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by

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Abstract

If M is a factor of type III_λ , $0 < \lambda < 1$, we classify the subfactors which contain the centralizer of a periodic state or a generalized trace.

1. Introduction. Let M be a factor of type III_λ , $0 < \lambda < 1$, with separable predual. Let ϕ be a periodic state, i.e. a faithful normal state whose modular automorphism group satisfies $\sigma_{T_0} = id$, where $T_0 = -\frac{2\pi}{\log \lambda}$. In the present paper we classify the von Neumann subalgebras of M which properly contain the centralizer M_ϕ of ϕ . They form a sequence $(P_k)_{k \in \mathbf{N}}$, where P_k is the unique subfactor of type III_{λ^k} containing M_ϕ . A similar result holds for subfactors containing the centralizer of a generalized trace.

With M and ϕ as above let $M_0 = M_\phi$ and $M_n = \{x \in M : \sigma_t^\phi(x) = \lambda^{\text{int} t} x, t \in \mathbf{R}\}$, $n \in \mathbf{Z}$. Let E_n denote the orthogonal projection of M onto M_n , i.e.

$$E_n(x) = \frac{1}{T_0} \int_0^{T_0} \sigma_t^\phi(x) \lambda^{-\text{int} t} dt.$$

By a result of Takesaki [6, Thm. 1.27] there is an isometry $v \in M_1$ such that

$$M_n = M_0 v^n, \quad M_{-n} = v^{*n} M_0, \quad n \in \mathbf{N},$$

hence M is the von Neumann algebra generated by $M_\phi = M_0$ and v . We can then write M instructively as a direct sum

$$M \text{ " = " } \cdots \oplus v^{*n} M_0 \oplus \cdots \oplus v^* M_0 \oplus M_0 \oplus M_0 v \oplus \cdots \oplus M_0 v^n \oplus \cdots,$$

where " = " means that for any $x \in M$, $x = \sum_{n \in \mathbf{Z}} x(n)$ under the Hilbert space topology induced by ϕ , and $x(n) = E_n(x)$.

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Let P_k denote the von Neumann subalgebra of M generated by M_ϕ and v^k , $k \in \mathbf{N}$. By [1, Thm. 4.2.6] $M'_\phi \cap M = \mathbf{C}$, hence the same holds for P_k , so in particular P_k is a factor which is clearly of type III_{λ^k} . Our main result shows that these factors are the only intermediate subfactors between M_ϕ and M .

Theorem 1. With the above notation let P be a von Neumann algebra such that $M_\phi \subsetneq P \subset M$. Then there exists $k \in \mathbf{N}$ such that $P = P_k$.

2. The proof.

The proof will be divided into some lemmas. We denote by $\mathcal{U}(N)$ the unitary group in a von Neumann algebra N , and for $A \subset M$ we write A^{-stg} and $A^{-\|\cdot\|_\phi}$ for the closure of A in the strong topology and the metric topology induced by the metric $\|x\|_\phi = \phi(x^*x)^{\frac{1}{2}}$ respectively.

Throughout the section the notation is the same as in the introduction.

Lemma 1. If $x \in M$ then

$$E_0(x) \in \text{conv}(uxw^* : u, w \in \mathcal{U}(M_\phi))^{-stg}.$$

Proof. We first assume $E_0(x)$ is self-adjoint. Let A be a maximal abelian subalgebra of M_ϕ containing $E_0(x)$. By [1, Lem. 4.2.3] A is maximal abelian in M . We assert that

$$E_0(x) \in \text{conv}(uxu^* : u \in \mathcal{U}(A))^{-\|\cdot\|_\phi} \cap A.$$

Indeed, since $A \subset M_\phi$ each $u \in \mathcal{U}(A)$ defines a unitary operator U on $L^2(M, \phi)$ by $Ux = uxu^*$, $x \in M$. Let E_A denote the orthogonal projection

$$E_A : L^2(M, \phi) \rightarrow \{\xi \in L^2(M, \phi) : U\xi = \xi, \forall u \in \mathcal{U}(A)\}$$

It follows from the fixed point theorem of Alaoglu-Birkhoff, see e.g. [4, Lem. 7.12.3] that

$$E_A \in \text{conv}(U : u \in \mathcal{U}(A))^{-stg},$$

hence

$$E_A(x) \in \text{conv}(uxu^* : u \in \mathcal{U}(A))^{-\|\cdot\|_\phi}.$$

Consider E_0 as the orthogonal projection of $L^2(M, \phi)$ onto $L^2(M_\phi, \phi)$, and consider M as imbedded in $L^2(M, \phi)$. If $y \in M$ and $y \in E_A(M)$ then $y \in A' \cap M = A$. Hence we have $E_0 E_A = E_A$, so $E_0 \geq E_A$ as projections in $B(L^2(M, \phi))$. Now $x = E_0(x) + (x - E_0(x))$. Since $x - E_0(x)$ is orthogonal to M_ϕ , it follows that it is orthogonal to A . Since furthermore $E_0(x) \in A$ it follows that $E_0(x) = E_A(x)$, proving our assertion.

Let now x be a general element in M , and put $x_0 = E_0(x)$. Since M_ϕ is a finite factor x_0 has a polar decomposition $x_0 = u|x_0|$ with $u \in \mathcal{U}(M_\phi)$, $|x_0| \in M_\phi$. Thus

$$E_0(u^*x) = u^*E_0(x) = |x_0|.$$

From the first part of the proof

$$|x_0| = E_0(u^*x) \in \text{conv}(wu^*xw^* : w \in \mathcal{U}(M_\phi))^{-\|\cdot\|_\phi}.$$

Thus we have

$$E_0(x) = u|x_0| \in \text{conv}(uwu^*xw^* : w \in \mathcal{U}(M_\phi))^{-\|\cdot\|_\phi}.$$

Since uwu^* and w^* belong to $\mathcal{U}(M_\phi)$ for all w , and the closure in the topology of $\|\cdot\|_\phi$ of a bounded set equals the strong closure [2, Ch. I, §4. Prop. 4] the proof is complete.

Lemma 2. Let v be an isometry in M_1 , and let $x \in M$. Then

$$E_n(x) = E_0(x(v^*)^n)v^n, \quad n \in \mathbf{N}.$$

Proof. Since the linear span of the spectral subspaces $M_m, m \in \mathbf{Z}$, is strongly dense in M , it suffices to show

$$E_0(x(v^*)^n)v^n = \begin{cases} x & \text{if } x \in M_n \\ 0 & \text{if } x \in M_m, m \neq n \end{cases}$$

If $x \in M_n$ then $x(v^*)^n \in M_0$, hence

$$E_0(x(v^*)^n)v^n = x(v^*)^nv^n = x.$$

If $x \in M_m, m \neq n$, then $x(v^*)^n \in M_{m-n}$, hence $E_0(x(v^*)^n) = 0$, proving the lemma.

Lemma 3. Let P be a von Neumann algebra such that $M_\phi \subset P \subset M$. Then $\sigma_t^\phi(P) = P$ for all $t \in \mathbf{R}$.

Proof. Let $x \in P$, then $E_n(x) \in P$. Indeed, by Lemma 1

$$E_0(x(v^*)^n) \in \text{conv}(ux(v^*)^nw : u, w \in \mathcal{U}(M_\phi))^{-stg}.$$

Thus by Lemma 2

$$E_n(x) \in \text{conv}(u(xv^*)^nwv^n : u, w \in \mathcal{U}(M_\phi))^{-stg}.$$

But $(v^*)^nwv^n \in M_\phi$ for all $w \in \mathcal{U}(M_\phi)$. Since $M_\phi \subset P$ and $x \in P$ the assertion follows. Let K_k be the Fejer's kernel

$$K_k(t) = \sum_{n=-k}^k \left(1 - \frac{|n|}{k+1}\right) e^{int}.$$

If $x \in P$ let

$$x_k = \sum_{n=-k}^k \left(1 - \frac{|n|}{k+1}\right) E_n(x).$$

From [3, Ch.I§2] it follows that $\|x_k\| \leq \|x\|$, and $\|x_k - x\|_\phi \rightarrow 0$, hence x_k converges strongly to x .

By the first part of the proof $E_n(x) \in P$ for all $n \in \mathbf{N}$. Thus for all $t \in \mathbf{R}$

$$\sigma_t^\phi(x_k) = \sum_{n=-k}^k \left(1 - \frac{|n|}{k+1}\right) \lambda^{\text{int}} E_n(x) \in P,$$

hence $\sigma_t^\phi(x) = \text{strong lim } \sigma_t^\phi(x_k) \in P$.

Proof of Theorem 1. By Lemma 3 and Takesaki's theorem [5] there is a faithful normal ϕ -invariant conditional expectation of M onto P . Considering M as acting in the GNS-representation of ϕ we may assume ϕ is the vector state defined by a separating and cyclic vector ξ . Then ξ is separating and cyclic for P acting on the subspace $[P\xi]$. Let Δ_M be the modular operator for M with respect to ξ , and let Δ_P be the modular operator of P as acting on $[P\xi]$. Then by the proof of Takesaki's theorem it follows that the projection $[P\xi]$ commutes with Δ_M and $\Delta_P = \Delta_M[P\xi]$.

Since $M'_\phi \cap P = M'_\phi \cap M = \mathbf{C}$ by [1, Thm. 4.2.6] it follows from the same theorem that

$$S(P) = \text{Spec}(\Delta_P) \subset \text{Spec}(\Delta_M) = S(M) = \{\lambda^n : n \in \mathbf{Z}\}.$$

It follows that $S(P)$ is the cyclic group $\{(\lambda^k)^n : n \in \mathbf{Z}\}$ for some $k \in \mathbf{N}$, hence P is a factor of type III_{λ^k} . If we compose the conditional expectation of M onto P with that of P onto the centralizer P_ϕ of $\phi|_P$, we get a conditional expectation of M onto P_ϕ . Clearly $P_\phi \supset M_\phi$. By [1, Thm. 4.2.6] M_ϕ is maximal among the semifinite subfactors of M which are images of faithful normal conditional expectations. Thus $P_\phi = M_\phi$. Applying Takesaki's result [6, Thm. 1.27] there is an isometry $w \in P$ such that P is generated by M_ϕ and w , and the spectral subspaces are of the form $(w^*)^n M_\phi$ or $M_\phi w^n$. Since $\sigma_t^{\phi|P} = \sigma_t^\phi|_P$ and P is of type III_{λ^k} , we have $\sigma_t^\phi(w) = \lambda^{ikt} w$, $t \in \mathbf{R}$. Thus $w \in M_k$. By [6, Lem. 1.18] there is a partial isometry $u \in M_k$ such that $w = uv^k$. It follows that P is generated by M_ϕ and v^k , hence is equal to P_k . QED.

3. The generalized trace.

We next apply the theorem to the case when the subfactor contains the centralizer of a generalized trace.

Theorem 2. Let M be a factor of type III_λ , $0 < \lambda < 1$, with separable predual. Let ψ be a generalized trace with centralizer M_ψ . Then there exists a sequence $\{N_k, k \in \mathbf{N}\}$, of subfactors of M of type III_{λ^k} containing M_ψ , such that if N is a von Neumann subalgebra of M properly containing M_ψ , then $N = N_k$ for some k .

Proof. By [1, Cor. 4.3.3] there is a unitary operator U in the spectral subspace $M(\sigma^\psi, \lambda^{-1})$ such that $UM_\psi U^{-1} = M_\psi$, and M is the von Neumann algebra generated by U and M_ψ .

Let N_k denote the von Neumann subalgebra of M generated by U^k and M_ψ . Then N_k is a subfactor of type III_{λ^k} containing M_ψ . Let now N be a von Neumann subalgebra of M containing M_ψ . By [1, Thm. 4.2.6] N is a factor. Let e be a finite projection in M_ψ , and let N_e denote the reduced von Neumann algebra eNe , acting on eH , H being the underlying Hilbert space, and similarly for M and M_ψ . Then $M = M_e \otimes F$, $N = N_e \otimes F$, $M_\psi = M_{\psi e} \otimes F$, where F is a type I_∞ -factor. By Theorem 1 applied to $M_{\psi e} \subset N_e \subset M_e$ there exists a unique subfactor P_k such that $N_e = P_k$ with P_k of type III_{λ^k} . Let $N_k = P_k \otimes F$. Then $N = N_k$. Suppose Q is another subfactor of M of type III_{λ^k} containing M_ψ . Then $Q_e \supset M_{\psi e}$, so by uniqueness of P_k , $Q_e = P_k$. Thus $Q = N_k$ as well, and N_k is the unique subfactor of type III_{λ^k} containing M_ψ . QED

With U the unitary operator introduced in the above proof we can identify M with the crossed product $M_\psi \times_{\text{Ad}U} \mathbf{Z}$, and similarly the crossed products $M_\psi \times_{\text{Ad}U^n} \mathbf{Z}$ as subfactors of $M_\psi \times_{\text{Ad}U} \mathbf{Z}$. Then we have:

Corollary. Let M, ψ, U be as above. Then the crossed products $M_\psi \times_{\text{Ad}U^n} \mathbf{Z}$, $n \in \mathbf{N}$, are the only von Neumann subalgebras of M containing M_ψ .

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