OPTIMAL HARVESTING FROM A POPULATION IN A STOCHASTIC CROWDED ENVIRONMENT

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Abstract

We study the (Ito) stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \alpha X_t(K - X_t)dB_t, \quad X_0 = x > 0$$

as a model for population growth in a stochastic environment with finite carrying capacity $K > 0$. Here r, α are constants and B_t denotes Brownian motion. If $r \geq 0$ we show that this equation has a unique strong global solution for all $x > 0$ and we study some of its properties.

Then we consider the following problem: What harvesting strategy maximizes the expected total discounted amount harvested (integrated over all future times)? We formulate this as a stochastic control problem. Then we show that there exists a constant optimal "harvest trigger value" $x^* \in (0, K)$ such that the optimal strategy is to do nothing if $X_t < x^*$ and to harvest $X_t - x^*$ if $X_t > x^*$. This leads to an optimal population process X_t being reflected downwards at x^* . We find x^* explicitly.

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1 Introduction

A classic model for population growth in a (deterministic) crowded environment is to represent the size of the population at time t as the solution of the *logistic differential equation*

$$\begin{aligned} \frac{dX_t}{dt} &= \lambda X_t(K - X_t); \quad t \geq 0 \\ X_0 &= x \geq 0 \end{aligned} \quad (1)$$

where $\lambda \in \mathbf{R}$, $K > 0$ and $x \geq 0$ are given constants. The constant K is called the *carrying capacity* of the environment (for this population). The parameter λ may be regarded as a measure of the quality of the environment.

In many situations, however, λ is not constant but subject to random fluctuations. To model this mathematically we put

$$\lambda = r + \alpha W_t \quad (2)$$

where $W_t = W_t(\omega)$; $t \geq 0$, $\omega \in \Omega$ denotes *white noise* and r, α are (real) constants. We interpret r as the mean value of λ and α as the size of noise. Substituting (2) into (1) we get the *stochastic logistic differential equation*

$$\begin{aligned} dX_t &= rX_t(K - X_t)dt + \alpha X_t(K - X_t)dB_t; \quad t \geq 0 \\ X_0 &= x \geq 0 \end{aligned} \quad (3)$$

where $B_t = B_t(\omega)$; $t \geq 0$, $\omega \in \Omega$ is 1 - dimensional Brownian motion and dB_t is the Ito differential. (See e.g. [Ø] for more information about stochastic differential equations). The coefficients of this equation,

$$b(x) = rx(K - x) \quad \text{and} \quad \sigma(x) = \alpha x(K - x), \quad (4)$$

do not satisfy the linear growth conditions

$$|b(x)| \leq C(1 + |x|), \quad |\sigma(x)| \leq C(1 + |x|) \quad \text{for all } x \quad (5)$$

for some constant C , and therefore it is not straightforward from the general theory that (3) has a global solution, i.e. a solution defined for all $t \geq 0$. Nevertheless, in Section 2 we prove that if $r > 0$ then a global solution of (3) does exist, for all initial values $x \geq 0$. Moreover, we investigate some of its properties. For example, we prove that if $r - \frac{1}{2}\alpha^2 K > 0$ then

$$\lim_{t \rightarrow \infty} X_t(\omega) = K \quad \text{a.s. (i.e. for almost all } \omega \in \Omega), \quad (6)$$

regardless of initial value $X_0 = x > 0$. (See Theorem 2.4)

In Section 3 we study the following question: Suppose we have a population, e.g. a fish population in a lake, whose size X_t at time t is described by the stochastic differential equation (3). At what times - and how much - should we harvest from the population in order to maximize the expected, total discounted volume harvested in all future? We formulate this as a stochastic control problem, which turns out to have a singular solution, in the sense that the optimal amount harvested up to time t is singular with respect to Lebesgue measure dt (if $\alpha \neq 0$). We find the solution using variational inequalities and we investigate some of its properties and its relation to the deterministic case ($\alpha = 0$).

In the deterministic case the solution is well known. See e.g. [C, §2.5] and the references there. However, to the best of our knowledge this is the first time a rigorous treatment of this stochastic case is given.

2 Solution of the stochastic logistic equation

In this section we prove that a unique global solution X_t of (3) exists and we investigate some of its properties.

First note that if $X_0 = x = 0$, then by strong uniqueness $X_t = 0$ for all $t \geq 0$. Similarly, if $x = K$ then $X_t = K$ for all $t \geq 0$. Moreover, if a solution X_t exists for all $t < \tau(\omega) \leq \infty$ for some $x > 0$ and $X_T = 0$ (or $X_T = K$) for some $T = T(\omega) < \tau(\omega)$ then by the strong Markov property $X_t = 0$ (or $X_t = K$, respectively) for all $t \in [T(\omega), \tau(\omega))$. In particular we get

$$0 \leq x \leq K \Rightarrow 0 \leq X_t \leq K \quad \text{for all } t < \tau(\omega), \quad (7)$$

and

$$K \leq x \Rightarrow K \leq X_t \quad \text{for all } t < \tau(\omega). \quad (8)$$

For n a natural number, $n > K$, we put

$$b_n(y) = \begin{cases} ry(K - y) & \text{if } 0 \leq y \leq n \\ rn(K - n) & \text{if } y > n \end{cases} \quad (9)$$

and

$$\sigma_n(y) = \begin{cases} \alpha y(K - y) & \text{if } 0 \leq y \leq n \\ \alpha n(K - n) & \text{if } y > n. \end{cases} \quad (10)$$

Then for each n , b_n and σ_n are Lipschitz continuous functions with at most linear growth, so for each n the equation

$$\begin{aligned} dX_t &= b_n(X_t)dt + \sigma_n(X_t)dB_t, \quad t > 0 \\ X_0 &= x \geq 0 \end{aligned} \quad (11)$$

has a unique, strong solution $X_t = X_t^{(n)}$ defined for all t .

Define

$$\tau_n = \inf\{t > 0; X_t^{(n)} = n\}; \quad n = 1, 2, \dots \quad (12)$$

Then by strong uniqueness we have that

$$X_t^{(n)}(\omega) = X_t^{(n+1)}(\omega) \quad \text{for all } t < \tau_n, \text{ a.s.} \quad (13)$$

Therefore

$$\tau_n = \inf\{t > 0; X_t^{(n)} = n\} < \inf\{t > 0; X_t^{(n+1)} = n+1\} = \tau_{n+1}, \quad (14)$$

so $\{\tau_n\}$ is an increasing sequence of stopping times. Put

$$\tau(\omega) = \lim_{n \rightarrow \infty} \tau_n(\omega) \leq \infty. \quad (15)$$

Then for $t < \tau(\omega)$ we can define a process X_t by setting

$$X_t(\omega) = X_t^{(n)}(\omega), \quad \text{if } t < \tau_n(\omega). \quad (16)$$

For if $t < \tau(\omega)$ then $t < \tau_n(\omega)$ for some n , and then by (13) this defines $X_t(\omega)$ uniquely.

Hence we have

$$X_t = x + \int_0^t r X_s (K - X_s) ds + \int_0^t \alpha X_s (K - X_s) dB_s \quad \text{for } t < \tau(\omega). \quad (17)$$

If $0 \leq x \leq K$ we have as in (7) that $0 \leq X_t^{(n)} \leq K$ for all $t < \tau_n(\omega)$. But then $\tau_n(\omega) \equiv \infty$ since $n > K$ and hence $\tau(\omega) = \infty$ for *a.a.* ω .

Next consider the case when $K < x$. Assume that $r \geq 0$. Fix $t > 0$. Note that by (17) and (8) we have

$$E^x[X_{t \wedge \tau_n}] = x + r E^x\left[\int_0^{t \wedge \tau_n} X_s (K - X_s) ds\right] \leq x \quad (18)$$

for all t and n , where E^x denotes expectation with respect to the law Q^x of X_t when starting at x .

Moreover,

$$\lim_{n \rightarrow \infty} X_{t \wedge \tau_n} = \begin{cases} X_t & \text{if } t < \tau \\ \infty & \text{if } t \geq \tau. \end{cases} \quad (19)$$

The Fatou lemma states that

$$E^x[\lim_{n \rightarrow \infty} X_{t \wedge \tau_n}] \leq \lim_{n \rightarrow \infty} E^x[X_{t \wedge \tau_n}].$$

So by (18) and (19) we conclude that

$$Q^x[t \geq \tau(\omega)] = 0.$$

Since this holds for arbitrary $t > 0$ we conclude that $\tau \equiv \infty$. We summarize what we have proved so far as follows:

LEMMA 2.1 Assume that either

$$r \geq 0 \quad (\text{and } x \geq 0 \text{ is arbitrary}) \quad (20)$$

or

$$0 \leq x \leq K \quad (\text{and } r \in \mathbf{R} \text{ is arbitrary}). \quad (21)$$

Then the stochastic logistic equation

$$dX_t = rX_t(K - X_t)dt + \alpha X_t(K - X_t)dB_t; \quad t > 0, \quad X_0 = x \quad (22)$$

has a unique, strong solution X_t defined for all $t \geq 0$.

Moreover, we have

$$x = 0 \Rightarrow X_t = 0 \text{ for all } t > 0 \quad (23)$$

$$x = K \Rightarrow X_t = K \text{ for all } t > 0 \quad (24)$$

$$0 < x < K \Rightarrow 0 \leq X_t \leq K \quad \text{for all } t > 0 \quad (25)$$

$$K < x \Rightarrow K \leq X_t \quad \text{for all } t > 0. \quad (26)$$

At this point it is useful to recall the well known solution in the deterministic case, i.e. when $\alpha = 0$. Denote the solution in this case by $x(t)$. Then we have

$$0 \leq x(0) \leq K \Rightarrow x(t) = \frac{Kx(0)}{x(0) + (K - x(0)) \exp(-rKt)}; \quad t \geq 0 \quad (27)$$

and

$$K < x(0) \Rightarrow x(t) = \frac{Kx(0)}{x(0) + (K - x(0)) \exp(-rKt)}; \quad t \geq 0 \text{ if } r \geq 0. \quad (28)$$

Figure 1. The deterministic case ($r > 0$).

Note in particular that

$$0 < x(0) < K \Rightarrow 0 < x(t) < K \text{ for all } t > 0 \quad (29)$$

$$K < x(0) \text{ and } r \geq 0 \Rightarrow K < x(t) \text{ for all } t > 0 \quad (30)$$

$$r > 0 \Rightarrow \lim_{t \rightarrow \infty} x(t) = K \text{ for all } x_0 > 0 \quad (31)$$

$$0 \leq x_0 < K \text{ and } r < 0 \Rightarrow \lim_{t \rightarrow \infty} x(t) = 0. \quad (32)$$

We now investigate to what extent these properties still hold in the stochastic case, i.e for X_t defined by (22).

Suppose $X_0 = x \neq K$, $x > 0$ and put

$$T(\omega) = \inf\{t > 0; X_t(\omega) = 0 \text{ or } X_t(\omega) = K\}.$$

Then for $t < T(\omega)$ define

$$Y_t = \ln \frac{X_t}{|K - X_t|} = \ln X_t - \ln |K - X_t|. \quad (33)$$

By Ito's formula we have

$$\begin{aligned} dY_t &= \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} (dX_t)^2 + \frac{1}{K - X_t} dX_t + \frac{1}{2} \frac{1}{(K - X_t)^2} (dX_t)^2 \\ &= \left[\frac{1}{X_t} + \frac{1}{K - X_t} \right] dX_t + \frac{1}{2} \left[\frac{1}{(K - X_t)^2} - \frac{1}{X_t^2} \right] \alpha^2 X_t^2 (K - X_t)^2 dt \\ &= \frac{K}{X_t(K - X_t)} dX_t + \frac{1}{2} \alpha^2 K [2X_t - K] dt. \end{aligned} \quad (34)$$

Hence

$$\frac{K dX_t}{X_t(K - X_t)} = dY_t - \frac{1}{2} \alpha^2 K [2X_t - K] dt \text{ for } t < T. \quad (35)$$

On the other hand, from (22) we have

$$\frac{dX_t}{X_t(K - X_t)} = r dt + \alpha dB_t \text{ for } t < T. \quad (36)$$

Combining (35) and (36) we get

$$dY_t = \left(r - \frac{1}{2} \alpha^2 K + \alpha^2 X_t \right) K dt + \alpha K dB_t \quad (37)$$

or

$$\ln \frac{X_t}{|K - X_t|} = \ln \frac{x}{|K - x|} + (r - \frac{1}{2}\alpha^2 K)Kt + \alpha^2 K \int_0^t X_s ds + \alpha K B_t \quad \text{for } t < T \quad (38)$$

where we have assumed that $B_0 = 0$.

Now note that since the right hand side of (38) is finite for all t , we must have $X_t \neq 0$ for all t , and $X_t \neq K$ for all t a.s. Hence $T = \infty$ and (38) holds for all $t \geq 0$. We have proved:

LEMMA 2.2 Assume either (20) or (21) and let X_t be the solution of (22). Then we have

$$0 < x < K \Rightarrow 0 < X_t < K \quad \text{for all } t \geq 0, \quad a.s. \quad (39)$$

and

$$K < x \quad \text{and} \quad r \geq 0 \Rightarrow K < X_t \quad \text{for all } t \geq 0, \quad a.s. \quad (40)$$

Hence properties (29) and (30) also hold in the stochastic case. What about properties (31) and (32)?

Here is a partial answer:

LEMMA 2.3

(a) Suppose $r - \frac{1}{2}\alpha^2 K > 0$. Then

$$\lim_{t \rightarrow \infty} X_t = K \quad a.s., \quad \text{for all } x > 0.$$

(b) Suppose $r + \frac{1}{2}\alpha^2 K < 0$. Then

$$\lim_{t \rightarrow \infty} X_t = 0 \quad a.s., \quad \text{for all } x \in (0, K).$$

Proof. Recall that *the law of iterated logarithm* for Brownian motion states that

$$\overline{\lim}_{t \rightarrow \infty} \frac{B_t}{\sqrt{2t \ln(\ln t)}} = 1 \quad a.s. \quad (41)$$

(See e.g. [L ; §22] for a proof). In particular, B_t grows slower than ct in the long run, for all constants $c > 0$. Combining this with (38) we get that if $r - \frac{1}{2}\alpha^2 K > 0$ then

$$\overline{\lim}_{t \rightarrow \infty} \frac{X_t}{|K - X_t|} \geq \ln \frac{x}{|K - x|} + \overline{\lim}_{t \rightarrow \infty} ((r - \frac{1}{2}\alpha^2 K)Kt + \alpha B_t) = \infty \quad a.s.,$$

which is possible only if $\lim_{t \rightarrow \infty} X_t = K$ *a.s.* This proves (a).

Similarly, if $r + \frac{1}{2}\alpha^2 K < 0$ and $x \in (0, K)$ then $X_t < K$ for all t and by (38) we get

$$\overline{\lim}_{t \rightarrow \infty} \frac{X_t}{|K - X_t|} \leq \ln \frac{x}{|K - x|} + \overline{\lim}_{t \rightarrow \infty} ((r + \frac{1}{2}\alpha^2 K)Kt + \alpha B_t) = -\infty \quad a.s.,$$

which implies that $\lim_{t \rightarrow \infty} X_t = 0$ *a.s.* This proves (b).

Next, consider the case when

$$-\frac{1}{2}\alpha^2 K < r < \frac{1}{2}\alpha^2 K.$$

Put

$$\beta = \frac{2r}{\alpha^2 K} \in (-1, 1)$$

and define

$$P^+(x) = \frac{\int_0^x y^{-\beta} (K - y)^\beta dy}{\int_0^K y^{-\beta} (K - y)^\beta dy} \quad \text{for } x \in [0, K]. \quad (42)$$

Then one gets the following result, which was pointed out to us by I. Helland:

PROPOSITION 2.4 [H] :

Suppose

$$-\frac{1}{2}\alpha^2 K < r < \frac{1}{2}\alpha^2 K.$$

If $X_0 = x \in (0, K)$ then

$X_t \rightarrow K$ as $t \rightarrow \infty$ with probability $P^+(x)$

and

$X_t \rightarrow 0$ as $t \rightarrow \infty$ with probability $1 - P^+(x)$,

where $P^+(x)$ is given by (42).

We summarize what we have found in one Theorem:

THEOREM 2.5

Assume that either

$$r \geq 0 \quad (\text{with } x \geq 0 \text{ arbitrary}).$$

or

$$0 \leq x \leq K \quad (\text{with } r \in \mathbf{R} \text{ arbitrary}).$$

Then the logistic stochastic differential equation

$$dX_t = rX_t(K - X_t)dt + \alpha X_t(K - X_t)dB_t; \quad X_0 = x$$

has a unique strong global solution X_t ; $t \geq 0$. The solution has the following properties:

$$x = 0 \Rightarrow X_t = 0 \text{ for all } t \tag{43}$$

$$x = K \Rightarrow X_t = K \text{ for all } t \tag{44}$$

$$0 < x < K \Rightarrow 0 < X_t < K \text{ for all } t \tag{45}$$

$$r \geq 0 \ \& \ K < x \Rightarrow K < X_t \text{ for all } t \tag{46}$$

$$\frac{1}{2}\alpha^2 K < r \implies \lim_{t \rightarrow \infty} X_t = K \text{ a.s., for all } x > 0 \quad (47)$$

$$r < -\frac{1}{2}\alpha^2 K \text{ \& } 0 < x < K \implies \lim_{t \rightarrow \infty} X_t = 0 \text{ a.s.} \quad (48)$$

If

$$-\frac{1}{2}\alpha^2 K < r < \frac{1}{2}\alpha^2 K \text{ \& } 0 < x < K \quad (49)$$

then

$$\left. \begin{aligned} \lim_{t \rightarrow \infty} X_t &= K \text{ with probability } P^+(x) \\ \lim_{t \rightarrow \infty} X_t &= 0 \text{ with probability } 1 - P^+(x) \end{aligned} \right\} \quad (50)$$

where $P^+(x)$ is given by (42)

3 A variational inequality formulation

Suppose that when we do not make any interventions, the population we consider has size X_t at time t given by the stochastic logistic equation

$$dX_t = rX_t(K - X_t)dt + \alpha X_t(K - X_t)dB_t, \quad t \geq s; \quad X_s = x > 0. \quad (51)$$

If $x > K$ we assume that $r \geq 0$, so that (51) has a unique strong solution defined for all $t \geq 0$ (Theorem 2.5).

Suppose that the population is, say, a fish population in a lake, and that we decide to harvest (fish) from the population at the rate $u_t = u_t(\omega) \geq 0$ at time t . We assume that u_t is non-anticipating, i.e. that $u(t, \omega)$ is adapted to the filtration $\mathcal{F}_t$ of Brownian motion. Then the population $X_t = X_t^u$ will satisfy the equation

$$dX_t = (rX_t(K - X_t) - u_t)dt + \alpha X_t(K - X_t)dB_t, \quad t \geq s; \quad X_s = x. \quad (52)$$

(see, for example, [Ø] for more details about stochastic differential equations). If the discounting exponent is ρ , then the total expected discounted harvested volume from time s on, is given by

$$I^u(s, x) = E^{s, x} \left[\int_s^T e^{-\rho t} u_t dt \right], \quad (53)$$

where

$$T = \inf\{t > s; \quad X_t^u = 0\} \quad (54)$$

(so if $X_t^u > 0$ for all t , then $T = \infty$). Here $E^{s,x}$ denotes the expectation with respect to the probability law of $Y_t := (s + t, X_{s+t})$; $t \geq 0$. With this setup the stochastic harvesting problem becomes

PROBLEM 3.1 Find $\Lambda(s, x)$ and $u_t^* \geq 0$ such that

$$\Lambda(s, x) = \sup_{u_t \geq 0} I^u(s, x) = I^{u^*}(s, x). \quad (55)$$

It turns out, however, that this stochastic control problem is singular, in the sense that the optimal control rate u_t^* does not exist. Therefore, it is necessary to increase the class of admissible controls as follows:

Note that if $u_t \geq 0$ is a given harvesting rate, then $\gamma_t^u := \int_s^t u_\tau d\tau$, representing the total amount harvested from time s to time t , is an absolutely continuous, nondecreasing $\mathcal{F}_t$ -adapted process. More generally, let our space Γ of all admissible controls consist of all $\mathcal{F}_t$ -adapted, nondecreasing processes. For technical reasons (stemming from the theory for stochastic integration) we also assume that $t \rightarrow \gamma_t$ is right-continuous. Since for each ω the function $t \rightarrow \gamma_t$ is nondecreasing, we can associate to γ_t the measure $d\gamma_t$ with the property that

$$\int_{s^+}^t d\gamma_\tau := \int_{(s,t]} d\gamma_\tau = \gamma_t - \gamma_s. \quad (56)$$

Note that with this notation we have

$$\int_s^t d\gamma_\tau = \Delta\gamma_s + \gamma_t - \gamma_s, \quad (57)$$

where in general

$$\Delta\gamma_{t_i} = \gamma_{t_i} - \gamma_{t_i^-} \quad (58)$$

is the jump of γ_t at $t = t_i$.

If $\gamma \in \Gamma$ is chosen then the population $X_t = X_t^\gamma$ satisfies the equation

$$X_t = x + \int_s^t rX_\tau(K - X_\tau)d\tau + \int_s^t \alpha X_\tau(K - X_\tau)dB_\tau - \int_s^t d\gamma_\tau; \quad t \geq s, \quad (59)$$

where

$$x = X_{s^-} \quad (60)$$

is called the *starting point* of X_t at time s . (Note that we need not have $X_s = x$, because X_t could possibly jump at $t = s$).

The expectation with respect to the probability law $P^{s,x}$ of $Y_t := (s+t, X_{s+t})_{t \geq 0}$ starting at (s, x) at time $t = 0$ is denoted by $E^{s,x}$. With this notation the expected total discounted amount harvested from time s is

$$J^\gamma(s, x) := E^{s,x} \left[\int_s^T e^{-\rho\tau} d\gamma_\tau \right] = E^{s,x} \left[e^{-\rho s} \Delta\gamma_s + \int_{s+}^T e^{-\rho\tau} d\gamma_\tau \right], \quad (61)$$

where

$$T = \inf\{t > 0; \quad X_t^\gamma = 0\} \leq \infty \quad (62)$$

is the time when the population dies out (if it does). (Note that from (52) we have by the strong Markov property that if $X_\tau = 0$ for some stopping time τ then $X_t = 0$ for all $t > \tau$).

This leads to the following problem:

PROBLEM 3.2 Find the function $\Phi(s, x)$ and an optimal harvesting strategy $\gamma^* \in \Gamma$ such that

$$\Phi(s, x) = \sup_{\gamma \in \Gamma} J^\gamma(s, x) = J^{\gamma^*}(s, x). \quad (63)$$

At this point it is useful to recall the construction of a *reflected* Markov process:

Let Y_t be an Ito diffusion in $\mathbf{R}^n$, i.e. Y_t is the unique strong solution of a stochastic differential equation of the form

$$dY_t = b(Y_t)dt + \sigma(Y_t)dB_t, \quad t \geq 0: \quad Y_0 = y \in \mathbf{R}^n \quad (64)$$

where $b : \mathbf{R}^n \rightarrow \mathbf{R}^n$, $\sigma : \mathbf{R}^n \rightarrow \mathbf{R}^{n \times m}$ are given functions and B_t is Brownian motion in $\mathbf{R}^n$. Let D be an open domain in $\mathbf{R}^n$ whose boundary ∂D has a tangent at every point. For each $x \in \partial D$ choose a vector $\eta(x) \in \mathbf{R}^n$ pointing into D . Choose $y \in \overline{D}$, the closure of D . Then consider the problem of finding a pair $(\hat{Y}_t, \xi_t)$ of continuous, $\mathcal{F}_t$ -adapted stochastic processes with the following properties

$$\hat{Y}_t \in \overline{D} \quad \text{for all } t \geq 0, \quad (65)$$

$$\left. \begin{array}{l} \xi_t \text{ is a nondecreasing process, increasing only when } \\ t \in \Lambda := \{t : \hat{Y}_t \in \partial D\}, \end{array} \right\} \quad (66)$$

$$\Lambda = \Lambda(\omega) \subset (0, \infty) \text{ has Lebesgue measure } 0 \text{ for a.a. } \omega \quad (67)$$

and

$$\begin{aligned} d\hat{Y}_t &= b(\hat{Y}_t)dt + \sigma(\hat{Y}_t)dB_t + \eta(\hat{Y}_t)d\xi_t, \quad t \geq 0 \\ \hat{Y}_0 &= y \in \overline{D} \text{ and } \xi_0 = 0. \end{aligned} \quad (68)$$

The equation (68) (with conditions (65) - (67)) is called a *Skorohod stochastic differential equation*. If it has a unique solution then $\hat{Y}_t$ is called *the reflection of Y_t at ∂D* (in the direction of the vector fields $\eta(x)$; $x \in \partial D$) and ξ_t is called *the local time of Y_t at ∂D* . For more information on this, see [F] and the references there.

Note that Y_t behaves like $\hat{Y}_t$ as long as $Y_t \in D$. When $\hat{Y}_t$ reaches ∂D it gets a "kick" $d\xi_t$ in the direction $\eta(\hat{Y}_t)$ and this keeps the process in $\overline{D}$.

In the following we will let $Y_t = Y_t^\gamma$ be the time-space process

$$Y_t^\gamma = \begin{bmatrix} s+t \\ X_{s+t}^\gamma \end{bmatrix}; \quad t \geq 0, \quad (69)$$

where $\{X_\tau^\gamma\}_{\tau \geq 0}$ is the process given by (59). In particular, we will apply the reflection construction above to the process Y_t^0 obtained by choosing $\gamma \equiv 0$.

THEOREM 3.3 Suppose we can find $\varphi \in C^2(\mathbf{R}^2)$ such that

$$\frac{\partial \varphi}{\partial t} \geq e^{-\rho t} \text{ everywhere} \quad (70)$$

and

$$L\varphi(t, x) := \frac{\partial \varphi}{\partial t} + rx(K-x)\frac{\partial \varphi}{\partial x} + \frac{1}{2}\alpha^2 x^2 (K-x)^2 \frac{\partial^2 \varphi}{\partial x^2} \leq 0 \text{ everywhere.} \quad (71)$$

Moreover, suppose further that

$$E^{s,x}[\varphi(T_R, X_{T_R}^\gamma)] \rightarrow 0 \text{ as } R \rightarrow \infty, \text{ for all } \gamma \in \Gamma, \quad (72)$$

where

$$T_R = T \wedge R \wedge \inf\{t > 0; X_t^\gamma \geq R\}. \quad (73)$$

Then

$$\varphi(s, x) \geq \Phi(s, x) \quad \text{for all } (s, x). \quad (74)$$

Now define *the non-intervention region*

$$D = \{(t, x); \quad \frac{\partial \varphi}{\partial t} > e^{-\rho t}\}. \quad (75)$$

Assume that

$$L\varphi(t, x) = 0 \quad \text{in } D \quad (76)$$

and assume also the following:

$$\left. \begin{array}{l} \text{For all } y \in \overline{D} \text{ there exists a unique solution } (Y_t^0, \xi_t) \text{ of the} \\ \text{Skorohod equation (65 - 68), where } D \text{ is given by (75) and} \\ \text{where the vector fields are } \eta(x) = (0, \pm 1) \\ \text{with the signs } \pm \text{ chosen so that } \eta(x) \text{ points into } D. \end{array} \right\} \quad (77)$$

Then

$$\varphi(s, x) = \Phi(s, x) \quad \text{for all } (s, x) \quad (78)$$

and the following control γ_t^* is optimal:

$$\gamma_t^* = \begin{cases} (x - \bar{x})^+ & \text{for } t = s \\ \xi_{t-s} & \text{for } t > s \end{cases} \quad (79)$$

where $\bar{x} = \bar{x}_{s,D} = \sup\{x; \quad (s, x) \in \overline{D}\}$ and we have used the notation

$$z^+ = \begin{cases} z & \text{if } z > 0 \\ 0 & \text{if } z \leq 0 \end{cases} \quad (80)$$

for $z \in \mathbf{R}$.

REMARK Note that if we initially, at time $t = s$, have a population x which is bigger than $\bar{x}_{s,D}$, then according to this theorem it is optimal to start with an immediate harvest of the amount $x - \bar{x}$. This brings the population down to the level $\bar{x} \in \bar{D}$, from which we use the optimal harvesting rule $\gamma_t^* = \xi_{t-s}$.

Proof of Theorem 3.3

Choose $\gamma \in \Gamma$ and assume that $\varphi \in C^2(\mathbf{R}^2)$ satisfies (70 - 72). Then by Ito's formula for semimartingales (see, for example, [P, Theorem II.7.33])

we have, by (59)

$$\begin{aligned}
E^{s,x}[\varphi(T_R, X_{T_R})] &= E^{s,x}[\varphi(s, X_s) + \int_{s^+}^{T_R} \frac{\partial \varphi}{\partial t}(\tau, X_\tau) d\tau + \int_{s^+}^{T_R} \frac{\partial \varphi}{\partial x}(\tau, X_\tau) dX_\tau \\
&+ \frac{1}{2} \int_{s^+}^{T_R} \frac{\partial^2 \varphi}{\partial x^2}(\tau, X_\tau) \alpha^2 X_\tau^2 (K - X_\tau)^2 d\tau \\
&+ \sum_{s < t_i \leq T_R} \{ \varphi(t_i, X_{t_i}) - \varphi(t_i, X_{t_i}^-) - \frac{\partial \varphi}{\partial x}(t_i, X_{t_i}^-) \Delta X_{t_i} \}
\end{aligned} \tag{81}$$

where the sum is taken over all jumping times $t_i \in (s, T_R]$. Using (71) this gives

$$\begin{aligned}
E^{s,x}[\varphi(T_R, X_{T_R})] &= E^{s,x}[\varphi(s, X_s) + \int_{s^+}^{T_R} L\varphi(\tau, X_\tau) d\tau - \int_{s^+}^{T_R} \frac{\partial \varphi}{\partial x}(\tau, X_\tau) d\gamma_\tau \\
&+ \sum_{s < t_i \leq T_R} \{ \varphi(t_i, X_{t_i}) - \varphi(t_i, X_{t_i}^-) + \frac{\partial \varphi}{\partial x}(t_i, X_{t_i}^-) \Delta \gamma_{t_i} \}] \\
&\leq \varphi(s, x) - E^{s,x}[\int_{s^+}^{T_R} \frac{\partial \varphi}{\partial x}(\tau, X_\tau) d\gamma_\tau] \\
&+ E^{s,x}[\sum_{s \leq t_i \leq T_R} \{ \Delta \varphi(t_i, X_{t_i}) + \frac{\partial \varphi}{\partial x}(t_i, X_{t_i}^-) \Delta \gamma_{t_i} \}] \\
&- E^{s,x}[\frac{\partial \varphi}{\partial x}(s, x) \Delta \gamma_s]
\end{aligned} \tag{82}$$

where

$$\Delta \varphi(t_i, X_{t_i}) = \varphi(t_i, X_{t_i}) - \varphi(t_i, X_{t_i}^-)$$

(recall that $X_{s^-} = x$).

Let γ_t^c denote the continuous part of γ_t , i.e.

$$\gamma_t^c = \gamma_t - \sum_{0 < s \leq t} \Delta \gamma_{t_i}.$$

Then (82) can be written

$$E^{s,x}[\varphi(T_R, X_{T_R})] \leq \varphi(s, x) - E^{s,x}\left[\int_s^{T_R} \frac{\partial \varphi}{\partial x}(\tau, X_\tau) d\gamma_\tau^c\right] + E^{s,x}\left[\sum_{s \leq t_i \leq T_R} \Delta \varphi(t_i, X_{t_i})\right]. \quad (83)$$

By the mean value property we have

$$\Delta \varphi(t_i, X_{t_i}) = \frac{\partial \varphi}{\partial x}(t_i, Z_i) \Delta X_{t_i} = -\frac{\partial \varphi}{\partial x}(t_i, Z_i) \Delta \gamma_{t_i} \quad (84)$$

for some $Z_i \in (X_{t_i}^-, X_{t_i})$.

Hence, combining (83), (84) with conditions (65) we get

$$E^{s,x}[\varphi(T_R, X_{T_R})] \leq \varphi(s, x) - E^{s,x}\left[\int_s^{T_R} e^{-\rho\tau} d\gamma_\tau\right].$$

Hence

$$\begin{aligned} \varphi(s, x) &\geq E^{s,x}[\varphi(T_R, X_{T_R})] + E^{s,x}\left[\int_s^{T_R} e^{-\rho\tau} d\gamma_\tau\right] \\ &\rightarrow E^{s,x}\left[\int_s^T e^{-\rho\tau} d\gamma_\tau\right] \text{ as } R \rightarrow \infty, \text{ by (72).} \end{aligned} \quad (85)$$

Since this holds for arbitrary $\gamma \in \Gamma$ we conclude that $\varphi(s, x) \geq \Phi(s, x)$, which is (74).

Next, assume that D is given by (75) and that (76) - (77) hold also. Let $\bar{\gamma}_t$ be the control defined by the right hand side of (79). We want to prove that $\bar{\gamma}_t$ is optimal. Let the corresponding process be $\bar{Y}_t = Y_t^{\bar{\gamma}} = (s+t, \bar{X}_{s+t})$. Note that $\bar{Y}_t$ jumps immediately to a point on ∂D and from then on $\bar{Y}_t$ behaves like the reflected process $\hat{Y}_t$ in D . Hence

$$L\varphi(t, \bar{X}_t) = 0 \quad \text{for almost all } t < T_R \text{ (with respect to Lebesgue measure).}$$

Moreover, for $t > s$, $d\bar{\gamma}_t$ only gives a contribution when $(t, \bar{X}_t) \in \partial D$ and then $\frac{\partial \varphi}{\partial x}(t, \bar{X}_t) = e^{-\rho t}$. Therefore

$$\int_{s^+}^{T_R} \frac{\partial \varphi}{\partial x}(t, \bar{X}_t) d\bar{\gamma}_t = \int_{s^+}^{T_R} e^{-\rho t} d\bar{\gamma}_t.$$

Finally we see that

$$\Delta \varphi(s, X_s) = -e^{-\rho s} \Delta \gamma_s,$$

because $\frac{\partial \varphi}{\partial x}(t, \bar{X}_t) = e^{-\rho t}$ outside D .

Hence, applying the calculations (81) - (85) above to $\gamma = \bar{\gamma}$ we get *equality* everywhere. Therefore,

$$\begin{aligned}\varphi(s, x) &= E^{s,x}[\varphi(T_R, \bar{X}_{T_R})] + E^{s,x}[\int_s^{T_R} e^{-\rho t} d\bar{\gamma}_t] \\ &\rightarrow J^{\bar{\gamma}}(s, x) \text{ as } R \rightarrow \infty\end{aligned}\quad (86)$$

Combining this with (74) we get (78) and hence (79).

EXAMPLE 3.4

To illustrate the content of Theorem 3.3, let us first apply it in the (much simpler) deterministic case, i.e. when $\alpha = 0$. In this case it seems reasonable to guess that the non-intervention region D has the form

$$D = \{(t, x); \quad 0 < x < x^*\}$$

for some $x^* < K$.

So we look for a function $\varphi(t, x) \in C^2(\mathbf{R}^2)$ such that

$$L_0\varphi(t, x) := \frac{\partial \varphi}{\partial t} + rx(K - x)\frac{\partial \varphi}{\partial x} = 0 \quad \text{for } 0 < x < x^* \quad (87)$$

$$\frac{\partial \varphi}{\partial x} > e^{-t} \quad \text{for } 0 < x < x^* \quad (88)$$

and

$$L_0\varphi(t, x) \leq 0 \quad \text{for } x^* < x \quad (89)$$

$$\frac{\partial \varphi}{\partial x} = e^{-t} \quad \text{for } x^* < x. \quad (90)$$

If we put

$$\varphi(t, x) = e^{-\rho t}\psi(x)$$

we get the inequalities

$$- \rho\psi(x) + rx(K - x)\psi'(x) = 0; \quad 0 < x < x^* \quad (91)$$

$$\psi'(x) > 1; \quad 0 < x < x^* \quad (92)$$

and

$$- \rho\psi(x) + rx(K - x)\psi'(x) \leq 0; \quad x^* < x \quad (93)$$

$$\psi'(x) = 1; \quad x^* < x. \quad (94)$$

The general solution $f(x)$ of the equation

$$-\rho f(x) + rx(K - x)f'(x) = 0$$

is

$$f(x) = C_1 \left(\frac{x}{|K - x|} \right)^{\frac{\rho}{rK}} ; \quad C_1 \text{ an arbitrary constant.}$$

Therefore, our candidate for a solution ψ of (91) - (94) is of the form

$$\psi(x) = \begin{cases} C_1 \left(\frac{x}{|K - x|} \right)^{\frac{\rho}{rK}} ; & 0 < x < x^* \\ C_2 + x ; & x^* < x. \end{cases} \quad (95)$$

To find the unknown parameters C_1 , C_2 and x^* , we use the requirement that ψ is C^2 at $x = x^*$:

(a) Continuity at x^* : $f(x^*) = C_2 + x^*$ i.e.

$$C_1 \left(\frac{x^*}{|K - x^*|} \right)^{\frac{\rho}{rK}} = C_2 + x^* \quad (96)$$

(b) C^1 at x^* : $f'(x^*) = 1$

Since $f'(x) = \frac{\rho}{rx(K-x)}f(x)$ this gives

$$\frac{\rho}{rx^*(K - x^*)} C_1 \left(\frac{x^*}{|K - x^*|} \right)^{\frac{\rho}{rK}} = 1. \quad (97)$$

(c) C^2 at x^* : $f''(x^*) = 0$.

Since $\frac{f''(x)}{f'(x)} = -\frac{1}{x} + \frac{1}{K-x} + \frac{f'(x)}{f(x)}$ this gives

$$-\frac{1}{x^*} + \frac{1}{K - x^*} + \frac{\rho}{rx^*(K - x^*)} = 0. \quad (98)$$

From the last equation we get, if $\rho < rK$,

$$x^* = \frac{rK - \rho}{2r} = \frac{1}{2}K - \frac{\rho}{2r}, \quad (99)$$

which substituted in (96) and (97) gives

$$C_1 = \frac{K^2 r^2 - \rho^2}{4r\rho} \left(\frac{Kr + \rho}{Kr - \rho} \right)^{\frac{\rho}{rK}} \quad (100)$$

and

$$C_2 = \frac{K^2 r}{4\rho} + \frac{\rho}{4r} - \frac{K}{2} = \frac{(Kr - \rho)^2}{4r\rho} \quad (101)$$

We verify that with these values of x^* , C_1 , C_2 the function ψ defined by (95) satisfies (91)-(94):

(91): follows from the construction of ψ

(92): We know that $\psi'(x^*) = 1$ and $\psi''(x^*) = 0$. Moreover, from the calculation in (98) we see that $\psi''(x) < 0$ for $0 < x < x^*$. Therefore, $\psi'(x) > 1$ for $0 < x < x^*$.

(93): For $x > x^*$ we have $\psi(x) = C_2 + x$ and hence

$$-\rho\psi(x) + rx(K - x)\psi'(x) = -\rho(C_2 + x) + rx(K - x) =: g(x).$$

Note that

$$g'(x) = -\rho + r(K - x) - rx = -\rho + rK - 2rx < 0 \text{ for } x > \frac{1}{2}\left(K - \frac{\rho}{r}\right) = x^*.$$

Therefore

$$g(x) < g(x^*) = 0 \text{ for } x > x^*.$$

(94): follows from the construction of ψ .

We conclude that if $\rho < rK$ then the function

$$\varphi(t, x) = e^{-\rho t} \psi(x)$$

actually coincides with $\Phi(t, x)$ and the optimal harvesting strategy is $\gamma_t^* = \int_s^t u_\tau^* d\tau + (x - x^*)^+$, where

$$u_t^* = \begin{cases} rx^*(K - x^*) & \text{if } X_t = x^* \\ 0 & \text{if } X_t \neq x^*. \end{cases}$$

Note that since there is no stochastics when $\alpha = 0$, then the corresponding optimal population process X_t will remain constant at the value x^* once this value is reached.

4 Optimal harvesting

In this section we apply the general theory of Section 3 to the problem of finding the optimal harvesting strategy when the population size X_t at time t is described by the logistic stochastic differential equation

$$\begin{aligned} dX_t &= rX_t(K - X_t)dt + \alpha X_t(K - X_t)dB; \quad t \geq s \\ X_s &= x > 0. \end{aligned} \tag{102}$$

Referring to the notation in Problem 3.2 we want to find the value function $\Phi(s, x)$ and the optimal harvesting strategy $\gamma^* \in \Gamma$. Theorem 3.3 tells us that if we can find a function $\varphi(t, x)$ with the required properties (71) - (77), then we can conclude that $\varphi = \Phi$ and we also have an explicit description of γ^* .

To find φ we first proceed as in Example 3.4, i.e. we *try* to look for a φ of the form

$$\varphi(t, x) = e^{-\rho t} \psi(x) \tag{103}$$

for some function $\psi \in C^2(\mathbf{R})$.

The conditions (69) - (76) on φ then transform into the following condition on ψ :

$$\psi'(x) \geq 1 \quad \text{everywhere} \quad (104)$$

$$L_0\psi := \frac{1}{2}\alpha^2 x^2 (K-x)^2 \psi''(x) + rx(K-x)\psi'(x) - \rho\psi(x) \leq 0 \quad \text{everywhere} \quad (105)$$

$$E^x[e^{-\rho T_R} \psi(X_{T_R}^\gamma)] \rightarrow 0 \quad \text{as } R \rightarrow \infty \quad \text{for all } \gamma \in \Gamma \quad (106)$$

$$L_0\psi(x) = 0 \quad \text{for } x \in D, \quad (107)$$

where D is the *non-intervention region*

$$D = \{x; \quad \psi'(x) > 1\}. \quad (108)$$

We emphasize that Theorem 3.3 implies that it is sufficient to find any such function ψ . If we do, then we automatically have that

$$\Phi(t, x) = e^{-\rho t} \psi(x).$$

So we are allowed to guess and use intuition etc. in our search for ψ , as long as we can verify that (104) - (108) hold.

It is natural to assume that D has the form

$$D = (0, x^*) \subset (0, K)$$

for some (unknown) x^* . So in D ψ must coincide with a solution f of the differential equation

$$L_0 f(x) = P(x)f''(x) + Q(x)f'(x) + R(x)f(x) = 0, \quad (109)$$

where

$$P(x) = \frac{1}{2}\alpha^2 x^2 (K-x)^2, \quad Q(x) = rx(K-x) \quad \text{and} \quad R(x) = -\rho. \quad (110)$$

Note that

$$\frac{xQ(x)}{P(x)} = \frac{2r}{\alpha^2(K-x)} = \frac{2r}{\alpha^2 K} + b_1 x + b_2 x^2 + \dots \quad (111)$$

and

$$\frac{x^2 R(x)}{P(x)} = -\frac{2\rho}{\alpha^2(K-x)^2} = -\frac{2\rho}{\alpha^2 K^2} + c_1 x + c_2 x^2 + \dots \quad (112)$$

are both analytic for $|x| < K$. Therefore the point $x = 0$ is a *regular singular point* for the differential equation (109) and we can apply the Frobenius theory to find its solutions (see e.g. [B, pp. 194-197]). Let γ_1, γ_2 be the solutions of the equation

$$\gamma(\gamma - 1) + b_0\gamma + c_0 = 0 \quad (113)$$

where (see (111) and (112) above)

$$b_0 = \frac{2r}{\alpha^2 K} \quad \text{and} \quad c_0 = -\frac{2\rho}{\alpha^2 K^2}. \quad (114)$$

In other words, let

$$\gamma_1 = \frac{1}{2} \left(1 - \frac{2r}{\alpha^2 K} + \sqrt{\left(\frac{2r}{\alpha^2 K} - 1\right)^2 + \frac{8\rho}{\alpha^2 K^2}} \right) \quad (115)$$

and

$$\gamma_2 = \frac{1}{2} \left(1 - \frac{2r}{\alpha^2 K} - \sqrt{\left(\frac{2r}{\alpha^2 K} - 1\right)^2 + \frac{8\rho}{\alpha^2 K^2}} \right). \quad (116)$$

Then clearly $\gamma_2 < 0 < \gamma_1$.

Moreover, note that

$$\gamma_1 < 1 \iff \rho < rK. \quad (117)$$

According to the Frobenius theory we can now conclude

$$\left. \begin{aligned} (i) \quad & \text{If } \gamma_1 - \gamma_2 = \sqrt{\left(\frac{2r}{\alpha^2(K-1)}\right)^2 + \frac{8\rho}{\alpha^2 K^2}} \notin \mathbf{N}, \\ & \text{where } \mathbf{N} \text{ is the set of natural numbers, then the general} \\ & \text{solution } f(x) \text{ of (109) in } (0, K) \text{ has the form} \\ & f(x) = Ax^{\gamma_1} g_1(x) + Bx^{\gamma_2} g_2(x) \quad (A, B \text{ arbitrary constants}) \\ & \text{where } g_i(x) = \sum_{k=0}^{\infty} a_k^{(i)} x^k \text{ are analytic for } |x| < K \\ & \text{and } g_i(0) = a_0^{(i)} = 1 \text{ for } i = 1, 2. \end{aligned} \right\} \quad (118)$$

$$\left. \begin{array}{l} (ii) \text{ If } \gamma_1 - \gamma_2 \in \mathbf{N}, \text{ then } f(x) = Ax^{\gamma_1}g_1(x) \text{ (} A \text{ arbitrary constant)} \\ \text{is always a solution of (109). (A second solution may be found} \\ \text{using the method of reduction of order).} \end{array} \right\} \quad (119)$$

Since the solution we are looking for is a candidate for the value function $\Phi(0, x)$ in the interval $(0, x^*)$, we require that our solution $f(x)$ must be *positive* and *bounded* near $x = 0$. This leads us to choose a solution $f(x)$ of the form

$$f(x) = Ax^{\gamma_1}g_1(x) \quad \text{where } A > 0. \quad (120)$$

Assume from now on that

$$\rho < rK. \quad (121)$$

(This is a natural assumption in view of Example 3.4). Then $\gamma_1 < 1$ by (117) and hence

$$f'(x) = \gamma_1 Ax^{\gamma_1-1}g_1(x) + Ax^{\gamma_1}g_1'(x) \rightarrow \infty \quad \text{as } x \rightarrow 0^+ \quad (122)$$

and

$$\begin{aligned} f''(x) &= \gamma_1(\gamma_1 - 1)Ax^{\gamma_1-2}g_1(x) + 2A\gamma_1x^{\gamma_1-1}g_1'(x) + Ax^{\gamma_1}g_1''(x) \\ &\rightarrow -\infty \quad \text{as } x \rightarrow 0^+. \end{aligned} \quad (123)$$

By the Feynman-Kac formula (see e.g. [Ø, Chapter 8]) we can for each given $y_0 \in (0, K)$ express the solution $f(x)$ for (109) in $(0, y_0)$ by

$$f(x) = E^x[f(X_{\tau(y_0)}) \exp(-\rho\tau(y_0))] = f(y_0)E^x[\exp(-\rho\tau(y_0))] \quad (124)$$

where

$$\tau(y_0) = \tau(y_0, x) = \inf\{t > 0; \quad X_t(\omega) \geq y_0\} \leq \infty \quad (125)$$

is the first exit time from $(0, y_0)$ of X_t when $X_0 = x$.

This implies that

$$f(x) > 0 \quad \text{for all } x \in (0, K), \quad (126)$$

for if there exists $y_0 \in (0, K)$ with $f(y_0) = 0$, then (124) gives that $f(x) = 0$ for all $x \in (0, y_0)$. It also implies that

$$f(x) \text{ is strictly increasing in } (0, K). \quad (127)$$

because if $0 < x_1 < x_2 < y_0 < K$ then $\tau(y_0, x_1) > \tau(y_0, x_2)$ and therefore $E^{x_1}[\exp(-\rho\tau(y_0))] < E^{x_2}[\exp(-\rho\tau(y_0))]$, which gives $f(x_1) < f(x_2)$.

Moreover, since $\tau(y) \rightarrow \infty$ if $y \rightarrow K^-$, we deduce that

$$f(y) \rightarrow \infty \text{ as } y \rightarrow K^-. \quad (128)$$

Combining this with (122) we get that the function $h(x) := f'(x)$ must have a minimum in $(0, K)$. Therefore there exists at least one point $x \in (0, K)$ such that $h'(x) = f''(x) = 0$.

Define

$$x^* = \inf\{x > 0; f''(x) = 0\}. \quad (129)$$

and put

$$D = (0, x^*).$$

Then

$$f''(x^*) = 0 \quad (130)$$

and

$$f''(x) < 0 \text{ for all } x \in D. \quad (131)$$

Note that from (127) we have $f'(x) \geq 0$ for all $x \in (0, K)$ and since $L_0 f(x) = 0$ we cannot have $f'(x^*) = 0$. Hence $f'(x^*) > 0$ and therefore we can choose the constant $A > 0$ in (120) such that

$$f'(x^*) = 1. \quad (132)$$

Then by (131) we have

$$f'(x) > 1 \text{ for } x \in D. \quad (133)$$

Finally choose C such that

$$f(x^*) = x^* + C. \quad (134)$$

Note that $C > 0$ since $f(0) = 0$ and $f'(x) > 1$ for $x \in (0, x^*)$.

Since $f''(x) < 0$ for $x < x^*$, we get

$$\begin{aligned} 0 = L_0 f(x) &= \frac{1}{2} \alpha^2 x^2 (K - x)^2 f''(x) + r x (K - x) f'(x) - \rho f(x) \\ &< r x (K - x) f'(x) - \rho f(x) \text{ for } x < x^*. \end{aligned}$$

Hence, if we put

$$F(x) = r x (K - x) f'(x) - \rho f(x)$$

we have

$$F(x) > 0 \text{ for } x < x^* \text{ and } F(x^*) = 0.$$

Therefore

$$F'(x^*) \leq 0,$$

i.e.

$$(rK - 2rx)f'(x) + rx(K - x)f''(x) - \rho f'(x) \leq 0 \text{ for } x = x^*.$$

Since $f''(x^*) = 0$ and $f'(x^*) = 1$, this gives

$$x^* \geq \frac{K}{2} - \frac{\rho}{2r}. \quad (\text{Compare with (99)}). \quad (135)$$

Now define

$$\psi(x) = \begin{cases} f(x) & \text{for } 0 \leq x < x^* \\ x + C & \text{for } x^* \leq x. \end{cases} \quad (136)$$

Then $\psi \in C^2$ by (130) and (132) and we see that (104), (107) and (108) hold by construction. To verify (105) it suffices to consider the case $x \geq x^*$. Then

$$L_0\psi(x) = rx(K - x) - \rho(x + C).$$

Note that

$$\frac{d}{dx}(L_0\psi(x)) = rK - 2rx - \rho < 0 \text{ for } x > \frac{K}{2} - \frac{\rho}{2r}.$$

Since $x^* \geq \frac{K}{2} - \frac{\rho}{2r}$ by (135) we conclude that

$$\sup_{x > x^*} L_0\psi(x) = L_0\psi(x^*) = 0.$$

Hence

$$L_0\psi(x) \leq 0 \text{ for all } x > x^*$$

and (105) is verified.

Finally, to verify (106) choose $\gamma \in \Gamma$ and let

$$\Omega_0 = \{\omega ; T(\omega) = \infty\}.$$

For $\omega \in \Omega_0$ we see that $\exp(-\rho T_R(\omega)) \rightarrow 0$ as $R \rightarrow \infty$ and since ψ is bounded on $[0, K]$ we get that

$$\exp(-\rho T_R(\omega))\psi(X_{T_R(\omega)}^\gamma) \rightarrow 0 \text{ as } R \rightarrow \infty,$$

pointwise boundedly for $\omega \in \Omega_0$.

For $\omega \notin \Omega_0$ we know that $X_{T_R}^\gamma(\omega) \rightarrow 0$ as $R \rightarrow \infty$ and since $\psi(0) = 0$ we again obtain that

$$\exp(-\rho T_R(\omega))\psi(X_{T_R}^\gamma(\omega)) \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

pointwise boundedly for $\omega \notin \Omega_0$.

Hence by bounded convergence

$$E^x[\exp(-\rho T_R)\psi(X_{T_R}^\gamma)] \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

and (106) is proved.

Thus the function φ defined by

$$\varphi(t, x) := e^{-\rho t}\psi(x)$$

satisfies all the requirements of Theorem 3.3 and we conclude that φ coincides with the value function Φ of Problem 3.2.

Next we note that if we combine the identities $f''(x^*) = 0$, $f'(x^*) = 1$, $f(x^*) = x^* + C$ with $L_0 f(x^*) = 0$ we get

$$rx^*(K - x^*) - \rho(x^* + C) = 0$$

or

$$x^* = \frac{K}{2} - \frac{\rho}{2} \pm \frac{1}{2r} \sqrt{(rK - \rho)^2 - 4r\rho C}. \quad (137)$$

By (135) we see that only the $+$ sign is possible. Also note that since $x^* \in \mathbf{R}$ exists we must have

$$C \leq \frac{(rK - \rho)^2}{4r\rho} \quad (\text{Compare with (101)}). \quad (138)$$

Finally, to apply Theorem 3.3 it suffices to show that there is a unique solution $(\hat{X}_t, \xi_t)$ of the Skorohod equation

$$\begin{aligned} d\hat{X}_t &= b(\hat{X}_t)dt + \sigma(\hat{X}_t)dB_t - d\xi_t \quad \text{for } t \geq 0 \\ \hat{X}_0 &= x \in [0, x^*], \quad \xi_0 = 0 \end{aligned} \quad (139)$$

such that $0 < \hat{X}_t \leq x^*$ for all $t \geq 0$, ξ_t is a nondecreasing process increasing only when $t \in \Lambda := \{t; \hat{X}_t = x^*\}$ and $\Lambda = \Lambda(\omega)$ has Lebesgue measure 0 for *a.a.* ω . The process $\hat{X}_t$ is called the *reflected X_t - process* at $x = x^*$ and ξ_t is called the *local time* of $\hat{X}_t$ at $x = x^*$. The existence and uniqueness of the pair $(\hat{X}_t, \xi_t)$ with the above properties follow from the general theory for Skorohod equations. (See e.g. Theorem 1.6.1 p. 89 in [F]).

Figure 2. The optimal harvesting rule and the corresponding population process (reflection at $x = x^*$).

We summarize what we have proved in the following theorem:

THEOREM 4.1 – (Optimal harvesting under uncertainty)

Let the population size X_t at time t be described by the logistic stochastic differential equation (102). Assume that

$$0 < \rho < rK. \quad (140)$$

Then the solution Φ of Problem 3.2 is given by

$$\Phi(s, x) = e^{-\rho s} \psi(x), \quad (141)$$

where $\psi(x)$ is defined as in (136), i.e.

$$\psi(x) = \begin{cases} f(x) & \text{for } 0 \leq x < x^* \\ x + C & \text{for } x \geq x^*, \end{cases} \quad (142)$$

with $f(x)$ and C uniquely determined by (120), (129)–(130), (132) and (134) and with x^* given by

$$x^* = \frac{K}{2} - \frac{\rho}{2r} + \frac{1}{2r} \sqrt{(rK - \rho)^2 - 4r\rho C}, \quad (143)$$

where

$$C \leq \frac{(rK - \rho)^2}{4r\rho}. \quad (144)$$

Moreover, the optimal harvesting strategy $\gamma_t^*(\omega)$ coincides with the local time ξ_t at $x = x^*$ for the process $\hat{X}_t$ reflected at $x = x^*$ given by (139). And hence the corresponding optimal population process $X_t^{\gamma^*}$ coincides with $\hat{X}_t$.

Concluding Remarks

The problem considered in this paper may be regarded as a mathematical formulation of the problem of *finding the best sustainable harvesting strategy under uncertainty*. We have proved that the optimal strategy is to wait until the population reaches the size x^* given by (143) before starting the harvesting (or harvest the population down to x^* immediately if it initially is above x^*) and from then on to harvest only when $X_t \geq x^*$ and then

exactly the (minimum) amount needed to prevent the process from exceeding the level x^* .

It is a consequence of our results that the trigger level x^* for harvesting is at least as big in the case with uncertainty as in the case without uncertainty (compare (143) with (99).) Thus it is clear from this discussion that one should harvest at a *higher* population level when there is uncertainty than when there is no uncertainty. Unfortunately, this optimal sustainable harvesting policy is often violated, for example in the fishing industry. The consequence is that we all get a lower total harvest than we could have achieved with the optimal policy.

The following questions arise:

- (i) Can we find C explicitly?
- (ii) Does x^* increase with $|\alpha|$?
- (iii) Is

$$\lim_{|\alpha| \rightarrow \infty} C(|\alpha|) = 0 \quad \text{and} \quad \lim_{|\alpha| \rightarrow \infty} x^*(|\alpha|) = K - \frac{\rho}{r}?$$

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