CALCULATION OF TWO-PRIMARY ALGEBRAIC K-THEORY OF SOME GROUP RINGS

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ABSTRACT. In this paper we use an excision result in algebraic K-theory and recent two-primary calculations of algebraic K-theory of number rings to obtain results on the two-primary algebraic K-theory of some group rings.

1. INTRODUCTION AND SUMMARY OF RESULTS

This section will serve as a motivation for our work and a summary of the results in this paper. Recently Rognes and Weibel [RW] obtained strong two-primary information on the algebraic K-theory of number rings. Inspired by their results we will in this paper focus on the two-primary algebraic K-theory of a large class of group rings. The link between number rings and group rings is provided by the Cartesian square of rings

$$\begin{array}{ccc} \mathbb{Z}[C_p] & \longrightarrow & \mathbb{Z}[\zeta_p] \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{F}_p \end{array}$$

where C_p denotes the cyclic group of order p , ζ_p a primitive p -th root of unity, and $\mathbb{F}_p$ the field of p elements. Results of Charney [Ch] and Weibel [W1] tell us that the algebraic K-theory diagram

$$\begin{array}{ccc} K(\mathbb{Z}[C_p]; \mathbb{Z}[\frac{1}{p}]) & \longrightarrow & K(\mathbb{Z}[\zeta_p]; \mathbb{Z}[\frac{1}{p}]) \\ \downarrow & & \downarrow \\ K(\mathbb{Z}; \mathbb{Z}[\frac{1}{p}]) & \longrightarrow & K(\mathbb{F}_p; \mathbb{Z}[\frac{1}{p}]) \end{array}$$

is a homotopy Cartesian square. Hence the iterated homotopy fibre, say X , in the diagram has trivial $\mathbb{Z}[\frac{1}{p}]$ -homotopy. So for p odd we get that $\pi_n(X)_2 = 0$ for all n .

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Here $\widehat{}$ indicates that the group is completed at the prime 2. This tells us that the algebraic K-theory diagram

$$\begin{array}{ccc} K(\mathbb{Z}[C_p])\widehat{}_2 & \longrightarrow & K(\mathbb{Z}[\zeta_p])\widehat{}_2 \\ \downarrow & & \downarrow \\ K(\mathbb{Z})\widehat{}_2 & \longrightarrow & K(\mathbb{F}_p)\widehat{}_2 \end{array}$$

is a homotopy Cartesian square. All the homotopy groups of the spaces in this square are finitely generated Abelian groups. Proofs of these facts for the group ring, the finite field and the number ring can be found in [Ku], [Q1] and [Q2], respectively. Furthermore we obtain the exact sequence

$$0 \rightarrow K_{2n-1}(\mathbb{Z}[C_p])\widehat{}_2 \rightarrow K_{2n-1}(\mathbb{Z}[\zeta_p])\widehat{}_2 \oplus K_{2n-1}(\mathbb{Z})\widehat{}_2 \rightarrow K_{2n-1}(\mathbb{F}_p)\widehat{}_2 \rightarrow$$

$$\rightarrow K_{2n-2}(\mathbb{Z}[C_p])\widehat{}_2 \rightarrow K_{2n-2}(\mathbb{Z}[\zeta_p])\widehat{}_2 \oplus K_{2n-2}(\mathbb{Z})\widehat{}_2 \rightarrow 0$$

for all $n \geq 2$.

For an Abelian group A we let $A\{2\}$ denote the maximal two-torsion subgroup of A . Next we will give the main results in this paper.

Theorem 1.1. *Let $n \geq 2$. Assume that*

- 1) $p \equiv 3 \pmod{4}$, or
- 2) $p \equiv 5 \pmod{8}$ and n is even.

Then the map $K_{2n-1}(\mathbb{Z}[\zeta_p])\{2\} \rightarrow K_{2n-1}(\mathbb{F}_p)\{2\}$ is an isomorphism. Thus we have isomorphisms

$$K_{2n-1}(\mathbb{Z}[C_p])\{2\} \cong K_{2n-1}(\mathbb{Z})\{2\}$$

and

$$K_{2n-2}(\mathbb{Z}[C_p])\{2\} \cong K_{2n-2}(\mathbb{Z}[\zeta_p])\{2\} \oplus K_{2n-2}(\mathbb{Z})\{2\}.$$

We call a prime number p two-regular if the 2-rank of the Picard group of the two-integers in $\mathbb{Q}(\zeta_p)$ equals zero and 2 is a primitive root modulo p . The last condition excludes the cases $p \equiv 1, 7 \pmod{8}$.

Theorem 1.2. *Let $p \equiv 5 \pmod{8}$, and n odd. Then we have an isomorphism*

$$K_{2n-1}(\mathbb{Z}[C_p])\{2\} \cong K_{2n-1}(\mathbb{Z})\{2\}.$$

Furthermore for $n \equiv 3 \pmod{4}$ we have an isomorphism

$$K_{2n-2}(\mathbb{Z}[C_p])\{2\} \cong K_{2n-2}(\mathbb{Z}[\zeta_p])\{2\} \oplus K_{2n-2}(\mathbb{Z})\{2\}.$$

If $n \equiv 1 \pmod{4}$, then the latter result is true provided p is a two-regular prime.

We can make these results more explicit in the case p is a two-regular prime. The exact 2-power of an integer i will be denoted by $\text{ord}_2(i)$. Theorem 1.3 will be used in a forthcoming paper on the cyclotomic trace map $K(\mathbb{Z}[C_p])\widehat{}_2 \rightarrow TC(\mathbb{Z}[C_p])\widehat{}_2$.

Theorem 1.3. *We have the following calculation of the two-completed algebraic K-theory for the group ring $\mathbb{Z}[C_p]$ where p is a two-regular prime and $n > 1$.*

$n \bmod 8$	1	2	3	4
$K_n(\mathbb{Z}[C_p])_2^\wedge$	$\mathbb{Z}_2^{\frac{p+1}{2}} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}_2^{\frac{p-1}{2}} \oplus \mathbb{Z}/16$	0
$n \bmod 8$	5	6	7	8
$K_n(\mathbb{Z}[C_p])_2^\wedge$	$\mathbb{Z}_2^{\frac{p+1}{2}}$	0	$\mathbb{Z}_2^{\frac{p-1}{2}} \oplus \mathbb{Z}/2^a$	0

Here $a = 1 + \text{ord}_2(n + 1)$. Furthermore the map from $K_n(\mathbb{Z})_2^\wedge$ to $K_n(\mathbb{Z}[C_p])_2^\wedge$ is an isomorphism on the torsion part.

Now some words on the methods we will use. We obtain Theorem 1.1 and the first part of Theorem 1.2 by comparing with étale cohomology groups, and exposing the results in [RW]. The étale cohomology groups that we consider enjoy nice properties, e.g., we can compute them and also determine the maps between them. However, to establish the surjections $K_{2n-1}(\mathbb{Z}[\zeta_p])_2^\wedge \rightarrow K_{2n-1}(\mathbb{F}_p)_2^\wedge$ for $n \equiv 1, 3 \bmod 4$ in Theorem 1.2 we have to appeal to homotopy theory. The reason being that the torsion group in $K_{2n-1}(\mathbb{Z}[\zeta_p])_2^\wedge$ is too small in order to have any chance to surject onto $K_{2n-1}(\mathbb{F}_p)_2^\wedge$. For $n = 1$ this is known to hold, and then we use the Adams element A to obtain the claim for all $n \equiv 1 \bmod 4$. The $n \equiv 3 \bmod 4$ case is shown to be a consequence of the surjectivity of the map $K_3(\mathbb{Z})\{2\} \rightarrow K_3(\mathbb{F}_p)\{2\}$.

2. PROOF OF THEOREM 1.1

In this section we work with the same assumptions as in Theorem 1.1.

We let $W(n)$ denote the colimit over the coefficient injections $\mathbb{Z}/2^\nu(n) \rightarrow \mathbb{Z}/2^{\nu+1}(n)$ where the integer n indicates the n -th Tate twist.

Proof of Theorem 1.1. The two-torsion in $K_{2n-1}(\mathbb{Z}[\zeta_p])_2^\wedge$ is isomorphic to the étale cohomology group $H_{\text{ét}}^0(\mathbb{Q}(\zeta_p); W(n))$ by Theorem 6.13 [RW]. The latter group is calculated in the following way. Let F be a field. Define the integer $w_n^{(2)}(F)$ by

$$w_n^{(2)}(F) = \max \{2^\nu \mid \text{Gal}(F(\zeta_{2^\nu})/F) \text{ has exponent dividing } n\}$$

for each integer n . If no maximum can be achieved, we put $w_n^{(2)}(F) = 2^\infty$. It is well known that $H_{\text{ét}}^0(F; W(n)) \cong \mathbb{Z}/w_n^{(2)}(F)$.

Moreover, we have the isomorphism $K_{2n-1}(\mathbb{F}_p)\{2\} \cong H_{\text{ét}}^0(\mathbb{F}_p; W(n))$. Thus to study the map from $K_{2n-1}(\mathbb{Z}[\zeta_p])\{2\}$ to $K_{2n-1}(\mathbb{F}_p)\{2\}$ amounts to studying the map $H_{\text{ét}}^0(\mathbb{Q}(\zeta_p); W(n)) \rightarrow H_{\text{ét}}^0(\mathbb{F}_p; W(n))$. The imposed assumptions ensure that these two groups are in fact isomorphic. In result the map between them must be an isomorphism, since the map is compatible with the inclusion of the two groups into $H_{\text{ét}}^0(\mathbb{C}; W(n)) \cong \mathbb{Z}/2^\infty$. $\square$

3. PROOF OF THEOREM 1.2

In this section we work with the same assumptions as in Theorem 1.2, i.e., $p \equiv 5 \bmod 8$ and n odd. The isomorphism $K_{2n-1}(\mathbb{Z}[C_p])\{2\} \cong K_{2n-1}(\mathbb{Z})\{2\}$ is an

immediate consequence of the following calculations, and the arguments given in the proof of Theorem 1.1. We have that $K_{2n-1}(\mathbb{F}_p)\{2\} \cong \mathbb{Z}/4$, $K_{2n-1}(\mathbb{Z}[\zeta_p])\{2\} \cong \mathbb{Z}/2$ and $K_{2n-1}(\mathbb{Z})\{2\} \cong \mathbb{Z}/2$ if $n \equiv 1 \pmod{4}$ and trivial otherwise.

The problem now is to establish the surjectivity of the map

$$K_{2n-1}(\mathbb{Z}[\zeta_p])_2^\wedge \longrightarrow K_{2n-1}(\mathbb{F}_p)_2^\wedge$$

for $n \equiv 1, 3 \pmod{4}$.

For K_1 this is a classical result of Rim, see pp.29-32 in [Mi]. We will use the K_1 result and then multiply by the Adams element A in order to prove that the same holds for all degrees congruent 1 modulo 8. We have the commutative diagram:

$$\begin{array}{ccc} K_1(\mathbb{Z}[\zeta_p, \frac{1}{2}]; \mathbb{Z}/2) & \longrightarrow & K_1(\mathbb{F}_p; \mathbb{Z}/2) \\ A^n \downarrow & & \cong \downarrow A^n \\ K_{8n+1}(\mathbb{Z}[\zeta_p, \frac{1}{2}]; \mathbb{Z}/2) & \longrightarrow & K_{8n+1}(\mathbb{F}_p; \mathbb{Z}/2) \\ \cong \uparrow & & \cong \uparrow \\ K_{8n+1}(\mathbb{Z}[\zeta_p, \frac{1}{2}])/2 & \longrightarrow & K_{8n+1}(\mathbb{F}_p)/2 \end{array}$$

The element A constructed in [Ad] acts on mod 2 homotopy shifting the degree by eight. Furthermore A acts isomorphically on the mod 2 homotopy of BU from Bott periodicity. A famous result of Quillen [Q1] tells us that

$$K(\mathbb{F}_p) \longrightarrow \mathbb{Z} \times BU \xrightarrow{\psi^p - 1} BU$$

is a fibration. Using the 5-lemma we see that A acts isomorphically on the mod 2 homotopy of $K(\mathbb{F}_p)$ as well. The isomorphisms in the lower square come from the calculations of $K_{8n}(\mathbb{Z}[\zeta_p, \frac{1}{2}])$ and $K_{8n}(\mathbb{F}_p)$ in [RØ] and [Q1], respectively. See also Theorem 4.1 in this paper. None of these groups have any two-torsion, remember that p is assumed to be two-regular, so an application of the universal coefficient theorem in algebraic K-theory gives us the isomorphisms. The diagram shows that

$$K_{8n+1}(\mathbb{Z}[\zeta_p])_2^\wedge \longrightarrow K_{8n+1}(\mathbb{F}_p)_2^\wedge$$

is surjective.

Next we consider the maps in degrees 5 modulo 8. We point out that we do not need the assumption that p is two-regular for the following arguments to be valid. We use the diagram

$$\begin{array}{ccc} & K_n(\mathbb{Z})_2^\wedge & \\ \swarrow & & \searrow \\ K_n(\mathbb{Z}[\zeta_p])_2^\wedge & \longrightarrow & K_n(\mathbb{F}_p)_2^\wedge \end{array}$$

induced by the triangle

$$\begin{array}{ccc} & \mathbb{Z} & \\ \swarrow & & \searrow \\ \mathbb{Z}[\zeta_p] & \xrightarrow{\quad} & \mathbb{F}_p \end{array}$$

to conclude that it suffices to prove that the map

$$K_{8n+5}(\mathbb{Z})_2 \longrightarrow K_{8n+5}(\mathbb{F}_p)_2$$

is surjective. From [St] p.45 the image of $K_3(\mathbb{Z})$ in $K_3(\mathbb{F}_p)$ is of order 24 for $p \geq 5$. In result we derive that

$$K_3(\mathbb{Z})_2 \longrightarrow K_3(\mathbb{F}_p)_2$$

is surjective. Consider the commutative diagram:

$$\begin{array}{ccc} K_3(\mathbb{Z})_2 & \longrightarrow & K_3(\mathbb{F}_p)_2 \\ \circ \bar{\eta}_2 \downarrow & & \downarrow \circ \bar{\eta}_2 \\ K_5(\mathbb{Z}; \mathbb{Z}/2) & \longrightarrow & K_5(\mathbb{F}_p; \mathbb{Z}/2) \end{array}$$

Here $\circ \bar{\eta}_2$ indicates pre-composition with the map $\bar{\eta}_2$. This map is defined by the diagram

$$\begin{array}{ccccc} S^4 & \xrightarrow{2} & S^4 & \xrightarrow{i} & S^4/2 \\ & \searrow & \downarrow \Sigma \eta & \swarrow \bar{\eta}_2 & \\ & 2\Sigma \eta & S^3 & & \end{array}$$

where η denotes the Hopf map and $S^4/2 = S^4 \cup_2 D^5$ the mod 2 Moore space. Recall that $2\Sigma \eta$ is null homotopic making it possible to define $\bar{\eta}_2$.

Once again we appeal to the fibre sequence

$$\Omega BU \xrightarrow{\delta} K(\mathbb{F}_p) \longrightarrow \mathbb{Z} \times BU \xrightarrow{\psi^p - 1} BU$$

from [Q1]. The map δ combined with $\bar{\eta}_2$ gives the commutative diagram:

$$\begin{array}{ccc} \pi_4(BU) & \xrightarrow{\delta} & K_3(\mathbb{F}_p) \\ \circ \bar{\eta}_2 \downarrow & & \downarrow \circ \bar{\eta}_2 \\ \pi_6(BU; \mathbb{Z}/2) & \xrightarrow{\delta} & K_5(\mathbb{F}_p; \mathbb{Z}/2) \end{array}$$

The horizontal maps are surjective according to [Q1]. Further the map on the left hand side is surjective by Bott periodicity and the fact that the complex e -invariant of η equals $1/2$, see [Q3]. We conclude that the map from $K_3(\mathbb{F}_p)$ to $K_5(\mathbb{F}_p; \mathbb{Z}/2)$ is surjective. We can also derive this from Proposition 2.6 in [Br].

Hence the map from $K_5(\mathbb{Z}; \mathbb{Z}/2)$ to $K_5(\mathbb{F}_p; \mathbb{Z}/2)$ is surjective as well. Since $K_4(\mathbb{Z})$ is the trivial group we deduce from the universal coefficient theorem in algebraic K-theory that the map from $K_5(\mathbb{Z})_2$ to $K_5(\mathbb{F}_p)_2$ is onto. The next step is to use the Adams element A to derive the same conclusion in all degrees 5 modulo 8. We have already seen that trick for the algebraic K-groups in degree 1 modulo 8, and the proof is complete.

4. PROOF OF THEOREM 1.3

Next we turn to the explicit calculation of the algebraic K-theory of the group ring $\mathbb{Z}[C_p]$ for p a two-regular prime in Theorem 1.3. The two-primary algebraic K-theory of $\mathbb{Z}[\zeta_p]$ and $\mathbb{Z}$ are determined in [RØ] and [W2], respectively.

Theorem 4.1. *Let $n > 1$. We have the following calculation of the two-completed algebraic K-theory of $\mathbb{Z}$ and $\mathbb{Z}[\zeta_p]$ where $a = 1 + \text{ord}_2(n + 1)$ and p is a two-regular prime.*

$n \bmod 8$	1	2	3	4
$K_n(\mathbb{Z}[\zeta_p])_2^\wedge$	$\mathbb{Z}_2^{\frac{p-1}{2}} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_2^{\frac{p-1}{2}} \oplus \mathbb{Z}/8$	0
$K_n(\mathbb{Z})_2^\wedge$	$\mathbb{Z}_2 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0
$n \bmod 8$	5	6	7	8
$K_n(\mathbb{Z}[\zeta_p])_2^\wedge$	$\mathbb{Z}_2^{\frac{p-1}{2}} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_2^{\frac{p-1}{2}} \oplus \mathbb{Z}/2^a$	0
$K_n(\mathbb{Z})_2^\wedge$	$\mathbb{Z}_2$	0	$\mathbb{Z}/2^a$	0

The rank of $K_n(\mathbb{Z}[G])$ can easily be extracted from the ranks of $K_n(\mathbb{Z}[\zeta_p])$ and $K_n(\mathbb{Z})$ by using the exact sequence in advance of Theorem 1.1 and Borel's result on the rank of algebraic K-groups of number rings [Bo]. More generally, the rank of $K_n(\mathbb{Z}[G])$ is known where G is a finite group, see [Ja].

Theorem 1.3 is now immediate from Theorems 1.1 and 1.2.

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