t, x)(\xi)$ is given by (3.4). The uniqueness follows.

ACKNOWLEDGEMENTS

This work is supported by VISTA, a research cooperation between The Norwegian Academy of Science and Letters and Den Norske Stats Oljeselskap (Statoil).

REFERENCES

- [B] F.E. Benth: Integrals in the Hida distribution space $(S)^*$. In T. Lindstrøm, B. Øksendal and A.S. Ustunel (editors): *Stochastic Analysis and Related Topics*. Gordon and Breach 1993, pp. 89-99.
- [Ch] P.L.Chow: Generalized solution of some parabolic equation with a random drift.
J.Appl.Math.Optimization 20 (1989), 81-96.
- [Gj] J. Gjerde: Multidimensional noise. Cand.Scient Thesis, Univ. of Oslo 1993.
- [GHLØUZ] H. Gjessing, H. Holden, T. Lindstrøm, J. Ubøe and T.-S. Zhang: The Wick product. In H.Niemi et al. (editors): *Frontiers in Pure and Applied Probability*, Vol. 1. TVP Publishers, Moscow, 29-67, 1993.
- [HKPS] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit: *White Noise Analysis*. Kluwer 1993.
- [HLØUZ 1] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T.-S. Zhang: Stochastic boundary value problems: A white noise functional approach. *Probab. Th. Rel. Fields* 95 (1993), 391-419.
- [HLØUZ 2] H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe and T.-S. Zhang: The Burgers equation with a noisy force. To appear in *Communications PDE*.
- [LØU 1] T. Lindstrøm, B. Øksendal and J. Ubøe: Stochastic differential equations involving positive noise. In M. Barlow and N. Bingham (editors): *Stochastic Analysis*. Cambridge Univ. Press 1991, 261-303.
- [LØU 2] T. Lindstrøm, B. Øksendal and J. Ubøe: Wick multiplication and Ito-Skorohod stochastic differential equations. In S. Albeverio et al (editors): *Ideas and Methods in Mathematical Analysis, Stochastics, and Applications*. Cambridge Univ. Press 1992, pp. 183-206.
- [LØU 3] T. Lindstrøm, B. Øksendal and J. Ubøe: Stochastic modelling of fluid flow in porous media. In S. Chen and J. Yong (editors): *Control Theory, Stochastic Analysis and Applications*. World Scientific 1991, pp. 156-172.
- [NZ] D. Nualart and M. Zakai: Generalized Brownian functionals and the solution to a stochastic partial differential equation. *J. Functional Analysis* 84 (1989), 279-296.
- [Po] J. Potthoff: White noise methods for stochastic partial differential equations. In B.L. Rozovskii and R.B. Sowers (eds.): *Stochastic Partial Differential Equations and Their Applications*. Springer-Verlag 1992.

- [PS] J.Potthoff and L.Streit: A characterization of Hida distributions.
J.Func.Anal. 101 (1991), 212-229.