Optimal portfolio in a fractional Black & Scholes market

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Dedicated to Prof. Ludwig Streit on the occasion of his 60th birthday

Abstract

We use the martingale method of Cox and Huang to solve explicitly the optimal portfolio problem in a Black & Scholes type of market driven by fractional Brownian motion $B_H(t)$ with Hurst parameter $H \in (\frac{1}{2}, 1)$. The results are compared to the corresponding well-known results in the standard Black & Scholes market.

1 Introduction

If H is a constant, $0 < H < 1$, then the *fractional Brownian motion with Hurst parameter H* is the Gaussian process $B_H(t) = B_H(t, \omega)$; $t \geq 0$, $\omega \in \Omega$ with mean $E[B_H(t)] = 0$ for all $t \geq 0$ and covariance

$$E[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$$

for all $s, t \geq 0$. Here E denotes the expectation with respect to the probability law μ_H for B_H on Ω . We assume that $B_H(0) = 0$.

If $H = \frac{1}{2}$ then $B_H(t)$ coincides with the standard Brownian motion $B(t)$. If $H > \frac{1}{2}$ then $B_H(t)$ has a *long range dependence* in the sense that if we let

$$\rho(n) = \text{cov}(B_H(1), B_H(n+1) - B_H(n))$$

then

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

$B_H(t)$ is *self-similar* in the sense that $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$, for any $\alpha > 0$. Because of these properties fractional Brownian motion has been suggested as a useful tool in finance and other applications. See e.g. [M]. However, $B_H(t)$ is neither a semimartingale nor a Markov process, so many of the powerful techniques from stochastic analysis are not available when dealing with $B_H(t)$. On the other hand, recently a white noise theory for $B_H(t)$ for $\frac{1}{2} < H < 1$ was developed [HØ] (see also [DHP]) and this was applied to prove that if the corresponding integration theory (in the Itô sense, i.e. based on the Wick product rather than

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the pathwise product) is used, then the corresponding fractional Black & Scholes market is without arbitrage, it is complete and an explicit fractional option pricing and hedging formula can be given.

We now describe this fractional Black & Scholes market more precisely and refer to [HØ] and the references therein for more information. We assume throughout that $\frac{1}{2} < H < 1$.

Suppose we have the following two investment possibilities:

(i) A *bank account* or a *bond*, where the price $A(t)$ at time $t \geq 0$ is given by

$$(1.1) \quad dA(t) = rA(t)dt; \quad A(0) = 1,$$

where $r > 0$ is a constant.

(ii) A *stock*, where the price $S(t)$ at time $t \geq 0$ is given by

$$(1.2) \quad dS(t) = aS(t)dt + \sigma S(t)dB_H(t); \quad S(0) = s > 0,$$

where $a > r > 0$ and $\sigma \neq 0$ are constants.

Here the differential $dB_H(t)$ is the Itô type fractional Brownian motion differential used in [HØ].

Suppose an investor chooses a portfolio $\theta(t) = (\alpha(t), \beta(t))$ giving the number of units $\alpha(t), \beta(t)$ held at time t of bonds and stocks, respectively. We assume that $\alpha(t), \beta(t)$ are $\mathcal{F}_t^{(H)}$ -adapted processes, where $\mathcal{F}_t^{(H)}$ is the σ -algebra generated by $\{B_H(s)\}_{s \leq t}$. Let

$$(1.3) \quad Z(t) = Z^\theta(t, \omega) = \alpha(t)A(t) + \beta(t)S(t)$$

be the *value process* corresponding to this portfolio. Following [HØ] we will assume that θ is *self-financing*, in the sense that

$$(1.4) \quad dZ(t) = \alpha(t)dA(t) + \beta(t)dS(t).$$

We say that θ is *admissible* if, in addition, $Z^\theta(t)$ is nonnegative, a.s. We let $\mathcal{A}$ denote the set of admissible portfolios.

With a given initial value $z > 0$ consider the problem to find $V(z)$ and $\theta^* \in \mathcal{A}$ such that

$$(1.5) \quad V(z) = V_H(z) = \sup_{\theta \in \mathcal{A}} E^z[u(Z^\theta(T))] = E^z[u(Z^{\theta^*}(T))],$$

where $T > 0$ is a given constant, E^z denotes expectation w.r.t. μ_H when $Z^\theta(0) = z$ and $u: (0, \infty) \rightarrow \mathbb{R}$ is a given utility function, assumed to be nondecreasing and concave. An example of such a utility function is

$$(1.6) \quad u(x) = \frac{1}{\gamma} x^\gamma \quad \text{where } \gamma \in (0, 1) \text{ is constant.}$$

The constant $1 - \gamma$ is interpreted as the risk aversion.

In the case of standard Brownian motion $B(t)$ a natural approach to this problem would be dynamic programming, which leads to the Hamilton-Jacobi-Bellman equation (see e.g. [Ø, Ch. 11]). However, with $B(t)$ replaced by $B_H(t)$ it is no longer possible to use such a method, because we cannot make the system Markovian.

Another commonly used approach in the standard case ($H = \frac{1}{2}$) is the *martingale approach*, introduced by Cox and Huang [CH1], [CH2]. See also the presentation in [KLS]. The purpose of this note is to show how this approach – in spite of the fact that $B_H(t)$ is not a martingale (not even a semimartingale) – can be adapted to solve (1.5). We give an explicit solution and compare it to the well-known solution in the standard case.

2 Explicit solution of the optimal portfolio problem

We now show in detail how the martingale approach of Cox and Huang can be used to solve problem (1.5) for an Itô type fractional Black & Scholes market with Hurst parameter $H \in (\frac{1}{2}, 1)$. To this end, note that if we substitute (from (1.3))

$$(2.1) \quad \alpha(t) = A(t)^{-1}(Z(t) - \beta(t)S(t))$$

into (1.4) we get

$$(2.2) \quad dZ(t) = rZ(t)dt + \sigma\beta(t)S(t) \left[\frac{a-r}{\sigma}dt + dB_H(t) \right].$$

As in [HØ] we rewrite this as

$$(2.3) \quad e^{-rt}Z(t) = z + \int_0^t \exp(-rs)\sigma\beta(s)S(s)d\hat{B}_H(s),$$

where, by the fractional Girsanov theorem [HØ, Theorem 3.18], the process

$$(2.4) \quad \hat{B}_H(t) := \frac{a-r}{\sigma}t + B_H(t)$$

is a fractional Brownian motion with respect to the measure $\hat{\mu}_H$ defined on $\mathcal{F}_T^{(H)}$ by

$$(2.5) \quad \frac{d\hat{\mu}_H}{d\mu_H} = \exp \left(- \int_0^T K(s)dB_H(s) - \frac{1}{2}|K|_\varphi^2 \right) =: \exp^\circ \left(- \int_0^T K(s)dB_H(s) \right) =: \eta,$$

where

$$|K|_\varphi^2 = \int_0^T \int_0^T K(s)K(t)\varphi(s,t)ds dt, \quad \varphi(s,t) = H(2H-1)|s-t|^{2H-2}$$

and

$$(2.6) \quad K(s) = \frac{(a-r)(Ts-s^2)^{\frac{1}{2}-H} \mathcal{X}_{[0,T]}(s)}{2\sigma H\Gamma(2H)\Gamma(2-2H)\cos(\pi(H-\frac{1}{2}))},$$

where Γ is the gamma function. As proved in [HØ] any given $\mathcal{F}_T^{(H)}$ -measurable $F(\omega) \geq 0$ can be achieved as the terminal value $Z^\theta(T, \omega)$ a.s. for some $\theta \in \mathcal{A}$ and with $Z^\theta(0) = z$ if and only if

$$(2.7) \quad z = E_{\hat{\mu}_H}[e^{-rT}F].$$

Therefore the problem (1.5) can be reformulated as follows

$$(2.8) \quad V(z) = \sup_F \{E[u(F)]; E[e^{-rT}\eta F] = z\},$$

where the supremum is taken over all $\mathcal{F}_T^{(H)}$ -measurable nonnegative random variables F . To solve this constrained maximization problem we consider, for a given $\lambda > 0$, the unconstrained problem

$$(2.9) \quad \Phi_\lambda(z) := \sup_F \{E[u(F)] - \lambda E[e^{-rT}\eta F]\}.$$

Suppose F_λ^* solves this unconstrained problem. Then for any F such that $E[e^{-rT}\eta F] = z$ we have

$$(2.10) \quad E[u(F_\lambda^*) - \lambda e^{-rT}\eta F_\lambda^*] \geq E[u(F) - \lambda e^{-rT}\eta F] = E[u(F)] - \lambda z.$$

In particular, if there exists $\lambda_0 > 0$ such that

$$(2.11) \quad E[e^{-rT}\eta F_{\lambda_0}^*] = z$$

then we get from (2.10)

$$E[u(F_{\lambda_0}^*)] \geq E[u(F)].$$

Hence $F_{\lambda_0}^*$ solves the constrained problem (2.8).

First let us assume that, as in (1.6),

$$(2.12) \quad u(x) = \frac{1}{\gamma} x^\gamma \quad \text{for some } \gamma \in (0, 1).$$

Then the unconstrained problem (2.9) becomes

$$(2.13) \quad \Phi_\lambda(z) = \sup_F E \left[\frac{1}{\gamma} F^\gamma(\omega) - \lambda e^{-rT} \eta(\omega) F(\omega) \right].$$

We solve this ω -wise by maximizing

$$g(F) := \frac{1}{\gamma} F^\gamma - \lambda e^{-rT} \eta F$$

with respect to $F \geq 0$:

We have

$$g'(F) = F^{\gamma-1} - \lambda e^{-rT} \eta = 0$$

for

$$F = \left(\lambda e^{-rT} \eta \right)^{\frac{1}{\gamma-1}}.$$

By concavity we conclude that $g(F)$ is maximal when

$$(2.14) \quad F = F_\lambda^* = \left(\lambda e^{-rT} \eta \right)^{\frac{1}{\gamma-1}}.$$

We now seek λ_0 such that (2.11) holds, i.e.

$$(2.15) \quad E \left[e^{-rT} \eta (\lambda_0 e^{-rT} \eta)^{\frac{1}{\gamma-1}} \right] = z.$$

This gives

$$(2.16) \quad \lambda_0 = \frac{z^{\gamma-1} e^{r\gamma T}}{\left(E \left[\eta^{\frac{\gamma}{\gamma-1}} \right] \right)^{\gamma-1}}.$$

If we substitute $\lambda = \lambda_0$ in (2.14) we get

$$(2.17) \quad F_{\lambda_0}^* = \frac{z e^{rT} \eta^{\frac{1}{\gamma-1}}}{E \left[\eta^{\frac{\gamma}{\gamma-1}} \right]}.$$

This is the optimal value of F in the constrained problem (2.8). Hence

$$(2.18) \quad V(z) = E \left[\frac{1}{\gamma} (F_{\lambda_0}^*)^\gamma \right] = \frac{z^\gamma e^{r\gamma T} E \left[\eta^{\frac{\gamma}{\gamma-1}} \right]}{\gamma (E \left[\eta^{\frac{\gamma}{\gamma-1}} \right])^\gamma} = \frac{1}{\gamma} z^\gamma e^{r\gamma T} (E \left[\eta^{\frac{\gamma}{\gamma-1}} \right])^{1-\gamma}.$$

By (2.5) we have

$$\begin{aligned} \eta^{\frac{\gamma}{\gamma-1}} &= \exp \left(\frac{\gamma}{1-\gamma} \int_0^T K(s) dB_H(s) + \frac{\gamma}{2(1-\gamma)} |K|_\varphi^2 \right) \\ &= \exp \left(\frac{\gamma}{1-\gamma} \int_0^T K(s) dB_H(s) - \frac{1}{2} \frac{\gamma^2}{(1-\gamma)^2} |K|_\varphi^2 + \frac{1}{2} \frac{\gamma^2}{(1-\gamma)^2} |K|_\varphi^2 + \frac{\gamma}{2(1-\gamma)} |K|_\varphi^2 \right) \\ &= \exp^\diamond \left(\frac{\gamma}{1-\gamma} \int_0^T K(s) dB_H(s) \right) \cdot \exp \left(\frac{\gamma}{2(1-\gamma)^2} |K|_\varphi^2 \right). \end{aligned}$$

Since $E[\exp^\diamond(\int_0^T f(s) dB_H(s))] = 1$ for all $f \in L_\varphi^2$, (see [HØ]) we get

$$(2.19) \quad E \left[\eta^{\frac{\gamma}{\gamma-1}} \right] = \exp \left(\frac{\gamma}{2(1-\gamma)^2} |K|_\varphi^2 \right).$$

Hence by (2.18)

$$(2.20) \quad V(z) = V_H(z) = \frac{1}{\gamma} z^\gamma \exp \left(r\gamma T + \frac{\gamma}{2(1-\gamma)} |K|_\varphi^2 \right).$$

Similarly, from (2.17) and (2.19) we get the corresponding optimal terminal value $Z^{\theta^*}(T)$:

$$\begin{aligned} Z^{\theta^*}(T) &= F_{\lambda_0}^* \\ &= z \exp \left(rT + \frac{1}{1-\gamma} \int_0^T K(s) dB_H(s) + \frac{1}{2(1-\gamma)} |K|_\varphi^2 - \frac{\gamma}{2(1-\gamma)^2} |K|_\varphi^2 \right) \\ (2.21) \quad &= z \exp \left(\frac{1}{1-\gamma} \int_0^T K(s) dB_H(s) + rT + \frac{1-2\gamma}{2(1-\gamma)^2} |K|_\varphi^2 \right). \end{aligned}$$

To compute $|K|_\varphi^2$ we use that

$$(2.22) \quad \int_0^T K(s) \varphi(s, t) ds = \frac{a-r}{\sigma} \quad \text{for } 0 \leq t \leq T$$

(see [HØ, (5.13)]). This gives, using (2.6),

$$\begin{aligned} (2.23) \quad |K|_\varphi^2 &= \int_0^T \int_0^T K(s) K(t) \varphi(s, t) ds dt = \frac{a-r}{\sigma} \int_0^T K(t) dt \\ &= \frac{(a-r)^2}{2\sigma^2 H \cdot \Gamma(2H) \cdot \Gamma(2-2H) \cdot \cos(\pi(H - \frac{1}{2}))} \int_0^T (Tt - t^2)^{\frac{1}{2}-H} dt \end{aligned}$$

$$(2.24) \quad = \frac{(a-r)^2 \cdot \Gamma^2(\frac{3}{2} - H) \cdot T^{2-2H}}{2\sigma^2 H \cdot (2-2H) \cdot \Gamma(2H) \cdot \Gamma^2(2-2H) \cos(\pi(H - \frac{1}{2}))},$$

where we have use that (see [HØ, Section 6])

$$(2.25) \quad \int_0^T (Tt - t^2)^{\frac{1}{2}-H} dt = \frac{\Gamma^2(\frac{3}{2} - H)}{\Gamma(3 - 2H)} T^{2-2H}$$

plus the basic identity

$$\Gamma(x + 1) = x\Gamma(x) .$$

Therefore, if we put

$$(2.26) \quad \Lambda_H = \frac{\Gamma^2(\frac{3}{2} - H)}{2H \cdot (2 - 2H) \cdot \Gamma(2H) \cdot \Gamma(2 - 2H) \cdot \cos(\pi(H - \frac{1}{2}))}$$

we obtain from (2.24) that

$$(2.27) \quad |K|_\varphi^2 = \left(\frac{a - r}{\sigma} \right)^2 \Lambda_H \cdot T^{2-2H} .$$

Therefore, by (2.20) and (2.21) we get

Theorem 2.1 *The value function $V(z)$ of the optimal portfolio problem (1.5)–(1.6) is given by*

$$(2.28) \quad V(z) = V_H(z) = \frac{1}{\gamma} z^\gamma \exp \left(r\gamma T + \frac{\gamma}{2(1-\gamma)} \cdot \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H \cdot T^{2-2H} \right) .$$

The corresponding optimal terminal value $Z^{\theta^}(T)$ is given by:*

$$(2.29) \quad Z^{\theta^*}(T) = z \exp \left(\frac{1}{1-\gamma} \int_0^T K(s) dB_H(s) + rT + \frac{1-2\gamma}{2(1-\gamma)^2} \cdot \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H} \right) .$$

Remark. It is natural to ask how the value function $V = V_H(z)$ in (2.28) is related to the value function $V_{1/2}(z)$ for the corresponding problem for standard Brownian motion ($H = \frac{1}{2}$). In this case it is well-known that (see e.g. [Ø, (11.2.53)])

$$(2.30) \quad V_{1/2}(z) = \frac{1}{\gamma} z^\gamma \exp \left(r\gamma T + \frac{\gamma}{2(1-\gamma)} \cdot \left(\frac{a-r}{\sigma} \right)^2 T \right) .$$

Therefore we see that, as was to be expected,

$$(2.31) \quad \lim_{H \rightarrow \frac{1}{2}+} V_H(z) = V_{1/2}(z) .$$

Next we turn to the question of finding the corresponding optimal portfolio $\theta^* = (\alpha^*, \beta^*)$. Because the system is not Markovian we cannot use the traditional PDE method to find θ^* . However, we can use the Clark-Ocone formula in [HØ, Theorem 4.15]. Applying this formula to our situation we get, with $Z = Z^{\theta^*}(T)$,

$$(2.32) \quad e^{-rT} Z(\omega) = E_{\hat{\mu}_H} [e^{-rT} Z] + \int_0^T \tilde{E}_{\hat{\mu}_H} [e^{-rT} \hat{D}_t Z \mid \mathcal{F}_t^{(H)}] d\hat{B}_H(t) .$$

where $\widehat{D}_t$ denotes the stochastic gradient with respect to the measure $\widehat{\mu}_H$ and $\widetilde{E}[\cdot|\cdot]$ denotes the *quasiconditional expectation*.

If we compare (2.32) with (2.3) we get by uniqueness that

$$\exp(-rt)\sigma\beta^*(t)S(t) = \widetilde{E}_{\widehat{\mu}_H}[e^{-rT}\widehat{D}_t Z \mid \mathcal{F}_t^{(H)}]$$

or

$$(2.33) \quad \beta^*(t) = e^{r(t-T)}\sigma^{-1}S^{-1}(t)\widetilde{E}_{\widehat{\mu}_H}[\widehat{D}_t Z \mid \mathcal{F}_t^{(H)}].$$

By (2.29) and (2.4) we have

$$(2.34) \quad \begin{aligned} Z = z \exp & \left(\frac{1}{1-\gamma} \int_0^T K(s) d\widehat{B}_H(s) - \frac{1}{1-\gamma} \int_0^T K(s) \frac{a-r}{\sigma} ds \right. \\ & \left. + rT + \frac{1-2\gamma}{2(1-\gamma)^2} \cdot \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H} \right) \end{aligned}$$

and therefore, by the chain rule,

$$(2.35) \quad \widehat{D}_t Z = \frac{K(t)}{1-\gamma} Z.$$

To facilitate the computation of the quasiconditional expectation we write, using (2.34) and (2.23),

$$(2.36) \quad \begin{aligned} Z = z \exp & \left(\frac{1}{1-\gamma} \int_0^T K(s) d\widehat{B}_H(s) - \frac{1}{2(1-\gamma)^2} |K|_\varphi^2 + \frac{1}{2(1-\gamma)^2} |K|_\varphi^2 \right. \\ & \left. - \frac{1}{1-\gamma} \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H} + rT + \frac{1-2\gamma}{2(1-\gamma)^2} \cdot \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H} \right) \\ = z \exp &^\circ \left(\frac{1}{1-\gamma} \int_0^T K(s) d\widehat{B}_H(s) \right) \cdot M, \end{aligned}$$

where

$$(2.37) \quad M = \exp \left(rT - \frac{\gamma}{2(1-\gamma)^2} \cdot \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H} \right).$$

Define

$$(2.38) \quad \begin{aligned} \rho_H(t) = & \frac{H(2H-1)}{4\Gamma^2(2H)\Gamma^2(2-2H)\cos^2(\pi(H-1/2))} \\ & \int_0^t \int_0^t |u-v|^{2H-2} (u-u^2)^{1/2-H} (v-v^2)^{1/2-H} dudv. \end{aligned}$$

Then

$$(2.39) \quad \int_0^t \int_0^t \phi(u,v) K(u) K(v) dudv = T^{2-2H} \rho_H \left(\frac{t}{T} \right) \left(\frac{a-r}{\sigma} \right)^2.$$

It is obvious that

$$\rho_H(1) = \Lambda_H.$$

Therefore,

$$\begin{aligned}
 (2.40) \quad \tilde{E}_{\hat{\mu}_H}[D_t Z \mid \mathcal{F}_t^{(H)}] &= \frac{K(t)}{1-\gamma} M \exp^\circ \left(\frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) \right) \\
 &= \frac{K(t)}{1-\gamma} M \exp \left[\frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) - \frac{1}{2(1-\gamma)^2} \int_0^t \int_0^t \phi(u, v) K(u) K(v) du dv \right] \\
 &= \frac{K(t)}{1-\gamma} \exp \left\{ \frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) - \frac{(a-r)^2 T^{2-2H}}{2(1-\gamma)^2 \sigma^2} \left[\gamma \Lambda_H + \rho_H \left(\frac{t}{T} \right) + rT \right] \right\}.
 \end{aligned}$$

Combining (2.40), (2.33) and (2.1)–(2.4) we get

Theorem 2.2 *The optimal portfolio $\theta^* = (\alpha^*, \beta^*)$ for problem (1.5)–(1.6) is given by*

$$\begin{aligned}
 (2.41) \quad \beta^*(t) &= e^{r(t-T)} \sigma^{-1} S^{-1}(t) \frac{K(t)}{1-\gamma} \exp \left\{ \frac{1}{1-\gamma} \int_0^t K(s) d\hat{B}_H(s) \right. \\
 &\quad \left. - \frac{(a-r)^2 T^{2-2H}}{2(1-\gamma)^2 \sigma^2} \left[\gamma \Lambda_H + \rho_H \left(\frac{t}{T} \right) + rT \right] \right\}
 \end{aligned}$$

and

$$(2.42) \quad \alpha^*(t) = A^{-1}(t) (Z^{\theta^*}(t) - \beta^*(t) S(t)),$$

where

$$(2.43) \quad e^{-rt} Z^{\theta^*}(t) = z + \int_0^t \exp(-rs) \sigma \beta^*(s) S(s) d\hat{B}_H(s).$$

Now let us consider the following utility function

$$(2.44) \quad u(x) = \log x, \quad x > 0.$$

This is called the Kelly criterion, see e.g. [A] and [Ø]. Then the unconstrained problem (2.9) becomes

$$(2.45) \quad \Phi_\lambda(z) = \sup_F E \left[\log F - \lambda e^{-rT} \eta F \right].$$

Let

$$g(F) = \log F - \lambda e^{-rT} \eta F.$$

The maximum of g is attained when

$$(2.46) \quad F = F_\lambda^* = \frac{1}{\lambda} e^{rT} \eta^{-1}.$$

In this case the equation (analogous to (2.15))

$$z = E \left[e^{-rT} \eta F_{\lambda_0}^* \right] = E \left[e^{-rT} \eta \frac{1}{\lambda_0} e^{rT} \eta^{-1} \right]$$

has a solution

$$\lambda_0 = 1/z.$$

Thus the optimal value of (2.8) with (2.44) is attained when

$$(2.47) \quad F_{\lambda_0}^* = ze^{rT}\eta^{-1}.$$

The value function is given by

$$(2.48) \quad \begin{aligned} V(z) &= E[\log F_{\lambda_0}^*] = \log z + rT - E \log \eta \\ &= \log z + rT + \frac{1}{2} \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H}. \end{aligned}$$

When $H \rightarrow 1/2+$, we obtain

$$V_{1/2}(z) = \log z + rT + \frac{1}{2} \left(\frac{a-r}{\sigma} \right)^2 T.$$

Similar to the argument for (2.40) we have

$$(2.49) \quad \exp^\diamond \left(\int_0^T K(s) d\hat{B}_H(s) \right) = 1 + \int_0^T H_t d\hat{B}_H(t),$$

where

$$(2.50) \quad \begin{aligned} H_t &= \tilde{E}_{\hat{\mu}_H} \left[\hat{D}_t \left(\exp^\diamond \left(\int_0^T K(s) d\hat{B}_H(s) \right) \right) | \mathcal{F}_t^{(H)} \right] \\ &= K(t) \exp^\diamond \left\{ \int_0^t K(s) d\hat{B}_H(s) \right\} \\ &= K(t) \exp \left\{ \int_0^t K(s) d\hat{B}_H(s) - \frac{1}{2} \left(\frac{a-r}{\sigma} \right)^2 T^{2-2H} \rho_H \left(\frac{t}{T} \right) \right\}. \end{aligned}$$

The optimal terminal value $Z^{\theta^*}(T)$ is

$$(2.51) \quad \begin{aligned} Z^{\theta^*}(T) &= F_{\lambda_0}^* \\ &= z \exp \left\{ rT + \int_0^T K(s) dB_H(s) + \frac{1}{2} |K|_\phi^2 \right\} \\ &= ze^{rT} \exp \left\{ |K|_\phi^2 - \int_0^T K(s) \frac{a-r}{\sigma} ds + \int_0^T K(s) d\hat{B}_H(s) - \frac{1}{2} |K|_\phi^2 \right\} \\ &= z \exp \left\{ rT + \int_0^T K(s) d\hat{B}_H(s) - \frac{1}{2} |K|_\phi^2 \right\} \\ &= ze^{rT} \exp^\diamond \left\{ \int_0^T K(s) d\hat{B}_H(s) \right\}. \end{aligned}$$

Thus by (2.50)

$$\tilde{E}_{\hat{\mu}_H} \left[\hat{D}_t Z^{\theta^*}(T) | \mathcal{F}_t^{(H)} \right] = ze^{rT} K(t) \exp \left\{ \int_0^t K(s) d\hat{B}_H(s) - \frac{1}{2} \left(\frac{a-r}{\sigma} \right)^2 T^{2-2H} \rho_H \left(\frac{t}{T} \right) \right\}.$$

By (2.33), we have

Theorem 2.3 *The value function $V(z)$ for problem (1.5) and (2.44) is given by*

$$(2.52) \quad V(z) = \log z + rT + \frac{1}{2} \left(\frac{a-r}{\sigma} \right)^2 \Lambda_H T^{2-2H}.$$

The optimal portfolio $\theta^ = (\alpha^*, \beta^*)$ for problem (1.5) and (2.44) is given by*

$$(2.53) \quad \begin{aligned} \beta^*(t) = & z e^{rt} \sigma^{-1} S^{-1}(t) z e^{rT} K(t) \\ & \exp \left\{ \int_0^t K(s) d\hat{B}_H(s) - \frac{1}{2} \left(\frac{a-r}{\sigma} \right)^2 T^{2-2H} \rho_H \left(\frac{t}{T} \right) \right\} \end{aligned}$$

and

$$(2.54) \quad \alpha^*(t) = A^{-1}(t)(Z^{\theta^*}(t) - \beta^*(t)S(t)),$$

where

$$(2.55) \quad e^{-rt} Z^{\theta^*}(t) = z + \int_0^t \exp(-rs) \sigma \beta^*(s) S(s) d\hat{B}_H(s).$$

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