WICK APPROXIMATION OF QUASILINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract: For a $\varepsilon > 0$ let $\{W_s^\varepsilon\}_{s \geq 0}$ be a smooth approximation to 1-dimensional Brownian motion $\{W_s\}_{s \geq 0}$. We consider the equation

$$1) \quad X_t^\varepsilon = \eta + \int_0^t \sigma(s, X_s^\varepsilon) \diamond W_s^\varepsilon ds + \int_0^t b(s, X_s^\varepsilon) ds; \quad 0 \leq s < \infty$$

where $\diamond$ denotes the Wick product. It is conjectured that (with reasonable conditions on b and σ) a unique strong solution X_t^ε exists for all ε and that $X_t^\varepsilon \rightarrow X_t$ as $\varepsilon \rightarrow 0$ (i.e., $W_s^\varepsilon \rightarrow W_s$), where X_t is the solution of the Itô differential equation

$$2) \quad X_t = \eta + \int_0^t \sigma(s, X_s) dW_s + \int_0^t b(s, X_s) ds; \quad 0 \leq s < \infty$$

We prove the conjecture in the quasilinear case, i.e., where $\sigma(s, x) = \sigma_s x$, where σ_s is independent of x .

The conjecture should be compared to the Wong-Zakai theorem, which says that if we let Y_t^ε be the solution of the stochastic equation (with $\diamond$ replaced by ordinary product)

$$1)' \quad Y_t^\varepsilon = \eta + \int_0^t \sigma(s, Y_s^\varepsilon) W_s^\varepsilon ds + \int_0^t b(s, Y_s^\varepsilon) ds; \quad 0 \leq s < \infty$$

then $Y_t^\varepsilon \rightarrow Y_t$, where Y_t is the solution of the Stratonovitch differential equation

$$2)' \quad Y_t = \eta + \int_0^t \sigma(s, Y_s) \circ dW_s + \int_0^t b(s, Y_s) ds; \quad 0 \leq s < \infty$$

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1. Introduction

Let (Ω, H, P) be the classical canonical Wiener space, where $\Omega = \{\omega : \mathbb{R}_+ \rightarrow \mathbb{R}; \omega \text{ is continuous, } \omega(0) = 0\}$ is the space continuous functions starting at 0, $H = \{h \in \Omega \mid h \text{ is absolutely continuous and } |h|_H^2 := \int_0^\infty |\dot{h}|^2 dt < \infty\}$ is the Cameron-Martin Hilbert space and P is the Wiener measure. Let $W : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ defined by $W_t(\omega) = \omega(t)$ be the canonical Wiener process on the time interval $\mathbb{R}_+$ starting at the origin.

Let $\rho \geq 0$ be a smooth (C^∞) function on the real line $\mathbb{R}$ with support $\text{supp}(\rho) \subset [0, 1]$ and such that

$$\int_0^1 \rho(s) ds = 1. \quad (1.1)$$

Define $\tilde{\rho}(s) := \rho(-s)$ and for $k = 1, 2, \dots$ put

$$\rho^{(k)}(s) = k\rho(ks), \quad \tilde{\rho}^{(k)}(s) = k\rho(-ks). \quad (1.2)$$

Define

$$\begin{aligned} W_t^{(k)}(\omega) &:= (W_t(\omega) * \tilde{\rho}^{(k)})(t) \\ &= \int_{\mathbb{R}} W_s(\omega) \tilde{\rho}^{(k)}(t-s) ds = \int_t^{1+t} W_s(\omega) \rho^{(k)}(s-t) ds. \end{aligned} \quad (1.3)$$

In other words, $W_t^{(k)}$ is a t -smoothed version of W_t , obtained by taking the convolution of (each component of) W_t with $\tilde{\rho}^{(k)}$.

Now we consider an n -dimensional stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t; \quad X_0 = x \in \mathbb{R}^n, \quad (1.4)$$

where the coefficients $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ satisfy the usual Lipschitz requirement

$$|b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| \leq C|x_1 - x_2|; \quad x_1, x_2 \in \mathbb{R}^n, \quad (1.5)$$

where C is a constant. The first version of the following well-known result was proved by Wong and Zakai [WZ65] (see e.g. [IW89], Theorem 7.2 and the references there):

The Wong-Zakai theorem: If we replace W_t by the smoothed version $W_t^{(k)}$ and for each fixed $\omega \in \Omega$ let $Y_t^{(k)} = Y_t^{(k)}(\omega)$ be the solution of the corresponding ordinary equation

$$\frac{dY_t^{(k)}}{dt} = b(Y_t^{(k)}) + \sigma(Y_t^{(k)})\dot{W}_t^{(k)}(\omega); \quad Y_0^{(k)} = x, \quad (1.6)$$

(where $\dot{W}_t^{(k)} = \frac{dW_t^{(k)}}{dt}$), then $Y_t^{(k)} \rightarrow Y_t$ as $k \rightarrow \infty$, where Y_t is the solution of the Stratonovitch stochastic differential equation

$$dY_t = b(Y_t)dt + \sigma(Y_t) \circ dW_t; \quad Y_0 = x. \quad (1.7)$$

The convergence is uniform on bounded t -intervals, for a.e. ω . In fact, we have

$$\mathbb{E}[\sup_{0 \leq t \leq T} |Y_t(\cdot) - Y_t^{(k)}(\cdot)|^2] \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (1.8)$$

for all $T < \infty$, where $\mathbb{E}$ denotes the expectation with respect to the probability law for W_t .

At first glance it is rather surprising that the limit of $Y_t^{(k)}$ is Y_t and not the solution X_t of the Itô equation (1.4). It is natural to ask if there is an analogous approximation procedure which converges to X_t . We propose the following procedure: Replace the ordinary, ω -wise (matrix) product in (1.6) by the *Wick product* $\diamond$, i.e., let $X_t^{(k)}$ be the solution of the stochastic equation

$$\frac{dX_t^{(k)}}{dt} = b(X_t^{(k)}) + \sigma(X_t^{(k)}) \diamond \dot{W}_t^{(k)}; \quad X_0^{(k)} = x. \quad (1.9)$$

Then we conjecture that (possibly under some extra conditions) $X_t^{(k)}$ converges to X_t as $k \rightarrow \infty$, in the same sense as above, i.e.,

$$\mathbb{E}[\sup_{0 \leq t \leq T} |X_t(\cdot) - X_t^{(k)}(\cdot)|^2] \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (1.10)$$

for all $T < \infty$.

The motivation for this conjecture comes from the following remarkable connection between Itô/Skorohod integrals and Wick products (see e.g. [LØU92], [HKPS93, ch.8] and [B93] for proofs and discussions): If $Y_s(\omega)$ is a Skorohod integrable stochastic process then

$$\int_0^t Y_s \delta W_s = \int_0^t Y_s \diamond N_s ds \quad \text{for all } t \geq 0,$$

where δW_s denotes Skorohod integration and N_s is white noise. In particular, if $Y_s(\cdot)$ is also adapted with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by $\{W_t(\cdot)\}_{t \geq 0}$, then

$$\int_0^t Y_s dW_s = \int_0^t Y_s \diamond N_s ds. \quad (1.12)$$

The smoothed version of this is that

$$\int_{\mathbb{R}} (\rho^{(k)} * Y)_t \delta W_t = \int_{\mathbb{R}} Y_t \diamond W_t^{(k)} dt \quad (1.13)$$

for all $k = 1, 2, \dots$, where

$$(\rho^{(k)} * Y)(t) = \int_{\mathbb{R}} Y_s \rho^{(k)}(t-s) ds.$$

It seems natural to guess that a similar connection between Itô/Skorohod integrals and Wick products also holds in the setting of Itô stochastic differential equation, in the sense of our conjecture (1.9)-(1.10) above.

The purpose of this paper is to prove this conjecture for a class of 1-dimensional quasilinear stochastic differential equations of the form

$$dX_t = b(t, X_t, \omega)dt + \sigma(t, \omega)X_t\delta W_t; \quad X_0 = \eta,$$

where $b : \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$, $\eta : \Omega \rightarrow \mathbb{R}$ are given functions, possibly anticipating (see exact conditions below). This is done by reducing the equation to a quasilinear (hyperbolic) partial differential equation in infinitely many variables and then applying a characteristic curve method (see §3).

We emphasize however, that our results in this paper are far from a solution to the general conjecture. In general it is not even clear that the Wick approximate equation (1.9) has a solution (unique or not) for each k , nor is it clear that this solution -if it exists- converges in any sense and to what.

Now we explain the main results of our paper in details:

Let σ and $b : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ satisfy the following conditions:

- (A) For any fixed $x \in \mathbb{R}$, $\sigma(t, x, \omega)$ and $b(t, x, \omega)$ are (real valued) stochastic processes (not necessarily adapted);
- B) For a.e. $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $\sigma(t, x, \omega)$ and $b(t, x, \omega)$ are continuously differentiable with respect to x .

We will impose more conditions as we go on. Let η be a random variable on $(\Omega, \mathcal{F}, P)$. Consider the following (anticipating) stochastic differential equation:

$$X_t = \eta + \int_0^t \sigma(s, X_s, \omega) dW_s + \int_0^t b(s, X_s, \omega) ds, \quad 0 \leq t < \infty, \quad (1.15)$$

Here and in the following dW_s is the Skorohod integral and we use $\sigma(s, X_s)$ or sometimes $\sigma_s(X_s)$ to represent $\sigma(s, X_s, \omega)$. When σ and b are adapted, the above equation has been studied since decades. When σ and b are not adapted (anticipating), several special cases have been studied recently. Let us point out two particular cases:

- I) When σ and b are linear and deterministic and η is not necessarily in $\mathcal{F}_0$, this equation was studied by Shiota [Sh86], Ustunel [Us88] etc. A global solution (a solution for all times $t \in \mathbb{R}_+$) was proved to exist.

II) When σ and b are linear but possibly anticipating, this equation was studied by Buckdahn [91] and Pardoux [Pa90] etc. Under some mild conditions, a global solution exists in this case too.

III) When σ and b are nonlinear but deterministic and time independent, i.e., $\sigma(t, x, \omega) = \sigma(x)$ and $b(t, x, \omega) = b(x)$, a local solution exists up to some stopping time. Under some more restrictive conditions, a global solution exists. See, e.g. [Bu92] etc.

In this paper, we study the approximation of Equation (1.1) by Wick product equations.

Let now $\{\phi_\varepsilon; \varepsilon > 0\}$ be a family of piecewise differentiable functions on $\mathbb{R}_+ \times \mathbb{R}_+$ such that

$$\phi_\varepsilon(s, t) \rightarrow \delta(s - t) \quad (1.16)$$

as $\varepsilon \rightarrow 0$, where δ is the Dirac delta function on $\mathbb{R}_+$, i.e., $\int_{\mathbb{R}_+} \delta(t - s) f(s) ds = f(t)$ for all C_0^∞ functions (C^∞ functions with compact support) f on $\mathbb{R}_+$. Let

$$W_s^\varepsilon = \int_{\mathbb{R}_+} \phi_\varepsilon(s, t) dW_t \quad (1.17)$$

be an approximation of the (one dimensional) Brownian motion. We consider the following stochastic equation:

$$X_t^\varepsilon = \eta + \int_0^t \sigma(s, X_s^\varepsilon) \diamond W_s^\varepsilon ds + \int_0^t b(s, X_s^\varepsilon) ds, \quad 0 \leq t < \infty, \quad (1.18)$$

where $\diamond$ means the Wick product, which we will recall in Section 2, see e.g. [GHLØUZ92], [Me93] etc. As explained above, it is natural to guess that $X_t^\varepsilon \rightarrow X_s$ as $\varepsilon \rightarrow 0$. We will prove that it turns out to be true in the linear case and also in the quasilinear case.

Unlike the Wong-Zakai approximation [WZ65] etc, in which case, the approximate equations are well-posed, our approximate equation (1.18) is not a familiar one. So we should first study this equation, in particular, the problems such as the existence and the uniqueness of the solution.

Our strategy to solve Equation (1.18) is to reduce it to a first order quasilinear (hyperbolic) partial differential equation in infinitely many variables. The latter in the case of finitely many variables is also hard and has been studied by many people. It is well-known that a strong solution fails to exist generally. There is a classical method, called the *characteristic curve method* [Jo78] for solving the above reduced equation in finite dimensional case. We will adopt it for our purpose. The characteristic curve method is particularly powerful for the linear case and for what we would like to call the quasilinear case. In these two cases, we can find the “explicit” solution of (1.18) using this approach. Formally, Equation (1.15) can be considered as the limiting case ($\varepsilon \rightarrow 0$) of (1.18). In the linear

case, this characteristic method coincides with the so-called (anticipating) Girsanov transformation method.

We feel that the approximate equation of type (1.18) is easier than Equation (1.15) because we have, for instance, an “energy integral” inequality for it. We state it in Section 4 and there we also use it to prove the uniqueness.

We will also give a direct approach to a particular quasilinear case in Section 6 and prove a strong convergence there.

Here is a summary of each section:

In Section 2 we introduce some notations and study some properties of the Wick product. In particular, we state two formulas which express the Wick product through ordinary (Wiener) product and vice versa, see (2.3) and (2.4) below. From this formula we deduce a formula (2.6) obtained recently by Benth and Gjessing [BG94]. We prove a special case which is needed in this paper. We also prove an integration by parts formula which is of useful in the following.

In Section 3 we reduce Equation (1.18) to a first order (hyperbolic) equation in infinitely many variables and then we explain the characteristic curve method used to solve the equation. We also write formally a corresponding equation for (1.15).

In Section 4 we deal with the linear case. We first obtain the explicit solution of (1.18) by solving the characteristic equations. We also prove that the density of the nonlinear transformation induced by one characteristic equation solves the equation. We obtain in this way an “explicit” expression of the density. This also means that our method coincides with the Girsanov transformation method in the limit. We establish an analogue of the energy inequality and use it to prove the uniqueness of the solution of Equation (1.18) (Section 4.2).

In Section 5 we deal with a quasilinear case. We prove that the characteristic curve method works here too. We prove the existence and solve the approximation problem.

In Section 6 we discuss a particular case (the adapted case) of the above quasilinear equation. In this case we can solve the equation using a simpler method (called the reduction method). A stronger approximation result is also obtained here.

2. The Wick product

Starting with the family of L^p spaces $\mathcal{L}^p$ over the Wiener space (Ω, H, P) and the basic differential operators ∇ (the gradient), ∇^* (the dual of ∇ , the divergence) and $L = -\nabla^* \nabla$ (the Ornstein-Uhlenbeck operator), a family of Sobolev spaces $\mathbb{D}_\alpha^p$, $1 \leq p \leq \infty, \alpha \in \mathbb{R}$, can be introduced as

$$\mathbb{D}_\alpha^p = (I - L)^{-\alpha/2}(\mathcal{L}^p)$$

with the norm

$$\|F\|_{\alpha,p} = \|(I - L)^{\alpha/2} F\|_p, \quad F \in \mathbb{D}_\alpha^p,$$

where $\|\cdot\|_p$ is the L_p norm on $\mathcal{L}^p$. Then obviously,

$$\mathbb{D}_0^p = \mathcal{L}^p, \quad \mathbb{D}_\alpha^p \subset \mathbb{D}_{\alpha'}^{p'} \quad \text{if } p > p' \text{ and } \alpha > \alpha'$$

and

$$(\mathbb{D}_\alpha^p)' = \mathbb{D}_{-\alpha}^{p/(p-1)}.$$

More generally, when we consider E -valued functionals, E being a separable real Hilbert space, the corresponding Sobolev spaces are denoted by $\mathbb{D}_\alpha^p[E]$. We will also use the notation $D_t F$ defined by

$$\int_{\mathbb{R}_+} D_t F \dot{h}_t dt = \nabla_h F \quad \text{for any } h \in H.$$

This means that if we let $\mathcal{H}_t$ be the Heaviside function

$$\mathcal{H}_t(s) = \begin{cases} 1 & s > t \\ 0 & s < r, \end{cases}$$

then $D_t F = \nabla_{\mathcal{H}_t} F$. Let $f_k : \mathbb{R}_+^k \rightarrow \mathbb{R}$ be such that $f_k \in L^2(\mathbb{R}_+^{\otimes k})$. Then the multiple Itô-Wiener integral

$$I_k(f_k) = \int_{\mathbb{R}_+^k} f_k(t_1, \dots, t_k) dW_{t_1} \cdots dW_{t_k}$$

is well-defined. As usual, we identify $\nabla^n F$ as a mapping from $\Omega \rightarrow H^{\otimes n}$. It is easy to see that if $F = I_k(f_k)$ with $f_k \in L^2(\mathbb{R}_+^{\otimes k})$, then $\nabla^n F$ exists and

$$\nabla^n F = \begin{cases} (k!/n!) \int_{\mathbb{R}_+^{k-n}} f_k(t_1, \dots, t_{k-n}; \cdot, \dots, \cdot) dW_{t_1} \cdots dW_{t_{k-n}} & \text{if } n \leq k \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION. Let $F = I_k(f_k)$ and $G = I_l(g_l)$. We define the Wick product of F and G by

$$F \diamond G = I_{k+l}(f_k \otimes g_l), \quad (2.1)$$

where $f_k \otimes g_l$ is a function of $k + l$ variables, which is the symmetrization of $f_k(s_1, \dots, s_k)g_l(s_{k+1}, \dots, s_{k+l})$. Then we define the Wick product of two arbitrary random variables by linearity.

Given two E -valued Wiener functionals F and G we define their Wick scalar product by

$$\langle F \diamond G \rangle_E = \sum_{n=1}^{\infty} \langle F, e_n \rangle \diamond \langle G, e_n \rangle \quad (2.2)$$

if each $\langle F, e_n \rangle \diamond \langle G, e_n \rangle$ exists and the above limit exists, where $\{e_n, n = 1, 2, \dots\}$ is an orthonormal basis of E .

We are going to deduce two formulas formally.

For $f, g \in L^2(\mathbb{R}_+; dt)$ we define $\tilde{f} = \int_0^\infty f_s dW_s$, $\langle f, g \rangle = \int_0^\infty f_s g_s ds$ and $\|f\| = \langle f, f \rangle^{1/2}$.

Set $F = \exp\{s\tilde{f} - s^2\|f\|^2/2\}$ and $G = \exp\{t\tilde{g} - t^2\|g\|^2/2\}$. Then according to the definition of Wick product, $F \diamond G$ exists and

$$\begin{aligned} F \diamond G &= \exp\{s\tilde{f} + t\tilde{g} - \|sf + tg\|^2/2\} \\ &= \exp\{s\tilde{f}g - s^2\|f\|^2/2\} \exp\{t\tilde{g} - t^2\|g\|^2/2\} \exp\{st \langle f, g \rangle\} \\ &= \sum_{m,n=1}^{\infty} s^m t^n \sum_{p \leq m \wedge n} \frac{(-1)^p I_{m-p}(f^{\otimes m-p}) I_{n-p}(g^{\otimes n-p}) \langle f, g \rangle^p}{p!(n-p)!(m-p)!}. \end{aligned}$$

On other hand, we have

$$F \diamond G = \sum_{n,m=1}^{\infty} s^m t^n I_m(f^{\otimes m}) I_n(g^{\otimes n}) / m! n! .$$

Comparing the coefficients of $s^m t^n$, we can write the above formula as

$$I_m(f^{\otimes m}) \diamond I_n(g^{\otimes n}) = \sum_{p \leq m \wedge n} \frac{(-1)^p \langle f, g \rangle^p}{p!(n-p)!(m-p)!} I_{m-p}(f^{\otimes m-p}) I_{n-p}(g^{\otimes n-p}).$$

Using the notation of derivative, we obtain

$$I_m(f^{\otimes m}) \diamond I_n(g^{\otimes n}) = \sum_{p \leq m \wedge n} \frac{(-1)^p}{p!} \langle \nabla^p I_m(f^{\otimes m}), \nabla^p I_n(g^{\otimes n}) \rangle_{H^{\otimes p}},$$

where $\nabla^p F$ is identified as a mapping from Ω to $H^{\otimes p}$. By polarization, we have

$$I_m(f_m) \diamond I_n(g^{\otimes n}) = \sum_{p \leq m \wedge n} \frac{(-1)^p}{p!} \langle \nabla^p I_m(f_m), \nabla^p I_n(g_n) \rangle_{H^{\otimes p}} .$$

This is a formula for the single chaos. For the general functionals, we have

$$F \diamond G = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \langle \nabla^p F, \nabla^p G \rangle_{H^{\otimes p}} . \quad (2.3)$$

This general formula represents the Wick product by the ordinary product. We can also obtain similarly a formula to represent the ordinary product by the Wick product, namely

$$FG = \sum_{p=0}^{\infty} \frac{1}{p!} \langle \nabla^p F \diamond \nabla^p G \rangle_{H^{\otimes p}} . \quad (2.4)$$

Summarizing the above formal results, we have

Under some conditions on F and G and a suitable sense of convergence of the concerned series, the formulas (2.3) and (2.4) are true.

There is no unique way to give a rigorous justification of the above formulas (2.3) and (2.4), which involves to identify the spaces where FG and $F \diamond G$ belong. In this paper we only need the formula (2.3) in the case $F \in \mathbb{D}_1^2$ and $G = \int_0^\infty h_t dW_t$ for $h \in L^2(\mathbb{R}_+; dt)$. In this case we can state a rigorous result as

LEMMA 2.1. If $F \in \mathbb{D}_1^p$, $p > 1$ and $H = \int_0^\infty h_t dW_t$ for $h \in L^2(\mathbb{R}_+; dt)$. Then $F \diamond H$ is well-defined as an element of $L^q(\Omega; P)$ for any $q < p$ and

$$F \diamond H = FH - \nabla_h F = FH - \int_0^\infty h_t D_t F dt. \quad (2.5)$$

PROOF. When $F = \exp\{\tilde{f} - \|\tilde{f}\|^2/2\}$, the formula is true by preceding computation. But the set of finite linear combinations of such F 's is dense in $\mathbb{D}_1^p$. So using a limit argument, we see (2.5) is true for $F \in \mathbb{D}_1^p$. ■

Another particular case of (2.3) is a formula obtained recently by Benth and Gjessing [BG94], which we will deduce as follows

Let $F = \exp\{\int_0^\infty f_s dW_s - \frac{1}{2} \int_0^\infty |f_s|^2 ds\}$, $f \in L^2$. Then it is easy to see that $\nabla^p F = F f^{\otimes p}$. According (2.3), we have

$$\begin{aligned} F \diamond G &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \langle \nabla^p F, \nabla^p G \rangle_{H^{\otimes p}} = F \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \langle f^{\otimes p}, \nabla^p G \rangle_{H^{\otimes p}} \\ &= F \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \nabla_f^p G = FG \left(\cdot - \int_0^\infty f_s ds \right). \end{aligned} \quad (2.6)$$

Formula (2.6) appeared in [BG94].

From Lemma 2.1, we can deduce the following “integration by parts” formula which will be useful in Section 4.

LEMMA 2.2. If $F \in \mathbb{D}_1^p$, $p > 1$ and $H = \int_0^\infty h_t dW_t$ for $h \in L^2(\mathbb{R}_+; dt)$. Let $G \in \mathbb{D}_1^q$, $q > 1$. Then

$$\mathbb{E}\{(F \diamond H)G\} = \mathbb{E}\{F \nabla_h G\}. \quad (2.7)$$

PROOF. By Lemma 2.1,

$$\begin{aligned} \mathbb{E}\{(F \diamond H)G\} &= \mathbb{E}\{[FH - \nabla_h F]G\} = \mathbb{E}[FHG] - \mathbb{E}[\nabla_h FG] \\ &= \mathbb{E}[FHG] - \mathbb{E}[F \nabla^*(h(\cdot)G)]. \end{aligned}$$

From the well-known fact $\nabla^*(h(\cdot)G) = HG - \nabla_h G$, we prove the lemma. ■

REMARK. H. Gjessing communicated us that formulas (2.3) and (2.4) are known, see e.g. (3.8) and (3.9) of [NZ93], pp.137-138.

3. Reduction to a hyperbolic equation

For simplicity we will omit the index ε in this subsection. Let $\phi(t, s)$ be a (deterministic) piecewise differentiable function and let $W_\phi(t) = \int_0^\infty \phi(t, s) dW_s$. Let σ and $b: \mathbb{R}_+ \times R \times \Omega \rightarrow R$ satisfy the conditions (A) and (B) stated in the introduction. Consider the following equation

$$\begin{cases} \dot{X}_t = \sigma(t, X_t) \diamond W_\phi(t) + b(t, X_t), & 0 \leq t < \infty \\ X_0 = \eta. \end{cases} \quad (3.1)$$

For any fixed $t \in \mathbb{R}_+$, $\hat{\phi}_t(\cdot) = \int_0^\infty \phi(t, s) ds$ is an element of H . Using Lemma 2.1, Equation (3.1) is equivalent to

$$\begin{aligned} \dot{X}_t &= \sigma(t, X_t) W_\phi(t) + b(t, X_t) - \nabla_{\hat{\phi}_t} \sigma(t, X_t) \\ &= \sigma(t, X_t) W_\phi(t) + b(t, X_t) - (\nabla_{\hat{\phi}_t} \sigma)(t, X_t) - \sigma'(t, X_t) \nabla_{\hat{\phi}_t} X_t \\ &= F(t, X_t) + \sigma'(t, X_t) \int_0^\infty \phi(t, r) D_r X_t dr = F(t, X_t) + \nabla_{\hat{G}(t, X_t)} X_t, \end{aligned} \quad (3.2)$$

where

$$F(t, x, \omega) = b(t, x, \omega) + \sigma(t, x, \omega) W_\phi(t) - \nabla_{\hat{\phi}_t} \sigma(t, x, \omega) \quad (3.3)$$

$$\equiv b(t, x, \omega) + \sigma(t, x, \omega) W_\phi(t) - \int_0^\infty \phi(t, r) D_r \sigma_t dr$$

$$G(t, x, \omega, \cdot) = -\sigma'(t, x, \omega) \phi(t, \cdot) \quad (3.4)$$

$$\hat{G}(t, \cdot; \omega) = \int_0^\infty G(t, x, \omega, r) dr = -\sigma'(t, x, \omega) \hat{\phi}_t. \quad (3.5)$$

For almost every $(t, \omega) \in \mathbb{R}_+ \times \Omega$, $\hat{G}(t, \cdot; \omega)$ can be considered an element of $H \subset \Omega$. We also write it as $\hat{G}(t, \omega)$. So $\hat{G}$ is a mapping from $\mathbb{R}_+ \times \mathbb{R} \times \Omega$ to $H \subset \Omega$.

Now let $\sigma(t, x) \in \mathbb{ID}_1^\infty$. If $X_t \in \mathbb{ID}_1^p$ for some $p > 1$, then it is easy to see that $\sigma(t, X_t) \in \mathbb{ID}_1^p$. According to Lemma 2.1, $\sigma(t, X_t) \diamond W_\phi(t)$ is well-defined.

DEFINITION. A (not necessarily adapted) stochastic process X_t , $0 \leq t < \infty$ is called a strong solution of Equation (3.1) if $X_t \in \mathbb{ID}_1^p$ for some $p > 1$ and X_t is differentiable with respect to t such that $X_0 = \eta$ a.s. and (3.1) holds almost surely on $\mathbb{R}_+ \times \Omega$.

From the above computation, we have

LEMMA 3.1. X_t is a strong solution of Equation (3.1) if and only if it satisfies (3.2) with the data given by (3.3)-(3.5).

Equation (3.2) is a first order quasilinear (hyperbolic) partial differential equation of variable coefficients in infinitely many variables. To see this clearly, let $\{e_n, n = 1, 2, \dots\}$ be an orthonormal basis of $L^2(\mathbb{R}_+; dt)$. Then formally we can write

$$X_t = X(t, \tilde{e}_1, \dots, \tilde{e}_n, \dots), \quad F_t = F(t, x, \tilde{e}_1, \dots, \tilde{e}_n, \dots)$$

and $G(t, x) = \sum_{n=1}^{\infty} G_n(t, x) e_n(t)$ with

$$G_n = \int_0^{\infty} e_n(t) G(t, x) dt = G_n(\tilde{e}_1, \dots, \tilde{e}_n, \dots).$$

Then Equation (3.2) becomes

$$\frac{\partial X}{\partial t} = F(t, X) + \sum_{n=1}^{\infty} G_n(t, X_t) \frac{\partial X}{\partial \tilde{e}_n}, \quad (3.6)$$

where we consider $\tilde{e}_1, \dots, \tilde{e}_n, \dots$ as independent variables.

We will establish some rules of differentiation adapted to our situation.

LEMMA 3.2. Let $g : \mathbb{R}_+ \times \Omega \rightarrow E$ be a continuous mapping, E being an arbitrary Banach space. If $K : \mathbb{R}_+ \rightarrow \Omega$ is continuous and differentiable such that $\frac{d}{dt}K(t) \in H$ and $g \in \mathbb{D}_1^p$ for some $p \geq 1$. Then $g(t, K(t))$ is differentiable with respect to t and we have the following chain rule:

$$\frac{d}{dt}g(t, K(t)) = \frac{\partial}{\partial t}g(t, K(t)) + \nabla g(t, K(t))\left(\frac{d}{dt}K(t)\right), \quad (3.7)$$

where $\nabla g(\omega)$, a mapping from H to E , is the derivative of g with respect to ω .

PROOF. Easy. $\blacksquare$

We will explain how to use the so-called characteristic curve method to solve the above equation.

Let $\Gamma_t : \Omega \rightarrow \Omega$, $0 \leq t < \infty$ and $v_t : \Omega \rightarrow \mathbb{R}$, $0 \leq t < \infty$ satisfy the following ordinary differential equations on the Banach space $\Omega \times \mathbb{R}$:

$$\Gamma_t = \omega - \int_0^t G(s, v_s, \Gamma_s) \hat{\phi}_s ds, \quad (3.8)$$

$$v_t = \eta + \int_0^t F(s, v_s, \Gamma_s) ds. \quad (3.9)$$

THEOREM 3.3. Suppose Equations (3.8)-(3.9) have a unique solution (Γ_t, v_t) , $0 \leq t < \infty$ such that for any $t \geq 0$, $\Gamma_t : \Omega \rightarrow \Omega$ is invertible, i.e., there is a $\Lambda_t : \Omega \rightarrow \Omega$ such that for a.e. $\omega \in \Omega$, $\Gamma_t(\Lambda_t(\omega)) = \omega$ and $\Lambda_t(\Gamma_t(\omega)) = \omega$ and $\Lambda_t(\omega)$ is differentiable for a.e. $\omega \in \Omega$. Then

$$X(t, \omega) = v_t(\Lambda_t(\omega)) \quad (3.10)$$

is a solution of (3.2), i.e., a solution of (3.1).

PROOF. Differentiating the identity $\Gamma(t, \Lambda_t(\omega)) = \omega$ with respect to t , we get

$$\frac{\partial \Gamma_t}{\partial t}(\Lambda_t) + (\nabla \Gamma_t)(\Lambda_t) \frac{\partial \Lambda_t}{\partial t} = 0,$$

i.e.,

$$\frac{\partial \Lambda_t}{\partial t} = -\{(\nabla \Gamma_t)(\Lambda_t)^{-1} \frac{\partial \Gamma_t}{\partial t}\}(\Lambda_t).$$

Differentiating the identity $\Gamma(t, \Lambda_t(\omega)) = \omega$ with respect to ω , we get

$$[(\nabla \Gamma_t)(\Lambda_t)]^{-1} = \nabla \Lambda_t. \quad (3.11)$$

By Lemma 3.2, we get

$$\begin{aligned} \dot{X}_t &= \frac{\partial v}{\partial t}(t, \Lambda_t) + (\nabla v)(t, \Lambda_t) \frac{d}{dt} \Lambda_t \\ &= \frac{\partial v}{\partial t}(t, \Lambda_t) - (\nabla v)(t, \Lambda_t) \{(\nabla \Gamma_t)(\Lambda_t)^{-1} \frac{\partial \Gamma_t}{\partial t}\}(\Lambda_t) \\ &= \frac{\partial v}{\partial t}(t, \Lambda_t) - (\nabla v)(t, \Lambda_t) \{ \nabla \Lambda_t \frac{\partial \Gamma_t}{\partial t} \}(\Lambda_t) = \frac{\partial v}{\partial t}(t, \Lambda_t) - \langle \nabla X_t, \hat{G} \rangle_H \\ &= F(t, v_t, \Gamma_t)|_{\omega=\Lambda_t} + \nabla_{\hat{G}} X_t = F(t, X_t) + \nabla_{\hat{G}} X_t. \end{aligned}$$

This proves the theorem. ■

REMARK. The main difficulty in applying this theorem is to prove that the solution Γ_t has inverse, i.e., $\Gamma_t : \Omega \rightarrow \Gamma$ is one-to-one for any $t \in \mathbb{R}_+$, since the characteristic equations are in general coupled. In the linear and quasilinear case, it turns out there is no coupling at all. So we can solve Equation (3.8) and then (3.9) separately. This is why we can handle these two cases easily.

In the finite dimensional case, there is a necessary and sufficient condition [ML85] for Γ_t to be invertible. But it seems too hard to apply it in our case.

REMARK. It is natural to think that the stochastic differential equation (1.15) is a particular case of the equation of Wick type with the Dirac function. The characteristic curve method makes sense in this singular case also.

4. The linear case

In this section we will solve the characteristic curve Equations (3.8) and (3.9) and find the explicit solution of (3.1) for linear case. Then we prove an energy integral inequality which implies the uniqueness easily. We also establish the relation between the solution and the Girsanov density which permits us to use the existing theory.

4.1 Existence

We use the same notation as in Section 3. For the simplicity of notation we will omit the index ε in this subsection.

Let $\sigma = \sigma_s(\omega)$ and $b = b_s(\omega)$ be two anticipating processes and let η be a random variable. Consider the following linear anticipating stochastic differential equation:

$$X_t = \eta + \int_0^t \sigma_s X_s dW_s + \int_0^t b_s X_s ds, \quad 0 \leq t < \infty \quad (4.1)$$

and its approximate equation

$$\dot{Y}_t = (\sigma_s Y_s) \diamond W_\phi(t) + b_s Y_s, \quad 0 \leq t < \infty, \quad Y_0 = \eta. \quad (4.2)$$

Equation (4.2) can be reduced as in Section 3 to the following one

$$\dot{Y}_t = F_t Y_t + \nabla_{\hat{G}_t} Y_t, \quad Y_0 = \eta,$$

where (we omit the explicit dependence on ε)

$$F_t = b_t + \sigma_t W_\phi(t) - \nabla_{\hat{\phi}_t} \sigma_t \quad (4.3)$$

$$\hat{G}(t, \cdot; \omega) = \int_0^\cdot G(t, r; \omega) dr = -\sigma_t(\omega) \hat{\phi}_t. \quad (4.4)$$

Equation (4.2) is a **linear** first order (hyperbolic) partial differential equation of variable coefficients in infinitely many variables.

We are going to solve the characteristic equations to find the explicit solution. On the Banach space Ω , endowed with the sup norm, Equation (3.8) becomes

$$\Gamma_t = \omega - \int_0^t \hat{G}(s, \Gamma_s) ds = \omega + \int_0^t \sigma_s(\Gamma_s) \hat{\phi}_s ds, \quad 0 \leq t < \infty. \quad (4.5)$$

There is no v_t in this equation and we can solve it in a usual way.

LEMMA 4.1. Let $\sigma \in \mathbb{ID}_1^\infty$ uniformly on $t \in \mathbb{R}_+$, i.e.,

$$\sup_{0 \leq t \leq T} \text{ess sup}_{\omega \in \Omega} |\nabla \sigma_t(\omega)|_H < \infty.$$

Then Equation (4.5) has a unique solution $\Gamma : \mathbb{R}_+ \times \Omega \rightarrow \Omega$. Moreover, for any fixed $t \in \mathbb{R}_+$, $\Gamma(t, \cdot)$ is invertible as a mapping from Ω to Ω . Denote its inverse by $\Lambda(t, \omega)$, i.e.,

$$\Gamma(t, \Lambda(t, \omega)) = \omega, \quad \Lambda(t, \Gamma(t, \omega)) = \omega. \quad (4.6)$$

Furthermore, $\Lambda(t, \omega)$ is given by $\Lambda(t, \omega) = \Lambda(0, t; \omega)$, where $\Lambda(s, t; \omega)$ satisfies the following equation:

$$\Lambda(s, t; \omega) = \omega + \int_s^t \hat{G}(r, \Lambda(r, t; \omega)) dr = \omega - \int_s^t \sigma_r(\Lambda(r, t; \omega)) \hat{\phi}_r dr, \quad 0 \leq s \leq t. \quad (4.7)$$

PROOF. The proof is a routine one. We will not provide the details here, see for instance, [Bu91], the proof of Lemma 3.1 there. ■

Now substituting Γ_t into (3.9), we get

$$\frac{\partial v(t, \omega)}{\partial t} = F(t, \Gamma(t, \omega))v(t, \omega).$$

The explicit solution of this equation is given by

$$v = v(0, \omega) \exp\left\{\int_0^\infty F(s, \Gamma(s, \Lambda(t, \omega))) ds\right\}.$$

To determine $v(0, \omega)$ we shall use the initial condition $X_0 = \eta$. Because $\Lambda(0, \omega) = \omega$, we have

$$\eta(\omega) = Y(0, \omega) = v(0, \Lambda(0, \omega)) = v(0, \omega). \quad (4.8)$$

Hence we obtain

THEOREM 4.2. If σ has bounded Malliavin derivative and $F(s, \Lambda(s, \omega))$ is well-defined and is Bochner integrable, then the solution of (4.2) exists and is given by

$$Y(t, \omega) = \eta(\Lambda(t, \omega)) \exp\left\{\int_0^\infty F(s, \Gamma(s, \Lambda(t, \omega))) ds\right\}. \quad (4.9)$$

To obtain the expression for the solution of Equation (4.2) from the above formula, we have to compute the expression $W_\phi(s)$ (which is a functional of ω) when we replace ω by $\Gamma(s, \Lambda(t, \omega))$, i.e., we want to compute

$$W_\phi(s)|_{\omega=\Gamma(s, \Lambda(t, \omega))}.$$

We shall make use of the following lemma whose proof is obvious.

LEMMA 4.3. If $\tilde{\omega} = \omega + \int_0^\infty h_r(\omega) dr$, then

$$W_\phi(s)|_{\omega=\tilde{\omega}} = W_\phi(s) + \int_0^\infty \phi(s, r) h_r(\omega) dr.$$

From (4.5), we have

$$W_\phi(s)|_{\omega=\Gamma(s,\omega)} = W_\phi(s) + \int_0^\infty \int_0^s \phi(s,r)\phi(u,r)\sigma_u(\Gamma(u,\omega))dudr. \quad (4.10)$$

Using (4.7) and Lemma 4.3, we then have

$$\begin{aligned} W_\phi(s)|_{\omega=\Gamma(s,\Lambda(t,\omega))} &= (W_\phi(s)|_{\omega=\Gamma(s,\omega)})|_{\omega=\Lambda(t,\omega)} \\ &= W_\phi(s)|_{\omega=\Lambda(t,\omega)} + \int_0^\infty \int_0^s \phi(s,r)\phi(u,r)\sigma_u(\Gamma(u,\Lambda(t,\omega)))dudr \\ &= W_\phi(s) - \int_0^\infty \int_0^\infty \phi(s,r)\phi(v,r)\sigma_v(\Lambda(v,t;\omega))dvdr \\ &\quad + \int_0^\infty \int_0^s \phi(s,r)\phi(u,r)\sigma_u(\Gamma(u,\Lambda(t,\omega)))dudr. \end{aligned} \quad (4.11)$$

From the above computation we get

THEOREM 4.4. The explicit solution of Equation (4.2) is given by

$$\begin{aligned} X(t,\omega) &= \eta(\Lambda(t,\omega)) \exp\left\{\int_0^t b(s,\Gamma(s,\Lambda(t,\omega)))ds\right\} \\ &\quad \times \exp\left\{\int_0^t \sigma(s,\Gamma(s,\Lambda(t,\omega)))W_\phi(s)ds - \int_0^t (\nabla_{\hat{\phi}_s}\sigma)(s,\Gamma(s,\Lambda(t,\omega)))ds\right\} \\ &\quad \times \exp\left\{-\int_0^t \int_0^T \int_0^s \sigma(s,\Gamma(s,\Lambda(t,\omega)))\sigma(v,\Gamma(s,\Lambda(v,t;\omega)))\phi(s,r)\phi(v,r)dsdvdr\right\} \\ &\quad \times \exp\left\{\int_0^t \int_0^T \int_0^s \sigma(s,\Gamma(s,\Lambda(t,\omega)))\sigma(u,\Gamma(u,\Lambda(t,\omega)))\phi(u,r)\phi(s,r)dsdudr\right\}, \end{aligned} \quad (4.12)$$

where Γ_t and Λ are given respectively by (4.5) and (4.7).

4.2. Energy integral and uniqueness

Let $W_h = \int_0^\infty h_s dW_s$ for some $h \in H$. Let X_t be the solution of the following equation:

$$\dot{X}_t = b_t X_t + (\sigma_t X_t) \diamond W_h, \quad 0 \leq t \leq T. \quad (4.13)$$

We want to deduce an energy integral inequality of the following type, see for instance [Fr56].

THEOREM 4.5. Let $p \geq 1$ be an integer. Assume that there is a constant $0 < K < \infty$ such that

$$\sup_{0 \leq t < \infty} |\nabla_h(\sigma_t)| \leq K, \quad \sup_{0 \leq t < \infty} |W_h \sigma_t| \leq K, \quad \sup_{0 \leq t < \infty} |b_t| \leq K. \quad (4.14)$$

Then

$$\mathbb{E}|X_t|^{2p} \leq \exp\{3(2p-1)Kt\}\mathbb{E}|X_0|^{2p}. \quad (4.15)$$

PROOF. Differentiating $\mathbb{E}|X_t|^{2p}$ with respect to t , we have

$$\frac{d}{dt}\mathbb{E}|X_t|^{2p} = 2p\mathbb{E}\{X_t^{2p-1}\dot{X}_t\} = 2p\mathbb{E}\{X_t^{2p-1}b_tX_t\} + 2p\mathbb{E}\{X_t^{2p-1}[(\sigma_t, X_t) \diamond W_h]\}.$$

But by the integration by parts formula (2.7), we have

$$\begin{aligned} 2p\mathbb{E}\{X_t^{2p-1}[(\sigma_t, X_t) \diamond W_h]\} &= 2p\mathbb{E}\{\sigma_t X_t \nabla_h(X_t^{2p-1})\} \\ &= 2p(2p-1)\mathbb{E}\{\sigma_t X_t^{2p-1} \nabla_h X_t\} = (2p-1)\mathbb{E}\{\sigma_t \nabla_h(X_t^{2p})\} \\ &= (2p-1)\mathbb{E}\{(\nabla_h^* \sigma_t)(X_t^{2p})\} = (2p-1)\mathbb{E}\{(W_h \sigma_t - \nabla_h \sigma_t)(X_t^{2p})\}. \end{aligned} \quad (4.16)$$

Thus

$$\frac{d}{dt}\mathbb{E}|X_t|^{2p} = (2p-1)\mathbb{E}\{(W_h \sigma_t - \nabla_h \sigma_t)(X_t^{2p})\}.$$

If the conditions of the theorem are satisfied, then

$$\frac{d}{dt}\mathbb{E}|X_t|^{2p} \leq 3(2p-1)K\mathbb{E}|X_t|^{2p}.$$

This implies the theorem by the Grownwall lemma. $\blacksquare$

From (4.15), we deduce

COROLLARY 4.6. Let the coefficients satisfy the conditions of Theorem 4.5. Then Equation (4.13) has a unique solution once X_0 is given in some L^2 space.

4.3 Approximation

The solution of (4.1) is already known and is given by (see [Bu], Theorem 4.1)

$$X_t = \eta(A(t, \omega)) \exp\left\{\int_0^t b(s, T(s, A(s, \omega)))ds\right\} L_t, \quad (4.17)$$

where T_t and $A_t = A_{0,t}$ are given by the following ordinary differential equations in Ω :

$$\begin{aligned} T_t \omega &= \omega + \int_0^{t \wedge \cdot} \sigma_s(T_s \omega) ds, \quad 0 \leq t \leq T \\ A_s \omega &= \omega - \int_{s \wedge \cdot}^{t \wedge \cdot} \sigma_s(A_{r,t} \omega) dr, \quad 0 \leq t \leq T \end{aligned}$$

and L_t is the density of T_t . The explicit expression for L_t can be obtained as

$$L_t(\omega) = [\mathcal{L}_t(A_t(\omega))]^{-1}$$

with

$$\begin{aligned}\mathcal{L}_t(\omega) &= \frac{dP \circ A_t^{-1}}{dP}(\omega) = \exp\left\{-\int_0^t \sigma_s(T_s) dW_s \right. \\ &\quad \left. - \frac{1}{2} \int_0^t \sigma_s(T_s)^2 ds - \int_0^t \int_0^s (D_r \sigma_s)(T_s) (D_s[\sigma_r(T_r)]) dr ds \right\}.\end{aligned}$$

Using (4.10) to compute $\int_0^\infty \sigma_s(T_s \omega) dW_s|_{\omega=A_t(\omega)}$, we obtain from the above formula $L_t(\omega) = \exp\{I_1 + I_2 + I_3 + I_4\}$, where

$$I_1 = \int_0^\infty \sigma_s(T_s A_t \omega) dW_s \quad (4.18)$$

$$I_2 = - \int_0^t \sigma_s(T_s A_t \omega) \sigma_s(T_s A_{s,t} \omega) ds \quad (4.19)$$

$$I_3 = \frac{1}{2} \int_0^\infty [\sigma_s(T_s A_t \omega)]^2 ds \quad (4.20)$$

$$I_4 = \int_0^t \int_0^s (D_r \sigma_s)(T_s A_t \omega) ([D_s \sigma_r(T_r)](A_t \omega)) dr ds. \quad (4.21)$$

We write this formula to point out that it seems that there is a mild error in the remark of p.55 of [Bu94], where the term (4.19) is missing.

REMARK. Note the apparent relation between the above Girsanov transformation method and the characteristic curve method.

We want to prove that the solution $X^\varepsilon := Y$ (We resume the dependence on ε) of (4.2) is also the density L_t^ε of the transformation of Γ given by (4.5):

$$\int_\Omega L_t^\varepsilon(\omega) G(\omega) P(d\omega) = \int_\Omega G(\Gamma_t^\varepsilon(\omega)) P(d\omega). \quad (4.22)$$

LEMMA 4.6. X_t^ε is the solution of (4.2) with $\eta = 1$ iff $X_t^\varepsilon = L_t^\varepsilon$.

PROOF. Let L_t^ε satisfy (4.22). Then by Lemma 2.2, for any $G \in \mathbb{D}_1^\infty$,

$$\begin{aligned}\mathbb{E} \int_0^t \{[(\sigma_s L_s^\varepsilon) \diamond W_\phi(s)] G\} ds &= \int_0^t \mathbb{E}[\sigma_s L_s^\varepsilon \nabla_{\phi_s} G] ds \\ &= \int_0^t \mathbb{E}[\sigma_s(\Gamma_s) (\nabla_{\phi_s} G)(\Gamma_s^\varepsilon)] ds.\end{aligned}$$

Now

$$\frac{d}{ds} G(\Gamma_s^\varepsilon) = (\nabla_{\frac{d}{ds} \Gamma_s^\varepsilon} G)(\Gamma_s^\varepsilon) = \sigma_s(\Gamma_s^\varepsilon) (\nabla_{\phi_s} G)(\Gamma_s^\varepsilon). \quad (4.23)$$

Thus

$$\begin{aligned}\mathbb{E} \int_0^t (\sigma_s L_s^\varepsilon) \diamond W_\phi(s) G ds &= \mathbb{E} \int_0^t \frac{d}{ds} G(\Gamma_s^\varepsilon) ds \\ &= \mathbb{E} G(\Gamma_t^\varepsilon) - \mathbb{E} G = \mathbb{E}\{[L_t^\varepsilon - 1]G\}.\end{aligned}$$

Since the above formula is true for any G , we prove that

$$\int_0^t (\sigma_s L_s^\varepsilon) \diamond W_\phi(s) ds = L_t^\varepsilon - 1.$$

This proves that L_t^ε is the solution of (4.2). By the uniqueness of the solution of (4.2), the lemma is proved. $\blacksquare$

Then by the results of [Bu91], under some mild conditions on the coefficients b and σ , L_t^ε converges to L_t in the topology $\sigma(L^1, L^\infty)$. We will not repeat this work here.

5. The quasilinear equation I

Let $b : \mathbb{R}_+ \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ satisfy the conditions (A) and (B) in the introduction. Consider the following quasilinear stochastic differential equation

$$X_t = \eta + \int_0^t b(s, X_s, \omega) ds + \int_0^t \sigma(s, \omega) X_s dW_s, \quad 0 \leq t < \infty \quad (5.1)$$

and its approximation equation

$$\frac{dX_t^\varepsilon}{dt} = b(t, \omega, X_t^\varepsilon) + (\sigma_t X_t^\varepsilon) \diamond W_{\phi^\varepsilon}(t), \quad X_0 = \eta. \quad (5.2)$$

We will discuss Equation (5.2) first. By (3.2)-(3.5), the solution of (5.2) is given by the solution of the following equation

$$\frac{\partial X_t^\varepsilon}{\partial t} = F(t, X_t^\varepsilon, \omega) + \nabla_{\hat{G}(t, X_t^\varepsilon, \omega)} X_t^\varepsilon, \quad (5.3)$$

where

$$F(t, \omega, x) = b(t, x, \omega) + \sigma(t, \omega) W_\phi(t) - \nabla_{\phi_t^\varepsilon} \sigma(t, \omega). \quad (5.4)$$

As in the linear case we can solve this equation by the “characteristic curve method”.

Let $\Lambda(t, \omega)$ and $\Gamma(t, \omega)$ be constructed as in preceding section and let the solution of (5.3) have the form (4.10), i.e.,

$$X^\varepsilon(t, \omega) = g(t, \Lambda(t, \omega)) \quad (5.5)$$

for some $g : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$. Then

$$\frac{\partial g(t, \Lambda(t, \omega))}{\partial t} = F(t, g(t, \Lambda(t, \omega)), \omega).$$

Making the transform $\omega \rightarrow \Gamma(t, \omega)$, we get

$$\frac{\partial g(t, \omega)}{\partial t} = F(t, g(t, \omega), \Gamma(t, \omega)).$$

From the explicit expression of F we have

$$\frac{\partial g(t, \omega)}{\partial t} = b(t, g(t, \omega), \Gamma(t, \omega))g(t, \omega) + h_t(\omega)g(t, \omega), \quad (5.6)$$

where

$$h_t(\omega) = \{\sigma(t, \omega)W_{\phi^\varepsilon}(t) - \nabla_{\phi^\varepsilon_t} \sigma(t, \omega)\}|_{\omega=\Gamma(t, \omega)}. \quad (5.7)$$

Introduce $Y_t(\omega) = g(t, \omega)e^{-\int_0^t h_s ds}$ to simplify (5.6) to

$$\frac{dY_t}{dt} = e^{-\int_0^t h_s ds} b(t, e^{\int_0^t h_s ds} Y_t, \Gamma(t, \omega)). \quad (5.8)$$

(5.8) is an ordinary equation with ω as a parameter. It can be solved in the usual way. Let us summarize the above computation as

PROPOSITION 5.1. Let h be given by (5.7) and make the above transformation. Then (5.2) becomes (5.8).

Let $b = 0$ and $\eta = 1$. Then $Y = 1$. So it is easy to see that

$$L_t^\varepsilon = e^{\int_0^t h_s ds}|_{\omega=\Lambda(t, \omega)} \quad (5.9)$$

is the solution of the equation $X_t = (\sigma_t X_t) \diamond W_\phi(t)$. Equation (5.8) becomes

$$\frac{dY_t}{dt} = (L_t^\varepsilon)^{-1}|_{\omega=\Gamma_t(\omega)} b(t, L_t^\varepsilon|_{\omega=\Gamma_t(\omega)} Y_t, \Gamma(t, \omega)). \quad (5.10)$$

Summarizing the above results, we have

THEOREM 5.2. Let $Y_t(\omega)$ be given by the unique solution of (5.10). Put

$$X^\varepsilon(t, \omega) = Y(t, \Lambda(t, \omega))L_t^\varepsilon. \quad (5.11)$$

Then X_t^ε is a solution of (5.2).

Formally letting $\varepsilon \rightarrow 0$, we can obtain the solution of (5.1) this way. Let T_t and $A_{s,t}$ be defined as in Section 4.3. Let L_t be the density of the transformation of T given by (4.5), i.e.,

$$\int_{\Omega} L_t F(\omega) P(d\omega) = \int_{\Omega} F(T_t(\omega)) P(d\omega).$$

THEOREM 5.3. Let $Y_t(\omega)$ be given by the unique solution of the following ordinary differential equation

$$\frac{dY_t}{dt} = (L_t(A_t))^{-1}b(t, L_t(A_t)Y_t, T_t). \quad (5.12)$$

Put

$$X(t, \omega) = Y(t, A(t, \omega))L_t. \quad (5.13)$$

Then X_t is a solution of (5.1).

PROOF. We will use the same idea as in Lemma 4.6. For any $G \in \mathbb{D}_1^\infty$,

$$\begin{aligned} \mathbb{E}[(\int_0^t \sigma_s X_s dW_s)G] &= \int_0^t \mathbb{E}\{\sigma_s X_s D_s G\} ds = \int_0^t \mathbb{E}\{\sigma_s (Y_s(T_s) L_s) D_s G\} ds \\ &= \int_0^t \mathbb{E}\{\sigma_s(A_s) Y_s(D_s G)(A_s)\} ds = \int_0^t \mathbb{E}\{Y_s \frac{d}{ds}[G(A_s)]\} ds \\ &= \mathbb{E}\{Y_t G(A_t) - \eta G\} - \int_0^t \mathbb{E}[G(A_s) \frac{d}{ds} Y_s] ds \\ &= \mathbb{E}\{Y_t G(A_t) - \eta G\} - \int_0^t \mathbb{E}[G(A_s) (L_s(A_s))^{-1} b(s, L_s(A_s) Y_s, T_s)] ds \\ &= \mathbb{E}\{Y_t G(A_t) - \eta G\} - \int_0^t \mathbb{E}[G(L_s)^{-1} b(s, L_s Y_s(T_s) L_s)] ds \\ &= \mathbb{E}\{Y_t(T_t) L_t G - \eta G\} - \int_0^t \mathbb{E}[G b(s, L_s Y_s(T_s))] ds = \mathbb{E}\{X_t G - \eta G\} - \int_0^t \mathbb{E}[G b(s, X_s)] ds. \end{aligned}$$

Since G is arbitrary, this proves the theorem. $\blacksquare$

It is easy to see that the solution of (5.8) tends to the solution of (5.12) in probability. So we can prove the convergence of X_t^ε to X_t in probability.

6. The quasilinear equation II: A simple approach

6.1. Reduction of the equation

Consider the nonlinear stochastic equation involving the Wick product

$$\frac{dX_t}{dt} = b(t, X_t, \omega) + \sigma_t X_t \diamond W_\phi(t). \quad (6.1)$$

Suppose now that for any $x \in \mathbb{R}$, $b(t, x, \omega)$ is an adapted process and σ_t is deterministic. Define

$$Y_t = X_t \diamond J_\phi(t), \quad (6.2)$$

where

$$J_\phi(t) = \exp^\diamond \left\{ - \int_0^t \sigma_s W_\phi(s) ds \right\} = \exp \left\{ - \int_{\mathbb{R}_+} \psi_s^t dW_s - \frac{1}{2} \int_{\mathbb{R}_+} |\psi_s^t|^2 ds \right\} \quad (6.3)$$

with

$$\psi_s^t = \int_0^t \sigma_s \phi(t-s) ds. \quad (6.4)$$

With this notation, (6.1) can be written as

$$\frac{dY_t}{dt} = \frac{dX_t}{dt} \diamond J_\phi(t) - \sigma_t J_\phi(t) \diamond X_t \diamond W_\phi(t) = J_\phi(t) \diamond b(t, \omega, X_t). \quad (6.5)$$

By formula (5) in [BG94] (see also (2.6) of this paper), if we define

$$\tau_\eta f(\omega) = f(\omega - \int_0^\cdot \eta_s ds),$$

then we have

$$J_\phi \diamond F = J_\phi \cdot (\tau_{-\psi_t} F).$$

Therefore from (6.3)

$$\frac{dY_t}{dt} = J_\phi(t) \tau_{-\psi_t} b(t, \tau_{-\psi_t} X_t). \quad (6.6)$$

On the other hand

$$\begin{aligned} \tau_{-\psi_t} X_t(t, \omega) &= \tau_{-\psi_t} [J_\phi(t) \diamond Y_t(t, \omega)] \\ &= \tau_{-\psi_t} [J_\phi(t) \tau_{\psi_t} Y_t] = \tau_{-\psi_t} J_\phi(t) Y_t \end{aligned}$$

Since $\tau_{-\psi_t} J_\phi(t) = [J_\phi(t)]^{-1}$, we obtain the equivalent equation of Y_t for (6.1):

$$\frac{dY_t}{dt} = J_\phi(t) b(t, \omega - \hat{\psi}_t, [J_\phi(t)]^{-1} Y_t). \quad (6.7)$$

This equation can be used to prove the existence and approximation theorem.

In fact, the equation for Y_t can also be obtained from the characteristic curve method. To see this, note that when σ_t is deterministic we have

$$\Gamma_t = \omega. + \int_0^t \sigma_s \hat{\delta}_\varepsilon(s - \cdot) ds$$

and

$$\Lambda_t = \omega. - \int_0^t \sigma_s \hat{\delta}_\varepsilon(s - \cdot) ds.$$

These equations are easy to solve and we can also check that $L_t|_{\omega=\Gamma_t(\omega)} = (J_\phi(t))^{-1}$, i.e., (5.10) gives (6.7).

6.2 Approximation

Now suppose that b is adapted and satisfies

$$|b(t, \omega_1, x) - b(t, \omega_2, x)| \leq C|\omega_1 - \omega_2|_H.$$

We will sketch how to obtain a stronger convergence theorem.

First note that the stochastic differential equation

$$dX_t = b(t, \omega, X_t)dt + \sigma_t X_t dW_t \quad (6.8)$$

has a unique strong solution. Let

$$K_t = \exp\left\{-\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t |\sigma_s|^2 ds\right\}$$

and

$$Y_t = K_t X_t \left(\omega - \int_0^{t \wedge \cdot} \sigma_s ds\right).$$

First of all, $\tilde{X}_t := X_t(\omega - \int_0^{t \wedge \cdot} \sigma_s ds)$ satisfies

$$d\tilde{X}_t = b(t, \omega - \int_0^{t \wedge \cdot} \sigma_s ds, \tilde{X}_t)dt + \sigma_t \tilde{X}_t dW_t - \sigma_t^2 \tilde{X}_t dt.$$

Applying the Itô formula, we obtain

$$\frac{dY_t}{dt} = K_t b(t, \omega - \int_0^{t \wedge \cdot} \sigma_s ds, K_t^{-1} Y_t). \quad (6.9)$$

Thus

$$\begin{aligned} Y_t^\varepsilon - Y_t &= Y_0^\varepsilon - Y_0 + \int_0^t [K_s^\varepsilon b(t, \omega - \int_0^{t \wedge \cdot} \phi_s^\varepsilon ds, (K_s^\varepsilon)^{-1} Y_s^\varepsilon) \\ &\quad - K_s b(s, \omega - \int_0^{s \wedge \cdot} \sigma_u du, K_s^{-1} Y_s)] ds \\ &= Y_0^\varepsilon - Y_0 + \int_0^t [K_s^\varepsilon - K_s] b(t, \omega - \int_0^{t \wedge \cdot} \psi_s^\varepsilon ds, (K_s^\varepsilon)^{-1} Y_s^\varepsilon) \\ &\quad + \int_0^t K_s [b(t, \omega - \int_0^{t \wedge \cdot} \psi_s^\varepsilon ds, (K_s^\varepsilon)^{-1} Y_s^\varepsilon) - b(t, \omega - \int_0^{t \wedge \cdot} \phi_s ds, (K_s^\varepsilon)^{-1} Y_s^\varepsilon)] ds \\ &\quad + \int_0^t K_s [b(t, \omega - \int_0^{t \wedge \cdot} \phi_s ds, (K_s^\varepsilon)^{-1} Y_s^\varepsilon) - b(t, \omega - \int_0^{t \wedge \cdot} \phi_s ds, K_s^{-1} Y_s)] ds \\ &\leq C_1(\varepsilon) + C_2 \int_0^t K_s |(K_s^\varepsilon)^{-1} Y_s^\varepsilon - K_s^{-1} Y_s| ds \\ &\leq C_1(\varepsilon) + C_2 \int_0^t K_s (K_s^\varepsilon)^{-1} |Y_s^\varepsilon - Y_s| ds, \end{aligned}$$

where $C_1(\varepsilon)$ is a random variable which converges to 0 in any L^p norm and C_2 is a constant (with respect to ε) which may differ from line to line. So

$$\sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t| \leq C_1(\varepsilon) \exp\{C_2 \int_0^T K_t(K_t^\varepsilon)^{-1} dt\}.$$

This implies that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t|^p = 0.$$

Therefore, we obtain

THEOREM 6.1. Let $b(\cdot, \cdot, x)$ is adapted and globally Lipschitz continuous, bounded and H continuous on ω . Then for any $p > 1$,

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \sup_{0 \leq t \leq T} |Y_t^\varepsilon - Y_t|^p = 0.$$

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