Construction and Regularity of Solutions to Stochastic Partial Differential Equations with Irregular Drift Coefficients

by

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1 Preface

This thesis consists of four papers produced in the course of the last four years. The purpose of this introduction is to show the connection between the papers and briefly explain the contents without using much technical detail. The papers are presented chronologically with respect to when they were finished.

To put it simple, this thesis deals with the regularization effect of noise on differential equations with 'bad' coefficients.

The discovery of differential equations is arguably the most revolutionary moment in the history of mathematics. A very general, first order Ordinary Differential Equation (ODE) can be written

$$\frac{dX_t}{dt} = b(X_t), \quad X_0 = x \quad (1.1)$$

where b is a given function and we are searching for a differentiable function, $t \mapsto X_t$ satisfying the above relation. The most widespread assumption on b one uses to study ODE's is the Lipschitz-continuity, namely

$$|b(x) - b(y)| \leq L|x - y|$$

where L is a given constant. Under this assumption it is well known that there exists a unique solution to the equation (1.1). If the Lipschitz condition is not satisfied, existence and/or uniqueness may be violated in (1.1). Consider for example $b(t) = 2\text{sign}(t)\sqrt{|t|}$ and $x = 0$. It is then easy to verify that $X_t = 0$ for all t is a solution. Furthermore $X_t^\pm = \pm t^2$ also solves the problem. In general, proving existence of a solution to (1.1) is also a challenging problem.

It is an interesting fact that adding noise to the equation has a regularizing effect. Namely, if one replaces (1.1) with its corresponding Stochastic Differential Equation (SDE)

$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x \quad (1.2)$$

where $B = (B_t)_{t \geq 0}$ is a Brownian motion, existence and uniqueness holds (in the stochastic sense) as long as b is *bounded and measurable*. This means that we can allow for discontinuous coefficients, which clearly violates the Lipschitz condition.

This result was first discovered by Zvonkin in [52] for the one-dimensional case and later generalized by Veretennikov in [49] to multiple dimensions. Both of these proofs utilizes the Yamada-Watanabe theorem, i.e. they show that there is strong uniqueness in (1.2) in connection with weak existence which gives the result.

This thesis tries to reach further than existence and uniqueness in the sense of regularity. Indeed, using a different technique, not relying on the Yamada-Watanabe theorem, we have studied the Malliavin differentiability and the stochastic flow of (1.2) when b is bounded and measurable. This is quite counter intuitive, as the common trend in the study of differential equations and dynamical systems is that the solutions 'inherits' the regularity from the driving coefficients. It is important to remark that this technique proves existence and uniqueness as well, i.e. it 'stands alone' in the sense that it does not rely on the results by Zvonkin or Veretennikov. This gives us reason to believe that the method presented in this thesis is in some sense more closely linked to the true nature of these equations.

In more detail, the content of the thesis is as follows:

Chapter 2

In this chapter we present the mentioned new method for proving existence and uniqueness of a solution to (1.2) when b is just bounded and measurable. As a by-product we get that the solution is Malliavin differentiable.

More precisely, we approximate a given function b by a sequence of smooth functions $\{b_n\}$, uniformly bounded in L^∞ . It is then well known that the corresponding solution, $X^n = (X_t^n)_{t \geq 0}$ to (1.2) is Malliavin differentiable and its Malliavin derivative satisfies

$$D_s X_t^n = I_{d \times d} + \int_s^t b'_n(u, X_u^n) D_s X_u^n du \quad (1.3)$$

which can be regarded as linear ODE with a random coefficient. We then use Picard iteration to write

$$D_s X_t^n = I_{d \times d} + \sum_{k \geq 1} \int_{s < u_1 < \dots < u_k < t} b'_n(u_1, X_{u_1}) \dots b'_n(u_k, X_{u_k}) du_1 \dots du_k.$$

The trick is then to find a bound on $D_s X_t^n$ which does not depend on the size of b'_n , but rather $\|b_n\|_{L^\infty}$. Using a compactness criterion in [6] one can show convergence of $\{X_t^n\}$ in the strong topology of $L^2(\Omega)$. Then, using an explicit representation formula for the solution developed in [3] we can show that X_t^n is converging to a process $X = (X_t)_{t \geq 0}$ which solves (1.2). Since the norm of $D_s X_t^n$ is uniformly bounded, this is enough to guarantee Malliavin differentiability of X_t .

Chapter 3

In this chapter we consider the same equation (1.2) with a bounded and measurable b , but to emphasize its dependence on the initial condition, x , we write X_t^x . In this chapter we show that the mapping $x \mapsto X_t^x$ is weakly differentiable.

Using the same approximation of b by $\{b_n\}_{n \geq 1}$, we note that the corresponding solution $X_t^{n,x}$ is differentiable in x and that the derivative satisfies

$$\frac{\partial}{\partial x} X_t^{n,x} = I_{d \times d} + \int_0^t b'_n(u, X_u^n) \frac{\partial}{\partial x} X_u^{n,x} du \quad (1.4)$$

which is the same equation as (1.3) when $s = 0$.

Similarly as in Chapter 2 we find a bound of the norm of $\frac{\partial}{\partial x} X_t^{n,x}$ depending only on $\|b_n\|_{L^\infty}$. Using weak compactness in connection with the strong compactness from Chapter 2 we can conclude that the mapping $x \mapsto X_t^x$ is weakly differentiable, almost surely.

This fact is applied to study the Stochastic Transport Equation,

$$\begin{aligned} \partial_t u(t, x) + b(t, x) \partial_x u(t, x) + \partial_x u(t, x) \circ dB_t &= 0 \\ u(0, x) &= u_0(x). \end{aligned} \quad (1.5)$$

Notice that, since b is not differentiable, the usual trick of integrating against a test-function and using integration by parts does not make sense for the product $b(t, x) \partial_x u(t, x)$. This chapter shows, however, that the solution is weakly differentiable so that one does not need to use integration by parts on this term, and this leads to a meaningful definition of a weak solution even for discontinuous b . Indeed, the solution to (1.5) is given by $u(t, x) = u_0(\phi_t^{-1}(x))$ where $\phi_t(x)$ is the flow map associated with (1.2). This relation is easily checked

by using the usual method of characteristic for (1.5) in connection with the Itô-Ventzell formula.

Chapter 4

This chapter deals with the special case of one-dimensional equation and autonomous b . However, it opens up for the study of an unbounded coefficient, where existence and uniqueness is not covered by Zvonking or Veretennikov.

More precisely, we consider a sub-linear coefficient, $|b(x)| \leq C(1 + |x|)$, which is enough to guarantee weak existence of a solution which does not have a finite blow-up time, see [2].

The advantage of one-dimension lies in the fact that in this case (1.3) can be solved explicitly by

$$D_s X_t^{n,x} = \exp\left\{\int_s^t b'_n(X_u^{n,x}) du\right\}.$$

The trick is this time is to use Itô's formula to get rid of the derivative of b_n . Namely, let $\tilde{b}_n$ be an anti-derivative of b_n and use Itô's formula to write

$$\tilde{b}_n(X_t^{n,x}) = \tilde{b}_n(X_s^{n,x}) + \int_s^t b_n(X_u^{n,x}) dX_u^{n,x} + \frac{1}{2} \int_s^t b'_n(X_u^{n,x}) du.$$

We can then find an explicit formula for the spatial derivative of the flow of (1.2), which does not use the derivative of the coefficient but rather a local time integral:

$$\frac{\partial}{\partial x} X_t^x = \exp\left\{-\int_{\mathbb{R}} b(y) dL_t^y(X^x)\right\}.$$

Using this representation it follows in a straightforward (but technical) application of Kolmogorov's continuity lemma that we get a continuous derivative of $x \mapsto X_t^x$ when b is a step function.

Chapter 5

In the final chapter we consider a Hilbert-space valued SDE,

$$dX_t = (AX_t + B(t, X_t))dt + \sqrt{Q}dW_t, \quad X_0 = x \in H \quad (1.6)$$

where the coefficient B is Hölder continuous, i.e.

$$|B(t, x) - B(t, y)| \leq C|x - y|^\alpha$$

for some $\alpha \in (0, 1)$. Here, A is a linear (discontinuous) operator, W is a cylindrical Brownian motion on H and Q a non-negative selfadjoint bounded operator on H . We show existence and uniqueness of a solution to this equation which is Malliavin differentiable. Notice that the assumption of Hölder continuity is substantially weaker than Lipschitz continuity.

The proof of this assertion combines the method introduced in Chapter 2 with estimates on the solution of the following infinite-dimensional PDE for every $n \in \mathbb{N}$

$$\begin{aligned} \frac{\partial U_n}{\partial t} + \frac{1}{2} \text{Tr} (D^2 U_n Q) + \langle Ax, DU_n \rangle + \langle B, DU_n \rangle &= B_n \\ U_n(T, x) &= 0 \end{aligned} \quad (1.7)$$

where B_n is the n -th component of B . The presense of D^2 has a regularizing effect in the sense that if B is Hölder continuous, the solution to (1.7) is twice differentiable. Letting $U(t, x) = \sum_{n \in \mathbb{N}} U_n(t, x) e_n$ we can apply Itô's formula to U and get

$$dU(t, X_t) = B(t, X_t)dt + DU(t, X_t)\sqrt{Q}dW_t.$$

Then we can rewrite (1.6) as

$$\begin{aligned} X_t &= e^{tA}(x - U(0, x)) + U(t, X_t) + \int_0^t A e^{(t-s)A} U(s, X_s) ds \\ &\quad + \int_0^t e^{(t-s)A} Q^{1/2} dW_s - \int_0^t e^{(t-s)A} DU(s, X_s) Q^{1/2} dW_s. \end{aligned}$$

In this new formulation all the right hand side terms have good regularity properties.

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2 A Variational Approach to the construction and Malliavin Differentiability of Strong solutions of SDE's

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Abstract

In this article we develop a new approach to construct solutions of stochastic equations with merely measurable drift coefficients. We aim at demonstrating the principles of our technique by analyzing strong solutions of stochastic differential equations driven by Brownian motion. An important and rather surprising consequence of our method which is based on Malliavin calculus is that the solutions derived by A. Y. Veretennikov [49] for Brownian motion with bounded and measurable drift in $\mathbb{R}^d$ are Malliavin differentiable. Further, a strength of our approach, which doesn't rely on a pathwise uniqueness argument, is that it can be transferred and applied to the analysis of various other types of stochastic equations: We prove Sobolev differentiability of stochastic differential equations in their initial conditions. The latter result is used to derive a Bismut-Elworthy-Li formula [10] for spatial derivatives of solutions to the Kolmogorov equation under weak conditions on the drift coefficient. Another application of our technique is the construction of unique solutions of the stochastic transport equation with irregular drift coefficients. Moreover, it is conceivable that our approach is also applicable to the construction of solutions of stochastic evolution equations on Hilbert spaces.

2.1 Introduction

In this paper we are mainly interested to study the following stochastic differential equation (SDE) given by

$$dX_t = b(t, X_t)dt + dB_t, \quad 0 \leq t \leq T, \quad X_0 = x \in \mathbb{R}^d, \quad (2.1)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable function and B_t is a d -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \pi)$. We denote by $\mathcal{F}_t$ the augmented filtration generated by B_t .

If b in (2.1) is of linear growth and (globally) fulfills a Lipschitz condition it is well known that there exists a unique global strong solution to the SDE (2.1). More precisely, there exists a continuous $\mathcal{F}_t$ -adapted process X_t solving (2.1) such that

$$E \left[\int_0^T X_t^2 dt \right] < \infty.$$

Important applications, however, of SDE's of the type (2.1) to physics or stochastic control theory show that Lipschitz continuity imposed on the drift coefficient b is a rather severe restriction. For example, in statistical mechanics, where one is interested in solutions of (2.1) as functionals of the driving noise (i.e. strong solutions) to model interacting infinite particle systems, the drift b is typically discontinuous or singular. See e.g. [21] and the references therein.

Strong solutions of SDE's with non-Lipschitz coefficients have been investigated by many authors in the past decades. To begin with we mention the work of Zvonkin [52], where the author obtains unique strong solutions of (2.1) in the one-dimensional case, when b is merely bounded and measurable. The latter result can be regarded as a milestone in the theory of SDE's. Subsequently, this result was generalized by Veretennikov [49] to the multidimensional case. The tools used by these authors to derive strong solutions are based on estimates of solutions of parabolic partial differential equations and a pathwise uniqueness argument.

Other important and more recent results in this direction based on a pathwise uniqueness argument (in connection with other techniques due to Portenko [36] or the Skorohod embedding) can be e.g. found in Krylov, Röckner [21], Gyöngy, Krylov [16] or Gyöngy, Martínez [17]. We also refer to [11], where the authors employ a modified version of Gronwall's Lemma. In this context we shall also point out the paper of Davie [7], who even establishes uniqueness of strong solutions of (2.1) for almost all Brownian paths in the case of bounded and measurable drift coefficients.

In this paper we further develop the new approach devised in [31] to construct strong solutions of SDE's with irregular drift coefficients which additionally yields the important insight that these solutions are Malliavin differentiable. See also [29] and [38]. More precisely, we derive the results in [31] without assuming a certain symmetry condition [31, Definition 3] on the drift b in (2.1), which severely restricts the class of SDE's to be studied. In particular, one of our main results is the extension of [31, Theorem 4] on the Malliavin differentiability of solutions of (2.1) for merely bounded Borel functions b from the one-dimensional to the multidimensional case.

Our approach is mainly based on Malliavin calculus. To be more precise, our technique relies on a compactness criterion based on Malliavin calculus and an approximation argument for certain generalized processes in the Hida distribution space which we directly verify to be strong solutions of (2.1). We remark that our construction method is different from the above mentioned authors' ones. The technique proposed in this paper is not based on a pathwise

uniqueness argument (or the Yamada-Watanabe theorem). In fact we tackle the construction problem from the "opposite" direction and prove that strong existence in connection with uniqueness in law of solutions of SDE's enforces strong uniqueness.

The additional information that strong solutions of SDE's with merely measurable drift coefficients are Malliavin differentiable has important and interesting implications. Further, one major strength of our approach is that it exhibits great flexibility to be applied and generalized to the analysis of various important aspects of solutions of a broader range of stochastic equations with irregular coefficients besides finite dimensional SDE's. In the last section of this paper we illustrate this by first showing how our techniques imply Sobolev differentiability of the strong solution X_t of SDE (2.1) in the initial condition x . Together with the Malliavin differentiability of X_t this is then used to derive a useful stochastic representation of spatial derivatives of solutions to the Kolmogorov equation - known as Bismut-Elworthy-Li formula - which does not involve derivatives of the initial condition of the Kolmogorov equation. Secondly, we present the applicability of our techniques and results to the problem of well-posedness of stochastic transport equations with singular coefficients (see also [32]).

The paper is organized as follows: In Section 2.2 we recall basic concepts of Malliavin calculus and Gaussian white noise theory. Section 2.3 is devoted to the study of the SDE (2.1). The main results of the paper are Theorem 2.3, Lemma 2.5, Corollary 2.6, and Theorem 2.17. Section 2.4 concludes by considering the above mentioned applications of techniques and results of the previous sections.

2.2 Framework

In this section we recall some facts from Gaussian white noise analysis and Malliavin calculus, which we aim at employing in Section 2.3 to construct strong solutions of SDE's. See [8, 35, 23] for more information on white noise theory. As for Malliavin calculus the reader is referred to [34, 25, 26, 8].

2.2.1 Basic Facts of Gaussian White Noise Theory

A building block of our proof for the construction of strong solutions (see Section 2.3) is based on a generalized stochastic process in the Hida distribution space which we verify to be a SDE solution. In the following, we shall give the definition of this space which goes back to T. Hida (see [8]).

From now on we fix a time horizon $0 < T < \infty$. Consider a (positive) self-adjoint operator A on $L^2([0, T])$ with $\text{Spec}(A) > 1$. Let us require that A^{-r} is of Hilbert-Schmidt type for some $r > 0$. Denote by $\{e_j\}_{j \geq 0}$ a complete orthonormal basis of $L^2([0, T])$ in $\text{Dom}(A)$ and let $\lambda_j > 0$, $j \geq 0$ be the eigenvalues of A such that

$$1 < \lambda_0 \leq \lambda_1 \leq \dots \longrightarrow \infty.$$

Let us assume that each basis element e_j is a continuous function on $[0, T]$. Further let $O_\lambda, \lambda \in \Gamma$, be an open covering of $[0, T]$ such that

$$\sup_{j \geq 0} \lambda_j^{-\alpha(\lambda)} \sup_{t \in O_\lambda} |e_j(t)| < \infty$$

for $\alpha(\lambda) \geq 0$.

In what follows let $\mathcal{S}([0, T])$ denote the standard countably Hilbertian space constructed from $(L^2([0, T]), A)$. See [35]. Then $\mathcal{S}([0, T])$ is a nuclear subspace of $L^2([0, T])$. We denote by $\mathcal{S}'([0, T])$ the corresponding conuclear space, that is the topological dual of $\mathcal{S}([0, T])$. Then the Bochner-Minlos theorem provides the existence of a unique probability measure π on $\mathcal{B}(\mathcal{S}'([0, T]))$ (Borel σ -algebra of $\mathcal{S}'([0, T])$) such that

$$\int_{\mathcal{S}'([0, T])} e^{i\langle \omega, \phi \rangle} \pi(d\omega) = e^{-\frac{1}{2} \|\phi\|_{L^2([0, T])}^2}$$

holds for all $\phi \in \mathcal{S}([0, T])$, where $\langle \omega, \phi \rangle$ is the action of $\omega \in \mathcal{S}'([0, T])$ on $\phi \in \mathcal{S}([0, T])$. Set

$$\Omega_i = \mathcal{S}'([0, T]), \quad \mathcal{F}_i = \mathcal{B}(\mathcal{S}'([0, T])), \quad \mu_i = \pi,$$

for $i = 1, \dots, d$. Then the product measure

$$\mu = \bigotimes_{i=1}^d \mu_i \tag{2.2}$$

on the measurable space

$$(\Omega, \mathcal{F}) := \left(\prod_{i=1}^d \Omega_i, \bigotimes_{i=1}^d \mathcal{F}_i \right) \tag{2.3}$$

is referred to as *d-dimensional white noise probability measure*.

Consider the Doleans-Dade exponential

$$\tilde{e}(\phi, \omega) = \exp \left(\langle \omega, \phi \rangle - \frac{1}{2} \|\phi\|_{L^2([0, T]; \mathbb{R}^d)}^2 \right),$$

for $\omega = (\omega_1, \dots, \omega_d) \in (\mathcal{S}'([0, T]))^d$ and $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}([0, T]))^d$, where $\langle \omega, \phi \rangle := \sum_{i=1}^d \langle \omega_i, \phi_i \rangle$.

In the following let $((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$ be the n -th completed symmetric tensor product of $(\mathcal{S}([0, T]))^d$ with itself. One verifies that $\tilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero. Hence there exist generalized Hermite polynomials $H_n(\omega) \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$ such that

$$\tilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \langle H_n(\omega), \phi^{\otimes n} \rangle \tag{2.4}$$

for ϕ in a certain neighbourhood of zero in $(\mathcal{S}([0, T]))^d$. It can be shown that

$$\left\{ \langle H_n(\omega), \phi^{(n)} \rangle : \phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}, \quad n \in \mathbb{N}_0 \right\} \tag{2.5}$$

is a total set of $L^2(\mu)$. Further one finds that the orthogonality relation

$$\int_{\mathcal{S}'} \langle H_n(\omega), \phi^{(n)} \rangle \langle H_m(\omega), \psi^{(m)} \rangle \mu(d\omega) = \delta_{n,m} n! \left(\phi^{(n)}, \psi^{(n)} \right)_{L^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})} \tag{2.6}$$

is valid for all $n, m \in \mathbb{N}_0$, $\phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$, $\psi^{(m)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} m}$ where

$$\delta_{n,m} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{else} \end{cases}.$$

Define $\widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ as the space of square integrable symmetric functions $f(x_1, \dots, x_n)$ with values in $(\mathbb{R}^d)^{\otimes n}$. Then the orthogonality relation (2.6) implies that the mappings

$$\phi^{(n)} \longmapsto \langle H_n(\omega), \phi^{(n)} \rangle$$

from $(\mathcal{S}([0, T]^d))^{\widehat{\otimes} n}$ to $L^2(\mu)$ possess unique continuous extensions

$$I_n : \widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n}) \longrightarrow L^2(\mu)$$

for all $n \in \mathbb{N}$. We remark that $I_n(\phi^{(n)})$ can be viewed as an n -fold iterated Itô integral of $\phi^{(n)} \in \widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a d -dimensional Wiener process

$$B_t = (B_t^{(1)}, \dots, B_t^{(d)}) \quad (2.7)$$

on the white noise space

$$(\Omega, \mathcal{F}, \mu) . \quad (2.8)$$

It turns out that square integrable functionals of B_t admit a Wiener-Itô chaos representation which can be regarded as an infinite-dimensional Taylor expansion, that is

$$L^2(\mu) = \bigoplus_{n \geq 0} I_n(\widehat{L}^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})). \quad (2.9)$$

We construct the Hida stochastic test function and distribution space by using the Wiener-Itô chaos decomposition (2.9). For this purpose let

$$A^d := (A, \dots, A), \quad (2.10)$$

where A was the operator introduced in the beginning of the section. We define the *Hida stochastic test function space* $(\mathcal{S})$ via a second quantization argument, that is we introduce $(\mathcal{S})$ as the space of all $f = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle \in L^2(\mu)$ such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| \left((A^d)^{\otimes n} \right)^p \phi^{(n)} \right\|_{L^2([0, T]^n; (\mathbb{R}^d)^{\otimes n})}^2 < \infty \quad (2.11)$$

for all $p \geq 0$. It turns out that the space $(\mathcal{S})$ is a nuclear Fréchet algebra with respect to multiplication of functions and its topology is given by the seminorms $\|\cdot\|_{0,p}$, $p \geq 0$. Further one observes that

$$\widetilde{e}(\phi, \omega) \in (\mathcal{S}) \quad (2.12)$$

for all $\phi \in (\mathcal{S}([0, T]^d))^d$.

In the sequel we refer to the topological dual of $(\mathcal{S})$ as *Hida stochastic distribution space* $(\mathcal{S})^*$. Thus we have constructed the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*.$$

The Hida distribution space $(\mathcal{S})^*$ exhibits the crucial property that it contains the *white noise* of the coordinates of the d -dimensional Wiener process B_t , that is the time derivatives

$$W_t^i := \frac{d}{dt} B_t^i, \quad i = 1, \dots, d, \quad (2.13)$$

belong to $(\mathcal{S})^*$.

We shall also recall the definition of the S -transform which is an important tool to characterize elements of the Hida test function and distribution space. See [37]. The S -transform of a $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined by the dual pairing

$$S(\Phi)(\phi) = \langle \Phi, \tilde{e}(\phi, \omega) \rangle \quad (2.14)$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$. Here $\mathcal{S}_{\mathbb{C}}([0, T])$ the complexification of $\mathcal{S}([0, T])$. We mention that the S -transform is a monomorphism from $(\mathcal{S})^*$ to $\mathbb{C}$. In particular, if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One checks that

$$S(W_t^i)(\phi) = \phi^i(t), \quad i = 1, \dots, d \quad (2.15)$$

for $\phi = (\phi^{(1)}, \dots, \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$.

Finally, we need the important concept of the *Wick* or *Wick-Grassmann product*, which we want to use in Section 2.3 to represent solutions of SDE's. The Wick product can be regarded as a tensor algebra multiplication on the Fock space and can be defined as follows: The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi)S(\Psi)(\phi) \quad (2.16)$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0, T]))^d$. As an example we find that

$$\langle H_n(\omega), \phi^{(n)} \rangle \diamond \langle H_m(\omega), \psi^{(m)} \rangle = \langle H_{n+m}(\omega), \phi^{(n)} \widehat{\otimes} \psi^{(m)} \rangle \quad (2.17)$$

for $\phi^{(n)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} n}$ and $\psi^{(m)} \in ((\mathcal{S}([0, T]))^d)^{\widehat{\otimes} m}$. The latter in connection with (2.4) shows that

$$\tilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \quad (2.18)$$

for $\phi \in (\mathcal{S}([0, T]))^d$. Here the Wick exponential $\exp^{\diamond}(X)$ of a $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \quad (2.19)$$

where $X^{\diamond n} = X \diamond \dots \diamond X$, if the sum on the right hand side converges in $(\mathcal{S})^*$.

2.2.2 Basic elements of Malliavin Calculus

In this Section we briefly elaborate a framework for Malliavin calculus.

Without loss of generality we consider the case $d = 1$. Let $F \in L^2(\mu)$. Then it follows from (2.9) that

$$F = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle \quad (2.20)$$

for unique $\phi^{(n)} \in \widehat{L}^2([0, T]^n)$. Assume that

$$\sum_{n \geq 1} n n! \left\| \phi^{(n)} \right\|_{L^2([0, T]^n)}^2 < \infty. \quad (2.21)$$

Then the *Malliavin derivative* D_t of F in the direction of B_t is defined by

$$D_t F = \sum_{n \geq 1} n \left\langle H_{n-1}(\cdot), \phi^{(n)}(\cdot, t) \right\rangle. \quad (2.22)$$

We introduce the stochastic Sobolev space $\mathbb{D}_{1,2}$ as the space of all $F \in L^2(\mu)$ such that (2.21) is fulfilled. The Malliavin derivative D is a linear operator from $\mathbb{D}_{1,2}$ to $L^2(\lambda \times \mu)$, where λ denotes the Lebesgue measure. We mention that $\mathbb{D}_{1,2}$ is a Hilbert space with the norm $\|\cdot\|_{1,2}$ given by

$$\|F\|_{1,2}^2 := \|F\|_{L^2(\mu)}^2 + \|D.F\|_{L^2([0,T] \times \Omega, \lambda \times \mu)}^2. \quad (2.23)$$

We obtain the following chain of continuous inclusions:

$$(\mathcal{S}) \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathbb{D}_{-1,2} \hookrightarrow (\mathcal{S})^*, \quad (2.24)$$

where $\mathbb{D}_{-1,2}$ is the dual of $\mathbb{D}_{1,2}$.

2.3 Main results

In this section, we want to further develop the ideas introduced in [31] to derive Malliavin differentiable strong solutions of stochastic differential equations with discontinuous coefficients. More precisely, we aim at analyzing the SDE's of the form

$$dX_t = b(t, X_t)dt + dB_t, \quad 0 \leq t \leq 1, \quad X_0 = x \in \mathbb{R}^d, \quad (2.25)$$

where the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Borel measurable function and B_t is a d -dimensional Brownian motion with respect to the stochastic basis

$$(\Omega, \mathcal{F}, \mu), \{\mathcal{F}_t\}_{0 \leq t \leq T} \quad (2.26)$$

for the μ -augmented filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ generated by B_t . At the end of this section we shall also apply our technique to equations with more general diffusions coefficients (Theorem 3.16).

Our method to construct strong solution is actually motivated by the following observation in [24] and [29] (see also [30]).

Proposition 2.1 *Suppose that the drift coefficient $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ in (2.25) is bounded and Lipschitz continuous. Then the unique strong solution $X_t = (X_t^1, \dots, X_t^d)$ of (2.25) allows for the explicit representation*

$$\varphi(t, X_t^i(\omega)) = E_{\tilde{\mu}} \left[\varphi(t, \tilde{B}_t^i(\tilde{\omega})) \mathcal{E}_T^\circ(b) \right] \quad (2.27)$$

for all $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(t, B_t^i) \in L^2(\mu)$ for all $0 \leq t \leq T$, $i = 1, \dots, d$. The object $\mathcal{E}_T^\circ(b)$ is given by

$$\begin{aligned} \mathcal{E}_T^\circ(b)(\omega, \tilde{\omega}) := & \exp^\circ \left(\sum_{j=1}^d \int_0^T \left(W_s^j(\omega) + b^j(s, \tilde{B}_s(\tilde{\omega})) \right) d\tilde{B}_s^j(\tilde{\omega}) \right. \\ & \left. - \frac{1}{2} \int_0^T \left(W_s^j(\omega) + b^j(s, \tilde{B}_s(\tilde{\omega})) \right)^{\circ 2} ds \right). \end{aligned} \quad (2.28)$$

Here $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mu}), (\tilde{B}_t)_{t \geq 0}$ is a copy of the quadruple $(\Omega, \mathcal{F}, \mu), (B_t)_{t \geq 0}$ in (2.26). Further $E_{\tilde{\mu}}$ denotes a Pettis integral of random elements $\Phi : \tilde{\Omega} \rightarrow (\mathcal{S})^*$ with respect to the measure $\tilde{\mu}$. The Wick product $\diamond$ in the Wick exponential of (2.28) is taken with respect to μ and W_t^j is the white noise of B_t^j in the Hida space $(\mathcal{S})^*$ (see (2.13)). The stochastic integrals $\int_0^T \phi(t, \tilde{\omega}) d\tilde{B}_s^j(\tilde{\omega})$ in (2.28) are defined for predictable integrands ϕ with values in the conuclear space $(\mathcal{S})^*$. See [19] for definitions. The other integral type in (2.28) is to be understood in the sense of Pettis.

Remark 2.2 Let $0 = t_1^n < t_2^n < \dots < t_{m_n}^n = T$ be a sequence of partitions of the interval $[0, T]$ with $\max_{i=1}^{m_n-1} |t_{i+1}^n - t_i^n| \rightarrow 0$. Then the stochastic integral of the white noise W^j can be approximated as follows:

$$\int_0^T W_s^j(\omega) d\tilde{B}_s^j(\tilde{\omega}) = \lim_{n \rightarrow \infty} \sum_{i=1}^{m_n} (\tilde{B}_{t_{i+1}^n}^j(\tilde{\omega}) - \tilde{B}_{t_i^n}^j(\tilde{\omega})) W_{t_i^n}^j(\omega)$$

in $L^2(\lambda \times \tilde{\mu}; (\mathcal{S})^*)$. For more information about stochastic integration on conuclear spaces the reader may consult [19].

In the sequel we shall use the notation $Y_t^{i,b}$ for the expectation on the right hand side of (2.27) for $\varphi(t, x) = x$, that is

$$Y_t^{i,b} := E_{\tilde{\mu}} \left[\tilde{B}_t^{(i)} \mathcal{E}_T^\diamond(b) \right]$$

for $i = 1, \dots, d$. We set

$$Y_t^b = \left(Y_t^{1,b}, \dots, Y_t^{d,b} \right). \quad (2.29)$$

The form of Formula (2.27) in Proposition 2.1 actually suggests that the expectation on the right hand side or Y_t^b in (2.29) may also represent solutions of (2.25) for merely measurable drift coefficients b . The latter naturally leads to the following question: Can one specify conditions on b under which one succeeds to directly verify the generalized process Y_t^b to be a (strong) solution of (2.25)? This question was successfully treated for the one-dimensional case using a comparison argument in [29] and for the multidimensional case under a rather strong symmetry condition on the drift b using Malliavin calculus in [31]. In this paper we considerably improve the results given in [31] by removing the symmetry condition on b . Our main result in this paper is the following theorem:

Theorem 2.3 Suppose that the drift coefficient $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ in (2.25) is a bounded Borel-measurable function. Then there exists a unique global strong solution X to Equation (2.25) such that X_t is Malliavin differentiable for all $0 \leq t \leq 1$.

Remark 2.4 In the one-dimensional case the existence and uniqueness of strong solutions to (2.25) for bounded and measurable drift coefficients was first obtained by Zvonkin in his celebrated paper [52]. The extension to the multi-dimensional case was given by [49]. We point out that our solution technique grants the important additional insight that such solutions are Malliavin differentiable. We remark that Theorem 2.3 is a generalization of [30, Theorem 5] from the one-dimensional to the multi-dimensional case. Let us also mention that we considerably improve the technique initiated in [31] (see also [29] and [38]) by removing a certain symmetry condition on the drift coefficients in (2.25) (see [30, Definition 3]), which

severely limits the class of SDE's to be analyzed. The removal of the latter condition, however, may actually pave the way for the construction of strong solutions of discontinuous infinite dimensional stochastic equations of the type (2.62) or SPDE's. See [28]. We point out that the methods of the authors mentioned in the introduction fail in this case.

To prove Theorem 2.3 we follow a procedure consisting of two steps (compare [31]). In the **first step**, we show for a sequence of uniformly bounded, smooth coefficients $b_n : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n \geq 1$, with compact support that for each $0 \leq t \leq 1$ the sequence of corresponding strong solutions $X_{n,t} = Y_t^{b_n}$, $n \geq 1$, is relatively compact in $L^2(\mu; \mathbb{R}^d)$ (Corollary 2.6). The main tool to prove compactness is the bound in Lemma 2.5 in connection with a compactness criteria in terms of Malliavin derivatives obtained in [6] (see Appendix 2.5). This step is one of the main contribution of this paper.

Given a merely measurable and bounded drift coefficient b , we then show in the **second step** that Y_t^b , $0 \leq t \leq 1$ is a generalized process in the Hida distribution space, and we apply the S -transform 2.14 to prove that for a given sequence of a.e. approximating, uniformly bounded, smooth coefficients b_n with compact support a subsequence of the corresponding strong solutions $X_{n_j,t} = Y_t^{b_{n_j}}$ fulfills

$$Y_t^{b_{n_j}} \rightarrow Y_t^b$$

in $L^2(\mu; \mathbb{R}^d)$ for $0 \leq t \leq 1$ (Lemma 2.14). Using a certain transformation property for Y_t^b (Lemma 2.16) we directly verify Y_t^b as a solution to (2.25) which in addition is Malliavin differentiable.

We now turn to the first step of our procedure. The successful completion of the first step relies on the following essential lemma:

Lemma 2.5 *Let $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth function with compact support. Then the corresponding strong solution X in (2.25) fulfills*

$$E [\|D_t X_s - D_{t'} X_s\|^2] \leq C_d(\|b\|_\infty) |t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$, $\alpha = \alpha(s) > 0$ and

$$\sup_{0 \leq t \leq 1} E [\|D_t X_s\|^2] \leq C_d(\|b\|_\infty)$$

where $C_d : [0, \infty) \rightarrow [0, \infty)$ is an increasing, continuous function, $\|\cdot\|$ a matrix-norm on $\mathbb{R}^{d \times d}$ and $\|\cdot\|_\infty$ the supremum norm.

From Lemma 2.5 together with Corollary 2.30 we immediately obtain the main result of step one of our procedure:

Corollary 2.6 *Let $b_n : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n \geq 1$, be a sequence of uniformly bounded, smooth coefficients with compact support. Then for each $0 \leq t \leq 1$ the sequence of corresponding strong solutions $X_{n,t} = Y_t^{b_n}$, $n \geq 1$, is relatively compact in $L^2(\mu; \mathbb{R}^d)$.*

In order to prove Lemma 2.5 we need the following estimate, which can be considered a generalization of a bound given in [7, Proposition 2.2]:

Proposition 2.7 *Let B be a d -dimensional Brownian Motion starting from the origin and $b_1, \dots, b_n$ be compactly supported continuously differentiable functions $b_i : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$. Let $\alpha_i \in \{0, 1\}^d$ be a multiindex such that $|\alpha_i| = 1$ for $i = 1, 2, \dots, n$. Then there exists a universal constant C (independent of $\{b_i\}_i$, n , and $\{\alpha_i\}_i$) such that*

$$\left| E \left[\int_{t_0 < t_1 < \dots < t_n < t} \left(\prod_{i=1}^n D^{\alpha_i} b_i(t_i, B(t_i)) \right) dt_1 \dots dt_n \right] \right| \leq \frac{C^n \prod_{i=1}^n \|b_i\|_{\infty} (t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)} \quad (2.30)$$

where Γ is the Gamma-function. Here D^{α_i} denotes the partial derivative with respect to the j 'th space variable, where j is the position of the 1 in α_i .

Proof. Without loss of generality, assume that $\|b_i\|_{\infty} \leq 1$ for $i = 1, 2, \dots, n$. Denote by $z = (z^{(1)}, \dots, z^{(d)})$ a generic element of $\mathbb{R}^d$ and by $\|\cdot\|$ the usual Euclidian norm. With $P(t, z) = (2\pi t)^{-d/2} e^{-\|z\|^2/2t}$, write the left hand side in (2.30) as

$$\left| \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n \right|.$$

Introduce the notation

$$J_n^{\alpha}(t_0, t, z_0) = \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^{nd}$. We shall show that $|J_n^{\alpha}(t_0, t, 0)| \leq C^n (t - t_0)^{n/2} / \Gamma(n/2 + 1)$, thus proving the proposition.

To do this, we will use integration by parts to shift the derivatives onto the Gaussian kernel. This will be done by introducing the alphabet

$$\mathcal{A}(\alpha) = \{P, D^{\alpha_1} P, \dots, D^{\alpha_n} P, D^{\alpha_1} D^{\alpha_2} P, \dots, D^{\alpha_{n-1}} D^{\alpha_n} P\}$$

where D^{α_i} , $D^{\alpha_i} D^{\alpha_{i+1}}$ denotes the derivatives in z on $P(t, z)$.

Take a string $S = S_1 \dots S_n$ in $\mathcal{A}(\alpha)$ and define

$$I_S^{\alpha}(t_0, t, z_0) = \int_{t_0 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n b_i(t_i, z_i) S_i(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n.$$

We will only need a special type of strings, and we say that a string is *allowed* if, when all the $D^{\alpha_i} P$'s are removed from the string, a string of the form $P \cdot D^{\alpha_s} D^{\alpha_{s+1}} P \cdot P \cdot D^{\alpha_{s+1}} D^{\alpha_{s+2}} P \dots P \cdot D^{\alpha_r} D^{\alpha_{r+1}} P$ for $s \geq 1$, $r \leq n - 1$ remains. Also, we will require that the first derivatives $D^{\alpha_i} P$ are written in an increasing order with respect to i .

Before we proceed with the proof of Proposition 2.7 we will need some intermediate results.

Lemma 2.8 *We can write*

$$J_n^{\alpha}(t_0, t, z_0) = \sum_{j=1}^{2^n - 1} \epsilon_j I_{S^j}^{\alpha}(t_0, t, z_0)$$

where each ϵ_j is either -1 or 1 and each S^j is an allowed string in $\mathcal{A}(\alpha)$.

Proof. The equation obviously holds for $n = 1$. Assume the equation holds for $n \geq 1$, and let b_0 be another function satisfying the requirements of the proposition. Likewise with α_0 . Then

$$\begin{aligned} J_{n+1}^{(\alpha_0, \alpha)}(t_0, t, z_0) &= \int_{t_0}^t \int_{\mathbb{R}^d} D^{\alpha_0} b_0(t_1, z_1) P(t_1 - t_0, z_1 - z_0) J_n^\alpha(t_1, t, z_1) dz_1 dt_1 \\ &= - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1, z_1) D^{\alpha_0} P(t_1 - t_0, z_1 - z_0) J_n^\alpha(t_1, t, z_1) dz_1 dt_1 \\ &\quad - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1, z_1) P(t_1 - t_0, z_1 - z_0) D^{\alpha_0} J_n^\alpha(t_1, t, z_1) dz_1 dt_1. \end{aligned}$$

Notice that

$$D^{\alpha_0} I_S^\alpha(t_1, t, z_1) = -I_{\tilde{S}}^{(\alpha_0, \alpha)}(t_1, t, z_1)$$

where

$$\tilde{S} = \begin{cases} D^{\alpha_0} P \cdot S_2 \cdots S_n & \text{if } S = P \cdot S_2 \cdots S_n \\ D^{\alpha_0} D^{\alpha_1} P \cdot S_2 \cdots S_n & \text{if } S = D^{\alpha_1} P \cdot S_2 \cdots S_n. \end{cases}$$

Here, $\tilde{S}$ is not an allowed string in $\mathcal{A}(\alpha)$. So from the induction hypothesis $D^{\alpha_0} J_n^\alpha(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{\tilde{S}}^{(\alpha_0, \alpha)}(t_0, t, z_0)$ this gives

$$J_{n+1}^{(\alpha_0, \alpha)} = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{D^{\alpha_0} P \cdot S^j}^{(\alpha_0, \alpha)} + \sum_{j=1}^{2^{n-1}} \epsilon_j I_{P \cdot \tilde{S}^j}.$$

It is easily checked that when S^j is an allowed string in $\mathcal{A}(\alpha)$, both $D^{\alpha_0} P \cdot S^j$ and $P \cdot \tilde{S}^j$ are allowed strings in $\mathcal{A}(\alpha_0, \alpha)$.

■

For the rest of the proof of Proposition 2.7 we will bound I_S^α when S is an allowed string, and the result will follow from the above representation.

Lemma 2.9 *Let $\phi, h : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable functions such that $|\phi(s, z)| \leq e^{-\|z\|^2/3s}$ and $\|h\|_\infty \leq 1$. Also let $\alpha, \beta \in \{0, 1\}^d$ be multiindices such that $|\alpha| = |\beta| = 1$. Then there exists a universal constant C (independent of ϕ, h, α and β) such that*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) dy dz ds dt \right| \leq C.$$

Proof. Let $l, m \in \mathbb{Z}^d$ and denote $[l, l+1] := [l^{(1)}, l^{(1)}+1] \times \cdots \times [l^{(d)}, l^{(d)}+1]$ and similarly for $[m, m+1]$. Define $\phi_l(s, z) = \phi(s, z) 1_{[l, l+1]}(z)$ and $h_m(t, y) = h(t, y) 1_{[m, m+1]}(y)$.

Denote the above integral by I , and $I_{l,m}$ the integral when ϕ, h is replaced by ϕ_l, h_m . Then we can write $I = \sum_{l, m \in \mathbb{Z}^d} I_{l,m}$. Below we let C be a generic constant that may vary from line to line.

Assume $\|l - m\|_\infty := \max_i |l^{(i)} - m^{(i)}| \geq 2$. For $z \in [l, l+1]$ and $y \in [m, m+1]$ we have $\|z - y\| \geq \|l - m\|_\infty - 1$. If $\alpha \neq \beta$ we have that

$$D^\alpha D^\beta P(t - s, z - y) = \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t - s)^2} P(t - s, y - z)$$

for a suitable choice of i, j . Then we can find C such that

$$|D^\alpha D^\beta P(t-s, z-y)| \leq C e^{-(\|l-m\|_\infty - 2)^2/4}.$$

If $\alpha = \beta$, we have

$$(D^\alpha)^2 P(t-s, y-z) = \left(\frac{(y^{(i)} - z^{(i)})^2}{t-s} - 1 \right) \frac{P(t-s, y-z)}{t-s}$$

and similarly we find C such that

$$|(D^\alpha)^2 P(t-s, y-z)| \leq C e^{-(\|l-m\|_\infty - 2)^2/4}.$$

In both cases we have $|I_{l,m}| \leq C e^{-\|l\|^2/8} e^{-(\|l-m\|_\infty - 2)^2/4}$ and it follows that

$$\sum_{\|l-m\|_\infty \geq 2} |I_{l,m}| \leq C.$$

Assume $\|l-m\|_\infty \leq 1$ and let $\hat{\phi}_l(s, u)$ and $\hat{h}_m(t, u)$ be the Fourier transform in the second variable, defined by

$$\hat{h}_m(t, u) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} h(t, x) e^{-i(u, x)} dx$$

and similar for $\hat{\phi}_l(s, u)$. By the Plancherel theorem we have that

$$\int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 du = \int_{\mathbb{R}^d} \phi_l(s, z)^2 dz \leq C e^{-\|l\|^2/6}$$

for all $s \in [0, 1]$ and

$$\int_{\mathbb{R}^d} \hat{h}_m(t, u)^2 du = \int_{\mathbb{R}^d} h_m(t, y)^2 dy \leq 1.$$

We can write

$$I_{l,m} = \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u) \hat{h}_m(t, -u) u^{(i)} u^{(j)} (t-s) e^{-(t-s)\|u\|^2/2} du ds dt. \quad (2.31)$$

To see this, start with the right hand side. Then we have by Fubini

$$\begin{aligned} & \int_{\mathbb{R}^d} \hat{h}_m(t, -u) \hat{\phi}_l(s, u) u^i u^j (t-s) e^{-(t-s)\|u\|^2/2} du \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, x) e^{i(u, x)} \phi_l(s, y) e^{-i(u, y)} u^i u^j (t-s) e^{-(t-s)\|u\|^2/2} du dx dy = \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, x) \phi_l(s, y) (t-s) \left[(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(u, x-y)} u^i u^j e^{-(t-s)\|u\|^2/2} du \right] dx dy. \end{aligned}$$

Now look at the expression in the square brackets. Substitute $v = \sqrt{t-s}u$ to get

$$\begin{aligned} & (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(u, x-y)} u^i u^j e^{-(t-s)\|u\|^2/2} du \\ &= (2\pi)^{-d} (t-s)^{-d/2} \int_{\mathbb{R}^d} e^{i(\frac{v}{\sqrt{t-s}}, x-y)} \frac{v^i}{\sqrt{t-s}} \frac{v^j}{\sqrt{t-s}} e^{-\|v\|^2/2} dv \\ &= (2\pi)^{-d} (t-s)^{-d/2} (t-s)^{-1} \int_{\mathbb{R}^d} e^{i(v, \frac{x-y}{\sqrt{t-s}})} v^i v^j e^{-\|v\|^2/2} dv. \end{aligned}$$

Now put $\widehat{f(v)} = e^{-\|v\|^2/2}$ and $p(v) = v^{(i)}v^{(j)}$. From properties of the Fourier transform we know that $\widehat{pf} = D^\alpha D^\beta \hat{f}$ and $\hat{f} = f$. This gives that the above expression is equal to

$$(2\pi)^{-d/2} (t-s)^{-d/2} (t-s)^{-1} D^\alpha D^\beta f \left(\frac{x-y}{\sqrt{t-s}} \right) = (t-s)^{-1} D^\alpha D^\beta P(t-s, x-y).$$

This gives the equation (2.31).

Applying $ab \leq \frac{1}{2}a^2c + \frac{1}{2}b^2c^{-1}$ with $a = \hat{\phi}_l(s, u)u^{(i)}$, $b = \hat{h}_m(t, -u)u^{(j)}$ and $c = e^{\|l\|^2/12}$ we get

$$\begin{aligned} |I_{l,m}| &\leq \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 (u^{(i)})^2 e^{\|l\|^2/12} e^{-(t-s)\|u\|^2/2} dudsdt \\ &\quad + \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 (u^{(j)})^2 e^{-\|l\|^2/12} e^{-(t-s)\|u\|^2/2} dudsdt \\ &\leq \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 \|u\|^2 e^{\|l\|^2/12} e^{-(t-s)\|u\|^2/2} dudsdt \\ &\quad + \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 \|u\|^2 e^{-\|l\|^2/12} e^{-(t-s)\|u\|^2/2} dudsdt. \end{aligned}$$

For the first term, integrate first with respect to t in order to get

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 \|u\|^2 e^{\|l\|^2/12} e^{-(t-s)\|u\|^2/2} dudsdt \leq C e^{-\|l\|^2/12}$$

and for the second term, integrate with respect to s first to get

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 \|u\|^2 e^{-\|l\|^2/12} e^{-(t-s)\|u\|^2/2} dudsdt \leq C e^{-\|l\|^2/12}$$

which gives $|I_{l,m}| \leq C e^{-\|l\|^2/12}$ and hence

$$\sum_{\|l-m\|_\infty \leq 1} |I_{l,m}| \leq C.$$

■

Corollary 2.10 *There exists an absolute constant C such that for measurable functions g and h bounded by 1*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \right| \leq C$$

and

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) D^\gamma P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) dy dz ds dt \right| \leq C.$$

Notice that we have $\int_{\mathbb{R}^d} P(t, z) dz = 1$ and that

$$\int_{\mathbb{R}^d} |D^\alpha P(t, z)| dz \leq C t^{-1/2}, \quad (2.32)$$

$$\int_{\mathbb{R}^d} |D^\alpha D^\beta P(t, z)| dz \leq C t^{-1}. \quad (2.33)$$

Lemma 2.11 *There is an absolute constant C such that for every Borel-measurable functions g and h bounded by 1, and $r \geq 0$*

$$\left| \int_{t_0}^t \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(t_2, z) P(t_2 - t_0, z) h(t_1, y) D^\alpha D^\beta P(t_1 - t_2, y - z) (t - t_1)^r dy dz dt_2 dt_1 \right| \leq C(1 + r)^{-1} (t - t_0)^{r+1}$$

and

$$\left| \int_{t_0}^t \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(t_2, z) D^\gamma E(t_2 - t_0, z) h(t_1, y) D^\alpha D^\beta P(t_1 - t_2, y - z) (t - t_1)^r dy dz dt_2 dt_1 \right| \leq C(1 + r)^{-1/2} (t - t_0)^{r+1/2}.$$

Proof. We begin by proving the estimate for $t = 1, t_0 = 0$. From Corollary 2.10 we have that for each $k \geq 0$

$$\left| \int_{2^{-k-1}}^{2^{-k}} \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \leq C(1 - 2^{-k-1})^r 2^{-k}.$$

To see this, make the substitutions $t' = 2^k t$ and $s' = 2^k s$. Use the easily verified fact that $P(at, z) = a^{-d/2} P(t, a^{-1/2} z)$ and substitute $z' = 2^{k/2} z$ and $y' = 2^{k/2} y$. Using $\tilde{h}(t, y) := \frac{(1-t)^r}{(1-2^{-k-1})^r} h(t, y)$ in Corollary 2.10, the result follows.

Summing this equation over k gives

$$\left| \int_0^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \leq C(1 + r)^{-1}$$

Moreover from the bound (2.33)

$$\begin{aligned} \left| \int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \\ \leq C \int_0^1 \int_0^{t/2} (t - s)^{-1} (1 - t)^r ds dt \leq C(1 + r)^{-1} \end{aligned}$$

and combining these bounds gives the first assertion for $t = 1, t_0 = 0$. For general t and t_0 use the change of variables $t'_1 = \frac{t_1 - t_0}{t - t_0}$, $t_2 = \frac{t_2 - t_0}{t - t_0}$, $y' = (t - t_0)^{-1/2}y$ and $z' = (t - t_0)^{-1/2}z$.

The second assertion is proved similarly. ■

We turn to the completion of the proof of Proposition 2.7 by showing that there exists a constant M such that for each allowed string S in the alphabet $\mathcal{A}(\alpha)$ we have

$$I_S^\alpha(t_0, t, z_0) \leq \frac{M^n(t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

We will prove this by induction on n . The case $n = 0$ is immediate, so assume $n > 0$ and that this holds for all allowed strings of length less than n . There are three cases

1. $S = D^{\alpha_1}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_2, \dots, \alpha_n)$
2. $S = P \cdot D^{\alpha_1}D^{\alpha_2}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_3, \dots, \alpha_n)$
3. $S = P \cdot D^{\alpha_1}P \dots D^{\alpha_m}P \cdot D^{\alpha_{m+1}}D^{\alpha_{m+2}}P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_{m+3}, \dots, \alpha_n)$.

In each case, S' is an allowed string in the given alphabet.

1. We use the inductive hypothesis to bound $I_{S'}^{\alpha'}(t_1, t, z_1)$ and the bound (2.32) to get

$$\begin{aligned} |I_S^\alpha(t_0, t, z_0)| &= \left| \int_{t_0}^t \int_{\mathbb{R}^d} b_1(t_1, z_1) D^{\alpha_1}P(t_1 - t_0, z_1 - z_0) I_{S'}^{\alpha'}(t_1, t, z_1) dz_1 dt_1 \right| \\ &\leq \frac{M^{n-1}}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} \int_{\mathbb{R}^d} |D^{\alpha_1}P(t_1 - t_0, z_1 - z_0)| dz_1 dt_1 \\ &\leq \frac{M^{n-1}C}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} (t_1 - t_0)^{-1/2} dt_1 \\ &= \frac{M^{n-1}C\sqrt{\pi}(t - t_0)^{k/2}}{\Gamma(\frac{n}{2} + 1)}. \end{aligned}$$

The result follows if M is large enough.

2. For this case we can write

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0}^t \int_{t_1}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b_1(t_1, z_1) b_2(t_2, z_2) \\ &\quad \times P(t_1 - t_0, z_1 - z_0) D^{\alpha_1}D^{\alpha_2}P(t_2 - t_1, z_2 - z_1) I_{S'}^{\alpha'}(t_2, t, z_2) dz_1 dz_2 dt_2 dt_1. \end{aligned}$$

We set $h(t_2, z_2) := b_2(t_2, z_2) I_{S'}^{\alpha'}(t_2, z_2) (t - t_2)^{1-n/2}$ so that by the inductive hypothesis we have

$$\|h\|_\infty \leq M^{n-2}/\Gamma(n/2).$$

Use this in the first part of Lemma 2.11 with $g = b_1$ and integrate with respect to t_2 first, to get

$$|I_S^\alpha(t_0, t, z_0)| \leq \frac{CM^{n-2}(t - t_0)^{n/2}}{n\Gamma(n/2)},$$

and the result follows if M is large enough.

3. We have

$$\begin{aligned}
I_S^\alpha(t_0, t, z_0) &= \int_{t_0 < \dots t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^{m+2} b_j(t_j, z_j) \\
&\quad \times \prod_{j=2}^m D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) \\
&\quad \times I_{S'}^{\alpha'}(t_{m+2}, t, z_{m+2}) dz_1 \dots dz_{m+2} dt_1 \dots dt_{m+2}.
\end{aligned}$$

Let $h(t_{m+2}, z_{m+2}) = b_{m+2}(t_{m+2}, z_{m+2}) I_{S'}^{\alpha'}(t_{m+2}, t, z)(t - t_{m+2})^{(2+m-n)/2}$, so that from the inductive hypothesis we have $\|h\|_\infty \leq M^{n-m-2}/\Gamma((n-m)/2)$. Write

$$\begin{aligned}
\Omega(t_m, z_m) &:= \int_{t_m}^t \int_{t_{m+1}}^t \int_{\mathbb{R}^{2d}} b_{m+1}(t_{m+1}, z_{m+1}) h(t_{m+2}, z_{m+2}) \\
&\quad \times (t - t_{m+2})^{(n-m-2)/2} D^{\alpha_m} P(t_{m+1} - t_m, z_{m+1} - z) \\
&\quad \times D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) dz_{m+1} dz_{m+2} dt_{m+1} dt_{m+2},
\end{aligned}$$

so that from Lemma (3.9) we have that

$$|\Omega(t_m, z_m)| \leq \frac{C(n-m)^{-1/2} M^{n-m-2} (t - t_m)^{(n-m-1)/2}}{\Gamma(\frac{n-m}{2})}.$$

Using this in

$$\begin{aligned}
I_S^\alpha(t_0, t, z_0) &= \int_{t_0 < \dots t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^m b_j(t_j, z_j) \\
&\quad \times \prod_{j=1}^{m-1} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) \Omega(t_m, z_m) dz_1 \dots dz_m dt_1 \dots dt_m,
\end{aligned}$$

and using the bound (2.32) several times gives

$$\begin{aligned}
|I_S^\alpha(t_0, t, z_0)| &\leq C^{m+1} (n-m)^{-1/2} \frac{M^{n-m-2}}{\Gamma((n-m)/2)} \\
&\quad \times \int_{t_0 < \dots t_m < t} (t_2 - t_1)^{-1/2} \dots (t_m - t_{m-1})^{-1/2} (t - t_m)^{(n-m-1)/2} dt_1 \dots dt_m \\
&= C^{m+1} (n-m)^{-1/2} \frac{M^{n-m-2} \pi^{(m-1)/2} \Gamma(\frac{n-m+1}{2})}{\Gamma(\frac{n-m}{2}) \Gamma(\frac{n}{2} + 1)} (t - t_0)^{n/2},
\end{aligned}$$

and the result follows when M is large enough, thus proving the induction step.

■

We are now ready to complete the proof of Lemma 2.5.

Proof of Lemma 2.5.

Using the chain-rule of the Malliavin derivative D_t (see [34]) we find that

$$D_t X_s = \mathcal{I}_d + \int_t^s b'(u, X_u) D_t X_u du \quad (2.34)$$

μ -a.e. for all $1 \geq t \geq s$, where $\mathcal{I}_d$ is the $d \times d$ identity matrix and $b' = \left(\frac{\partial}{\partial x_i} b^{(j)}(t, x) \right)_{1 \leq i, j \leq d}$ is the (bounded) space derivative of b .

Fix $0 \leq t' \leq t < 1$. Then, for $1 \geq s \geq t$ we have

$$\begin{aligned} D_{t'} X_s - D_t X_s &= \int_{t'}^s b'(u, X_u) D_{t'} X_u du - \int_t^s b'(u, X_u) D_t X_u du \\ &= \int_{t'}^t b'(u, X_u) D_{t'} X_u du + \int_t^s b'(u, X_u) (D_{t'} X_u - D_t X_u) du \\ &= D_{t'} X_t - \mathcal{I}_d + \int_t^s b'(u, X_u) (D_{t'} X_u - D_t X_u) du. \end{aligned}$$

Applying Picard iteration to the above equation we find that

$$\begin{aligned} &D_{t'} X_s - D_t X_s \\ &= \left(\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, X_{s_1}) : \dots : b'(s_n, X_{s_n}) ds_1 \dots ds_n \right) (D_{t'} X_t - \mathcal{I}_d) \end{aligned} \quad (2.35)$$

in $L^2(\mu)$, uniformly in s , where $:$ denotes (non-commutative) matrix multiplication. On the other hand we also observe that

$$D_{t'} X_t - \mathcal{I}_d = \sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, X_{s_1}) : \dots : b'(s_n, X_{s_n}) ds_1 \dots ds_n. \quad (2.36)$$

Denote by $\|\cdot\|$ the maximum norm on $\mathbb{R}^{d \times d}$. Then Girsanov's theorem, Hölder's inequality and the Novikov condition in connection with (2.35) and (2.36) yield

$$\begin{aligned} E [\|D_{t'} X_s - D_t X_s\|^2] &= E \left[\left\| \left(\mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n \right) \right. \right. \\ &\quad \times \left. \left(\sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n \right) \right\|^2 \\ &\quad \times \mathcal{E} \left(\sum_{j=1}^d \int_0^1 b^{(j)}(u, B_u) dB_u^{(j)} \right) \Big] \\ &\leq C_1 \left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R}^{d \times d})}^2 \\ &\quad \times \left\| \sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R}^{d \times d})}^2 \end{aligned}$$

where C_1 is a constant and $\mathcal{E}(M_t)$ denotes the Doleans-Dade exponential of a martingale M_t .

So we obtain that

$$\begin{aligned}
& E \left[\|D_{t'} X_s - D_t X_s\|^2 \right] \\
& \leq C_1 \left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{t < s_1 < \dots < s_n < s} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R}^{d \times d})}^2 \\
& \quad \times \left\| \sum_{n=1}^{\infty} \int_{t' < s_1 < \dots < s_n < t} b'(s_1, B_{s_1}) : \dots : b'(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R}^{d \times d})}^2 \\
& \leq C_1 \left(1 + \sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\| \int_{t < s_1 < \dots < s_n < s} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \right. \right. \\
& \quad \left. \left. \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R})} \right)^2 \\
& \quad \times \left(\sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\| \int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \right. \right. \\
& \quad \left. \left. \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R})} \right)^2. \tag{2.37}
\end{aligned}$$

Now, look at the expression

$$A := \int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \frac{\partial}{\partial x_{l_n}} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n. \tag{2.38}$$

Then, using (deterministic) integration by parts, repeatedly, one finds that A^2 can be written as a sum of at most 2^{2n} summands of the form

$$\int_{t' < s_1 < \dots < s_{2n} < t} g_1(s_1) \dots g_{2n}(s_{2n}) ds_1 \dots ds_{2n}, \tag{2.39}$$

where $g_l \in \left\{ \frac{\partial}{\partial x_j} b^{(i)}(\cdot, B_{\cdot}) : 1 \leq i, j \leq d \right\}$, $l = 1, 2, \dots, 2n$. Since $A^4 = A^2 A^2$, we can argue similarly and conclude that there are *at most* 2^{8n} such summands (of length $4n$). Using this principle once more we see that A^8 can be represented as a sum of at most 2^{32n} summands of the form (2.39) now with length $8n$.

Combining this with Proposition 2.7 we get that

$$\begin{aligned}
& \left\| \int_{t' < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, B_{s_2}) \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, B_{s_n}) ds_1 \dots ds_n \right\|_{L^8(\mu; \mathbb{R})} \\
& \leq \left(\frac{2^{32n} C^{8n} \|b\|_{\infty}^{8n} |t - t'|^{4n}}{\Gamma(4n + 1)} \right)^{1/8} \\
& \leq \frac{2^{4n} C^n \|b\|_{\infty}^n |t - t'|^{n/2}}{(4n!)^{1/8}}. \tag{2.40}
\end{aligned}$$

Then it follows from (2.37) that

$$\begin{aligned} E [\|D_t X_s - D_{t'} X_s\|^2] &\leq C_1 \left(1 + \sum_{n=1}^{\infty} \frac{d^{n+2} 2^{4n} C^n \|b\|_{\infty}^n |t-s|^{n/2}}{(4n!)^{1/8}} \right)^2 \\ &\quad \times \left(\sum_{n=1}^{\infty} \frac{d^{n+2} 2^{4n} C^n \|b\|_{\infty}^n |t-t'|^{(n-1)/2}}{(4n!)^{1/8}} \right)^2 |t-t'| \\ &\leq C_d (\|b\|_{\infty}) |t-t'| \end{aligned}$$

for a function C_d as claimed in the theorem.

Similarly, we deduce the estimate for $\sup_{0 \leq t \leq s} E[\|D_t X_s\|^2]$.

■

This concludes step one in our program and we are now coming to the second step. For a Borel-measurable, bounded coefficient b we gradually show the following:

- Y_t^b in (2.29) is a well-defined object in the Hida distribution space $(\mathcal{S})^*$, $0 \leq t \leq 1$, (Lemma 2.12).
- For any a.e. approximating sequence of uniformly bounded, smooth coefficients b_n with compact support a subsequence of the corresponding strong solutions $X_{n_j, t} = Y_t^{b_{n_j}}$, fulfills $Y_t^{b_{n_j}} \rightarrow Y_t^b$ in $L^2(\mu)$ for $0 \leq t \leq 1$ (in particular $Y_t^b \in L^2(\mu)$, $0 \leq t \leq 1$), (Lemma 2.14).
- We apply a transformation property for Y_t^b (Lemma 2.16) and identify Y_t^b as a Malliavin differential strong solution to (2.25).

The first lemma gives a criterion under which the process Y_t^b belongs to the Hida distribution space.

Lemma 2.12 *Suppose that*

$$E_{\mu} \left[\exp \left(36 \int_0^1 \|b(s, B_s)\|^2 ds \right) \right] < \infty, \quad (2.41)$$

where the drift $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable (in particular, (2.41) is valid for b bounded). Then the coordinates of the process Y_t^b , defined in (2.29), that is

$$Y_t^{i, b} = E_{\tilde{\mu}} \left[\tilde{B}_t^{(i)} \mathcal{E}_T^{\circ}(b) \right], \quad (2.42)$$

are elements of the Hida distribution space.

Proof. See [31] ■

Lemma 2.13 *Let $b_n : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a sequence of Borel measurable functions with $b_0 = b$ such that*

$$\sup_{n \geq 0} E \left[\exp \left(512 \int_0^1 \|b_n(s, B_s)\|^2 ds \right) \right] < \infty \quad (2.43)$$

holds. Then

$$\left| S(Y_t^{i,b_n} - Y_t^{i,b})(\phi) \right| \leq \text{const} \cdot E[J_n]^{\frac{1}{2}} \cdot \exp(34 \int_0^1 \|\phi(s)\|^2 ds)$$

for all $\phi \in (S_{\mathbb{C}}([0, 1]))^d$, $i = 1, \dots, d$, where the factor J_n is defined by

$$\begin{aligned} J_n = & \sum_{j=1}^d \left(2 \int_0^1 \left(b_n^{(j)}(u, B_u) - b^{(j)}(u, B_u) \right)^2 du \right. \\ & \left. + \left(\int_0^1 \left| (b_n^{(j)}(u, B_u))^2 - (b^{(j)}(u, B_u))^2 \right| du \right)^2 \right). \end{aligned} \quad (2.44)$$

In particular, if b_n approximates b in the following sense

$$E[J_n] \rightarrow 0 \quad (2.45)$$

as $n \rightarrow \infty$, it follows that

$$Y_t^{b_n} \rightarrow Y_t^b \text{ in } (\mathcal{S})^*$$

as $n \rightarrow \infty$ for all $0 \leq t \leq 1$, $i = 1, \dots, d$.

Proof. See [31] ■

Lemma 2.14 Let $b_n : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a sequence of Borel-measurable, uniformly bounded, smooth functions with compact support which approximates a Borel-measurable, bounded coefficient $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ a.e. with respect to the Lebesgue measure. Then for any $0 \leq t \leq 1$ there exists a subsequence of the corresponding strong solutions $X_{n_j, t} = Y_t^{b_{n_j}}$, $j = 1, 2, \dots$, such that

$$Y_t^{b_{n_j}} \rightarrow Y_t^b$$

for $j \rightarrow \infty$ in $L^2(\mu)$. In particular this implies $Y_t^b \in L^2(\mu)$, $0 \leq t \leq 1$.

Proof. By Corollary 2.6 we know that there exists a subsequence $Y_t^{b_{n_j}}$, $j = 1, 2, \dots$, converging in $L^2(\mu)$. Further, by boundedness obviously $E[J_{n_j}] \rightarrow 0$ in (2.45), and thus $Y_t^{b_{n_j}} \rightarrow Y_t^b$ in $(\mathcal{S})^*$. But then, by uniqueness of the limit, also $Y_t^{b_{n_j}} \rightarrow Y_t^b$ in $L^2(\mu)$. ■

Remark 2.15 Note that by well known approximation results there always exists a sequence of functions b_n , $n \geq 1$, fulfilling the assumptions in Lemma 2.14. Then Lemma 2.14 guarantees that we are now ready to state the following “transformation property” for Y_t^b .

Lemma 2.16 Assume that $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Borel-measurable and bounded. Then

$$\varphi^{(i)} \left(t, Y_t^b \right) = E_{\tilde{\mu}} \left[\varphi^{(i)} \left(t, \tilde{B}_t \right) \mathcal{E}_T^{\hat{\sigma}}(b) \right] \quad (2.46)$$

a.e. for all $0 \leq t \leq 1$, $i = 1, \dots, d$ and $\varphi = (\varphi^{(1)}, \dots, \varphi^{(d)})$ such that $\varphi(B_t) \in L^2(\mu; \mathbb{R}^d)$.

Proof. See [38, Lemma 16] or [29]. ■

Using the above auxiliary results we can finally give the proof of Theorem 2.3.

Proof of Theorem 2.3. We aim at employing the transformation property (2.46) of Lemma 2.16 to verify that Y_t^b is a unique strong solution of the SDE (2.25). To shorten notation we set $\int_0^t \varphi(s, \omega) dB_s := \sum_{j=1}^d \int_0^t \varphi^{(j)}(s, \omega) dB_s^{(j)}$ and $x = 0$. Also, let b_n , $n = 1, 2, \dots$, be a sequence of functions as required in Lemma 2.14 (see Remark 2.15).

We first remark that Y_t^b has a continuous modification. The latter can be checked as follows: Since each $Y_t^{b_n}$ is a strong solution of the SDE (2.25) with respect to the drift b_n we obtain from Girsanov's theorem and our assumptions that

$$\begin{aligned} E_\mu \left[\left(Y_t^{i, b_n} - Y_u^{i, b_n} \right)^4 \right] &= E_{\tilde{\mu}} \left[\left(\tilde{B}_t^{(i)} - \tilde{B}_u^{(i)} \right)^4 \mathcal{E} \left(\int_0^1 b_n(s, \tilde{B}_s) d\tilde{B}_s \right) \right] \\ &\leq \text{const} \cdot |t - u|^2 \end{aligned}$$

for all $0 \leq u, t \leq 1$, $n \geq 1$, $i = 1, \dots, d$. The above constant comes from the fact that $\left\{ \mathcal{E} \left(\int_0^1 b_n(s, \tilde{B}_s) d\tilde{B}_s \right) \right\}_{n \geq 1}$ is bounded in $L^2(\tilde{\mu}, \mathbb{R}^d)$. To see this, we note that the Doleans-Dade exponential is the solution of the SDE

$$dZ_t^n = Z_t^n b_n(t, \tilde{B}_t) d\tilde{B}_t$$

starting at $Z_0^n = 1$. So that from Itô's formula we have

$$\begin{aligned} E_\mu [(Z_t^n)^2] &= 1 + \int_0^t E_{\tilde{\mu}} [(Z_s^n)^2 \|b_n(s, \tilde{B}_s)\|^2] ds \\ &\leq 1 + \|b_n\|_\infty^2 \int_0^t E_{\tilde{\mu}} [(Z_s^n)^2] ds \\ &\leq \exp(t \|b_n\|_\infty^2) \\ &\leq \exp \left(t \sup_{k \geq 1} \|b_k\|_\infty^2 \right) \end{aligned}$$

where the second last inequality comes from Grönwall's lemma. The last expression is bounded by assumption. By Lemma 2.14 we know that

$$Y_t^{b_{n_j}} \longrightarrow Y_t^b \text{ in } L^2(\mu; \mathbb{R}^d)$$

for a subsequence and hence we have almost sure convergence for a further subsequence, $0 \leq t \leq 1$. So we get that by Fatou's lemma

$$E_\mu \left[\left(Y_t^{i, b} - Y_u^{i, b} \right)^4 \right] \leq \text{const} \cdot |t - u|^2 \quad (2.47)$$

for all $0 \leq u, t \leq 1$, $i = 1, \dots, d$. Then Kolmogorov's Lemma provides a continuous modification of Y_t^b .

Since $\tilde{B}_t$ is a weak solution of (2.25) for the drift $b(s, x) + \phi(s)$ with respect to the measure $d\mu^* = \mathcal{E} \left(\int_0^1 \left(b(s, \tilde{B}_s) + \phi(s) \right) d\tilde{B}_s \right) d\mu$ we obtain that

$$\begin{aligned} S(Y_t^{i,b})(\phi) &= E_{\tilde{\mu}} \left[\tilde{B}_t^{(i)} \mathcal{E} \left(\int_0^1 \left(b(s, \tilde{B}_s) + \phi(s) \right) d\tilde{B}_s \right) \right] \\ &= E_{\mu^*} \left[\tilde{B}_t^{(i)} \right] \\ &= E_{\mu^*} \left[\int_0^1 \left(b^{(i)}(s, \tilde{B}_s) + \phi^{(i)}(s) \right) ds \right] \\ &= \int_0^t E_{\tilde{\mu}} \left[b^{(i)}(s, \tilde{B}_s) \mathcal{E} \left(\int_0^1 \left(b(u, \tilde{B}_u) + \phi(u) \right) d\tilde{B}_u \right) \right] ds + S \left(B_t^{(i)} \right) (\phi). \end{aligned}$$

Hence the transformation property (2.46) applied to b gives

$$S(Y_t^{i,b})(\phi) = S \left(\int_0^t b^{(i)}(u, Y_u^{i,b}) du \right) (\phi) + S(B_t^{(i)})(\phi).$$

Then the injectivity of S implies that

$$Y_t^b = \int_0^t b(s, Y_s^b) ds + B_t.$$

The Malliavin differentiability of Y_t^b follows from the fact that

$$\sup_{n \geq 1} \left\| Y_t^{i,b_n} \right\|_{1,2} \leq M < \infty$$

for all $i = 1, \dots, d$ and $0 \leq t \leq 1$. See e.g. [34].

On the other hand our conditions allow the application of Girsanov's theorem to any other strong solution. Then the proof of Proposition 2.1 (see e.g. [38, Proposition 1]) shows that any other solution necessarily takes the form Y_t^b .

■

Finally, we give an extension of Theorem 2.3 to a class of non-degenerate d -dimensional Itô-diffusions.

Theorem 2.17 *Consider the time-homogeneous $\mathbb{R}^d$ -valued SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x \in \mathbb{R}^d, \quad 0 \leq t \leq T, \quad (2.48)$$

where the coefficients $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Require that there exists a bijection $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$, which is twice continuously differentiable. Let $\Lambda_x : \mathbb{R}^d \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ and $\Lambda_{xx} : \mathbb{R}^d \rightarrow L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ be the corresponding derivatives of Λ and assume that

$$\Lambda_x(y)\sigma(y) = id_{\mathbb{R}^d} \text{ for } y \text{ a.e.}$$

as well as

$$\Lambda^{-1} \text{ is Lipschitz continuous.}$$

Suppose that the function $b_* : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$b_*(x) := \Lambda_x(\Lambda^{-1}(x)) [b(\Lambda^{-1}(x))] + \frac{1}{2} \Lambda_{xx}(\Lambda^{-1}(x)) \left[\sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i] \right]$$

satisfies the conditions of Theorem 2.3, where e_i , $i = 1, \dots, d$, is a basis of $\mathbb{R}^d$. Then there exists a Malliavin differentiable solution X_t to (2.48).

Proof. The proof can be directly obtained from Itô's Lemma. See [31]. ■

2.4 Applications

In this section we consider various applications of the techniques and results of the previous sections. In Subsection 2.4.1 we show that our techniques can be used to analyse Sobolev differentiability of strong solutions X_t of the SDE (2.25) in the initial condition x . In Subsection 2.4.2 our results on Malliavin and Sobolev differentiability of strong solutions of SDE's are employed to derive a stochastic representation of spatial derivatives of solutions to the Kolmogorov equation (Bismut-Elworthy-Li formula, [10]) with irregular drift coefficients, which does not involve derivatives of the initial condition (2.51). The latter result might be also relevant from a numerical point of view. In Subsection 2.4.3 we account for the great flexibility of our approach to be generalized to the study of solutions of various other stochastic equations besides SDE's in finite dimension. In particular we present the applicability of our techniques and results to the problem of well-posedness of stochastic transport equations with singular coefficients.

2.4.1 Sobolev differentiability of strong solutions of SDE's

In the sequel we consider the solution process of the SDE (2.25) as a function of the initial value, $x \in \mathbb{R}^d$, denoted by X_t^x . We remark that the following analysis can be extended to the SDE (2.48) with non-trivial diffusion coefficient under the assumptions of Theorem 2.17. It is well known (see [22]) that when b is continuously differentiable with bounded derivative, the process X_s^x is μ -a.e. differentiable in x and its derivative satisfies the following linear ODE

$$\frac{\partial}{\partial x} X_s^x = \mathcal{I}_d + \int_0^s b'(u, X_u^x) \frac{\partial}{\partial x} X_u^x du. \quad (2.49)$$

Note that this equation is the same as (2.34) when $t = 0$. Using this observation in connection with the same technique as in the proof of Lemma 2.5, by replacing the Malliavin derivative of X_s^x by $\frac{\partial}{\partial x} X_s^x$, we get the following Lemma.

Lemma 2.18 *Let $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth function with compact support. Then the corresponding strong solution X in (2.25) fulfills*

$$\sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E \left[\left\| \frac{\partial}{\partial x} X_s^x \right\|^2 \right] \leq C_d (\|b\|_\infty),$$

where $C_d : [0, \infty) \rightarrow [0, \infty)$ is an increasing, continuous function, $\|\cdot\|$ a matrix-norm on $\mathbb{R}^{d \times d}$ and $\|\cdot\|_\infty$ the supremum norm.

Applying this Lemma we obtain the following result on the (local) Sobolev differentiability of $(x \mapsto X_t^x)$ in the case of *irregular* b , which we will use in our applications in the next subsections.

Proposition 2.19 *Let $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be bounded and measurable. Let U be an open, bounded subset of $\mathbb{R}^d$. For each $t \in [0, 1]$ we have*

$$(x \mapsto X_t^x) \in L^2(\Omega, W^{1,2}(U)),$$

where $W^{1,2}(U)$ is the space of square integrable Sobolev-differentiable functions.

The proof of Proposition 2.19 follows the same pattern as presented in Section 3. In the following, we retain the notation from the previous section.

Proof.

We start by observing that for any $\varphi \in C_0^\infty(U; \mathbb{R}^d)$ and $t \in [0, 1]$ the sequence of random variables

$$\langle X_t^n, \varphi \rangle := \int_U \langle X_t^{n,x}, \varphi(x) \rangle_{\mathbb{R}^d} dx$$

is relatively compact in $L^2(\Omega)$. To see this we use the compactness criterion of Corollary 4.22. Note that since the Malliavin derivative is a closed linear operator we have

$$\begin{aligned} E[|D_s^j \langle X_t^n, \varphi \rangle|^2] &= \sum_{i=1}^d \left(\int_U E[D_s^j X_t^{i,n,x}] \varphi_i(x) dx \right)^2 \\ &\leq \|\varphi\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}^2 \lambda\{\text{supp}(\varphi)\} \sup_{x \in \text{supp}(U), 0 \leq s \leq t} E[\|D_s X_t^{n,x}\|^2], \end{aligned}$$

where D^j is the Malliavin derivative in the direction of B^j and where λ denotes Lebesgue measure and $\text{supp}(\varphi)$ is the support of φ . Summing over $j = 1, \dots, d$ in connection with Lemma 2.5 gives

$$E[\|D_s \langle X_t^n, \varphi \rangle\|^2] \leq d \|\varphi\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}^2 \lambda\{\text{supp}(\varphi)\} C_d(\|b_n\|_\infty).$$

Similarly we get

$$E[\|D_s \langle X_t^n, \varphi \rangle - D_{s'} \langle X_t^n, \varphi \rangle\|^2] \leq d \|\varphi\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}^2 \lambda\{\text{supp}(\varphi)\} C_d(\|b_n\|_\infty) |s' - s|,$$

which shows that $\langle X_t^n, \varphi \rangle$ is relatively compact. Denote by $Y_t(\varphi)$ its limit after taking an (if necessary) subsequence.

Taking the S -transform of $\langle X_t^n, \varphi \rangle$ and $\langle X_t, \varphi \rangle$ we see that for any $\phi \in (\mathcal{S}_\mathbb{C}([0, 1]))^d$

$$\begin{aligned} |S(\langle X_t^n, \varphi \rangle)(\phi) - S(\langle X_t, \varphi \rangle)(\phi)|^2 &= |S(X_t^n - X_t)(\phi), \varphi|^2 \\ &\leq \|\varphi\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}^2 \int_U |S(X_t^{n,x} - X_t^x)(\phi)|^2 dx \\ &\leq \|\varphi\|_{L^2(\mathbb{R}^d; \mathbb{R}^d)}^2 \int_U C E[J_n(x)] \exp(68 \int_0^1 \|\phi(s)\|^2 ds) dx, \end{aligned}$$

where C is a constant and

$$J_n(x) := \sum_{j=1}^d \left(2 \int_0^1 \left(b_n^{(j)}(u, x + B_u) - b^{(j)}(u, x + B_u) \right)^2 du \right)$$

$$+ \left(\int_0^1 \left| (b_n^{(j)}(u, x + B_u))^2 - (b^{(j)}(u, x + B_u))^2 \right| du \right)^2.$$

See Lemma 2.13. Since $\{b_n\}$ is uniformly bounded, using dominated convergence, we get that

$$\langle X_t^n, \varphi \rangle \rightarrow \langle X_t, \varphi \rangle$$

in $(\mathcal{S})^*$, and thus in particular weakly in $L^2(\Omega, \mu)$. By uniqueness of the limits we can conclude that

$$Y(\varphi) = \langle X_t, \varphi \rangle \quad \mu\text{-a.s.},$$

thus proving the assertion.

Note that there exists a subsequence $n(k)$ such that $\langle X_t^{n(k)}, \varphi \rangle$ converges for every φ , that is, $n(k)$ is independent of φ . To see this, let $x = 0$ and choose $n(k)$ such that

$$X_t^{n(k),0} \rightarrow X_t^0$$

in $L^2(\Omega)$. If there exists $\varphi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and $\epsilon > 0$ such that $\|\langle X_t^{n(k)}, \varphi \rangle - \langle X_t, \varphi \rangle\| \geq \epsilon$ we may by the above extract a further subsequence $\langle X_t^{n(k(j))}, \varphi \rangle$ converging to $\langle X_t, \varphi \rangle$, which gives a contradiction. From now we denote this subsequence by n for simplicity.

We now proceed to prove that $(x \mapsto X_t^x) \in L^2(\Omega; W^{1,2}(U; \mathbb{R}^d))$: Because of Lemma 2.18 we get that $(x \mapsto X_t^{n,x})$ is bounded in $L^2(\Omega; W^{1,2}(U; \mathbb{R}^d))$, thus relatively compact in the weak topology. Then there exists a subsequence $n(k)$ such that $X_t^{n(k),\cdot}$ converges weakly to an $Y \in L^2(\Omega; W^{1,2}(U; \mathbb{R}^d))$. Then for all $A \in \mathcal{F}$ and $\varphi \in C_0^\infty$ we have

$$\begin{aligned} E[1_A \langle X_t, \varphi' \rangle] &= \lim_{k \rightarrow \infty} E[1_A \langle X_t^{n(k)}, \varphi' \rangle] \\ &= \lim_{k \rightarrow \infty} -E[1_A \langle \frac{\partial}{\partial x} X_t^{n(k)}, \varphi \rangle] = -E[1_A \langle Y, \varphi \rangle]. \end{aligned}$$

Hence we have

$$\langle X_t, \varphi' \rangle = -\langle Y, \varphi \rangle \quad \mu\text{-a.s.} \quad (2.50)$$

Finally, we need to show that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that X_t has a weak derivative on this subset. To this end choose a sequence $\{\varphi_n\}$ in $C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ dense in $W_0^{1,2}(U; \mathbb{R}^d)$. Choose a measurable subset Ω_n of Ω with full measure such that (2.50) holds on Ω_n with φ replaced by φ_n . Then $\Omega_0 := \cap_{n \geq 1} \Omega_n$ satisfies the desired property. ■

Remark 2.20 *By a similar argument as in the above proof, one can show that there exists a subsequence $n(k)$ such that $X_t^{n(k),x} \rightarrow X_t^x$ in $L^2(\Omega; \mathbb{R}^d)$ for all t and x , i.e. the choice of subsequence is independent of t and x . From now on, we shall always use this subsequence, for simplicity denoted by n .*

Remark 2.21 *We mention that Proposition 2.19 was first studied in [32]. There the authors could even show that the solution to the SDE (2.25) gives rise to a stochastic flow of Sobolev diffeomorphisms*

$$\mathbb{R}^d \ni x \mapsto \phi_{s,t}(x) \in \mathbb{R}^d, \quad s, t \in \mathbb{R},$$

where $\phi_{s,t}(x)$ denotes the solution at time t of the SDE (2.25) that starts in x at time s .

2.4.2 Stochastic representation of spatial derivatives of solutions to Kolmogorov equations

We will now study the Kolmogorov equation

$$\frac{\partial}{\partial t}v(t, x) = \sum_{j=1}^d b_j(t, x) \frac{\partial}{\partial x_j}v(t, x) + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}v(t, x) \quad (2.51)$$

with initial condition $v(0, x) = \Phi(x)$, where $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bounded and measurable.

It can be shown, see [50], that when Φ is continuous and bounded there exists a solution to (2.51) given by

$$v(t, x) = E[\Phi(X_t^x)], \quad (2.52)$$

where v is a solution to the Kolmogorov Equation (2.51) which is unique among all bounded solutions in the space $\cap_{p>1} W_{loc}^{(1,2),p}((0, 1] \times \mathbb{R}^d) \cap C([0, 1] \times \mathbb{R}^d)$. Here $W_{loc}^{(1,2),p}((0, 1] \times \mathbb{R}^d)$ denotes the space of functions that are once weakly differentiable on $(0, 1]$ and twice weakly differentiable on $\mathbb{R}^d$, and these derivatives are locally integrable to the p -th power.

In the following we are interested in the spatial derivative $\frac{\partial}{\partial x}v$, and our aim is to use the Mallavin derivative and the spatial derivative $\frac{\partial}{\partial x}X_t^x$ to find an explicit formula for $\frac{\partial}{\partial x}v$ which does not depend on the derivative of Φ .

Remark 2.22 *i) The spatial derivative $\frac{\partial}{\partial x}v(t, x)$ has various important interpretations. For example, if $v(t, x)$ describes the concentration of a certain type of molecules at time t and position x in a medium, where $b(t, x)$ is the macroscopic velocity of motion of the medium at t, x , then $\frac{\partial}{\partial x}v(t, x)$ can be regarded as a sensitivity measure of the concentration of molecules with respect to its location.*

ii) We remark that the representation (2.53) we derive below might be useful from a numerical point of view, since no derivative of the initial condition Φ is involved. It is e.g. conceivable to approximate $\frac{\partial}{\partial x}v(t, x)$ by using Monte-Carlo simulation. The latter is at least possible in the one-dimensional case, where $\frac{\partial}{\partial x}X_t^x$ can be written in terms of a backward stochastic integral or a local time integral without using the derivative of b . Compare [31].

Theorem 2.23 (Bismut-Elworthy-Li formula) *Assume $\Phi \in C_b(\mathbb{R}^d)$ and let U be an open, bounded subset of $\mathbb{R}^d$. Then the derivative of the solution to (2.51) takes the following (matrix notational) form*

$$\frac{\partial}{\partial x}v(t, x) = E[\Phi(X_t^x)t^{-1} \int_0^t \frac{\partial}{\partial x}X_s^x dB_s] \quad (2.53)$$

for almost all $x \in U$ and all $t \in (0, 1]$.

Proof.

Assume that $\Phi \in C_b^2(\mathbb{R}^d)$ (the general case of $\Phi \in C_b(\mathbb{R}^d)$ can be proved by approximation of Φ in relation (2.55)) and let b_n and $X_t^{n,x}$ be as in the previous section. If we replace b by b_n in (2.51) we have the unique solution given by

$$v_n(t, x) = E[\Phi(X_t^{n,x})].$$

By using Remark 4.11 we see that $v_n(t, x) \rightarrow v(t, x)$ for each t and x .

By [34, Page 109] we know that

$$D_s X_t^{n,x} \frac{\partial}{\partial x} X_s^{n,x} = \frac{\partial}{\partial x} X_t^{n,x},$$

where the above product is the usual matrix product. Then we have

$$\frac{\partial}{\partial x} X_t^{n,x} = t^{-1} \int_0^t D_s X_t^{n,x} \frac{\partial}{\partial x} X_s^{n,x} ds. \quad (2.54)$$

Interchanging integration and differentiation we get

$$\begin{aligned} \frac{\partial}{\partial x} v_n(t, x) &= E[\Phi'(X_t^{n,x}) \frac{\partial}{\partial x} X_t^{n,x}] \\ &= E[t^{-1} \int_0^t D_s \Phi(X_t^{n,x}) \frac{\partial}{\partial x} X_s^{n,x} ds] \\ &= E[\Phi(X_t^{n,x}) t^{-1} \int_0^t \frac{\partial}{\partial x} X_s^{n,x} dB_s], \end{aligned}$$

where we have used the chain rule and the duality formula for the Malliavin derivative in the last equality.

Fix $\varphi \in C_0^\infty(U)$. Using the matrix notation, we will show that

$$\int_{\mathbb{R}^d} \frac{\partial}{\partial x} \varphi(x) v(t, x) dx = - \int_{\mathbb{R}^d} \varphi(x) E[\Phi(X_t^x) t^{-1} \int_0^t \frac{\partial}{\partial x} X_s^x dB_s] dx. \quad (2.55)$$

Indeed, by dominated convergence and Remark 4.11 we have

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{\partial}{\partial x} \varphi(x) v(t, x) dx &= - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi(x) E[\Phi(X_t^{n,x}) t^{-1} \int_0^t \frac{\partial}{\partial x} X_s^{n,x} dB_s] dx \\ &= - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi(x) E[(\Phi(X_t^{n,x}) - \Phi(X_t^x)) t^{-1} \int_0^t \frac{\partial}{\partial x} X_s^{n,x} dB_s] dx \\ &\quad - \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi(x) E[\Phi(X_t^x) t^{-1} \int_0^t \frac{\partial}{\partial x} X_s^{n,x} dB_s] dx \\ &= - \lim_{n \rightarrow \infty} i)_n - \lim_{n \rightarrow \infty} ii)_n. \end{aligned}$$

For the first term we have

$$i)_n \leq \int_{\mathbb{R}^d} |\varphi(x)| \left\| \frac{\partial}{\partial x} \Phi \right\|_\infty \|X_t^{n,x} - X_t^x\|_{L^2(\Omega; \mathbb{R}^d)} t^{-1/2} \left(\sup_{k \geq 1, s \in [0, 1]} E[\left\| \frac{\partial}{\partial x} X_s^{k,x} \right\|_{\mathbb{R}^d \times d}^2] \right)^{1/2} dx,$$

which goes to zero as n tends to infinity by Remark 4.11 and Lebesgue dominated convergence theorem. For the second term, $ii)_n$ since X_t^x is *Malliavin differentiable* and $\Phi \in C_b^2(\mathbb{R}^d)$ we have by the Clark-Ocone formula in matrix notational form that (see e.g. [34])

$$\Phi(X_t^x) = E[\Phi(X_t^x)] + \int_0^t E[D_s \Phi(X_t^x) | \mathcal{F}_s] dB_s,$$

so that

$$ii)_n = \int_{\mathbb{R}^d} \varphi(x) E[\Phi(X_t^x) t^{-1} \int_0^t \frac{\partial}{\partial x} X_s^{n,x} dB_s] dx \quad (2.56)$$

$$= t^{-1} \int_{\mathbb{R}^d} \varphi(x) E\left[\left(E[\Phi(X_t^x)] + \int_0^t E[D_s \Phi(X_t^x) | \mathcal{F}_s] dB_s\right) \int_0^t \frac{\partial}{\partial x} X_s^{n,x} dB_s\right] dx \quad (2.57)$$

$$= t^{-1} \int_0^t \int_{\mathbb{R}^d} \varphi(x) E[D_s \Phi(X_t^x) \frac{\partial}{\partial x} X_s^{n,x}] dx ds. \quad (2.58)$$

It is easily verified by Lemma 2.5 that $\varphi(\cdot) D_s \Phi(X_t^x) = \varphi(\cdot) \Phi'(X_t^x) D_s X_t^x$ lies in $L^2(\mathbb{R}^d \times \Omega; \mathbb{R}^d)$ so that for each s , the function

$$g_n(s) = \int_{\mathbb{R}^d} \varphi(x) E[D_s \Phi(X_t^x) \frac{\partial}{\partial x} X_s^{n,x}] dx$$

converges to $\int_{\mathbb{R}^d} \varphi(x) E[D_s \Phi(X_t^x) \frac{\partial}{\partial x} X_s^x] dx$ by the weak convergence of $\frac{\partial}{\partial x} X_s^{n,x}$ (for a subsequence). Now

$$\begin{aligned} |g_n(s)| &\leq \int_{\mathbb{R}^d} |\varphi(x)| \|D_s \Phi(X_t^x)\|_{L^2(\Omega; \mathbb{R}^d)} \left\| \frac{\partial}{\partial x} X_s^{n,x} \right\|_{L^2(\Omega; \mathbb{R}^d)} dx \\ &\leq \sup_{y \in \mathbb{R}^d, u \leq t, k \in \mathbb{N}} \|D_u \Phi(X_t^y)\|_{L^2(\Omega; \mathbb{R}^d)} \left\| \frac{\partial}{\partial x} X_u^{k,y} \right\|_{L^2(\Omega; \mathbb{R}^d)} \int_{\mathbb{R}^d} |\varphi(x)| dx \end{aligned}$$

so that by Lebesgue dominated convergence theorem

$$\lim_{n \rightarrow \infty} ii)_n = t^{-1} \int_0^t \int_{\mathbb{R}^d} \varphi(x) E[D_s \Phi(X_t^x) \frac{\partial}{\partial x} X_s^x] dx ds.$$

Reversing equations (2.56), (2.57) and (2.58) with $\frac{\partial}{\partial x} X_s^x$ in place of $\frac{\partial}{\partial x} X_s^{n,x}$ gives the result. $\blacksquare$

Remark 2.24 *i) In the case of smooth coefficients b a similar proof based on the duality formula for Malliavin derivatives can be found in [14].*

ii) By using relation (2.55) and the existence of a probability density of X_t^x in the proof of Theorem 2.23 it can actually be shown that for merely bounded and measurable Φ we have

$$(x \mapsto E[\Phi(X_t^x)]) \in W^{1,2}(U)$$

for all t .

2.4.3 Well-posedness of stochastic transport equations with singular coefficients

In order to emphasize the applicability of the techniques and results of the previous sections to a variety of other problems in stochastic analysis, we present the following result on the problem of well-posedness of stochastic transport equations with singular coefficients, which is based on the estimate (2.30) in Proposition 2.7.

The stochastic transport equation is the following equation

$$\begin{cases} d_t u(t, x) + (b(t, x) \cdot \frac{\partial}{\partial x} u(t, x)) dt + \sum_{i=1}^d \frac{\partial}{\partial x_i} u(t, x) \circ dB_t^i = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (2.59)$$

where $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a given vector field and $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is a given initial data. The stochastic integration is understood in the Stratonovich sense.

By a differentiable, weak L^∞ -solution of the transport equation (2.59) we mean a stochastic process $u \in L^\infty(\Omega \times [0, 1] \times \mathbb{R}^d)$ such that, for every t , the function $u(t, \cdot)$ is weakly differentiable a.s. with $\sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E[|\frac{\partial}{\partial x} u(s, x)|^4] < \infty$ and for every test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d} \theta(x) u(t, x) dx$ has a continuous modification which is an $\mathcal{F}_t$ -semi martingale and

$$\begin{aligned} \int_{\mathbb{R}^d} \theta(x) u(t, x) dx &= \int_{\mathbb{R}^d} \theta(x) u_0(x) dx \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial x} u(s, x) \cdot b(s, x) \theta(x) dx ds \\ &\quad + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s, x) \frac{\partial}{\partial x_i} \theta(x) dx \right) \circ dB_s^i, \end{aligned} \quad (2.60)$$

where $\frac{\partial}{\partial x} u(t, x)$ is the weak derivative of $u(t, x)$ in the space-variable.

Theorem 2.25 *Let b be bounded and measurable, and let $u_0 \in C_b^1(\mathbb{R}^d)$. Then there exists a unique differentiable, weak L^∞ -solution $u(t, x)$ to (2.59). Moreover, for fixed t and x , this solution is Malliavin-differentiable.*

Proof. See [32]. ■

Remark 2.26 *We point out that the corresponding deterministic transport equation of (2.59) is in general not well-posed - even if b is continuous.*

Finally, we conclude this section with a remark on two further potential applications of our approach.

Remark 2.27 *i) Another interesting implication of our result that strong solutions of SDE's with merely measurable drift coefficients are Malliavin differentiable is that for all $0 \leq t \leq T$, $h \in L^2([0, T])$ the process*

$$\{X_t(\omega + u \int_0^\cdot h(s) ds)\}_{u \in \mathbb{R}} \quad (2.61)$$

has a version which is absolutely continuous for almost all $\omega \in \Omega = C_0([0, T])$ (Wiener space), see e.g. [34]. By considering the “initial condition” $y = x + B_t(\omega)$ in the ODE

$$\begin{aligned} \frac{d}{dt} X_t^y &= b(t, X_t^y) \\ X_0^y &= y, \end{aligned}$$

relation (2.61) in connection with (2.25) actually gives an interesting “link” to the flow property of solutions of ODE's with discontinuous coefficients. This may be of use in perturbation

problems of discontinuous ordinary differential equations and other applications. See e.g. [27] or [12].

ii) We also mention that our technique may be applied to examine strong solutions of

$$dX_t = b(t, X_t)dt + dB_t^Q, \quad X_0 = x \in H, \quad (2.62)$$

where B_t^Q is a Q -cylindrical Brownian motion on a Hilbert space H and Q a positive symmetric trace class operator. Applications to certain classes of SPDE's are also conceivable. See [28]. We point out that equations of the type (2.62) are not accessible within the framework of the authors of [52], [49], [36], [21], [16] and [17]. For example, the construction method of the authors in [17] heavily rests on an estimate of Krylov [20], which has no extension to infinite dimensions.

2.5 Appendix

The following result which is due to [6, Theorem 1] provides a compactness criterion for subsets of $L^2(\mu; \mathbb{R}^d)$ using Malliavin calculus.

Theorem 2.28 *Let $\{(\Omega, \mathcal{A}, P); H\}$ be a Gaussian probability space, that is $(\Omega, \mathcal{A}, P)$ is a probability space and H a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the σ -field $\mathcal{A}$. Denote by $\mathbf{D}$ the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \dots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \dots, h_n) h_i, \quad h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega; H)}.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1}\mathbf{D}G\|_{L^2(\Omega; H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [6].

Lemma 2.29 *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0, 1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0, 1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \quad \text{if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem 2.28 and Lemma 2.29 is now the following compactness criteria which is essential for the proof of Corollary 2.6:

Corollary 2.30 *Let a sequence of $\mathcal{F}_1$ -measurable random variables $X_n \in \mathbb{D}_{1,2}$, $n = 1, 2, \dots$, be such that there exist constants $\alpha > 0$ and $C > 0$ with*

$$\sup_n E[|X_n|^2] \leq C,$$

$$\sup_n E [\|D_t X_n - D_{t'} X_n\|^2] \leq C|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$ and

$$\sup_n \sup_{0 \leq t \leq 1} E [\|D_t X_n\|^2] \leq C.$$

Then the sequence X_n , $n = 1, 2, \dots$, is relatively compact in $L^2(\Omega)$.

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3 Sobolev Differentiable Stochastic Flows for SDE's with Singular Coefficients: Applications to the Transport Equation

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Preprint

Abstract

In this paper, we establish the existence of a stochastic flow of Sobolev diffeomorphisms

$$\mathbb{R}^d \ni x \mapsto \phi_{s,t}(x) \in \mathbb{R}^d, \quad s, t \in \mathbb{R},$$

for a stochastic differential equation (SDE) of the form

$$dX_t = b(t, X_t) dt + dB_t, \quad s, t \in \mathbb{R}, \quad X_s = x \in \mathbb{R}^d.$$

The above SDE is driven by a *bounded measurable* drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a d -dimensional Brownian motion B . More specifically, we show that the stochastic flow $\phi_{s,t}(\cdot)$ of the SDE lives in the space $L^2(\Omega; W^{1,p}(\mathbb{R}^d, w))$ for all s, t and all $p > 1$, where $W^{1,p}(\mathbb{R}^d, w)$ denotes a weighted Sobolev space with weight w possessing a p -th moment with respect to Lebesgue measure on $\mathbb{R}^d$. This result is counter-intuitive, since the dominant ‘culture’ in stochastic (and deterministic) dynamical systems is that the flow ‘inherits’ its spatial regularity from the driving vector fields.

The spatial regularity of the stochastic flow yields existence and uniqueness of a Sobolev differentiable weak solution of the (Stratonovich) stochastic transport equation

$$\begin{cases} d_t u(t, x) + (b(t, x) \cdot Du(t, x)) dt + \sum_{i=1}^d e_i \cdot Du(t, x) \circ dB_t^i = 0, \\ u(0, x) = u_0(x), \end{cases}$$

where b is *bounded and measurable*, u_0 is C_b^1 and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. It is well-known that the deterministic counter part of the above equation does not in general have a solution.

Using stochastic perturbations and our analysis of the above SDE, we establish a deterministic flow of Sobolev diffeomorphisms for classical one-dimensional (deterministic) ODE's driven by *discontinuous* vector fields. Furthermore, and as a corollary of the latter result, we construct a Sobolev stochastic flow of diffeomorphisms for one-dimensional SDE's driven by *discontinuous diffusion* coefficients.

3.1 Introduction

In this article we analyze the spatial regularity in the initial condition $x \in \mathbb{R}^d$ for strong solutions X_t^x to the d -dimensional stochastic differential equation (SDE)

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x}) du + B_t - B_s, \quad s, t \in \mathbb{R}. \quad (3.1)$$

In the above SDE, the drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is only *Borel measurable and bounded*, and the equation is driven by standard Brownian motion B , in $\mathbb{R}^d$.

More specifically, we construct a two-parameter pathwise Sobolev differentiable stochastic flow

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \mapsto \phi_{s,t}(x) \in \mathbb{R}^d$$

for the SDE (3.1) such that each flow map

$$\mathbb{R}^d \ni x \mapsto \phi_{s,t}(x) \in \mathbb{R}^d$$

is a Sobolev diffeomorphism in the sense that

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d; w)) \quad (3.2)$$

for all $s, t \in \mathbb{R}$, all $p > 1$. In (3.2) above, $W^{1,p}(\mathbb{R}^d, w)$ denotes a weighted Sobolev space of mappings $\mathbb{R}^d \rightarrow \mathbb{R}^d$ with any measurable weight function $w : \mathbb{R}^d \rightarrow [0, \infty)$ satisfying the integrability requirement

$$\int_{\mathbb{R}^d} (1 + |x|^p) w(x) dx < \infty. \quad (3.3)$$

In particular, $\phi_{s,t}(\cdot)$ is locally α -Hölder continuous for all $\alpha < 1$. When the SDE (3.1) is autonomous, we show further that the stochastic flow corresponds to a Sobolev differentiable perfect cocycle on $\mathbb{R}^d$. For precise statements of the above results, see Theorem 3.3 and Corollary 3.5 in the next section.

In this article we offer a novel approach for constructing a Sobolev differentiable stochastic flow for the SDE (1). Our approach is based on Malliavin calculus ideas coupled with new probabilistic estimates on the spatial weak derivatives of solutions of the SDE. A unique (pleasantly surprising) feature of these estimates is that they do not depend on the spatial regularity of the drift coefficient b . Needless to say, the existence of differentiable flows for SDE's with measurable drifts is counter-intuitive: The dominant ‘culture’ in stochastic (and deterministic) dynamical systems is that the flow ‘inherits’ its spatial regularity from the driving vector fields. Furthermore, in the stochastic setting, the stochastic flow is in general even a little ‘rougher’ in the space variable than the driving vector fields. (cf. [22], [28]).

The existence of a Sobolev differentiable stochastic flow for the SDE (1) is exploited (Section 3) to obtain a unique weak solution $u(t, x)$ of the (Stratonovich) stochastic transport equation

$$\begin{cases} d_t u(t, x) + (b(t, x) \cdot Du(t, x)) dt + \sum_{i=1}^d e_i \cdot Du(t, x) \circ dB_t^i = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (3.4)$$

when b is just bounded and measurable, $u_0 \in C_b^1(\mathbb{R}^d)$, and $\{e_i\}_{i=1}^d$ a basis for $\mathbb{R}^d$. This result is surprising since the corresponding deterministic transport equation is in general ill-posed.

Cf. [1], [8]. We also note that our result holds without the existence of the divergence of b ; and furthermore, our solutions are spatially (and also Malliavin) Sobolev differentiable (cf. [14]).

In Section 4, we apply the ideas of Section 2 to show the existence of a family of solutions $\tilde{X}_t^x$ of the one-dimensional ODE

$$\frac{d\tilde{X}_t}{dt} = b(\tilde{X}_t), \quad t \in \mathbb{R}, \quad \tilde{X}_0 = x \in \mathbb{R}, \quad (3.5)$$

which are locally of class $W^{1,2}$ in x (Theorem 3.27, Section 4). This result is obtained under the requirement that the coefficient b is monotone decreasing and is either bounded above or below. The proof of the result uses a stochastic perturbation argument via small Brownian noise coupled with local time techniques. As far as we know, it appears that the above result is new. Furthermore, solutions to the ODE (3.5) generate a one-parameter group of $W^{1,2}$ diffeomorphisms of $\mathbb{R}$ onto itself. As a consequence of the above result, we construct a $W^{1,2}$ perfect cocycle of diffeomorphisms for solutions of the one-dimensional Stratonovich SDE:

$$dX_t^x = b(X_t^x) \circ dB_t, \quad t \in \mathbb{R}, \quad X_0^x = x \in \mathbb{R}. \quad (3.6)$$

It is surprising that such regularity of the flow is feasible despite the inherent discontinuities in the driving vector field of in the ODE (3.5) and the SDE (3.6). SDE's with discontinuous coefficients and driven by Brownian motion (or more general noise) have been an important area of study in stochastic analysis and other related branches of mathematics. Important applications of this class of SDE's pertain to the modeling of the dynamics of interacting particles in statistical mechanics and the description of a variety of other random phenomena in areas such as biology or engineering. See e.g. [33] or [23] and the references therein.

Using estimates of solutions of parabolic PDE's and the Yamada-Watanabe principle, the existence of a global unique strong solution to the SDE (3.1) was first established by A.K. Zvonkin [41] in the 1-dimensional case, when b is bounded and measurable. The latter work is a significant development in the theory of SDE's. Subsequently, the result was generalized by A.Y. Veretennikov [39] to the multi-dimensional case. More recently, N.V. Krylov and M. Röckner employed local integrability criteria on the drift coefficient b to obtain unique strong solutions of (3.1) by using an argument of N. I. Portenko [33]. An alternative approach, which doesn't rely on a pathwise uniqueness argument and which also yields the Malliavin differentiability of solutions to (3.1) was recently developed in [27], [26]. We also refer to the recent article [5] for an extension of the previous results to a Hilbert space setting. In [5], the authors employ techniques based on solutions of infinite-dimensional Kolmogorov equations.

Another important issue in the study of SDE's with (bounded) measurable coefficients is the regularity of their solutions with respect to the initial data and the existence of stochastic flows. See [22], [28] for more information on the existence and regularity of stochastic flows for SDE's, and [29], [30] in the case of stochastic differential systems with memory. Using the method of stochastic characteristics, stochastic flows may be employed to prove uniqueness of solutions of stochastic transport equations under weak regularity hypotheses on the drift coefficient b . See for example [14], where the authors use estimates of solutions of backward Kolmogorov equations to show the existence of a stochastic flow of diffeomorphisms with α' -Hölder continuous derivatives for $\alpha' < \alpha$, where $b \in C([0, 1]; C_b^\alpha(\mathbb{R}^d))$, and $C_b^\alpha(\mathbb{R}^d)$ is the space of bounded α -Hölder continuous functions. A similar result also holds true, when $b \in L^q([0, 1]; L^p(\mathbb{R}^d))$ for p, q such that $p \geq 2, q > 2, \frac{d}{p} + \frac{2}{q} < 1$. See [12]. Here the authors

construct, for any $\alpha \in (0, 1)$, a stochastic flow of α -Hölder continuous homeomorphisms for the SDE (3.1). Furthermore, it is shown in [12] that the map

$$\mathbb{R}^d \ni x \mapsto X^x \in L^p([0, 1] \times \Omega; \mathbb{R}^d)$$

is differentiable in the $L^p(\Omega)$ -sense for every $p \geq 2$.

The approach used in [12] is based on a Zvonkin-type transformation [41] and estimates of solutions of an associated backward parabolic PDE. We also mention the recent related works [11], [10] and [2]. For an overview of this topic the reader may also consult the book [8].¹ In this connection, it should be noted that our method for constructing a stochastic flow for the SDE (3.1) is heavily dependent on Malliavin calculus ideas together with some difficult probabilistic estimates (cf. [26]).

Our paper is organized as follows: In Section, 2 we introduce basic definitions and notations and provide some auxiliary results that are needed to prove the existence of a Sobolev differentiable stochastic flow for the SDE (3.1). See Theorem 3.3 and Corollary 3.5 in Section 2. We also briefly discuss a specific extension of this result to SDE's with multiplicative noise. In Section 3 we give an application of our approach to the construction of a unique Sobolev differentiable solution to the (Stratonovich) stochastic transport equation (3.27). Ideas developed in Section 2 are used in Section 4 to show the existence and regularity of a deterministic flow for the one-dimensional ODE (3.5), and a perfect cocycle for the one-dimensional SDE (3.6).

3.2 Existence of a Sobolev Differentiable Stochastic Flow

Throughout this paper we denote by $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$, $t \in \mathbb{R}$, d -dimensional Brownian motion on the complete Wiener space $(\Omega, \mathcal{F}, \mu)$ where $\Omega := C(\mathbb{R}; \mathbb{R}^d)$ is given the compact open topology and $\mathcal{F}$ is its μ -completed Borel σ -field with respect to Wiener measure μ .

In order to describe the cocycle associated with the stochastic flow of our SDE, we define the μ -preserving (ergodic) Wiener shift $\theta(t, \cdot) : \Omega \rightarrow \Omega$ by

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad \omega \in \Omega, \quad t, s \in \mathbb{R}.$$

The Brownian motion is then a *perfect helix* with respect to θ : That is

$$B_{t_1+t_2}(\omega) - B_{t_1}(\omega) = B_{t_2}(\theta(t_1, \omega))$$

for all $t_1, t_2 \in \mathbb{R}$ and all $\omega \in \Omega$. The above helix property is a convenient pathwise expression of the fact that Brownian motion B has stationary ergodic increments.

Our main focus of study in this section is the d -dimensional SDE

$$X_t^{s,x} = x + \int_s^t b(u, X_u^{s,x}) du + B_t - B_s, \quad s, t \in \mathbb{R}, \quad x \in \mathbb{R}^d, \quad (3.7)$$

where the drift coefficient $b : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bounded Borel measurable function.

It is known that the above SDE has a unique strong global solution $X^{s,x}$ for each $x \in \mathbb{R}^d$ ([39] or [26], [27]).

Here, we will establish the existence of a *Sobolev-differentiable* stochastic flow of diffeomorphisms for the SDE (3.7).

¹After completing the preparation of this article, personal communication with F. Flandoli indicated work in preparation with E. Fedrizzi [13] on similar issues regarding the regularity of stochastic flows for SDE's, using a different approach.

Definition 3.1 A map $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x, \omega) \mapsto \phi_{s,t}(x, \omega) \in \mathbb{R}^d$ is a **stochastic flow of homeomorphisms** for the SDE (3.7) if there exists a universal set $\Omega^* \in \mathcal{F}$ of full Wiener measure such that for all $\omega \in \Omega^*$, the following statements are true:

- (i) For any $x \in \mathbb{R}^d$, the process $\phi_{s,t}(x, \omega)$, $s, t \in \mathbb{R}$, is a strong global solution to the SDE (3.7).
- (ii) $\phi_{s,t}(x, \omega)$ is continuous in $(s, t, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.
- (iii) $\phi_{s,t}(\cdot, \omega) = \phi_{u,t}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega)$ for all $s, u, t \in \mathbb{R}$.
- (iv) $\phi_{s,s}(x, \omega) = x$ for all $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$.
- (v) $\phi_{s,t}(\cdot, \omega) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are homeomorphisms for all $s, t \in \mathbb{R}$.

A stochastic flow $\phi_{s,t}(\cdot, \omega)$ of homeomorphisms is said to be *Sobolev-differentiable* if for all $s, t \in \mathbb{R}$, the maps $\phi_{s,t}(\cdot, \omega)$ and $\phi_{s,t}^{-1}(\cdot, \omega)$ are Sobolev-differentiable in the sense described below.

From now on we use $|\cdot|$ to denote the norm of a vector in $\mathbb{R}^d$ or a matrix in $\mathbb{R}^{d \times d}$.

In order to prove the existence of a Sobolev differentiable flow for the SDE (3.7), we need to introduce a suitable class of weighted Sobolev spaces. Fix $p \in (1, \infty)$ and let $w : \mathbb{R}^d \rightarrow (0, \infty)$ be a Borel measurable function satisfying

$$\int_{\mathbb{R}^d} (1 + |x|^p) w(x) dx < \infty. \quad (3.8)$$

Let $L^p(\mathbb{R}^d, w)$ denote the Banach space of all Borel measurable functions $u = (u_1, \dots, u_d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |u(x)|^p w(x) dx < \infty, \quad (3.9)$$

and equipped with the norm

$$\|u\|_{L^p(\mathbb{R}^d, w)} := \left[\int_{\mathbb{R}^d} |u(x)|^p w(x) dx \right]^{1/p}.$$

Furthermore, denote by $W^{1,p}(\mathbb{R}^d, w)$ the linear space of functions $u \in L^p(\mathbb{R}^d, w)$ with weak partial derivatives $D_j u \in L^p(\mathbb{R}^d, w)$ for $j = 1, \dots, d$. We equip this space with the complete norm

$$\|u\|_{1,p,w} := \|u\|_{L^p(\mathbb{R}^d, w)} + \sum_{i,j=1}^d \|D_j u_i\|_{L^p(\mathbb{R}^d, w)}. \quad (3.10)$$

We will show that the strong solution $X_t^{s,\cdot}$ of the SDE (3.7) is in $L^2(\Omega, L^p(\mathbb{R}^d, w))$ when $p > 1$ (see Corollary 3.13). In fact, the SDE (3.7) implies the following estimate:

$$|X_t^{s,x}|^p \leq c_p(|x|^p + |t - s|^p \|b\|_\infty^p + |B_t - B_s|^p).$$

for all $s, t \in \mathbb{R}$, $x \in \mathbb{R}^d$.

On the other hand, it is easy to see that the solutions $X_t^{s,\cdot}$ of SDE (3.7) are in general not in $L^p(\mathbb{R}^d, dx)$ with respect to Lebesgue measure dx on $\mathbb{R}^d$: Just consider the special trivial case $b \equiv 0$. This implies that solutions of the SDE (3.7) (if they exist) may not belong to the Sobolev space $W^{1,p}(\mathbb{R}^d, dx)$, $p > 1$. However, we will show that such solutions do indeed belong to the weighted Sobolev spaces $W^{1,p}(\mathbb{R}^d, w)$ for $p \geq 1$.

Remark 3.2 (i) Let $w : \mathbb{R}^d \rightarrow (0, \infty)$ be a weight function in Muckenhoupt's A_p -class ($1 < p < \infty$), that is a locally (Lebesgue) integrable function on $\mathbb{R}^d$ such that

$$\sup \left(\frac{1}{\lambda_d(B)} \int_B w(x) dx \right) \left(\frac{1}{\lambda_d(B)} \int_B (w(x))^{1/(1-p)} dx \right)^{p-1} =: c_{w,p} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$ and λ_d is Lebesgue measure on $\mathbb{R}^d$. For example the function $w(x) = |x|^\gamma$ is an A_p -weight iff $-d < \gamma < d(p-1)$. Other examples of weights are given by positive superharmonic functions. See e.g. [18] and [21] and the references therein. Denote by $H^{1,p}(\mathbb{R}^d, w)$ the completion of $C^\infty(\mathbb{R}^d)$ with respect to the norm $\|\cdot\|_{1,p,w}$ in (3.10). If w is a A_p -weight, then we have

$$W^{1,p}(\mathbb{R}^d, w) = H^{1,p}(\mathbb{R}^d, w)$$

for all $1 < p < \infty$. See e.g. [18].

(ii) Let $p_0 = \inf\{q > 1 : w \text{ is a } A_q\text{-weight}\}$ and let $u \in W^{1,p}(\mathbb{R}^d, w)$. If $p_0 < p/d$, then u is locally Hölder continuous with any exponent α such that $0 < \alpha < 1 - dp_0/p$.

We now state our main result in this section which gives the existence of a Sobolev differentiable stochastic flow for the SDE (3.7).

Theorem 3.3 In the SDE (3.7), assume that the drift coefficient b is Borel-measurable and bounded. Then the SDE (3.7) has a Sobolev differentiable stochastic flow $\phi_{s,t} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $s, t \in \mathbb{R}$: That is

$$\phi_{s,t}(\cdot) \text{ and } \phi_{s,t}^{-1}(\cdot) \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w))$$

for all $s, t \in \mathbb{R}$ and all $p > 1$.

Remark 3.4 If w is a A_p -weight then it follows from Remark 3.2 (ii) that a version of $\phi_{s,t}(\cdot)$ is locally Hölder continuous for all $0 < \alpha < 1$ and all s, t .

The following corollary is a consequence of Theorem 3.3 and the helix property of the Brownian motion.

Corollary 3.5 Consider the autonomous SDE

$$X_t^{s,x} = x + \int_s^t b(X_u^{s,x}) du + B_t - B_s, \quad s, t \in \mathbb{R}, \quad (3.11)$$

with bounded Borel-measurable drift $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Then the stochastic flow of the SDE (3.11) has a version which generates a perfect Sobolev-differentiable cocycle $(\phi_{0,t}, \theta(t, \cdot))$ where $\theta(t, \cdot) : \Omega \rightarrow \Omega$ is the μ -preserving Wiener shift. More specifically, the following perfect cocycle property holds **for all** $\omega \in \Omega$ and all $t_1, t_2 \in \mathbb{R}$:

$$\phi_{0,t_1+t_2}(\cdot, \omega) = \phi_{0,t_2}(\cdot, \theta(t_1, \omega)) \circ \phi_{0,t_1}(\cdot, \omega)$$

We will prove Theorem 3.3 through a sequence of lemmas and propositions. We begin by stating our main proposition:

Proposition 3.6 *Let $b : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be bounded and measurable. Let U be an open and bounded subset of $\mathbb{R}^d$. For each $t \in \mathbb{R}$ and $p > 1$ we have*

$$X_t \in L^2(\Omega; W^{1,p}(U))$$

We will prove Proposition 3.6 using two steps. In the **first step**, we show that for a bounded smooth function $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ with compact support, it is possible to estimate the norm of X_t in $L^2(\Omega, W^{1,p}(U))$ independently of the size of b' , with the estimate depending only on $\|b\|_\infty$. To do this we use the same technique as introduced in [26].

In the **second step**, we will approximate our bounded measurable coefficient b by a sequence $\{b_n\}_{n=1}^\infty$ of smooth compactly supported functions as in step 1. We then show that the corresponding sequence $X_t^{n,\cdot}$ of solutions is relatively compact in $L^2(\Omega)$ when integrated against a test function on $\mathbb{R}^d$. By step 1 we use weak compactness of the above sequence in $L^2(\Omega, W^{1,p}(U))$ to conclude that the limit point X_t of the above sequence must also lie in this space.

We now turn to the first step of our procedure. Note that if b is a compactly supported smooth function, the corresponding solution of the SDE (3.1) is (strongly) differentiable with respect to x , and the first order spatial Jacobian $\frac{\partial}{\partial x} X_t^x$ satisfies the linearized random ODE

$$\begin{cases} \frac{d}{dt} \frac{\partial}{\partial x} X_t^x = b'(t, X_t^x) \frac{\partial}{\partial x} X_t^x \\ \frac{\partial}{\partial x} X_0^x = \mathcal{I}_d \end{cases}, \quad (3.12)$$

where $\mathcal{I}_d$ is the $d \times d$ identity matrix and $b'(t, x) = \left(\frac{\partial}{\partial x_i} b^{(j)}(t, x) \right)_{1 \leq i, j \leq d}$ denotes the spatial Jacobian derivative of b .

A key estimate in the first step of the argument is provided by the following proposition:

Proposition 3.7 *Assume that b is a smooth function with compact support. Then for any $p \in [1, \infty)$ and $t \in \mathbb{R}$, we have the following estimate for the solution of the linearized equation (3.12):*

$$\sup_{x \in \mathbb{R}^d} E \left[\left| \frac{\partial}{\partial x} X_t^x \right|^p \right] \leq C_{d,p} (\|b\|_\infty)$$

where $C_{d,p}$ is an increasing continuous function depending only on d and p .

The proof of Proposition 3.7 relies on the following sequence of lemmas which provide estimates on expressions depending on the Gaussian distribution and its derivatives. To this end we define $P(t, z) := (2\pi t)^{d/2} e^{-|z|^2/2t}$, $t > 0$, where $|z|$ is the Euclidean norm of a vector $z \in \mathbb{R}^d$.

Lemma 3.8 *Let $\phi, h : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable functions such that $|\phi(s, z)| \leq e^{-|z|^2/3s}$ and $\|h\|_\infty \leq 1$. Also let $\alpha, \beta \in \{0, 1\}^d$ be multiindices such that $|\alpha| = |\beta| = 1$. Then there exists a universal constant C (independent of ϕ, h, α and β) such that*

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi(s, z) h(t, y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \right| \leq C.$$

Furthermore, there is a universal positive constant (also denoted by) C such that for measurable functions g and h bounded by 1, we have

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t-s, y-z) dy dz ds dt \right| \leq C$$

and

$$\left| \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) D^\gamma P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) dy dz ds dt \right| \leq C.$$

Proof.

We will only give a proof of the first estimate in the lemma. The proofs of the second and third estimates are left to the reader.

Denote the first integral in the lemma by I . Let $l, m \in \mathbb{Z}^d$ and define $[l, l+1) := [l^{(1)}, l^{(1)}+1) \times \dots \times [l^{(d)}, l^{(d)}+1)$ and similarly for $[m, m+1)$. Truncate the functions ϕ, h by setting $\phi_l(s, z) := \phi(s, z) 1_{[l, l+1)}(z)$ and $h_m(t, y) := h(t, y) 1_{[m, m+1)}(y)$.

In the first integral, we replace ϕ, h by ϕ_l, h_m respectively, and thus define

$$I_{l,m} := \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \phi_l(s, z) h_m(t, y) D^\alpha D^\beta P(t - s, y - z) dy dz ds dt$$

Therefore we can write $I = \sum_{l,m \in \mathbb{Z}^d} I_{l,m}$. Below we let C be a generic constant that may vary from line to line.

Assume $\|l - m\|_\infty := \max_i |l^{(i)} - m^{(i)}| \geq 2$. For $z \in [l, l+1)$ and $y \in [m, m+1)$ we have $|z - y| \geq \|l - m\|_\infty - 1$. If $\alpha \neq \beta$ we have that

$$D^\alpha D^\beta P(t - s, z - y) = \frac{(z^{(i)} - y^{(i)})(z^{(j)} - y^{(j)})}{(t - s)^2} P(t - s, y - z)$$

for a suitable choice of i, j . Then we can find C such that

$$|D^\alpha D^\beta P(t - s, z - y)| \leq C e^{-(\|l - m\|_\infty - 2)^2/4}.$$

If $\alpha = \beta$, we have

$$(D^\alpha)^2 P(t - s, y - z) = \left(\frac{(y^{(i)} - z^{(i)})^2}{t - s} - 1 \right) \frac{P(t - s, y - z)}{t - s}$$

and similarly we find C such that

$$|(D^\alpha)^2 P(t - s, y - z)| \leq C e^{-(\|l - m\|_\infty - 2)^2/4}.$$

In both cases we have $|I_{l,m}| \leq C e^{-|l|^2/8} e^{-(\|l - m\|_\infty - 2)^2/4}$ and it follows that

$$\sum_{\|l - m\|_\infty \geq 2} |I_{l,m}| \leq C.$$

Assume $\|l - m\|_\infty \leq 1$ and let $\hat{\phi}_l(s, u)$ and $\hat{h}_m(t, u)$ be the Fourier transform in the second variable, defined by

$$\hat{h}_m(t, u) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} h(t, x) e^{-i(u, x)} dx$$

and similarly for $\hat{\phi}_l(s, u)$. By the Plancherel theorem we have that

$$\int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 du = \int_{\mathbb{R}^d} \phi_l(s, z)^2 dz \leq C e^{-|l|^2/6}$$

for all $s \in [0, 1]$ and

$$\int_{\mathbb{R}^d} \hat{h}_m(t, u)^2 du = \int_{\mathbb{R}^d} h_m(t, y)^2 dy \leq 1.$$

We can write

$$I_{l,m} = \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u) \hat{h}_m(t, -u) u^{(i)} u^{(j)} (t-s) e^{-(t-s)|u|^2/2} du ds dt. \quad (3.13)$$

To see this, start with the right hand side. Then we have by Fubini's theorem

$$\begin{aligned} & \int_{\mathbb{R}^d} \hat{h}_m(t, -u) \hat{\phi}_l(s, u) u^i u^j (t-s) e^{-(t-s)|u|^2/2} du \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, x) e^{i(u,x)} \phi_l(s, y) e^{-i(u,y)} u^i u^j (t-s) e^{-(t-s)|u|^2/2} du dx dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_m(t, x) \phi_l(s, y) (t-s) \left[(2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(u,x-y)} u^i u^j e^{-(t-s)|u|^2/2} du \right] dx dy. \end{aligned}$$

Now look at the expression in the square brackets. Substitute $v = \sqrt{t-s}u$ to get

$$\begin{aligned} & (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(u,x-y)} u^i u^j e^{-(t-s)|u|^2/2} du \\ &= (2\pi)^{-d} (t-s)^{-d/2} \int_{\mathbb{R}^d} e^{i(\frac{v}{\sqrt{t-s}}, x-y)} \frac{v^i}{\sqrt{t-s}} \frac{v^j}{\sqrt{t-s}} e^{-|v|^2/2} dv \\ &= (2\pi)^{-d} (t-s)^{-d/2} (t-s)^{-1} \int_{\mathbb{R}^d} e^{i(v, \frac{x-y}{\sqrt{t-s}})} v^i v^j e^{-|v|^2/2} dv. \end{aligned}$$

Now put $f(v) = e^{-|v|^2/2}$ and $p(v) = v^{(i)} v^{(j)}$. From properties of the Fourier transform we know that $\widehat{pf} = D^\alpha D^\beta \hat{f}$ and $\hat{f} = f$. This gives that the above expression is equal to

$$(2\pi)^{-d/2} (t-s)^{-d/2} (t-s)^{-1} D^\alpha D^\beta f \left(\frac{x-y}{\sqrt{t-s}} \right) = (t-s)^{-1} D^\alpha D^\beta P(t-s, x-y).$$

This gives the equation (3.13).

Applying the inequality $ab \leq \frac{1}{2}a^2c + \frac{1}{2}b^2c^{-1}$ to (3.13) with $a = \hat{\phi}_l(s, u)u^{(i)}$, $b = \hat{h}_m(t, -u)u^{(j)}$ and $c = e^{|l|^2/12}$ we get

$$\begin{aligned} |I_{l,m}| &\leq \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 (u^{(i)})^2 e^{|l|^2/12} e^{-(t-s)|u|^2/2} du ds dt \\ &\quad + \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 (u^{(j)})^2 e^{-|l|^2/12} e^{-(t-s)|u|^2/2} du ds dt \\ &\leq \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 |u|^2 e^{|l|^2/12} e^{-(t-s)|u|^2/2} du ds dt \\ &\quad + \frac{1}{2} \int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 |u|^2 e^{-|l|^2/12} e^{-(t-s)|u|^2/2} du ds dt. \end{aligned}$$

For the first term, integrate first with respect to t in order to get

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{\phi}_l(s, u)^2 |u|^2 e^{-|l|^2/12} e^{-(t-s)|u|^2/2} du ds dt \leq C e^{-|l|^2/12}$$

and for the second term, integrate with respect to s first to get

$$\int_{1/2}^1 \int_{t/2}^t \int_{\mathbb{R}^d} \hat{h}_m(t, -u)^2 |u|^2 e^{-|l|^2/12} e^{-(t-s)|u|^2/2} du ds dt \leq C e^{-|l|^2/12}$$

which gives $|I_{l,m}| \leq C e^{-|l|^2/12}$ and hence

$$\sum_{\|l-m\|_\infty \leq 1} |I_{l,m}| \leq C.$$

■

Using the previous lemma we can show the following:

Lemma 3.9 *Let $g, h : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be Borel-measurable and bounded by 1 and let $r \geq 0$. As before we let α, β, γ be multiindexes with length 1. There exists a universal constant C such that*

$$\left| \int_{t_0}^t \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(t_2, z) P(t_2 - t_0, z) h(t_1, y) D^\alpha D^\beta P(t_1 - t_2, y - z) (t - t_1)^r dy dz dt_2 dt_1 \right| \leq C(1+r)^{-1} (t - t_0)^{r+1}$$

and

$$\left| \int_{t_0}^t \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(t_2, z) D^\gamma P(t_2 - t_0, z) h(t_1, y) D^\alpha D^\beta P(t_1 - t_2, y - z) (t - t_1)^r dy dz dt_2 dt_1 \right| \leq C(1+r)^{-1/2} (t - t_0)^{r+1/2}.$$

Proof.

We begin by proving the first estimate in the lemma for $t = 1, t_0 = 0$. The following estimate holds for each integer $k \geq 0$:

$$\left| \int_{2^{-k-1}}^{2^{-k}} \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \leq C(1 - 2^{-k-1})^r 2^{-k}.$$

To see this, use the fact $P(at, z) = a^{-d/2} P(t, a^{-1/2}z)$ and make the following substitutions in the second estimate in Lemma 3.8: $t' := 2^k t$ and $s' := 2^k s$, $z' := 2^{k/2} z$ and $y' := 2^{k/2} y$, $\tilde{h}(t, y) := \frac{(1-t)^r}{(1-2^{-k-1})^r} h(t, y)$.

Summing the above inequalities over k gives

$$\left| \int_0^1 \int_{t/2}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t - s, y - z) (1 - t)^r dy dz ds dt \right| \leq C(1+r)^{-1}.$$

Moreover, it is easy to see that

$$\left| \int_0^1 \int_0^{t/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(s, z) P(s, z) h(t, y) D^\alpha D^\beta P(t-s, y-z) (1-t)^r dy dz ds dt \right| \\ \leq C \int_0^1 \int_0^{t/2} (t-s)^{-1} (1-t)^r ds dt \leq C(1+r)^{-1},$$

and combining these bounds gives the first assertion of the lemma for $t = 1$, $t_0 = 0$. For general t and t_0 use the change of variables $t'_1 := \frac{t_1-t_0}{t-t_0}$, $t'_2 := \frac{t_2-t_0}{t-t_0}$, $y' := (t-t_0)^{-1/2}y$ and $z' := (t-t_0)^{-1/2}z$.

The second assertion of the lemma is proved similarly. ■

We now turn to the following key estimate (cf. [6, Proposition 2.2]):

Lemma 3.10 *Let B be a d -dimensional Brownian Motion starting from the origin and $b_1, \dots, b_n$ be compactly supported continuously differentiable functions $b_i : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ for $i = 1, 2, \dots, n$. Let $\alpha_i \in \{0, 1\}^d$ be a multiindex such that $|\alpha_i| = 1$ for $i = 1, 2, \dots, n$. Then there exists a universal constant C (independent of $\{b_i\}_i$, n , and $\{\alpha_i\}_i$) such that*

$$\left| E \left[\int_{t_0 < t_1 < \dots < t_n < t} \left(\prod_{i=1}^n D^{\alpha_i} b_i(t_i, x + B_{t_i}) \right) dt_1 \dots dt_n \right] \right| \\ \leq \frac{C^n \prod_{i=1}^n \|b_i\|_\infty (t-t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)} \quad (3.14)$$

where Γ is the Gamma-function and $x \in \mathbb{R}^d$. Here D^{α_i} denotes the partial derivative with respect to the j 'th space variable, where j is the position of the 1 in α_i .

Proof. Without loss of generality, assume that $\|b_i\|_\infty \leq 1$ for $i = 1, 2, \dots, n$. Using the Gaussian density we write the left hand side of the estimate (3.14) in the form

$$\left| \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, x + z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n \right|.$$

Introduce the notation

$$J_n^\alpha(t_0, t, z_0) = \int_{t_0 < t_1 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n D^{\alpha_i} b_i(t_i, x + z_i) P(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^{nd}$. We shall show that $|J_n^\alpha(t_0, t, z_0)| \leq C^n (t-t_0)^{n/2} / \Gamma(n/2+1)$, thus proving the proposition.

To do this, we will use integration by parts to shift the derivatives from the b_i 's onto the Gaussian kernel. This will be done by introducing the alphabet

$$\mathcal{A}(\alpha) = \{P, D^{\alpha_1} P, \dots, D^{\alpha_n} P, D^{\alpha_1} D^{\alpha_2} P, \dots, D^{\alpha_{n-1}} D^{\alpha_n} P\}$$

where D^{α_i} , $D^{\alpha_i} D^{\alpha_{i+1}}$ denotes the derivatives in z of $P(t, z)$.

Take a string $S = S_1 \cdots S_n$ in $\mathcal{A}(\alpha)$ and define

$$I_S^\alpha(t_0, t, z_0) = \int_{t_0 < \dots < t_n < t} \int_{\mathbb{R}^{dn}} \prod_{i=1}^n b_i(t_i, x + z_i) S_i(t_i - t_{i-1}, z_i - z_{i-1}) dz_1 \dots dz_n dt_1 \dots dt_n.$$

We will need only a special type of strings: Say that a string is *allowed* if, when all the $D^{\alpha_i} P$'s are removed from the string, a string of the form $P \cdot D^{\alpha_s} D^{\alpha_{s+1}} P \cdot P \cdot D^{\alpha_{s+1}} D^{\alpha_{s+2}} P \dots P \cdot D^{\alpha_r} D^{\alpha_{r+1}} P$ for $s \geq 1$, $r \leq n-1$ remains. Also, we will require that the first derivatives $D^{\alpha_i} P$ are written in an increasing order with respect to i .

We now claim that

$$J_n^\alpha(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} \epsilon_j I_{S^j}^\alpha(t_0, t, z_0)$$

where each ϵ_j is either -1 or 1 and each S^j is an allowed string in $\mathcal{A}(\alpha)$. To see this, we proceed by induction on $n \geq 1$:

The claim obviously holds for $n = 1$. Assume that it holds for $n \geq 1$, and let b_0 be another function satisfying the requirements of the lemma. Likewise with α_0 . Then

$$\begin{aligned} J_{n+1}^{(\alpha_0, \alpha)}(t_0, t, z_0) &= \int_{t_0}^t \int_{\mathbb{R}^d} D^{\alpha_0} b_0(t_1, x + z_1) P(t_1 - t_0, z_1 - z_0) J_n^\alpha(t_1, t, z_1) dz_1 dt_1 \\ &= - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1, x + z_1) D^{\alpha_0} P(t_1 - t_0, z_1 - z_0) J_n^\alpha(t_1, t, z_1) dz_1 dt_1 \\ &\quad - \int_{t_0}^t \int_{\mathbb{R}^d} b_0(t_1, x + z_1) P(t_1 - t_0, z_1 - z_0) D^{\alpha_0} J_n^\alpha(t_1, t, z_1) dz_1 dt_1. \end{aligned}$$

Notice that

$$D^{\alpha_0} I_S^\alpha(t_1, t, z_1) = -I_{\tilde{S}}^{(\alpha_0, \alpha)}(t_1, t, z_1)$$

where

$$\tilde{S} = \begin{cases} D^{\alpha_0} P \cdot S_2 \cdots S_n & \text{if } S = P \cdot S_2 \cdots S_n \\ D^{\alpha_0} D^{\alpha_1} P \cdot S_2 \cdots S_n & \text{if } S = D^{\alpha_1} P \cdot S_2 \cdots S_n. \end{cases}$$

Here, $\tilde{S}$ is not an allowed string in $\mathcal{A}(\alpha)$. So from the induction hypothesis $D^{\alpha_0} J_n^\alpha(t_0, t, z_0) = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{\tilde{S}}^{(\alpha_0, \alpha)}(t_0, t, z_0)$ this gives

$$J_{n+1}^{(\alpha_0, \alpha)} = \sum_{j=1}^{2^{n-1}} -\epsilon_j I_{D^{\alpha_0} P \cdot S^j}^{(\alpha_0, \alpha)} + \sum_{j=1}^{2^{n-1}} \epsilon_j I_{P \cdot \tilde{S}^j}.$$

It is easily checked that when S^j is an allowed string in $\mathcal{A}(\alpha)$, both $D^{\alpha_0} P \cdot S^j$ and $P \cdot \tilde{S}^j$ are allowed strings in $\mathcal{A}(\alpha_0, \alpha)$.

This proves the claim.

For the rest of the proof of Lemma 3.10 we will bound I_S^α when S is an allowed string, i.e. we will show that there is a positive constant M such that

$$I_S^\alpha(t_0, t, z_0) \leq \frac{M^n (t - t_0)^{n/2}}{\Gamma(\frac{n}{2} + 1)}.$$

for all integers $n \geq 1$ and for each allowed string S in the alphabet $\mathcal{A}(\alpha)$.

We proceed by induction on $n \geq 0$: The case $n = 0$ is immediate, so assume $n > 0$ and that this holds for all allowed strings of length less than n . We divide into three cases:

1. $S = D^{\alpha_1} P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_2, \dots, \alpha_n)$
2. $S = P \cdot D^{\alpha_1} D^{\alpha_2} P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_3, \dots, \alpha_n)$
3. $S = P \cdot D^{\alpha_1} P \dots D^{\alpha_m} P \cdot D^{\alpha_{m+1}} D^{\alpha_{m+2}} P \cdot S'$ where S' is a string in $\mathcal{A}(\alpha')$ and $\alpha' := (\alpha_{m+3}, \dots, \alpha_n)$.

In all the above cases, S' is an allowed string in the alphabet.

1. We use the inductive hypothesis to bound $I_{S'}^{\alpha'}(t_1, t, z_1)$ and the bound

$$\int_{\mathbb{R}^d} |D^\alpha P(t, z)| dz \leq C t^{-1/2} \quad (3.15)$$

to get

$$\begin{aligned} |I_S^\alpha(t_0, t, z_0)| &= \left| \int_{t_0}^t \int_{\mathbb{R}^d} b_1(t_1, z_1) D^{\alpha_1} P(t_1 - t_0, z_1 - z_0) I_{S'}^{\alpha'}(t_1, t, z_1) dz_1 dt_1 \right| \\ &\leq \frac{M^{n-1}}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} \int_{\mathbb{R}^d} |D^{\alpha_1} P(t_1 - t_0, z_1 - z_0)| dz_1 dt_1 \\ &\leq \frac{M^{n-1} C}{\Gamma(\frac{n+1}{2})} \int_{t_0}^t (t - t_1)^{(n-1)/2} (t_1 - t_0)^{-1/2} dt_1 \\ &= \frac{M^{n-1} C \sqrt{\pi} (t - t_0)^{k/2}}{\Gamma(\frac{n}{2} + 1)}. \end{aligned}$$

The result follows if $M \geq \max\{C\sqrt{\pi}, 1\}$.

2. For this case we can write

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0}^t \int_{t_1}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b_1(t_1, z_1) b_2(t_2, z_2) \\ &\quad \times P(t_1 - t_0, z_1 - z_0) D^{\alpha_1} D^{\alpha_2} P(t_2 - t_1, z_2 - z_1) I_{S'}^{\alpha'}(t_2, t, z_2) dz_1 dz_2 dt_2 dt_1. \end{aligned}$$

We set $h(t_2, z_2) := b_2(t_2, z_2) I_{S'}^{\alpha'}(t_2, z_2) (t - t_2)^{1-n/2}$ so that by the inductive hypothesis we have

$$\|h\|_\infty \leq M^{n-2} / \Gamma(n/2).$$

Use the above estimate in the first assertion of Lemma 3.9 with $g = b_1$ and integrate with respect to t_2 first, to get

$$|I_S^\alpha(t_0, t, z_0)| \leq \frac{C M^{n-2} (t - t_0)^{n/2}}{n \Gamma(n/2)},$$

and the result follows if $M \geq \max\{C, 1\}$.

3. We have

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0 < \dots < t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^{m+2} b_j(t_j, z_j) \\ &\quad \times \prod_{j=2}^m D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) \\ &\quad \times I_{S'}^{\alpha'}(t_{m+2}, t, z_{m+2}) dz_1 \dots dz_{m+2} dt_1 \dots dt_{m+2}. \end{aligned}$$

Set $h(t_{m+2}, z_{m+2}) := b_{m+2}(t_{m+2}, z_{m+2})I_{S'}^{\alpha'}(t_{m+2}, t, z)(t - t_{m+2})^{(2+m-n)/2}$. Then from the inductive hypothesis we have $\|h\|_\infty \leq M^{n-m-2}/\Gamma((n-m)/2)$. Define

$$\begin{aligned} A(t_m, z_m) &:= \int_{t_m}^t \int_{t_{m+1}}^t \int_{\mathbb{R}^{2d}} b_{m+1}(t_{m+1}, z_{m+1}) h(t_{m+2}, z_{m+2}) \\ &\quad \times (t - t_{m+2})^{(n-m-2)/2} D^{\alpha_m} P(t_{m+1} - t_m, z_{m+1} - z) \\ &\quad \times D^{\alpha_{m+1}} D^{\alpha_{m+2}} P(t_{m+2} - t_{m+1}, z_{m+2} - z_{m+1}) dz_{m+1} dz_{m+2} dt_{m+1} dt_{m+2}. \end{aligned}$$

Then Lemma 3.9 implies that

$$|A(t_m, z_m)| \leq \frac{C(n-m)^{-1/2} M^{n-m-2} (t - t_m)^{(n-m-1)/2}}{\Gamma(\frac{n-m}{2})}.$$

Using this in

$$\begin{aligned} I_S^\alpha(t_0, t, z_0) &= \int_{t_0 < \dots < t_{m+2} < t} \int_{\mathbb{R}^{(m+2)d}} P(t_1 - t_0, z_1 - z_0) \prod_{j=1}^m b_j(t_j, z_j) \\ &\quad \times \prod_{j=1}^{m-1} D^{\alpha_j} P(t_j - t_{j-1}, z_j - z_{j-1}) \Omega(t_m, z_m) dz_1 \dots dz_m dt_1 \dots dt_m, \end{aligned}$$

and using the bound (3.15) several times gives

$$\begin{aligned} |I_S^\alpha(t_0, t, z_0)| &\leq C^{m+1} (n-m)^{-1/2} \frac{M^{n-m-2}}{\Gamma((n-m)/2)} \\ &\quad \times \int_{t_0 < \dots < t_m < t} (t_2 - t_1)^{-1/2} \dots (t_m - t_{m-1})^{-1/2} (t - t_m)^{(n-m-1)/2} dt_1 \dots dt_m \\ &= C^{m+1} (n-m)^{-1/2} \frac{M^{n-m-2} \pi^{(m-1)/2} \Gamma(\frac{n-m+1}{2})}{\Gamma(\frac{n-m}{2}) \Gamma(\frac{n}{2} + 1)} (t - t_0)^{n/2}. \end{aligned}$$

We can choose M so large that the result holds. This completes the induction argument.

■

We are now ready to complete the proof of Proposition 3.7.

Proof of Proposition 3.7. Let $t \in [0, 1]$. Iterating the linearized equation (3.12) we obtain

$$\frac{\partial}{\partial x} X_t^x = \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, X_{s_1}^x) : \dots : b'(s_n, X_{s_n}^x) ds_1 \dots ds_n.$$

Let $p \in [1, \infty)$ and choose $r, s \in [1, \infty)$ such that $sp = 2^q$ for some integer q and $\frac{1}{r} + \frac{1}{s} = 1$. Then by Girsanov's theorem and Hölder's inequality

$$\begin{aligned} E \left[\left\| \frac{\partial}{\partial x} X_t^x \right\|^p \right] &= E \left[\left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|^p \right. \\ &\quad \times \mathcal{E} \left(\int_0^1 b(u, x + B_u) dB_u \right) \Big] \\ &\leq C_1 (\|b\|_\infty) \left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{sp}(\mu, \mathbb{R}^{d \times d})}^p, \end{aligned}$$

where $\mathcal{E}(\int_0^1 b(u, x + B_u) dB_u)$ is the Doleans-Dade exponential of the martingale

$$\int_0^1 b(u, x + B_u) dB_u = \sum_{j=1}^d \int_0^1 b^{(j)}(u, x + B_u) dB_u^j$$

and C_1 is a continuous increasing function.

Then we obtain

$$\begin{aligned} & E \left| \frac{\partial}{\partial x} X_t^x \right|^p \\ & \leq C_1(\|b\|_\infty) \left\| \mathcal{I}_d + \sum_{n=1}^{\infty} \int_{0 < s_1 < \dots < s_n < t} b'(s_1, x + B_{s_1}) : \dots : b'(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{sp}(\mu, \mathbb{R}^{d \times d})}^p \\ & \leq C_1(\|b\|_\infty) \left(1 + \sum_{n=1}^{\infty} \sum_{i,j=1}^d \sum_{l_1, \dots, l_{n-1}=1}^d \left\| \int_{t < s_1 < \dots < s_n < s} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \dots \right. \right. \\ & \quad \left. \left. \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{ps}(\mu; \mathbb{R})} \right)^p. \end{aligned}$$

Now consider the expression

$$A := \int_{0 < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \dots \frac{\partial}{\partial x_{l_n}} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n.$$

Then, using (deterministic) integration by parts, repeatedly, it is easy to see that A^2 can be written as a sum of at most 2^{2n} terms of the form

$$\int_{0 < s_1 < \dots < s_{2n} < t} g_1(s_1) \dots g_{2n}(s_{2n}) ds_1 \dots ds_{2n}, \quad (3.16)$$

where $g_l \in \left\{ \frac{\partial}{\partial x_j} b^{(i)}(\cdot, x + B_{\cdot}) : 1 \leq i, j \leq d \right\}$, $l = 1, 2, \dots, 2n$. Similarly, by induction it follows that A^{2^q} is the sum of at most $2^{2^q n}$ terms of the form

$$\int_{0 < s_1 < \dots < s_{2^q n} < t} g_1(s_1) \dots g_{2^q n}(s_{2^q n}) ds_1 \dots ds_{2^q n}, \quad (3.17)$$

Combining this with Lemma 3.10, we obtain the following estimate:

$$\begin{aligned} & \left\| \int_{0 < s_1 < \dots < s_n < t} \frac{\partial}{\partial x_{l_1}} b^{(i)}(s_1, x + B_{s_1}) \frac{\partial}{\partial x_{l_2}} b^{(l_1)}(s_2, x + B_{s_2}) \dots \frac{\partial}{\partial x_j} b^{(l_{n-1})}(s_n, x + B_{s_n}) ds_1 \dots ds_n \right\|_{L^{2^q}(\mu; \mathbb{R})} \\ & \leq \left(\frac{2^{2^q n} C^{2^q n} \|b\|_\infty^{2^q n} t^{2^q - 1} n}{\Gamma(2^{q-1} n + 1)} \right)^{2^{-q}} \leq \frac{2^{qn} C^n \|b\|_\infty^n}{((2^{q-1} n)!)^{2^{-q}}}. \end{aligned}$$

Then it follows that

$$\begin{aligned} & E \left[\left\| \frac{\partial}{\partial x} X_t^x \right\|^p \right] \\ & \leq C_1(\|b\|_\infty) \left(1 + \sum_{n=1}^{\infty} \frac{d^{n+2} 2^{qn} C^n \|b\|_\infty^n}{((2^{q-1} n)!)^{2^{-q}}} \right)^p = C_{d,p}(\|b\|_\infty). \end{aligned}$$

The right hand side of this inequality is independent of $x \in \mathbb{R}^d$, and the result follows. ■

For the remainder of the paper we will fix a bounded and measurable $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$. It is proved in [39] and [26] that the corresponding SDE (3.7) has a unique strong solution, denoted $X^{s,x}$. Suppose now $b_n : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a sequence of compactly supported smooth functions. Suppose that $b_n(t, x) \rightarrow b(t, x) dt \times dx$ -a.e. and $|b_n(t, x)| \leq M < \infty$ for all n, t, x for some constant M . Denote by $X^{n,s,x}$ the solution of (3.7) associated with the coefficient $b_n, n \geq 1$. We then have the following

Lemma 3.11 *The sequence $X_t^{n,s,x}$ converges weakly in $L^2(\Omega; \mathbb{R}^d)$ to $X_t^{s,x}$.*

Remark 3.12 *Assume $\psi \in L^\infty([0, 1] \times \Omega; \mathbb{R}^d)$ adapted to the filtration generated by the Brownian motion. Then we can bound the Doleans-Dade exponential $\mathcal{E}(\int_0^1 \psi(u) dB_u)$ in $L^p(\Omega)$ by an increasing continuous function of $\|\psi\|_{L^\infty([0,1] \times \Omega)}$.*

To see this, notice that $M_t := \mathcal{E}(\int_0^t \psi(u) dB_u)$ is the unique solution to the linear SDE

$$dM_t = M(t)\psi(t)dB_t, \quad M_0 = 1.$$

By Itô's formula we get

$$\begin{aligned} E[M_t^p] &= 1 + \frac{p(p-1)}{2} \int_0^t E[M_u^p |\psi(u)|^2] du \\ &\leq 1 + \frac{p(p-1)}{2} \|\psi\|_{L^\infty([0,1] \times \Omega)} \int_0^t E[M_u^p] du \leq \exp \left\{ \frac{tp(p-1)\|\psi\|_{L^\infty([0,1] \times \Omega)}}{2} \right\} \end{aligned}$$

where we have used Grönwall's lemma in the last inequality.

Proof of Lemma 3.11. For simplicity, consider $d = 1$ and $s = 0$. We start by noting that the set

$$\left\{ \mathcal{E} \left(\int_0^1 h(u) dB_u \right) : h \in C_b^1(\mathbb{R}) \right\}$$

spans a dense subspace of $L^2(\Omega; \mathbb{R})$. So it suffices to prove the convergence

$$E[X_t^{n,x} \mathcal{E} \left(\int_0^1 h(u) dB_u \right)] \rightarrow E[X_t^x \mathcal{E} \left(\int_0^1 h(u) dB_u \right)].$$

By the Cameron-Martin theorem we have

$$E[X_t^x \mathcal{E} \left(\int_0^1 h(u) dB_u \right)] = \int_{\Omega} X_t^x(\omega + h) d\mu(\omega).$$

The function $(u, x) \mapsto b(u, x) + h'(u)$ is still bounded, and so $X_t^x(\cdot + h)$ must coincide with the solution to (3.7) when b is replaced by $b + h'$. Hence by uniqueness in law of (3.7) we may write

$$E[X_t^x \mathcal{E} \left(\int_0^1 h(u) dB_u \right)] = E[(x + B_t) \mathcal{E} \left(\int_0^1 b(u, x + B_u) + h'(u) dB_u \right)],$$

and similarly for $X_t^{n,x}$. We thus get

$$E[X_t^{n,x} \mathcal{E} \left(\int_0^1 h(u) dB_u \right)] - E[X_t^x \mathcal{E} \left(\int_0^1 h(u) dB_u \right)]$$

$$= E[(x + B_t) \left(\mathcal{E} \left(\int_0^1 b_n(u, x + B_u) + h'(u) dB_u \right) - \mathcal{E} \left(\int_0^1 b(u, x + B_u) + h'(u) dB_u \right) \right)].$$

Using the inequality $|e^a - e^b| \leq |e^a + e^b||a - b|$, Hölders inequality and Burkholder-Davis-Gundy inequality we find a constant C such that the above is bounded by

$$C \left(E \left[\left(\mathcal{E} \left(\int_0^1 b_n(u, x + B_u) + h'(u) dB_u \right) + \mathcal{E} \left(\int_0^1 b(u, x + B_u) + h'(u) dB_u \right) \right)^4 \right] \right)^{1/4} \times \\ \left(E \left[\left(\int_0^1 (b_n(u, x + B_u) - b(u, x + B_u))^2 du \right)^2 \right] \right)^{1/4} \\ + \left(\int_0^1 (b(u, x + B_u) + h'(u))^2 - (b_n(u, x + B_u) + h'(u))^2 du \right)^4 \right)^{1/4}.$$

From Remark 3.12, since b_n is uniformly bounded we get that $\{\mathcal{E}(\int_0^1 b_n(u, x + B_u) + h'(u) dB_u)\}_n$ is bounded in $L^4(\Omega)$, so that the first factor above is uniformly bounded. The latter factor converges to zero by bounded convergence. ■

In [26] the authors show that the sequence $X_t^{n,s,x}$ is relatively compact in $L^2(\Omega; \mathbb{R}^d)$ (in the strong topology). Thus $\{X_t^{n,x}\}$ has a converging subsequence, which, by Lemma 3.11 has to converge to X_t^x . Since every subsequence of $\{X_t^{n,x}\}$ has a further subsequence converging to the same limit X_t^x , the full sequence is convergent in $L^2(\Omega; \mathbb{R}^d)$ (with strong topology).

As a consequence of Proposition 3.7 and the above discussion we obtain the following result:

Corollary 3.13 *Let $X^{s,x}$ be the unique strong solution to the SDE (3.7) and $q > 1$ an integer. Then there exists a constant $C = C(d, \|b\|_\infty, q) < \infty$ such that*

$$E \left[|X_{t_1}^{s_1, x_1} - X_{t_2}^{s_2, x_2}|^q \right] \leq C(|s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} + |x_1 - x_2|^q)$$

for all $s_1, s_2, t_1, t_2, x_1, x_2$.

In particular, there exists a continuous version of the random field $(s, t, x) \mapsto X_t^{s,x}$ with Hölder continuous trajectories of Hölder constant $\alpha < \frac{1}{2}$ in s, t and $\alpha < 1$ in x , locally (see [22]).

Proof. Retain the above notation. Without loss of generality let $0 \leq s_1 < s_2 < t_1 < t_2$.

Then

$$\begin{aligned}
& X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2} \\
&= x_1 - x_2 + \int_{s_1}^{t_1} b_n(u, X_u^{n,s_1,x_1}) du - \int_{s_2}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \\
&\quad + (B_{t_1} - B_{s_1}) - (B_{t_2} - B_{s_2}) \\
&= x_1 - x_2 + \int_{s_1}^{s_2} b_n(u, X_u^{n,s_1,x_1}) du + \int_{s_2}^{t_1} b_n(u, X_u^{n,s_1,x_1}) du \\
&\quad - \int_{s_2}^{t_1} b_n(u, X_u^{n,s_2,x_2}) du - \int_{t_1}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \\
&\quad + (B_{t_1} - B_{t_2}) + (B_{s_2} - B_{s_1}) \\
&= x_1 - x_2 + \int_{s_1}^{s_2} b_n(u, X_u^{n,s_1,x_1}) du - \int_{t_1}^{t_2} b_n(u, X_u^{n,s_2,x_2}) du \\
&\quad + \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2})) du \\
&\quad + \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2})) du \\
&\quad + (B_{t_1} - B_{t_2}) + (B_{s_2} - B_{s_1}).
\end{aligned}$$

So due to the uniform boundedness of $b_n, n \geq 1$ we get

$$\begin{aligned}
& E[|X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2}|^q] \\
&\leq C_q(|x_1 - x_2|^q + |s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} \\
&\quad E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2})) du\right|^q\right] \\
&\quad + E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2})) du\right|^q\right]). \tag{3.18}
\end{aligned}$$

Using the fact that $X_t^{n,\cdot,s}$ is a stochastic flow of diffeomorphisms (see e.g. [22]), the mean value theorem and Proposition 3.7, we get

$$\begin{aligned}
& E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1}) - b_n(u, X_u^{n,s_1,x_2})) du\right|^q\right] \\
&= |x_1 - x_2|^q E\left[\left|\int_{s_2}^{t_1} \int_0^1 (b_n(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}) \frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)}) d\tau du\right|^q\right] \\
&\leq |x_1 - x_2|^q \int_0^1 E\left[\left|\int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_1+\tau(x_2-x_1)}) \frac{\partial}{\partial x} X_u^{n,s_1,x_1+\tau(x_2-x_1)}) du\right|^q\right] d\tau \\
&= |x_1 - x_2|^q \int_0^1 E\left[\left|\frac{\partial}{\partial x} X_{t_1}^{n,s_1,x_1+\tau(x_2-x_1)} - \frac{\partial}{\partial x} X_{s_2}^{n,s_1,x_1+\tau(x_2-x_1)}\right|^q\right] d\tau \\
&\leq C_q |x_1 - x_2|^q \sup_{t \in [s_1, 1], x \in \mathbb{R}^d} E\left[\left|\frac{\partial}{\partial x} X_{t_1}^{n,s_1,x}\right|^q\right] \\
&\leq C_{d,q}(\|b\|_\infty) |x_1 - x_2|^q. \tag{3.19}
\end{aligned}$$

Finally we observe that estimation of the last term of the right hand side of (3.18) can be reduced to the previous case (3.19) by applying the Markov property, since

$$\begin{aligned}
& E \left[\left| \int_{s_2}^{t_1} (b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2})) du \right|^q \right] \\
& \leq \int_{s_2}^{t_1} E[|b_n(u, X_u^{n,s_1,x_2}) - b_n(u, X_u^{n,s_2,x_2})|^q] du \\
& = \int_{s_2}^{t_1} E[E[|b_n(u, X_u^{n,s_2,y}) - b_n(u, X_u^{n,s_2,x_2})|^q]_{y=X_{s_2}^{n,s_1,x_2}}] du \\
& \leq CE[|X_{s_2}^{n,s_1,x_2} - x_2|^q] = CE[|X_{s_2}^{n,s_1,x_2} - X_{s_1}^{n,s_1,x_2}|^q] \\
& \leq M_q |s_2 - s_1|^{q/2}
\end{aligned}$$

for a positive constant $M_q < \infty$.

Therefore, we have

$$E[|X_{t_1}^{n,s_1,x_1} - X_{t_2}^{n,s_2,x_2}|^q] \leq C_q(|s_1 - s_2|^{q/2} + |t_1 - t_2|^{q/2} + |x_1 - x_2|^q)$$

for a constant C_q independent of n .

To complete the proof, we use the fact that $X_{t_1}^{n,s_1,x_1} \rightarrow X_{t_1}^{s_1,x_1}$ and $X_{t_2}^{n,s_2,x_2} \rightarrow X_{t_2}^{s_2,x_2}$ in $L^2(\mu)$ for $n \rightarrow \infty$ (see [26]) together with Fatou's lemma applied to a.e. convergent subsequences of $\{X_{t_1}^{n,s_1,x_1}\}_{n=1}^\infty$ and $\{X_{t_2}^{n,s_2,x_2}\}_{n=1}^\infty$. ■

This concludes step one of our program.

We now proceed to Step 2. For simplicity we consider $s = 0$, i.e. we look at the sequence $\{X_t^{n,x}\}_{n \geq 1} := \{X_t^{n,0,x}\}_{n \geq 1}$ and $X_t^x := X_t^{0,x}$. The following lemma establishes convergence of the above sequence:

Lemma 3.14 *For any $\varphi \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $t \in [0, 1]$ the sequence*

$$\langle X_t^n, \varphi \rangle = \int_{\mathbb{R}^d} \langle X_t^{n,x}, \varphi(x) \rangle_{\mathbb{R}^d} dx$$

converges to $\langle X_t, \varphi \rangle$ in $L^2(\Omega, \mu)$.

Proof. Denote by D_s the Malliavin derivative (see the Appendix) and by U the compact support of φ . By noting the inequalities

$$\begin{aligned}
E[|D_s \langle X_t^n, \varphi \rangle|^2] &= E[|\langle D_s X_t^n, \varphi \rangle|^2] \\
&\leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2 |U| \sup_{x \in U} E[|D_s X_t^{n,x}|^2]
\end{aligned}$$

and

$$\begin{aligned}
& E[|D_s \langle X_t^n, \varphi \rangle_{L^2(\mathbb{R}^d)} - D_{s'} \langle X_t^n, \varphi \rangle|^2] \\
&= E[|\langle D_s X_t^n - D_{s'} X_t^n, \varphi \rangle|^2] \\
&\leq \|\varphi\|_{L^2(\mathbb{R}^d)}^2 |U| \sup_{x \in U} E[|D_s X_t^{n,x} - D_{s'} X_t^{n,x}|^2]
\end{aligned}$$

we can invoke Corollary 3.33 together with Lemma 3.5 [26] to obtain a subsequence $\langle X_t^{n(k)}, \varphi \rangle$ converging in $L^2(\Omega, \mu)$ as $k \rightarrow \infty$. Denote the limit by $Y(\varphi)$.

Similary to the proof of Lemma 3.11 one can show that $E[\langle X_t^n, \varphi \rangle \mathcal{E}(\int_0^1 h(u) dB_u)]$ converges to $E[\langle X_t, \varphi \rangle \mathcal{E}(\int_0^1 h(u) dB_u)]$ for all $h \in C_b^1(\mathbb{R}; \mathbb{R}^d)$. We then get that $\langle X_t^n, \varphi \rangle$ converges weakly to $\langle X_t, \varphi \rangle$, and so by uniqueness of the limits we can conclude that

$$Y(\varphi) = \langle X_t, \varphi \rangle.$$

To see that the full sequence converges we assume that there exists an $\epsilon > 0$ and a subsequence $\langle X_t^{n(k)}, \varphi \rangle$ such that

$$\|\langle X_t^{n(k)}, \varphi \rangle - \langle X_t, \varphi \rangle\| \geq \epsilon$$

for every k . Applying the above procedure to $\langle X_t^{n(k)}, \varphi \rangle$ gives a further subsequence converging to $\langle X_t, \varphi \rangle$ thus giving a contradiction. ■

We are now able to finalize the proof of Proposition 3.6.

Proof of Proposition 3.6.

Using Proposition 3.7, we have

$$\sup_n E \left[\left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p \right] < \infty.$$

Hence there exists a subsequence of $\frac{\partial}{\partial x} X_t^{n(k),x}$ converging in the weak topology of $L^2(\Omega, L^p(U))$ to an element Y . Then we have for any $A \in \mathcal{F}$ and $\varphi \in C_0^\infty(U; \mathbb{R}^d)$

$$\begin{aligned} E[1_A \langle X_t, \varphi' \rangle] &= \lim_{k \rightarrow \infty} E[1_A \langle X_t^{n(k)}, \varphi' \rangle] \\ &= \lim_{k \rightarrow \infty} -E[1_A \langle \frac{\partial}{\partial x} X_t^{n(k)}, \varphi \rangle] = -E[1_A \langle Y, \varphi \rangle]. \end{aligned}$$

Hence we have for $\varphi \in C_0^\infty$:

$$\langle X_t, \varphi' \rangle = -\langle Y, \varphi \rangle \quad (3.20)$$

P -a.s.. Finally, we need to show that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that X_t has a weak derivative on this subset. To this end choose a sequence $\{\varphi_n\}$ in $C^\infty(U; \mathbb{R}^d)$ dense in $W_0^{1,2}(U; \mathbb{R}^d)$. Choose a measurable subset Ω_n of Ω with full measure such that (3.20) holds on Ω_n with φ replaced by φ_n . Then $\Omega_0 := \cap_{n \geq 1} \Omega_n$ satisfies the desired property. ■

We now return to the weighted Sobolev spaces. Using the same techniques as in the above lemma, we prove the following

Lemma 3.15 *For all $p \in (1, \infty)$ we have*

$$X_t \in L^2(\Omega, W^{1,p}(\mathbb{R}^d, w))$$

Proof. For simplicity, we consider the case $d = 1$. It suffices to show that $E[(\int \frac{\partial}{\partial x} X_t^x |^p w(x) dx)^{2/p}]$ is finite. To this end, let $X_t^{n,x}$ denote the sequence approximating X_t^x as in the previous lemma. Assume first that $p \geq 2$. Then by Hölder's inequality w.r.t. the Wiener measure μ we have

$$E[(\int \frac{\partial}{\partial x} X_t^{n,x} |^p w(x) dx)^{2/p}]$$

$$\leq \left(E \left[\int \left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p w(x) dx \right] \right)^{2/p} \leq \left(\int w(x) dx \right)^{p/2} \left(\sup_{x \in \mathbb{R}} E \left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p \right)^{2/p}.$$

For $1 < p \leq 2$, by Hölder's inequality w.r.t. $w(x)dx$ we have

$$E \left[\left(\int \left| \frac{\partial}{\partial x} X_t^{n,x} \right|^p w(x) dx \right)^{2/p} \right] \leq \left(\int w(x) dx \right)^{(4-p)/2} \sup_{x \in \mathbb{R}^d} E \left[\left| \frac{\partial}{\partial x} X_t^{n,x} \right|^2 \right].$$

In both cases we can find a subsequence converging to an element $Y \in L^2(\Omega, L^p(\mathbb{R}^d, w))$ in the weak topology, in particular for every $A \in \mathcal{F}$ and $f \in L^q(\mathbb{R}^d, w)$ (q is the Sobolev conjugate of p) we have

$$\lim_{k \rightarrow \infty} E[1_A \int \frac{\partial}{\partial x} X_t^{n(k),x} f(x) w(x) dx] = E[1_A \int Y(x) f(x) w(x) dx]$$

by choosing f such that $fw \in L^q(\mathbb{R}, dx)$ (e.g. put $f(x) = e^{-w(x)}\varphi(x)$ for $\varphi \in C_0^\infty(\mathbb{R})$), it follows that Y must coincide with the weak derivative of X_t^x . This proves the lemma. ■

We now complete the proof of our main theorem in this section (Theorem 3.3) and its corollary:

Proof of Theorem 3.3. Denote by $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \ni (s, t, x) \mapsto \phi_{s,t}(x) \in \mathbb{R}^d$ the continuous version of the solution map $(s, t, x) \mapsto X_t^{s,x}$ provided by Corollary 3.13. Let Ω^* be the set of all $\omega \in \Omega$ such that the SDE (3.7) has a unique spatially Sobolev differentiable family of solutions. Then by completeness of the probability space $(\Omega, \mathcal{F}, \mu)$, it follows that $\Omega^* \in \mathcal{F}$ and $\mu(\Omega^*) = 1$. Furthermore, by uniqueness of solutions of the SDE (3.7), it is easy to check that the following two-parameter group property

$$\phi_{s,t}(\cdot, \omega) = \phi_{u,t}(\cdot, \omega) \circ \phi_{s,u}(\cdot, \omega), \quad \phi_{s,s}(x, \omega) = x, \quad (3.21)$$

holds for all $s, u, t \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and all $\omega \in \Omega^*$. Finally, we apply Lemma 3.15 and use the relation $\phi_{s,t}(\cdot, \omega) = \phi_{t,s}^{-1}(\cdot, \omega)$, to complete the proof of the theorem. ■

Proof of Corollary 3.5. Let Ω^* denote the set of full Wiener measure introduced in the above proof of Theorem 3.3. We claim that $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$. To see this, let $\omega \in \Omega^*$ and fix an arbitrary $t_1 \in \mathbb{R}$. Then from the autonomous SDE (3.11) it follows that

$$X_{t+t_1}^{t_1,x}(\omega) = x + \int_{t_1}^{t+t_1} b(X_u^{t_1,x}(\omega)) du + B_{t+t_1}(\omega) - B_{t_1}(\omega), \quad t_1, t \in \mathbb{R}, \quad (3.22)$$

By the helix property of B and a simple change of variable the above relation implies

$$X_{t+t_1}^{t_1,x}(\omega) = x + \int_0^t b(X_{u+t_1}^{t_1,x}(\omega)) du + B_t(\theta(t_1, \omega)), \quad t \in \mathbb{R}, \quad (3.23)$$

The above relation implies that the SDE (3.11) admits a Sobolev differentiable family of solutions when ω is replaced by $\theta(t_1, \omega)$. Hence $\theta(t_1, \omega) \in \Omega^*$. Thus $\theta(t_1, \cdot)(\Omega^*) \subseteq \Omega^*$, and since $t_1 \in \mathbb{R}$ is arbitrary, this proves our claim. Furthermore, using uniqueness in the integral equation (3.22) it follows that

$$X_{t_2+t_1}^{t_1,x}(\omega) = X_{t_2}^{0,x}(\theta(t_1, \omega)) \quad (3.24)$$

for all $t_1, t_2 \in \mathbb{R}$, all $x \in \mathbb{R}^d$ and $\omega \in \Omega^*$. To prove the following cocycle property for all $\omega \in \Omega^*$

$$\phi_{0,t_1+t_2}(\cdot, \omega) = \phi_{0,t_2}(\cdot, \theta(t_1, \omega)) \circ \phi_{0,t_1}(\cdot, \omega)$$

we rewrite the identity (3.24) in the form

$$\phi_{t_1,t_1+t_2}(x, \omega) = \phi_{0,t_2}(x, \theta(t_1, \omega)), \quad t_1, t_2 \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega^*, \quad (3.25)$$

replace x by $\phi_{0,t_1}(x, \omega)$ in the above identity and invoke the two-parameter flow property (3.21). This completes the proof of Corollary 3.5. ■

Finally, we give an extension of Theorem 3.3 to a class of non-degenerate d -dimensional Itô-diffusions.

Theorem 3.16 *Consider the time-homogeneous $\mathbb{R}^d$ -valued SDE*

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dB_t, \quad X_0^x = x \in \mathbb{R}^d, \quad 0 \leq t \leq 1, \quad (3.26)$$

where the coefficients $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Suppose that $\sigma(x)$ has an inverse $\sigma^{-1}(x)$ for all $x \in \mathbb{R}^d$. Further assume that $\sigma^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ is continuously differentiable such that

$$\frac{\partial}{\partial x_k} \sigma_{lj}^{-1} = \frac{\partial}{\partial x_j} \sigma_{lk}^{-1}$$

for all $l, k, j = 1, \dots, d$. In addition, require that the function $\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ defined by

$$\Lambda(x) := \int_0^1 \sigma^{-1}(tx) \cdot x dt$$

possesses a Lipschitz continuous inverse $\Lambda^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$. Let $D\Lambda : \mathbb{R}^d \rightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ and $D^2\Lambda : \mathbb{R}^d \rightarrow L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ be the existing corresponding derivatives of Λ . Assume that the function $b^* : \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$\begin{aligned} b^*(x) &:= D\Lambda(\Lambda^{-1}(x)) [b(\Lambda^{-1}(x))] \\ &+ \frac{1}{2} D^2\Lambda(\Lambda^{-1}(x)) \left[\sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(x)) [e_i] \right] \end{aligned}$$

is bounded and Borel measurable, where e_i , $i = 1, \dots, d$, is a basis of $\mathbb{R}^d$.

Then there exists a stochastic flow $(s, t, x) \mapsto \phi_{s,t}(x)$ of the SDE (3.26) such that

$$\phi_{s,t}(\cdot) \in L^2(\Omega, W^p(\mathbb{R}^d, w))$$

for all $0 \leq s \leq t \leq 1$ and all $p > 1$.

Proof. Because of our assumptions we see that Λ^{-1} is twice continuously differentiable and that

$$D\Lambda(y)\sigma(y) = \mathcal{I}_d$$

for all $y \in \mathbb{R}^d$.

Then Itô's Lemma applied to (3.7) implies that

$$\begin{aligned} dY_t^x &= D\Lambda(\Lambda^{-1}(Y_t^x)) [b(\Lambda^{-1}(Y_t^x))] \\ &\quad + \frac{1}{2} D^2\Lambda(\Lambda^{-1}(Y_t^x)) \left[\sum_{i=1}^d \sigma(\Lambda^{-1}(Y_t^x)) [e_i], \sum_{i=1}^d \sigma(\Lambda^{-1}(Y_t^x)) [e_i] \right] dt + dB_t, \\ Y_0^x &= \Lambda(x), \quad 0 \leq t \leq 1, \end{aligned}$$

where $Y_t^x = \Lambda(X_t^x)$. Because of Theorem 3.3 and a chain rule for functions in Sobolev spaces (see e.g. [40]) there exists a stochastic flow $(s, t, x) \mapsto \phi_{s,t}(x)$ of the SDE (3.26) such that $\phi_{s,t}(\cdot) \in L^2(\Omega, W^p(\mathbb{R}^d, w))$ for all $0 \leq s \leq t \leq 1$ and all $p > 1$. ■

3.3 Application to the Stochastic Transport Equation

In this section we will study the stochastic transport equation

$$\begin{cases} dt u(t, x) + (b(t, x) \cdot Du(t, x)) dt + \sum_{i=1}^d e_i \cdot Du(t, x) \circ dB_t^i = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (3.27)$$

where $e_1, \dots, e_d$ is the canonical basis of $\mathbb{R}^d$, $b : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a given bounded measurable vector field and $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ is a given initial data. The stochastic integration is understood in the Stratonovich sense.

In [22] it is proved that for smooth data and sufficiently regular vector field b , (3.27) has an explicit solution $u(t, x) = u_0(\phi_t^{-1}(x))$ where $\phi_t(x)$ is the flow map generated by the strong solutions $(X_t^x)_{t \geq 0}$ of the SDE (3.7). In fact this solution of the transport equation is strong in the sense that $u(t, \cdot)$ is differentiable everywhere in x almost surely for all t , and it satisfies the integral equation

$$u(t, x) + \int_0^t Du(s, x) \cdot b(s, x) ds + \sum_{i=1}^d \int_0^t e_i \cdot Du(s, x) \circ dB_s^i = u_0(x)$$

almost surely, for every t .

We shall use the following notion of weak solution (cf. Definition 12 in [14]).

Definition 3.17 *Let b be bounded and measurable and $u_0 \in L^\infty(\mathbb{R}^d)$. A differentiable, weak L^∞ -solution of the transport equation (3.27) is a stochastic process $u \in L^\infty(\Omega \times [0, 1] \times \mathbb{R}^d)$ such that, for every t , the function $u(t, \cdot)$ is weakly differentiable a.s. with $\sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E[|Du(s, x)|^4] < \infty$ and for every test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d} \theta(x) u(t, x) dx$ has a continuous modification which is an $\mathcal{F}_t$ -semi-martingale and*

$$\begin{aligned} \int_{\mathbb{R}^d} \theta(x) u(t, x) dx &= \int_{\mathbb{R}^d} \theta(x) u_0(x) dx \\ &\quad - \int_0^t \int_{\mathbb{R}^d} Du(s, x) \cdot b(s, x) \theta(x) dx ds \\ &\quad + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s, x) D_i \theta(x) dx \right) \circ dB_s^i, \end{aligned} \quad (3.28)$$

where $Du(t, x)$ is the weak derivative of $u(t, x)$ in the space-variable.

Our definition of weak solution differs slightly from that in [14] due to the fact that we do not require any regularity on the coefficient b except Borel measurability and boundedness. To compensate for it, the expression depends on the weak derivative of $u(t, x)$.

It is easy to see that equation (3.28) can be written in the equivalent Itô form:

Lemma 3.18 *A process $u \in L^\infty(\Omega \times [0, 1] \times \mathbb{R}^d)$ is a differentiable, weak L^∞ solution of the transport equation (3.27) if and only if, for every test function $\theta \in C_0^\infty(\mathbb{R}^d)$, the process $\int_{\mathbb{R}^d} \theta(x)u(t, x)dx$ has a continuous $\mathcal{F}_t$ -adapted modification and*

$$\begin{aligned} \int_{\mathbb{R}^d} \theta(x)u(t, x)dx &= \int_{\mathbb{R}^d} \theta(x)u_0(x)dx \\ &\quad - \int_0^t \int_{\mathbb{R}^d} Du(s, x) \cdot b(s, x)\theta(x)dxds \\ &\quad + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s, x)D_i\theta(x)dx \right) dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u(s, x)\Delta\theta(x)dxds. \end{aligned}$$

The main result of this section is the following existence and uniqueness theorem for solutions of the stochastic transport equation (3.27):

Theorem 3.19 *Let b be bounded and Borel measurable. Suppose $u_0 \in C_b^1(\mathbb{R}^d)$. Then there exists a unique $W^{1,\infty}$ weak solution $u(t, x)$ to the stochastic transport equation (3.27). (Moreover, for fixed t and x , this solution is Malliavin-differentiable.)*

Remark 3.20 *As noted in [14], the deterministic transport equation is generally ill-posed under the conditions of Theorem 3.19. It is remarkable that Brownian forcing on the transport equation induces uniqueness and regularity of the solution.*

We shall prove Theorem 3.19 using a sequence $b_n : [0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ of uniformly bounded sequence of smooth functions with compact support converging almost everywhere to b . We then study the corresponding sequence of solutions of the transport equation (3.27) when b is replaced by b_n .

For the rest of this section we denote by ϕ_t the flow of the SDE (3.7) driven by the vector field b , and by $\phi_{n,t}$ the flow of the SDE (3.7) with b_n in place of b .

We begin with the following lemma:

Lemma 3.21 *Let $u_0 \in C_b^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Then the sequence*

$$\left(\int_{\mathbb{R}^d} u_0(\phi_{n,s}^{-1}(x))f(x)dx \right)_{n \geq 1}$$

converges to $\int_{\mathbb{R}^d} u_0(\phi_s^{-1}(x))f(x)dx$ in $L^2(\Omega)$ for every $s \in [0, 1]$.

Proof. Consider

$$\left\| \int_{\mathbb{R}^d} u_0(\phi_{n,s}^{-1}(x))f(x)dx - \int_{\mathbb{R}^d} u_0(\phi_s^{-1}(x))f(x)dx \right\|_{L^2(\Omega)}$$

$$\leq \int_{\mathbb{R}^d} \|u_0(\phi_{n,s}^{-1}(x)) - u_0(\phi_s^{-1}(x))\|_{L^2(\Omega)} |f(x)| dx$$

We have $\|u_0(\phi_{n,s}^{-1}(x)) - u_0(\phi_s^{-1}(x))\|_{L^2(\Omega)} \leq \|Du_0\|_{\infty} \|\phi_{n,s}^{-1}(x) - \phi_s^{-1}(x)\|_{L^2(\Omega)}$ which goes to zero for every s and x . Now

$$\|u_0(\phi_{n,s}^{-1}) - u_0(\phi_s^{-1})\|_{L^2(\Omega)} |f| \leq 2\|u_0\|_{\infty} |f| \in L^1(\mathbb{R}^d)$$

and the result follows by dominated convergence. ■

We also need the following result (see Theorem 2 in [17] and also [35], [36]):

Theorem 3.22 *Let $\mathcal{U}$ be open subset of $\mathbb{R}^d$ and $f \in W^{1,d}(\mathcal{U})$ be a homeomorphism. Then f satisfies the Lusin's condition, that is*

$$E \subset \mathcal{U}, |E| = 0 \implies |f(E)| = 0.$$

Here $|A|$ stands for the Lebesgue measure of a set A .

Moreover, for every measurable function $g : \mathcal{U} \rightarrow [0, \infty)$ and measurable set $E \subset \mathcal{U}$ the following change of variable formula is valid:

$$\int_E (g \circ f) |\det Jf| dx = \int_{f(E)} g(y) dy,$$

where $\det Jf$ is the determinant of the Jacobian of f .

Remark 3.23 *The random diffeomorphisms $\phi_t(\cdot), \phi_t^{-1}(\cdot) \in W_{loc}^{1,p}(\mathbb{R}^d)$ a.e. satisfy the conditions of Theorem 3.22 on each bounded and open subset $\mathcal{U}$ of $\mathbb{R}^d$.*

We are now ready to prove Theorem 3.19:

Proof of Theorem 3.19.

1. Existence of a weak solution:

We consider the approximation $\{b_n\}$ of b as described in Corollary 13. Then we know that there exists a unique strong solution to the transport equation (3.27) when b is replaced by b_n , which is uniquely given by $u_n(t, x) = u_0(\phi_{n,t}^{-1}(x))$, $n \geq 1$. In particular, u_n is a differentiable, weak L^∞ -solution, such that for every $\theta \in C^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} \theta(x) u_n(t, x) dx &= \int_{\mathbb{R}^d} \theta(x) u_0(x) dx \\ &\quad - \int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds \\ &\quad + \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s, x) D_i \theta(x) dx \right) dB_s^i \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{R}^d} u_n(s, x) \Delta \theta(x) dx ds. \end{aligned} \tag{3.29}$$

Let's now define $u(t, x) := u_0(\phi_t^{-1}(x))$ so that $u \in L^\infty(\Omega \times [0, 1] \times \mathbb{R}^d)$, and $u(t, \cdot)$ is weakly differentiable, a.s. We now let n go to infinity to get that $u(t, x)$ is a solution of the transport equation.

The following two limits exist in $L^2(\Omega)$ by Lemma 3.21 and dominated convergence:

$$\begin{aligned} \int_{\mathbb{R}^d} \theta(x) u_n(t, x) dx &\rightarrow \int_{\mathbb{R}^d} \theta(x) u(t, x) dx \\ \int_0^t \int_{\mathbb{R}^d} u_n(s, x) \Delta \theta(x) dx ds &\rightarrow \int_0^t \int_{\mathbb{R}^d} u(s, x) \Delta \theta(x) dx ds \end{aligned}$$

By the Itô isometry we have

$$\sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u_n(s, x) D_i \theta(x) dx \right) dB_s^i \rightarrow \sum_{i=1}^d \int_0^t \left(\int_{\mathbb{R}^d} u(s, x) D_i \theta(x) dx \right) dB_s^i$$

in $L^2(\Omega)$. Finally, we claim that

$$\int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds \rightarrow \int_0^t \int_{\mathbb{R}^d} Du(s, x) \cdot b(s, x) \theta(x) dx ds$$

in $L^2(\Omega)$. To see this observe that

$$\left(\int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds \right)_n$$

is convergent in $L^2(\Omega)$ because of the convergence of the other terms in equality (3.29). Then the claim is proved once we show that $\int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds$ converges **weakly** to $\int_0^t \int_{\mathbb{R}^d} Du(s, x) \cdot b(s, x) \theta(x) dx ds$. Then the strong and weak limit must coincide.

To prove weak convergence, we write the difference in three parts, namely:

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du(s, x) \cdot b(s, x) \theta(x) dx ds \\ &= \int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b_n(s, x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot b(s, x) \theta(x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_{n,s}^{-1}(x)) D\phi_{n,s}^{-1}(x) \cdot b(s, x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) D\phi_{n,s}^{-1}(x) \cdot b(s, x) \theta(x) dx ds \\ &+ \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) D\phi_{n,s}^{-1}(x) \cdot b(s, x) \theta(x) dx ds - \int_0^t \int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) D\phi_s^{-1}(x) \cdot b(s, x) \theta(x) dx ds \\ &= (i)_n + (ii)_n + (iii)_n \end{aligned}$$

We shall deal with these terms separately.

(α): The first term $(i)_n$ converges to 0 strongly in $L^2(\Omega)$ as $n \rightarrow \infty$, since by Hölder's inequality and Fubini's theorem

$$E[(i)_n^2] = E \left[\left(\int_0^t \int_{\mathbb{R}^d} Du_n(s, x) \cdot (b_n(s, x) - b(s, x)) \theta(x) dx ds \right)^2 \right]$$

$$\leq \int_0^t \int_{\mathbb{R}^d} E[|Du_n(s, x)|^2] |b_n(s, x) - b(s, x)|^2 |\theta(x)| dx \|\theta\|_{L^1(\mathbb{R})}$$

We have that

$$E[|Du_n(s, x)|^2] \leq \|Du_0\|_\infty^2 E[|D\phi_{n,s}^{-1}(x)|^2]$$

which is uniformly bounded in n , s and x by Proposition 3.7. Then, using dominated convergence, we obtain $\lim_{n \rightarrow \infty} (i)_n = 0$.

(β): The second term converges strongly to 0 in $L^2(\Omega)$, because of the following estimates:

$$\begin{aligned} E[(ii)_n^2] &\leq \|b\|_\infty^2 E\left[\left(\int_0^t \int_{\mathbb{R}^d} |Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))| |D\phi_{n,s}^{-1}(x)| |\theta(x)| dx ds\right)^2\right] \\ &\leq \|b\|_\infty^2 t \|\theta\|_{L^1(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} E[|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))|^2 |D\phi_{n,s}^{-1}(x)|^2] |\theta(x)| dx ds \\ &\leq \|b\|_\infty^2 t \|\theta\|_{L^1(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d} (E[|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))|^4])^{1/2} (E[|D\phi_{n,s}^{-1}(x)|^4])^{1/2} |\theta(x)| dx ds \\ &\leq \|b\|_\infty^2 t \|\theta\|_{L^1(\mathbb{R}^d)} \sup_{k,r,y} \left(E[|D\phi_{k,r}^{-1}(y)|^4]\right)^{1/2} \\ &\quad \times \int_0^t \int_{\mathbb{R}^d} (E[|Du_0(\phi_{n,s}^{-1}(x)) - Du_0(\phi_s^{-1}(x))|^4])^{1/2} |\theta(x)| dx ds. \end{aligned}$$

The above estimates are consequences of Hölder's inequality. Since Du_0 is bounded and continuous, the right hand side of the above inequality converges to 0 by dominated convergence.

(γ): For the last term, let $X \in L^2(\Omega)$ and consider

$$E[(iii)_n X] = \int_0^t E\left[\int_{\mathbb{R}^d} Du_0(\phi_s^{-1}(x)) (D\phi_{n,s}^{-1}(x) - D\phi_s^{-1}(x)) \cdot b(s, x) \theta(x) X dx\right] ds$$

Now, for each s , since Du_0 , b and θ are bounded and $D\phi_s^{-1}$ is the weak limit of $D\phi_{n,s}^{-1}$, this expression tends to 0 as $n \rightarrow \infty$.

2. Uniqueness of weak solutions:

Let us assume that u is a weak solution to the stochastic transport equation (3.28) with $\sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E[|Du(s, x)|^4] < \infty$. We will show that

$$u(t, x) = u_0(\phi_t^{-1}(x)) \text{ a.e.}$$

This will guarantee uniqueness of the solution to the transport equation. So let V be a bounded and open subset of $\mathbb{R}^d$ and consider for the locally integrable function $u(t, \cdot)$ on $\mathbb{R}^d$ its mollification

$$u_\epsilon(t, x) = (u * \eta_\epsilon) = \int_{\mathbb{R}^d} u(t, y) \eta_\epsilon(x - y) dy,$$

with respect to the standard mollifier η .

We observe that u_ϵ satisfies the equation

$$u_\epsilon(t, x) = u_{0,\epsilon}(x) - \int_0^t (b \cdot Du)_\epsilon(s, x) ds - \int_0^t (Du)_\epsilon(s, x) \circ dB_s.$$

Then using the Itô-Ventzell formula applied to u_ϵ and $\phi_t(x)$ (see [22]) gives

$$u_\epsilon(t, \phi_t(x)) = u_{0,\epsilon}(x) + \int_0^t ((Du)_\epsilon(s, \phi_s(x)) \cdot b(s, \phi_s(x)) - (b \cdot Du)_\epsilon(s, \phi_s(x))) ds. \quad (3.30)$$

Now let $\tau \in L^\infty(\Omega)$ and θ be a smooth function with compact support in V . Then it follows from (3.30) that

$$\begin{aligned} & E[\tau \int_V \theta(x) u_\epsilon(t, \phi_t(x)) dx] \\ &= E[\tau \int_V \theta(x) u_{0,\epsilon}(x) dx] \end{aligned} \quad (3.31)$$

$$+ E[\tau \int_0^t \int_V \theta(x) ((Du)_\epsilon(s, \phi_s(x)) \cdot b(s, \phi_s(x)) - (b \cdot Du)_\epsilon(s, \phi_s(x))) dx ds]. \quad (3.32)$$

Using Theorem 3.22 applied to $\phi_t^{-1}(\cdot)$ we obtain

$$\begin{aligned} & E[\tau \int_0^t \int_V \theta(x) ((Du)_\epsilon(s, \phi_s(x)) \cdot b(s, \phi_s(x)) - (b \cdot Du)_\epsilon(s, \phi_s(x))) dx ds] \\ &= E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) ((Du)_\epsilon(s, x) \cdot b(s, x) - (b \cdot Du)_\epsilon(s, x)) |\det(J\phi_s^{-1}(x))| dx ds] \\ &= I_1 + I_2, \end{aligned} \quad (3.33)$$

where

$$I_1 := E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) ((Du)_\epsilon(s, x) \cdot b(s, x)) |\det(J\phi_s^{-1}(x))| dx ds] \quad (3.34)$$

and

$$I_2 := -E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) (b \cdot Du)_\epsilon(s, x) |\det(J\phi_s^{-1}(x))| dx ds]. \quad (3.35)$$

Since V is bounded, there exists a $n \in \mathbb{N}$ such that $V \subset \bar{V} \subset W := (-n, n)^d$. Then we get

$$\begin{aligned} \|(Du)_\epsilon\|_{L^2(\phi_s(V))} &\leq \|Du\|_{L^2(\phi_s(W))}, \\ \|(b \cdot Du)_\epsilon\|_{L^2(\phi_s(V))} &\leq \|b \cdot Du\|_{L^2(\phi_s(W))} \\ &\leq \|b\|_\infty \|Du\|_{L^2(\phi_s(W))}. \end{aligned} \quad (3.36)$$

Using (3.36), Hölder's inequality, Fubini's theorem and Theorem 3.22, we obtain

$$\begin{aligned}
I_1 &\leq CE \int_0^t \left(\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) b(s, x) |\det(J\phi_s^{-1}(x))|)^2 dx \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x) |Du(s, x)|^2 dx \right)^{\frac{1}{2}} ds \\
&\leq C \int_0^t E \left[\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) b(s, x) |\det(J\phi_s^{-1}(x))|)^2 dx \right]^{\frac{1}{2}} \\
&\quad \cdot E \left[\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x) |Du(s, x)|^2 dx \right]^{\frac{1}{2}} ds \\
&\leq C \int_0^t E \left[\int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) |\det(J\phi_s^{-1}(x))|^2 dx \right]^{\frac{1}{2}} \\
&\quad \cdot E \left[\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x) |Du(s, x)|^2 dx \right]^{\frac{1}{2}} ds \\
&\leq C \int_0^t \left(\int_{\mathbb{R}^d} E[\chi_{\phi_s(V)}(x)]^{\frac{1}{2}} E[|\det(J\phi_s^{-1}(x))|^4]^{\frac{1}{2}} dx \right)^{\frac{1}{2}} \\
&\quad \cdot \left(\int_{\mathbb{R}^d} E[\chi_{\phi_s(W)}(x)]^{\frac{1}{2}} E[|Du(s, x)|^4]^{\frac{1}{2}} dx \right)^{\frac{1}{2}} ds \\
&\leq C \sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E[|\det(J\phi_s^{-1}(x))|^4]^{\frac{1}{2}} \sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E[|Du(s, x)|^4]^{\frac{1}{2}} \\
&\quad \cdot \int_0^t \left(\int_{\mathbb{R}^d} E[\chi_{\phi_s(V)}(x)]^{\frac{1}{2}} dx \right) ds \\
&\leq C \int_0^t \left(\int_{\mathbb{R}^d} E[\chi_{\phi_s(V)}(x)]^{\frac{1}{2}} dx \right) ds
\end{aligned} \tag{3.37}$$

for a constant C depending on the sizes of V , θ and b , since

$$\sup_{0 \leq s \leq 1, x \in \mathbb{R}^d} E[|\det(J\phi_s^{-1}(x))|^4] \leq M < \infty$$

because of Lemma 3.7 applied to $\phi_s^{-1}(x)$.

Further, it follows from Girsanov's theorem, Hölder's inequality and the symmetry of the distribution of the Brownian motion that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^d} E[\chi_{\phi_s(W)}(x)]^{\frac{1}{2}} dx ds \\
&= \int_0^t \int_{\mathbb{R}^d} (\mu(\phi_s^{-1}(x) \in W))^{\frac{1}{2}} dx ds \\
&\leq C \int_0^t \int_{\mathbb{R}^d} (\mu(B_s + x \in W))^{\frac{1}{4}} dx ds \\
&= C \int_0^t \int_{\mathbb{R}^d} (\mu(B_s + x \in (-n, n)^d))^{\frac{1}{4}} dx ds \\
&\leq C \int_0^t \left(2 \int_0^\infty (1 - \Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy \right)^d ds,
\end{aligned} \tag{3.38}$$

where Φ is the standard normal distribution function.

On the other hand we know that

$$1 - \Phi(x) \leq \frac{1}{2\pi x} \exp(-x^2/2)$$

for all $x > 0$ (see [3]).

So

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} E[\chi_{\phi_s(W)}(x)]^{\frac{1}{2}} dx ds \\
& \leq C \int_0^t (2 \int_0^n (1 - \Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy + 2 \int_n^\infty (1 - \Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d ds \\
& \leq K \int_0^t ((\int_0^n (1 - \Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d + (\int_n^\infty (1 - \Phi(\frac{-n+y}{\sqrt{s}}))^{\frac{1}{4}} dy)^d) ds \\
& \leq M(1 + \int_0^t (\int_n^\infty (\frac{\sqrt{s}}{2\pi(y-n)} \exp(-(y-n)^2/2s))^{\frac{1}{4}} dy)^d ds) \\
& = M(1 + \int_0^t (\int_0^\infty (\frac{\sqrt{s}}{2\pi y} \exp(-y^2/2s))^{\frac{1}{4}} dy)^d ds) \\
& = M(1 + \int_0^t (\int_0^\infty \sqrt{s} (\frac{1}{2\pi y} \exp(-y^2/2))^{\frac{1}{4}} dy)^d ds) \\
& \leq L < \infty.
\end{aligned} \tag{3.39}$$

Furthermore, since

$$(Du)_\epsilon \longrightarrow Du \text{ in } L_{loc}^p(\mathbb{R}^d)$$

for all $p > 1$ and since

$$\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) b(s, x) |\det(J\phi_s^{-1}(x))|^2 dx < \infty \text{ a.e.}$$

because of the above estimates, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) ((Du)_\epsilon(s, x) \cdot b(s, x)) |\det(J\phi_s^{-1}(x))| dx \\
& \longrightarrow \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) ((Du)(s, x) \cdot b(s, x)) |\det(J\phi_s^{-1}(x))| dx
\end{aligned}$$

for $\epsilon \searrow 0$ $\mu \times ds$ -a.e.

On the other hand the latter expression w.r.t. ϵ is dominated by the integrable term

$$(\int_{\mathbb{R}^d} (\chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) b(s, x) |\det(J\phi_s^{-1}(x))|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^d} \chi_{\phi_s(W)}(x) |Du(s, x)|^2 dx)^{\frac{1}{2}}.$$

So using dominated convergence it follows from (3.37) and (3.39) that

$$\begin{aligned}
I_1 &= I_1(\epsilon) \longrightarrow \\
& E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) ((Du)(s, x) \cdot b(s, x)) |\det(J\phi_s^{-1}(x))| dx ds] \\
& \text{for } \epsilon \searrow 0.
\end{aligned} \tag{3.40}$$

Similarly to I_1 we also get

$$\begin{aligned} I_2 &= I_2(\epsilon) \longrightarrow \\ &\quad -E[\tau \int_0^t \int_{\mathbb{R}^d} \chi_{\phi_s(V)}(x) \theta(\phi_s^{-1}(x)) (b \cdot Du)(s, x) |\det(J\phi_s^{-1}(x))| dx ds] \\ \text{for } \epsilon &\searrow 0 \end{aligned} \tag{3.41}$$

and

$$E[\tau \int_V \theta(x) u_\epsilon(t, \phi_t(x)) dx] \longrightarrow E[\tau \int_V \theta(x) u(t, \phi_t(x)) dx] \tag{3.42}$$

as $\epsilon \searrow 0$.

In addition, because of the assumptions on u_0 it is clear that

$$E[\tau \int_V \theta(x) u_{0,\epsilon}(x) dx] \longrightarrow E[\tau \int_V \theta(x) u_0(x) dx]$$

as $\epsilon \searrow 0$.

Altogether we can conclude that

$$E[\tau \int_{\mathbb{R}^d} \theta(x) u(t, \phi_t(x)) dx] = E[\tau \int_{\mathbb{R}^d} \theta(x) u_0(x) dx]$$

for all $\tau \in L^\infty(\Omega)$ and compactly supported smooth functions θ . Hence

$$u(t, \phi_t(x)) = u_0(x)$$

$\mu \times dx$ -a.e.

Since $\phi_t^{-1}(\cdot)$ satisfies the Lusin condition in Theorem 3.22 on bounded open subsets we can find a Ω^* with $\mu(\Omega^*) = 1$ such that for all $\omega \in \Omega^*$

$$u(t, x) = u_0(\phi_t^{-1}(x)) \text{ } dx - \text{a.e.}$$

Due to the continuity of u with respect to time the latter relation also holds uniformly in t .

Finally, the Malliavin differentiability of (a version) of $u(t, x)$ is a consequence of the fact that $\phi_t^{-1}(x)$ is Malliavin differentiable (see [26]) and of the chain rule for Malliavin derivatives (see [31]). ■

3.4 Application to ODE's

In this section we employ the approach developed in Section 3.2 to study the existence of absolutely continuous solutions $x \mapsto X_t^x$ of the time-homogeneous (deterministic) ODE

$$dX_t^x = b(X_t^x)dt, X_0 = x \in \mathbb{R}, t \in \mathbb{R}, \tag{3.43}$$

where $b : \mathbb{R} \longrightarrow \mathbb{R}$ is a discontinuous function. More precisely, we show that the sequence of solutions $X_t^{n,x}, n \geq 1$ to the perturbed equation

$$dX_t^{n,x} = b(X_t^{n,x})dt + \frac{1}{n}dB_t, X_0^{n,x} = x \in \mathbb{R}, t \in \mathbb{R},$$

converge to a solution $X_t^{n,x}$ of the ODE (3.43). Furthermore, we show that this family of solutions to the ODE is absolutely continuous in $x \in \mathbb{R}$.

We begin with the following observation:

Proposition 3.24 *Let $b = (b_1, \dots, b_d) : [0, 1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a bounded Borel measurable function with at most countably many points of discontinuities in the space variable uniformly in time. Further, suppose that there exist constants $m_i > 0, i = 1, \dots, d$ such that for each i*

$$\text{either } m_i \leq b_i(t, y) \text{ for all } t, y \text{ or } b_i(t, y) \leq -m_i \text{ for all } t, y. \quad (3.44)$$

Then there exists for all initial values $x \in \mathbb{R}^d$ a solution to the ODE

$$dX_t^x = b(t, X_t^x)dt, X_0 = x, 0 \leq t \leq 1.$$

Proof. By a result of A. Y. Veretennikov [39] we know that the perturbed equation

$$dX_t^{n,x} = b(t, X_t^{n,x})dt + \frac{1}{n}dB_t, X_0^{n,x} = x \in \mathbb{R}^d, 0 \leq t \leq 1$$

has a unique strong solution $X_t^{n,x}$ with continuous paths for all $n \geq 1$.

On the other hand, we also know that the Brownian paths are α -Hölder continuous a.e. for all $\alpha < \frac{1}{2}$. See e.g. [20]. Let us fix a ω in some Ω^* with $\mu(\Omega^*) = 1$ on which all those solutions and Brownian paths are concentrated. Then there exists a constant $C = C(\omega) < \infty$ such that for all $0 \leq t_1, t_2 \leq 1$ and $n \geq 1$

$$\begin{aligned} |X_{t_1}^{n,x} - X_{t_2}^{n,x}| &\leq M |t_1 - t_2| + \frac{C}{n} |t_1 - t_2|^\alpha \\ &\leq M |t_1 - t_2| + C |t_1 - t_2|^\alpha \end{aligned}$$

for a constant $M < \infty$. Clearly, we also have

$$\sup_{0 \leq t \leq 1} |X_t^{n,x}| \leq M < \infty$$

uniformly in n for some M . So it follows from the theorem of Arzela-Ascoli that

$$X_{\cdot}^{n_k,x} \xrightarrow[k \rightarrow \infty]{} X_{\cdot}^x = (X_{\cdot}^{x,(1)}, \dots, X_{\cdot}^{x,(d)}) \text{ in } C([0, 1]; \mathbb{R}^d)$$

for some subsequence $\{n_k\}_{k=1}^\infty$ of $\{n\}_{n=1}^\infty$. Thus

$$X_t^{x,(j)} = x_j + \lim_{n \rightarrow \infty} \int_0^t b_j(s, X_s^{n_k,x})ds$$

for all $t \in [0, 1], j \in \{1, 2, \dots, d\}$. Thus for each $1 \leq j \leq d$, either

$$X_{t_1}^{x,(j)} - X_{t_2}^{x,(j)} = \lim_{n \rightarrow \infty} \int_{t_1}^{t_2} b_j(s, X_s^{n_k,x})ds \geq m_j > 0, t_1 < t_2$$

or

$$X_{t_1}^{x,(j)} - X_{t_2}^{x,(j)} = \lim_{n \rightarrow \infty} \int_{t_1}^{t_2} b_j(s, X_s^{n_k,x})ds \leq -m_j < 0, t_1 < t_2.$$

So any of the components $X_{\cdot}^{x,(j)}$ of $X_{\cdot}^x$ is a bijection on $[0, 1]$. Hence $X_t^{x,(j)}$ can only hit the j -th projection of the points of discontinuities of b in the space variable at most countably many times for $t \in [0, 1]$. Therefore X_t^x doesn't hit the discontinuity points of b t -a.e. Finally, using dominated convergence we get

$$X_t^x = x + \int_0^t b(s, X_s^x)ds, 0 \leq t \leq 1.$$

■

Remark 3.25 In [38] it is shown that even if $b : [0, \infty) \rightarrow [a, b]$ with $a > 0$ is Borel measurable, then the ODE (3.43) has a solution in $[0, \infty)$.

In the sequel let us denote by

$$\int_0^t \int_{\mathbb{R}} f(s, y) L^{X^x}(dy, ds) \quad (3.45)$$

the integral of a bounded measurable function $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ with respect to the local time $L^{X^x}(dy, ds)$ of $X^x := X^{0, x}$ (in time and space). For more information about local time-space integration the reader is referred to [3] or [37].

We also need the following auxiliary result:

Lemma 3.26 Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Borel measurable function and let $(b_n)_{n \geq 1}$ in $C_0^\infty(\mathbb{R})$ be a sequence of functions such that

$$b_n(y) \xrightarrow{n \rightarrow \infty} b(y) \text{ a.e.,}$$

and

$$|b_n(y)| \leq C$$

for all $n \geq 1, y \in \mathbb{R}$ and some finite positive constant C . Denote by $X^{n, x}$ the unique strong solution to

$$dX_t^{n, x} = b_n(X_t^{n, x})dt + \delta dB_t, X_0^{n, x} = x \in \mathbb{R}, 0 \leq t \leq 1,$$

where $\delta > 0$ is a constant. Let b'_n be the derivative of b_n for each $n \geq 1$.

Then the following convergence

$$\exp\left(\int_0^t b'_n(X_s^{n, x})ds\right) \xrightarrow{n \rightarrow \infty} \exp\left(-\frac{1}{\delta^2} \int_0^t \int_{\mathbb{R}} b(y) L^{X^x}(dy, ds)\right)$$

holds weakly in $L^2(\mathcal{U} \times [0, 1] \times \Omega, dx \times dt \times d\mu)$ for all bounded open sets $\mathcal{U} \subset \mathbb{R}$.

Proof.

We start by noting that the set of functions

$$\left(\varphi \otimes \exp\left\{\int_0^1 h(s)dB_s\right\}\right),$$

is total in $L^2(\mathcal{U} \times [0, 1] \times \Omega, dx \times dt \times d\mu)$ when φ ranges through $C_0^\infty(\mathcal{U} \times [0, 1])$ and h ranges through the step functions defined on $[0, 1]$.

By Girsanov's theorem we have

$$\begin{aligned}
& \left| \left(\varphi \otimes \exp\left\{\int_0^1 h(s)dB_s\right\}, \exp\left(\int_0^t b'_n(X_s^{n,x})ds\right) - \exp\left(-\frac{1}{\delta^2} \int_0^t \int_{\mathbb{R}} b(y)L^{X^x}(dy, ds)\right) \right) \right|_{L^2(\mathcal{U} \times [0,1] \times \Omega)} \\
&= \left| \int_{\mathbb{R}} \int_0^1 \varphi(x, r) E[\exp\left\{\int_0^1 h(s)dX_s^{n,x}\right\} \exp\left\{-\int_0^t b'_n(x+B_s)ds\right\} \mathcal{E}\left(\int_0^t b_n(x+B_u)dB_u\right)] dr dx \right. \\
&\quad - \left. \int_{\mathbb{R}} \int_0^1 \varphi(x, r) E[\exp\left\{\int_0^t h(s)dX_s^x\right\} \exp\left\{-\frac{1}{\delta^2} \int_0^t \int_{\mathbb{R}} b(y)L^{\delta B \cdot +x}(ds, dy)\right\} \mathcal{E}\left(\int_0^t b(x+B_u)dB_u\right)] dr dx \right| \\
&\leq \left| \int_{\mathbb{R}} \int_0^1 \varphi(x, r) E\left[\left(\exp\left\{\int_0^1 h(s)dX_s^{n,x}\right\} - \exp\left\{\int_0^1 h(s)dX_s^x\right\}\right) \right. \right. \\
&\quad \times \left. \exp\left\{\int_0^t b'_n(x+B_s)ds\right\} \mathcal{E}\left(\int_0^1 b_n(x+B_u)dB_u\right)\right] dr dx \Big| \\
&+ \left| \int_{\mathbb{R}} \int_0^1 \varphi(x, r) E[\exp\left\{\int_0^1 h(s)dX_s^x\right\} \right. \\
&\quad \times \left. \left(\exp\left\{\int_0^t b'_n(x+B_s)ds\right\} - \exp\left\{-\frac{1}{\delta^2} \int_0^t \int_{\mathbb{R}} b(y)L^{\delta B \cdot +x}(ds, dy)\right\}\right) \mathcal{E}\left(\int_0^1 b_n(x+B_u)dB_u\right)] dr dx \right| \\
&+ \left| \int_{\mathbb{R}} \int_0^1 \varphi(x, r) E[\exp\left\{\int_0^1 h(s)dX_s^x\right\} \exp\left\{-\frac{1}{\delta^2} \int_0^t \int_{\mathbb{R}} b(y)L^{\delta B \cdot +x}(ds, dy)\right\} \right. \\
&\quad \times \left. \left(\mathcal{E}\left(\int_0^1 b_n(x+B_u)dB_u\right) - \mathcal{E}\left(\int_0^1 b(x+B_u)dB_u\right)\right)] dr dx \right| \\
&=: i)_n + ii)_n + iii)_n
\end{aligned}$$

For the first term, since

$$\exp\left\{\int_0^t b'_n(x+B_s)ds\right\} = 1 + \sum_{m \geq 1} \int_{0 < s_1 < \dots < s_m < t} \prod_{j=1}^m b'_n(x+B_{s_j}) ds_1 \dots ds_m,$$

we get that the sequence

$$\left\{ \exp\left\{\int_0^t b'_n(x+B_s)ds\right\} \mathcal{E}\left(\int_0^1 b_n(x+B_u)dB_u\right) \right\}_{n \geq 1}$$

is bounded in $L^2(\Omega)$ and we have

$$i)_n \leq \int_0^1 \int_{\mathbb{R}} |\varphi(x, r)| \|\exp\{\int_0^1 h(s) dX_s^{n,x}\} - \exp\{\int_0^1 h(s) dX_s^x\}\|_{L^2(\Omega)} \times \\ \|\exp\{\int_0^t b'_n(x + B_s) ds\} \mathcal{E}(\int_0^1 b_n(x + B_u) dB_u)\|_{L^2(\Omega)} dx dr.$$

We know that $X_t^{n,x} \rightarrow X_t^x$ in $L^2(\Omega)$ and since h is a step function we get by dominated convergence that

$$\lim_{n \rightarrow \infty} i)_n = 0.$$

For the second term, by [3, Theorem 3.1] we have

$$\int_0^t b_n^1(\delta B_s + x) ds = - \int_0^t \int_{\mathbb{R}} \frac{1}{\delta^2} b_n(y) L^{\delta B+x}(ds, dy)$$

for all t , μ -a.e.

On the other hand, we also know (see [37, p. 220]) that

$$- \int_0^t \int_{\mathbb{R}} \frac{1}{\delta^2} b_n(y) L^{\delta B+x}(ds, dy) \\ = 2(F_n(\delta B_t + x) - F_n(x) - \int_0^t \frac{1}{\delta^2} b_n(\delta B_s + x) \delta dB_s)$$

where $F_n(y) := \int_0^y \frac{1}{\delta^2} b_n(u) du$. The last expressions holds is true when b_n is replaced by b . We see that the convergence

$$\int_0^t b_n^1(\delta B_s + x) ds \rightarrow - \int_0^t \int_{\mathbb{R}} \frac{1}{\delta^2} b(y) L^{\delta B+x}(ds, dy)$$

holds μ almost surely (possibly on a subsequence). Similary as for $i)_n$ we may invoke dominated convergence to conclude

$$\lim_{n \rightarrow \infty} ii)_n = 0.$$

For the last term notice that $\mathcal{E}(\int_0^1 b_n(x + B_u) dB_u) \rightarrow \mathcal{E}(\int_0^1 b(x + B_u) dB_u)$ μ -almost surely (possibly on a subsequence). Since b_n is uniformly bounded we get that $\mathcal{E}(\int_0^1 b_n(x + B_u) dB_u) - \mathcal{E}(\int_0^1 b(x + B_u) dB_u)$ is bounded in, say, $L^4(\Omega)$, and thus the same sequence is uniformly integrable when squared. We then get that

$$\|\mathcal{E}(\int_0^1 b_n(x + B_u) dB_u) - \mathcal{E}(\int_0^1 b(x + B_u) dB_u)\|_{L^2(\Omega)} \rightarrow 0$$

by the Vitali Convergence theorem. By dominated convergence we get

$$\lim_{n \rightarrow \infty} iii)_n = 0.$$

■

Theorem 3.27 Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded decreasing function such that either $m \leq b(y)$ for all y or $b(y) \leq -m$ for all y for some constant $m > 0$. Then there exists a unique continuous function $(t, x) \mapsto \tilde{X}_t^x$ on $\mathbb{R} \times \mathbb{R}$ such that

$$\tilde{X}_t^x = x + \int_0^t b(\tilde{X}_s^x) ds, \quad (3.46)$$

for all $t, x \in \mathbb{R}$. Moreover, the map

$$(t, x) \mapsto \tilde{X}_t^x$$

belongs to $L^2([0, 1]; W^{1,2}(\mathcal{U}))$ for any bounded open interval $\mathcal{U}$ in $\mathbb{R}$. The family

$$\mathbb{R} \ni x \mapsto \tilde{X}_t^x \in \mathbb{R}, \quad t \in \mathbb{R},$$

is a group of $W^{1,2}$ Sobolev diffeomorphisms on $\mathbb{R}$.

Proof. Uniqueness is easy. Indeed, suppose Y_t^x is any solution of equation (3.46). Then

$$\frac{d}{dt}(\tilde{X}_t^x - Y_t^x)^2 = 2\left(\frac{d}{dt}\tilde{X}_t^x - \frac{d}{dt}Y_t^x\right)(\tilde{X}_t^x - Y_t^x) = 2(b(\tilde{X}_t^x) - b(Y_t^x))(\tilde{X}_t^x - Y_t^x) \leq 0$$

since b is decreasing. Integrating the above inequality from 0 to t , we get

$$(\tilde{X}_t^x - Y_t^x)^2 \leq (\tilde{X}_0^x - Y_0^x)^2 = 0$$

This proves uniqueness.

We next prove existence of the flow for the ODE (3.46). It is sufficient to prove existence for $x \in \mathcal{U}$, an open bounded interval in $\mathbb{R}$ and for $t \in [0, 1]$. Suppose $b_m, m \geq 1$ is a sequence of decreasing functions in $C_0^\infty(\mathbb{R})$ such that $b_m(y) \rightarrow b(y)$ a.e. $m \rightarrow \infty$ a.e. and $|b_m(y)| \leq C < \infty$ for all m, y and for some positive constant C .

Consider the solution $X^{m,n,x}$ of the SDE

$$X_t^{m,n,x} = x + \int_0^t b_m(X_s^{m,n,x}) ds + \frac{1}{n} B_t, \quad 0 \leq t \leq 1$$

for all $x \in \mathcal{U}, n, m \geq 1$ μ -a.e.

We have (see [22])

$$\frac{\partial}{\partial x} X_t^{m,n,x} = 1 + \int_0^t b'_m(X_s^{m,n,x}) \frac{\partial}{\partial x} X_s^{m,n,x} ds, \quad 0 \leq t \leq 1$$

for all $x \in \mathcal{U}, n, m \geq 1$ μ -a.e.

Therefore,

$$\frac{\partial}{\partial x} X_t^{m,n,x} = \exp\left(\int_0^t b'_m(X_s^{m,n,x}) ds\right). \quad (3.47)$$

Let $\varphi \in C_0^\infty(\mathcal{U}), \xi \in L^\infty(\Omega), h \in L^\infty([0, 1])$.

Now from the proof of Lemma 3.15 in Section 3.2, it follows that for each $n \geq 1$ and $p > 1$

$$\sup_{0 \leq t \leq 1} \sup_{x \in \mathcal{U}} \sup_{m \geq 1} E\left[\left|\frac{\partial}{\partial x} X_t^{m,n,x}\right|^p\right] \leq M < \infty,$$

where $M = M(n, p)$ is a positive constant.

The latter proof also shows that

$$\begin{aligned} & - \int_{\mathcal{U}} \int_0^1 E[X_t^{m,n,x} \xi] h(t) \frac{\partial}{\partial x} \varphi(x) dt dx \\ & \xrightarrow{m \rightarrow \infty} - \int_{\mathcal{U}} \int_0^1 E[X_t^{n,x} \xi] h(t) \frac{\partial}{\partial x} \varphi(x) dt dx. \end{aligned}$$

Since the map $(t, x) \mapsto X_t^{x,n}$ belongs to $L^2([0, 1] \times \Omega; W^{1,2}(\mathcal{U}))$, then it has a continuous version $(t, x) \mapsto \tilde{X}_t^{n,x}$ which is absolutely continuous in x .

Using 3.47 and Lemma 3.26 we find

$$\begin{aligned} & - \int_{\mathcal{U}} \int_0^1 E[X_t^{m,n,x} \xi] h(t) \frac{\partial}{\partial x} \varphi(x) dt dx \\ & = \int_{\mathcal{U}} \int_0^1 E\left[\frac{\partial}{\partial x} X_t^{m,n,x} \xi\right] h(t) \varphi(x) dt dx \\ & \xrightarrow{m \rightarrow \infty} \int_{\mathcal{U}} \int_0^1 E[\exp(-n^2 \int_0^t \int_{\mathbb{R}} b(y) L^{X^{n,x}}(ds, dy)) \xi] h(t) \varphi(x) dt dx. \end{aligned}$$

Hence

$$\frac{\partial}{\partial x} \tilde{X}_t^{n,x} = \exp(-n^2 \int_0^t \int_{\mathbb{R}} b(y) L^{X^{n,x}}(ds, dy)) \quad (3.48)$$

for all $n \geq 1, dt \times d\mu \times dx$ -a.e. So we may identify $\frac{\partial}{\partial x} \tilde{X}_t^{n,x}$ with the process on the right hand side of (3.48). Then $(t, x) \mapsto \tilde{X}_t^{n,x}$ is continuous μ -a.s.

Furthermore, since $b'_n(y) \leq 0, y \in \mathbb{R}$ in (3.47) we can argue by weak convergence that the right hand side of (3.48) is dominated by a constant $K \geq 0$ uniformly in n, x, t, μ -a.e.

Thus

$$\sup_{0 \leq t \leq 1} |\tilde{X}_t^{n,x_1} - \tilde{X}_t^{n,x_2}| \leq K |x_1 - x_2|$$

for all $x_1, x_2 \in \mathcal{U}, n \geq 1$ μ -a.e.

On the other hand we may assume by Corollary 3.13 that $(t, x) \mapsto X_t^{n,x}$ is continuous μ -a.e. Hence we have

$$\tilde{X}_t^{n,x} = x + \int_0^t b(\tilde{X}_s^{n,x}) ds + \frac{1}{n} B_t$$

for all $n \geq 1, t, x, \mu$ -a.e.

So using the α -Hölder continuity of Brownian paths, it follows that (for a fixed ω)

$$|\tilde{X}_{t_1}^{n,x} - \tilde{X}_{t_2}^{n,x}| \leq M |t_1 - t_2| + C(\omega) |t_1 - t_2|^\alpha \quad (3.49)$$

for all $0 \leq t_1, t_2 \leq 1, n \geq 1$ and all $x \in \mathcal{U}$, where $\alpha < \frac{1}{2}$, $C(\omega) = C(\omega, \alpha) < \infty$ and $M < \infty$.

Let $\mathcal{V}$ be a compact sub-interval of $\mathcal{U}$. Fix an appropriate $\omega \in \Omega$. Then by the Arzela-Ascoli theorem there is a subsequence (n_k) such that $(x \mapsto \tilde{X}_t^{n_k,x}), k \geq 1$ converges in $C(\mathcal{V}; C([0, 1]))$. Then repeated application of a weak compactness argument it follows that the limit, say $(t, x) \mapsto \bar{X}_t^x$ belongs to $L^2([0, 1]; W^{1,2}(\mathring{\mathcal{V}}))$ ($\mathring{\mathcal{V}}$ the interior of $\mathcal{V}$). Finally, and as in the proof of Proposition 3.24, it follows that $\bar{X}_t^x$ solves the ODE (3.46) for all x in $\mathring{\mathcal{V}}$. ■

Remark 3.28 *Using techniques of Malliavin calculus the authors in [25] prove that, for fixed $x \in \mathbb{R}$, the sequence $\{\tilde{X}_t^{n,x}\}_{n=1}^\infty$ in the proof of Theorem 3.3 converges to $\tilde{X}_t^x$ in $L^2(\mu)$ as $n \rightarrow \infty$.*

Curiously enough, the next theorem is a consequence of the deterministic result in Theorem 3.27 above. It establishes the existence of a perfect cocycle of $W_{loc}^{1,2}$ -Sobolev diffeomorphisms for solutions of the one-dimensional Stratonovich SDE:

$$dX_t^x = b(X_t^x) \circ dW(t), \quad t \in \mathbb{R}, X_0 = x \in \mathbb{R},$$

driven by a bounded decreasing *diffusion* coefficient $b : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the conditions of Theorem 3.27.

Theorem 3.29 *Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded decreasing function with a positive constant m such that either $m \leq b(y)$ for all $y \in \mathbb{R}$ or $b(y) \leq -m$ for all $y \in \mathbb{R}$. Suppose $W : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ is one-dimensional Brownian motion such that $W(0) = 0$, and $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ is the Wiener shift. Consider the Stratonovich SDE*

$$X_t^x = x + \int_0^t b(X_s^x) \circ dW(s), \quad t, x \in \mathbb{R}. \quad (3.50)$$

Then the above SDE has a family of strong pathwise continuous solutions

$$\mathbb{R} \times \mathbb{R} \ni (t, x) \mapsto X_t^x \in \mathbb{R}$$

such that $(X_t^x, \theta(t, \cdot))$ is a perfect cocycle of $W_{loc}^{1,2}$ -Sobolev diffeomorphisms on $\mathbb{R}$.

Proof. The idea of the proof is simple: We rescale the deterministic flow of the ODE (3.46) using the one-dimensional Brownian motion W . This is feasible by an appropriate application of Itô's formula due to Föllmer, Protter and Shiryaev [16].

To simplify notation, we denote by $Y(t, x) := \tilde{X}_t^x, t, x \in \mathbb{R}$, the deterministic flow of the ODE (3.46). Define the random field:

$$X_t^x := Y(W(t), x), \quad t, x \in \mathbb{R}. \quad (3.51)$$

We claim that the following equality

$$\int_0^{W(t)} b(Y(u, x)) du = \int_0^t b(X_u^x) \circ dW(u), \quad X_0 = x \in \mathbb{R}, \quad (3.52)$$

holds for all $x \in \mathbb{R}$ a.s. for all $t \in \mathbb{R}$. To prove (3.52), we apply Itô's formula using the absolutely continuous change of variable $F(z) := \int_0^z b(Y(u, x)) du, z \in \mathbb{R}$. Note that F is locally of class $W^{1,2}$ because $F'(z) = b(Y(z, x))$ for a.e. $z \in \mathbb{R}$. So by Itô's formula ([16]), it follows that

$$\begin{aligned} F(W(t)) &= \int_0^t F'(W(u)) dW(u) + \frac{1}{2}[F'(W), W](t) \\ &= \int_0^t F'(W(u)) \circ dW(u) \\ &= \int_0^t b(X_u^x) \circ dW(u) \end{aligned} \quad (3.53)$$

a.s. for all $t \geq 0$. In the above relation, the bracket $[\cdot, \cdot]$ stands for the quadratic covariation. This proves our claim (3.52). Using (3.51), (3.46) and (3.52), we get

$$\begin{aligned} X_t^x &= Y(W(t), x) \\ &= x + \int_0^{W(t)} b(Y(u, x)) du \\ &= x + \int_0^t b(X_u^x) \circ dW(u) \end{aligned} \quad (3.54)$$

a.s. for all $t \geq 0$. Hence $X_t^x, t, x \in \mathbb{R}$, is a family of solutions of the SDE (3.50). Since the map $[0, 1] \ni t \mapsto Y(t, \cdot) \in W^{1,2}(U)$ belongs to $C([0, 1], W^{1,2}(U))$, then so do the maps $[0, 1] \ni t \mapsto X_t^i(\omega) \in W^{1,2}(U)$ for a.a. $\omega \in \Omega$, where U is any bounded open interval in $\mathbb{R}$.

To prove the cocycle property for $(X_t^i, \theta(t, \cdot))$, we use the group property for the ODE (3.46):

$$Y(t_1, \cdot) \circ Y(t_2, \cdot) = Y(t_1 + t_2, \cdot) \quad t_1, t_2 \in \mathbb{R}. \quad (3.55)$$

Hence,

$$\begin{aligned} [X_{t_2}^i(\theta(t_1, \omega)) \circ X_{t_1}^i(\omega)](x) &= Y((\theta(t_1, \omega)(t_2), Y(\omega(t_1), x))) \\ &= Y(\omega(t_1 + t_2), x) \\ &= X_{t_1+t_2}^x(\omega), \quad x, t_1, t_2 \in \mathbb{R}, \omega \in \Omega. \end{aligned} \quad (3.56)$$

This completes the proof of the corollary. ■

Remark 3.30 *It is rather remarkable that the Stratonovich SDE (3.50) admits the existence of a perfect cocycle of $W_{loc}^{1,2}$ -Sobolev diffeomorphisms with respect to a discontinuous diffusion coefficient. On the other hand, it is conceivable that the SDE (3.50) has more than one solution. In fact, we haven't even been able to find similar examples to (3.50) in the literature.*

3.5 Appendix

The following result which is due to [4] provides a compactness criterion for subsets of $L^2(\mu; \mathbb{R}^d)$ using Malliavin calculus. See e.g. [31], [13] or [7] for more information about Malliavin calculus.

Theorem 3.31 *Let $\{(\Omega, \mathcal{A}, P); H\}$ be a Gaussian probability space, that is $(\Omega, \mathcal{A}, P)$ is a probability space and H a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the σ -field $\mathcal{A}$. Denote by $\mathbf{D}$ the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, \dots, h_n)) = \sum_{i=1}^n \partial_i f(h_1, \dots, h_n) h_i, \quad h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega; H)}.$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \|C^{-1} \mathbf{D}G\|_{L^2(\Omega; H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

In order to formulate compactness criteria useful for our purposes, we need the following technical result which also can be found in [4].

Lemma 3.32 *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0, 1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0, 1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt dt' \right)^{1/2} \right\}.$$

A direct consequence of Theorem 3.31 and Lemma 3.32 is now the following compactness criterion which is essential for the proof of Lemma 3.14:

Corollary 3.33 *Let $X_n \in \mathbb{D}_{1,2}$, $n = 1, 2, \dots$, be a sequence of $\mathcal{F}_1$ -measurable random variables such that there are constants $\alpha > 0$ and $C > 0$ with*

$$\sup_n E[\|X_n\|^2] \leq C,$$

$$\sup_n E[\|D_t X_n - D_{t'} X_n\|^2] \leq C|t - t'|^\alpha$$

for $0 \leq t' \leq t \leq 1$ and

$$\sup_n \sup_{0 \leq t \leq 1} E[\|D_t X_n\|^2] \leq C.$$

where D_t denotes Malliavin differentiation. Then the sequence X_n , $n = 1, 2, \dots$, is relatively compact in $L^2(\Omega)$ (D_t the Malliavin derivative).

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5 Malliavin differentiability and strong solutions for a class of SDE in Hilbert spaces

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Abstract

We consider a class of Hilbert-space valued SDE's where the drift coefficients are non-Lipschitzian in the sense of Hölder-continuity. Using a novel technique based on Malliavin calculus we show in this paper the existence and uniqueness of a mild solution to such equations. We emphasize that our approach does not rely on the Yamada-Watanabe principle. Moreover our method gives the important additional insight that the obtained solution is Malliavin differentiable - a property which was recently shown to play a crucial role in the study of the geometry of certain optimal causal transference plans, [12].

5.1 Introduction

In a separable Hilbert space H , consider the stochastic differential equation

$$dX_t = AX_t dt + B(t, X_t) dt + \sqrt{Q} dW_t, \quad X_0 = x \quad (5.1)$$

where $A : D(A) \subset H \rightarrow H$ is the infinitesimal generator of a C_0 -semigroup e^{tA} , $t \geq 0$; $B : [0, T] \times H \rightarrow H$ is continuous, W is a cylindrical Wiener process and Q is a non-negative selfadjoint bounded operator on H . If the operator $Q_t = \int_0^t e^{sA} Q e^{sA^*} ds$ is trace class, and suitable linear growth conditions on B are assumed, weak existence is known for equation (5.1), see [5].

The aim of this paper is to prove Malliavin differentiability and a direct proof of *strong existence*, under additional assumptions on (A, B) stated in section 5.1.1. On B we assume Hölder continuity in x uniformly in t . On A we assume certain non-degeneracy condition related to null-controllability. See [2] for the case of Hölder-coefficients. For merely bounded and measurable coefficients, see [9] and the recent work in [3].

5.1.1 Notations and assumptions

Norm and inner product in H will be denoted $|\cdot|$ and $\langle \cdot, \cdot \rangle$. A complete orthonormal system $\{e_n\}_{n \geq 1}$ in H is assumed to be fixed. If $\varphi : H \rightarrow H$, we shall denote its components with respect to $\{e_n\}_{n \geq 1}$ by φ_n : $\varphi_n(x) = \langle \varphi(x), e_n \rangle$.

Given $\alpha, T > 0$, we shall denote by $C([0, T]; C_b^\alpha(H, H))$ the space of all functions $G : [0, T] \times H \rightarrow H$ which are continuous and bounded in (t, x) , and such that there exists $C > 0$ such that

$$|G(t, x) - G(t, y)| \leq C|x - y|^\alpha, \quad x, y \in H, \quad t \in [0, T].$$

We denote by $\|G\|_{\alpha, T}$ or simply $\|G\|_\alpha$ the norm

$$\|G\|_\alpha = \sup_{t \in [0, T], x \in H} |G(t, x)| + \sup_{t \in [0, T]} \sup_{x \neq y \in H} \frac{|G(t, x) - G(t, y)|}{|x - y|^\alpha}$$

We use the notation $\|G_n\|_\alpha$ also for the similar norm of the components $G_n(t, x) = \langle G(t, x), e_n \rangle$.

Let us now list the assumptions of this paper:

1. The operator A is selfadjoint, with compact resolvent, and $Ae_n = -\alpha_n e_n$, with non-decreasing positive $\{\alpha_n\}_{n \geq 1}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha_n^{1-\delta}} < \infty \quad (5.2)$$

2. $B \in C([0, T]; C_b^\alpha(H, H))$ for some $\alpha, T > 0$.

- 3.

$$e^{tA}(H) \subset Q_t^{1/2}(H) \quad \text{for all } t > 0 \quad (5.3)$$

4. The well defined bounded operator $\Lambda_t = Q_t^{-1/2} e^{tA}$ satisfies

$$\int_0^T \|\Lambda_t\|^{1+\theta} dt < \infty \quad (5.4)$$

for some $\theta \geq \max(\alpha, 1 - \alpha)$.

Remark 5.1 From assumption (5.4) we have, in particular,

$$\int_0^T \|\Lambda_t\|^{1+\alpha} dt < \infty, \quad \int_0^T \|\Lambda_t\|^{2-\alpha} dt < \infty$$

5.2 Idea of the method

In this section we do not care about the rigor of the computations. The aim is to explain the idea.

For ever n , consider the following (backward) PDE in H of Kolmogorov type, on some interval $[0, T]$:

$$\begin{aligned} \frac{\partial U_n}{\partial t} + \frac{1}{2} \text{Tr} (D^2 U_n Q) + \langle Ax, DU_n \rangle + \langle B, DU_n \rangle &= B_n \\ U_n(T, x) &= 0 \end{aligned} \quad (5.5)$$

Notice it is a non-homogeneous equation, opposite to the usual equations of Kolmogorov type; the right-hand-side B_n is the n -component of B . If U_n is a sufficiently regular solution, from Itô's formula we get

$$\begin{aligned} dU_n(t, X_t) &= B_n(t, X_t)dt + \langle DU_n(t, X_t), Q^{1/2}dW_t \rangle \\ &= \langle B(t, X_t)dt + DU(t, X_t)Q^{1/2}dW_t, e_n \rangle \end{aligned}$$

namely

$$dU(t, X_t) = B(t, X_t)dt + DU(t, X_t)Q^{1/2}dW_t$$

where $U(t, x) = \sum_n U_n(t, x)e_n$ and where we have used the PDE above. About our vector-valued notations, let us stress that $U(t, \cdot) : H \rightarrow H$, hence $DU(t, X_t) \in L(H, H)$. Moreover, for every $v \in H$,

$$\langle DU(t, X_t)v, e_n \rangle = \langle DU_n(t, X_t), v \rangle.$$

Formally speaking, the previous identity gives us a formula for $B(t, X_t)dt$:

$$B(t, X_t)dt = dU(t, X_t) - DU(t, X_t)Q^{1/2}dW_t.$$

We put this formula in equation (5.1) and get

$$dX_t = AX_tdt + dU(t, X_t) - DU(t, X_t)Q^{1/2}dW_t + Q^{1/2}dW_t.$$

Now we follow the usual variation of constant method and get:

$$\begin{aligned} d_s e^{(t-s)A} X_s &= e^{(t-s)A} dU(s, X_s) \\ &\quad + e^{(t-s)A} Q^{1/2} dW_s - e^{(t-s)A} DU(s, X_s) Q^{1/2} dW_s \end{aligned}$$

namely

$$\begin{aligned} X_t - e^{tA}x &= \int_0^t e^{(t-s)A} dU(s, X_s) \\ &+ \int_0^t e^{(t-s)A} Q^{1/2} dW_s - \int_0^t e^{(t-s)A} DU(s, X_s) Q^{1/2} dW_s. \end{aligned}$$

Integrating by parts the first integral we finally get the equation

$$\begin{aligned} X_t &= e^{tA}(x - U(0, x)) + U(t, X_t) + \int_0^t A e^{(t-s)A} U(s, X_s) ds \\ &+ \int_0^t e^{(t-s)A} Q^{1/2} dW_s - I_t(X) \end{aligned} \quad (5.6)$$

where

$$I_t(X) := \int_0^t e^{(t-s)A} DU(s, X_s) Q^{1/2} dW_s. \quad (5.7)$$

The non-regular drift B has been removed from the equation, this is the point of the trick. Several new terms appear, which however will be proved to have good Lipschitz properties.

In order to make rigorous this program we need: i) to solve the PDE (5.5) in a sufficiently regular space to be able to perform the previous computations (one bounded derivative plus an approximation argument is sufficient for this); ii) to prove that all the terms in equation (5.6) are Lipschitz continuous in the space variable (for this we need a uniform control of first and second derivatives). Moreover, we need that the Lipschitz constant of the term $U(t, X_t)$ is small; we get this by taking small T and using the condition $U_n(T, x) = 0$.

5.3 H -valued Ornstein-Uhlenbeck semigroup

Let R_t be the Ornstein-Uhlenbeck semigroup, defined on $B_b(H)$ as

$$\begin{aligned} R_t \varphi(x) &= E[\varphi(Z_t^x)], \quad \varphi \in B_b(H) \\ dZ_t^x &= AZ_t^x dt + Q^{1/2} dW_t, \quad Z_0^x = x. \end{aligned}$$

See [6], Chapter 6, for an extensive analysis of it. We introduce the analogous semigroup on H -valued functions:

$$\mathcal{R}_t \Phi(x) = E[\Phi(Z_t^x)], \quad \Phi \in B_b(H, H).$$

We have

$$\langle \mathcal{R}_t \Phi(x), h \rangle = R_t \varphi_h(x), \quad \varphi_h(x) = \langle \Phi(x), h \rangle, \quad h \in H$$

Theorem 5.2 *Under the assumption (5.3), we have*

$$\Phi \in UC_b(H, H) \Rightarrow \mathcal{R}_t \Phi \in UC_b^2(H, H)$$

for all $t > 0$. The differential $D\mathcal{R}_t \Phi(x) \in L(H, H)$ at a given point $x \in H$ is the linear operator given by

$$D\mathcal{R}_t \Phi(x)g = \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \Phi(e^{At}x + y) N_{Q_t}(dy) \quad (5.8)$$

or

$$\begin{aligned}\langle D\mathcal{R}_t\Phi(x)g, h \rangle &= \langle DR_t\phi_h(x), g \rangle \\ &= \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \phi_h(e^{At}x + y) N_{Q_t}(dy)\end{aligned}$$

for all $t > 0$, $g, h \in H$. The second derivative $D^2\mathcal{R}_t\Phi(x) \in L(H, L(H, H))$ at a given point $x \in H$, is given by (recall that $D^2\mathcal{R}_t\Phi(x)$ is a linear operator in H , for every $g \in H$)

$$\begin{aligned}& [D^2\mathcal{R}_t\Phi(x)g]k \\ &= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \Phi(e^{tA}x + y) N_{Q_t}(dy)\end{aligned}\tag{5.9}$$

or

$$\begin{aligned}& \langle [D^2\mathcal{R}_t\Phi(x)g]k, h \rangle = \langle D^2R_t\phi_h(x)g, k \rangle \\ &= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \phi_h(e^{tA}x + y) N_{Q_t}(dy)\end{aligned}$$

for all $t > 0$, $g, k, h \in H$. If $\Phi \in UC_b^1(H, H)$ then

$$\langle [D^2\mathcal{R}_t\Phi(x)g]k, h \rangle = \int_H \langle \Lambda_t k, Q_t^{-1/2} y \rangle \langle D\phi_h(e^{tA}x + y), e^{tA}g \rangle N_{Q_t}(dy).$$

Finally,

$$\|D\mathcal{R}_t\Phi(x)\| \leq \|\Lambda_t\| \|\Phi\|_0 \tag{5.10}$$

$$\|D^2\mathcal{R}_t\Phi(x)\| \leq \sqrt{2} \|\Lambda_t\|^2 \|\Phi\|_0 \tag{5.11}$$

$$\|D^2\mathcal{R}_t\Phi(x)\| \leq \|e^{tA}\| \|\Lambda_t\| \|\Phi\|_1 \tag{5.12}$$

Proof. Step 1. Let us check that the right hand side of (5.8), namely the mapping

$$g \mapsto I_{t,x}g := \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \Phi(e^{At}x + y) N_{Q_t}(dy)$$

defines a linear bounded operator in H ; $x \in H$ and $t > 0$ are given. The integral is a well defined element of H , because

$$\begin{aligned}& \int_H \left| \langle \Lambda_t g, Q_t^{-1/2} y \rangle \Phi(e^{At}x + y) \right|^2 N_{Q_t}(dy) \\ & \leq \|\Phi\|_0^2 \int_H \left| \langle \Lambda_t g, Q_t^{-1/2} y \rangle \right|^2 N_{Q_t}(dy).\end{aligned}$$

Linearity of $I_{t,x}$ is clear; in addition, from this estimate it follows that $I_{t,x}$ is bounded, and $\|I_{t,x}\| \leq \|\Lambda_t\| \|\Phi\|_0$. So inequality (5.10) will be true when we can say that $I_{t,x} = D\mathcal{R}_t\Phi(x)$.

Step 2. Let us prove that $\mathcal{R}_t\Phi$ is differentiable at x and $I_{t,x}$ is the differential. We have, for $g, h \in H$,

$$\begin{aligned}& \langle \mathcal{R}_t\Phi(x+g) - \mathcal{R}_t\Phi(x) - I_{t,x}g, h \rangle \\ &= R_t\phi_h(x+g) - R_t\phi_h(x) - \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \phi_h(e^{At}x + y) N_{Q_t}(dy).\end{aligned}$$

Now, by Theorem 6.2.2 of [6],

$$\begin{aligned} R_t \phi_h(x+g) - R_t \phi_h(x) &= \int_0^1 \langle DR_t \phi_h(x+sg), g \rangle ds \\ &= \int_0^1 \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \phi_h(e^{At}(x+sg) + y) N_{Q_t}(dy) ds \\ &= \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \left(\int_0^1 \phi_h(e^{At}(x+sg) + y) ds \right) N_{Q_t}(dy). \end{aligned}$$

Thus we have

$$\begin{aligned} &\langle \mathcal{R}_t \Phi(x+g) - \mathcal{R}_t \Phi(x) - I_{t,x} g, h \rangle \\ &= \int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle \left[\int_0^1 \phi_h(e^{At}(x+sg) + y) - \phi_h(e^{tA}x + y) ds \right] N_{Q_t}(dy) \\ &\leq \left(\int_H \langle \Lambda_t g, Q_t^{-1/2} y \rangle^2 N_{Q_t}(dy) \right)^{1/2} \\ &\quad \left(\int_H \left[\int_0^1 \phi_h(e^{At}(x+sg) + y) - \phi_h(e^{tA}x + y) ds \right]^2 N_{Q_t}(dy) \right)^{1/2} \\ &\leq |\Lambda_t| \omega_t(g) \leq \|\Lambda_t\| |g| \omega_t(g) \end{aligned}$$

where

$$\omega_t(g) = \sup_{s \in [0,1], x, y \in H} |\Phi(e^{tA}(x+sg) + y) - \Phi(e^{tA}x + y)|.$$

Since $\Phi \in UC_b(H, H)$,

$$\lim_{g \rightarrow 0} \omega_t(g) = 0$$

and thus $\mathcal{R}_t \Phi$ is differentiable at x with differential $I_{t,x}$. One can check that the differential is uniformly continuous in x . Clearly, by (5.10), it is also bounded. Thus we have proved $\mathcal{R}_t \Phi \in UC_b^1(H, H)$ and all claims about $D\mathcal{R}_t \Phi$.

Step 3. For given t, x let us analyze the right-hand-side of (5.9). Following [6], Lemma 6.2.7, for every bounded measurable $\varphi : H \rightarrow \mathbb{R}$, let us introduce the linear operator $G_\varphi^{t,x}$ in H defined as (we use different notations for the Gaussian measure with respect to the quoted reference)

$$\langle G_\varphi^{t,x} \alpha, \beta \rangle = \int_H \left[\langle \alpha, Q_t^{-1/2} y \rangle \langle \beta, Q_t^{-1/2} y \rangle - \langle \alpha, \beta \rangle \right] \varphi(e^{tA}x + y) N_{Q_t}(dy).$$

It is prove in [6] that $G_\varphi^{t,x}$ is even Hilbert-Schmidt, with Hilbert-Schmidt norm bounded by $2\|\varphi\|_0$. Therefore, in particular, $G_\varphi^{t,x}$ is a bounded linear operator with norm

$$\|G_\varphi^{t,x}\| \leq 2\|\varphi\|_0.$$

To understand the right-hand-side of (5.9), let us introduce the linear mapping in H

$$k \mapsto J_{t,x,g} k := \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \Phi(e^{tA}x + y) N_{Q_t}(dy).$$

We have

$$\begin{aligned} |\langle J_{t,x,g}k, h \rangle| &= |\langle G_{\varphi_h}^{t,x} \Lambda_t g, \lambda_t k \rangle| \leq \|G_{\varphi_h}^{t,x}\| \|\Lambda_t g\| \|\Lambda_t k\| \leq 2\|\varphi_h\|_0 \|\Lambda_t\|^2 |g| |k| \\ &\leq 2\|\Lambda_t\|^2 \|\Phi\|_0 |g| |k| |h|. \end{aligned}$$

Thus $J_{t,x,g}$ is bounded and

$$\|J_{t,x,g}\| \leq 2\|\Lambda_t\|^2 \|\Phi\|_0 |g|.$$

Therefor $g \mapsto J_{t,x,g}$ is a bounded linear operator from H to $L(H, H)$, denoted by $J_{t,x}$ in the sequel of the proof (we have $J_{t,x}g = J_{t,x,g}$), and

$$\|J_{t,x}\|_{L(H, L(H, H))} \leq 2\|\Lambda_t\|^2 \|\Phi\|_0.$$

If we prove that $J_{t,x}$ is $D^2\mathcal{R}_t\Phi(x)$, we have also proved inequality (5.11).

The proof of (5.12) is similar and based on the Hilbert-Schmidt property mentioned above.

Step 4. Given t, x , let us prove that $D\mathcal{R}_t\Phi$ is differentiable at x , and its differential is $J_{t,x}$. Recall that $\langle D\mathcal{R}_t\Phi(x)g, h \rangle$ is equal to $\langle DR_t\phi_h(x), g \rangle$. We have, for $g, h, k \in H$,

$$\langle [D\mathcal{R}_t\Phi(x+g) - D\mathcal{R}_t\Phi(x)]k, h \rangle = \langle DR_t\phi_h(x+g) - DR_t\phi_h(x), k \rangle$$

hence, by Proposition 6.2.2 of [6] this equals,

$$\begin{aligned} &\int_0^1 \langle D^2 R_t \varphi_h(x+sg)g, k \rangle ds \\ &= \int_0^1 \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \phi_h(e^{tA}(xsg) + y) N_{Q_t}(dy) ds \\ &= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \left(\int_0^1 \phi_h(e^{tA}(xsg) + y) ds \right) N_{Q_t}(dy). \end{aligned}$$

Moreover,

$$\langle [J_{t,x}g]k, h \rangle = \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \phi_h(e^{tA}x + y) N_{Q_t}(dy).$$

Therefore

$$\begin{aligned} &\langle [D\mathcal{R}_t\Phi(x+g) - D\mathcal{R}_t\Phi(x) - J_{t,x}g]k, h \rangle \\ &= \int_H \left[\langle \Lambda_t g, Q_t^{-1/2} y \rangle \langle \Lambda_t k, Q_t^{-1/2} y \rangle - \langle \Lambda_t g, \Lambda_t k \rangle \right] \psi_{t,x,g,h}(y) N_{Q_t}(dy) \\ &= \langle G_{\psi_{t,x,g,h}} \Lambda_t g, \Lambda_t k \rangle \end{aligned}$$

where

$$\psi_{t,x,g,h}(y) = \int_0^1 [\varphi_h(e^{tA}(x+sg) + y) - \varphi_h(e^{tA}x + y)] ds$$

Hence, by Lemma 6.2.7 of [6],

$$\begin{aligned} |\langle [D\mathcal{R}_t\Phi(x+g) - D\mathcal{R}_t\Phi(x) - J_{t,x}g]k, h \rangle| &= |\langle G_{\psi_{t,x,g,h}} \Lambda_t g, \Lambda_t k \rangle| \\ &\leq 2\|\psi_{t,x,g,h}\|_0 \|\Lambda_t\|^2 |g| |k|. \end{aligned}$$

But

$$|\psi_{t,x,g,h}(y)| \leq |h| \int_0^1 [\Phi(e^{tA}(x+sg) + y) - \Phi(e^{tA}x + y)] ds \leq |h| \omega_t(g)$$

as in step 2. Therefore $D\mathcal{R}_t\Phi$ is differentiable at x and $D^2\mathcal{R}_t\Phi$ is $J_{t,x}$. One can check that $D^2\mathcal{R}_t\Phi$ is uniformly continuous in x . By (5.11), it is also bounded. We have proved $\mathcal{R}_t\Phi \in UC_b^2(H, H)$ and all claims about $D^2\mathcal{R}_t\Phi$ when $\Phi \in UC_b(H, H)$. The proof of the claims on $D^2\mathcal{R}_t\Phi$ when $\Phi \in UC_b^1(H, H)$ is similar and based on Proposition 6.2.9 of [6]. We do not give the details. The proof is complete.

■

5.4 Non homogenous Kolmogorov equation

In this section we assume the conditions on A, B, Q stated in the introduction and Section 5.1.1 and we study the sequence on non-homogeneous Kolmogorov equations in H

$$\begin{aligned} \frac{\partial U_n}{\partial t} &= \frac{1}{2} Tr(D^2 U_n Q) + \langle Ax, DU_n \rangle + \langle B, DU_n \rangle + G_n \\ U_n(0, x) &= 0 \end{aligned} \quad (5.13)$$

where G_n are the components of a function $G \in C([0, T]; C_b^\alpha(H, H))$. In this section we use forward notations for the PDE, for the sake of simplicity. The final result will apply to the backward PDE (5.5) (in particular, the assumption $B \in C([0, T]; C_b^\alpha(H, H))$ is invariant by time reversal).

We also show that the H -valued function $U(t, x) = \sum_n U_n(t, x)e_n$ has a meaning and we analyze its properties.

We interpret the PDE (5.13) as the integral equation

$$U_n(t, x) = \int_0^t R_{t-s}(\langle B(s), DU_n(s) \rangle + G_n(s))(x) ds. \quad (5.14)$$

Here we write $B(s)$ for $B(s, \cdot)$ and so on. Let us introduce also the H -valued equation

$$U(t, x) = \int_0^t \mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))(x) ds. \quad (5.15)$$

where we have denoted $\sum_n e_n \langle B(s), DU_n(s) \rangle$ by $\langle B(s), D \rangle U(s)$.

We can state the main result of this section. The regularity we prove for U is not optimal, and the theorem is restricted for simplicity of exposition to small T 's.

Theorem 5.3 *Under the assumptions of Section 5.1.1, given*

$$B, G \in C([0, T]; UC_b(H, H)),$$

for T small enough there exists a unique solution U of equation (5.15) in $C([0, T]; UC_b^1(H, H))$. If we put $K_T := \|DU\|_0$, then

$$\lim_{T \rightarrow 0} K_T = 0.$$

Moreover, $DU \in C([0, T]; C_b^\theta(H, H))$, θ such that assumption (5.4) hold.

If in addition $B, G \in C([0, T]; C_b^\alpha(H, H))$ for some $\alpha > 0$, then $U \in C([0, T]; UC_b^2(H, H))$. Finally, there is a constant $C_T > 0$ such that

$$\|D^2 U_n\|_0 \leq C_T \|G_n\|_\alpha \quad (5.16)$$

for every $n \in \mathbb{N}$.

Proof. Step 1. Consider the map $\mathcal{L}$ defined as

$$\mathcal{L}U(t, x) = \int_0^t \mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))(x) ds.$$

It is defined on functions $U \in C([0, T]; UC_b^1(H, H))$. It is easy to check that $\mathcal{L}U \in C([0, T]; UC_b(H, H))$. But we also have the bound

$$\begin{aligned} & \int_0^t \|\mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))(x)\| ds \\ & \leq \int_0^t \|\Lambda_{t-s}\| \|\langle B(s), D \rangle U(s) + G(s)\|_0 ds \\ & \leq (\|B\|_0 \|DU\|_0 + \|G\|_0) \int_0^t \|\Lambda_s\| ds \end{aligned}$$

which implies $\mathcal{L}U \in C([0, T]; UC_b^1(H, H))$ and

$$D\mathcal{L}U(t, x) = \int_0^t D\mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))(x) ds.$$

Since $\lim_{T \rightarrow 0} \int_0^T \|\Lambda\| ds = 0$, and the map $\mathcal{L}$ is linear, it is a contraction in $C([0, T]; UC_b^1(H, H))$ for sufficiently small T (one has to use also an estimate on U in the norm of $C([0, T]; UC_b(H, H))$). Moreover, if U is a solution, then

$$DU(t, x) = \int_0^t D\mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))(x) ds \quad (5.17)$$

hence

$$\|DU\|_0 \leq (\|B\|_0 \|DU\|_0 + \|G\|_0) \int_0^t \|\Lambda_s\| ds$$

hence, for T such that $\|B\|_0 \int_0^T \|\Lambda\| ds \leq 1/2$ we have

$$\frac{1}{2} \|DU\|_0 \leq \|G\|_0 \int_0^t \|\Lambda_s\| ds$$

which proves $\lim_{T \rightarrow 0} K_T = 0$. We have proved the first claims of the theorem.

Step 2. Let us recall a result from interpolation theory developed in [6], Chapter 2. From Theorem 2.3.3 and the remarks at the beginning of section 2.3.3, for every $\theta_1(0, 1)$ there is a constant $C_\theta > 0$ such that

$$\|\varphi\|_\theta \leq C_\theta \|\varphi\|_0^{1-\theta} \|\varphi\|_1^\theta$$

for every $\varphi \in UC_b^1(H, \mathbb{R})$. The same result is true for $\Phi \in UC_b^1(H, H)$. Indeed, for every $h \in H$ the function $\varphi_h = \langle \Phi(\cdot), h \rangle$ belongs to $UC_b^1(H, \mathbb{R})$, hence

$$|\langle \Phi(x) - \Phi(y), h \rangle| \leq C_\theta \|\varphi_h\|_0^{1-\theta} \|\varphi_h\|_1^\theta |h| |x - y|^\theta.$$

But $\|\varphi_h\|_0 \leq \|\Phi\|_0 |h|$ and $\|\varphi_h\|_1 \leq \|\Phi\|_1 |h|$. Hence

$$|\langle \Phi(x) - \Phi(y), h \rangle| \leq C_\theta \|\Phi\|_0^{1-\theta} \|\Phi\|_1^\theta |h| |x - y|^\theta$$

which implies

$$\|\Phi\|_\theta \leq C_\theta \|\Phi\|_0^{1-\theta} \|\Phi\|_1^\theta.$$

We also have

$$\|\Phi\|_\theta \leq C_\theta \|\Phi\|_0^{1-\theta} \|D\Phi\|_0^\theta + C_\theta \|\Phi\|_0.$$

Similary, if $\Phi \in UC_b^2(H, H)$, we have

$$\|D\Phi\|_\theta \leq C_\theta \|D\Phi\|_0^{1-\theta} \|D^2\Phi\|_0^\theta + C_\theta \|D\Phi\|_0.$$

Step 3. Let us apply the previous interpolation inequality to $\mathcal{R}_t\Phi$, $\Phi \in UC_b(H, H)$, $t \geq 0$, with $\theta \in (0, 1)$:

$$\begin{aligned} \|D\mathcal{R}_t\Phi\|_\theta &\leq C_\theta \|D\mathcal{R}_t\Phi\|_0^{1-\theta} \|D^2\mathcal{R}_t\Phi\|_0^\theta + C_\theta \|D\mathcal{R}_t\Phi\|_0 \\ &\leq C_\theta (\|\Lambda_t\| \|\Phi\|_0)^{1-\theta} (\sqrt{2} \|\Lambda_t\|^2 \|\Phi\|_0)^\theta + C_\theta \|\Lambda_t\| \|\Phi\|_0 \\ &\leq C'_\theta (\|\Lambda_t\|^{1+\theta} + 1) \|\Phi\|_0 \end{aligned}$$

for a new constant $C'_\theta > 0$, where we have used inequalities (5.10) and (5.11). Thus from (5.17) we have

$$\begin{aligned} \|DU(t)\|_\theta &\leq \int_0^t \|D\mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))\|_\theta ds \\ &\leq \int_0^t C'_\theta (\|\Lambda_t\|^{1+\theta} + 1) \|\langle B(s), D \rangle U(s) + G(s)\|_0 ds \\ &= C'_\theta \|\langle B, D \rangle U + G\|_0 \int_0^t (\|\Lambda_t\|^{1+\theta} + 1) ds. \end{aligned}$$

If θ satisfies the assumption of section 5.1.1, namely $\int_0^t \|\Lambda_t\|^{1+\theta} ds < \infty$, we deduce that $DU(t) \in C_b^\theta(H, H)$ for each $t \in [0, T]$. Easily one can check that $DU \in C([0, T], C_b^\theta(H, H))$.

Step 4. Assume now $B, G \in C([0, T], C_b^\alpha(H, H))$. Since $\theta \geq \alpha$ (see section 5.1.1), we know that $\langle B, D \rangle U + G \in C([0, T], C_b^\alpha(H, H))$. We use again an interpolation result of [6], see the proof of Lemma 6.4.1: there exists $C''_\alpha > 0$ such that

$$\|D^2\mathcal{R}_t\varphi(x)\| \leq C''_\alpha \|\Lambda_t\|^{2-\alpha} \|\varphi\|_\alpha$$

for all $\varphi \in C_b^\alpha(H, \mathbb{R})$. It follows that

$$\|D^2\mathcal{R}_t\Phi(x)\| \leq C''_\alpha \|\Lambda_t\|^{2-\alpha} \|\Phi\|_\alpha$$

for all $\Phi \in C_b^\alpha(H, H)$.

Using these fact, from (5.17) we have

$$\begin{aligned} \|D^2U(t)\| &\leq \int_0^t \|D^2\mathcal{R}_{t-s}(\langle B(s), D \rangle U(s) + G(s))\| ds \\ &\leq \int_0^t C_\theta'' \|\Lambda_t\|^{2-\alpha} \|\langle B(s), D \rangle U(s) + G(s)\|_\alpha ds \\ &= C_\theta' \|\langle B, D \rangle U + G\|_\alpha \int_0^t \|\Lambda_s\|^{2-\alpha} ds \end{aligned}$$

We have $\int_0^t \|\Lambda_s\|^{2-\alpha} ds < \infty$ (see section 5.1.1), hence $U \in C([0, T]; UC_b^2(H, H))$.

Step 5. From (5.17) or directly from equation (5.14) we have

$$DU_n(t, x) = \int_0^t DR_{t-s}(\langle B(s), DU_n(s) \rangle + G_n(s))(x) ds$$

and thus

$$D^2U_n(t, x) = \int_0^t D^2R_{t-s}(\langle B(s), DU_n(s) \rangle + G_n(s))(x) ds.$$

From the first one of these identities, with the same computations of step 1, we get (on the interval $[0, T]$ found in step 1)

$$\|DU_n\|_0 \leq C_1 \|G_n\|_0.$$

As in step 3, we get

$$\|DU_n\|_\alpha \leq C_2 \|\langle B, DU_n \rangle + G_n\|_0$$

and thus

$$\|DU_n\|_\alpha \leq (C_1 C_2 \|B\|_0 + C_2) \|G_n\|_0.$$

Finally, from the equation for $D^2U_n(t, x)$, exactly as in step 4, we prove

$$\|D^2U_n\|_0 \leq C_3 \|\langle B, DU_n \rangle + G_n\|_\alpha.$$

Putting together these estimates, we obtain (5.16). The proof is complete. $\blacksquare$

5.5 Malliavin Differentiability

5.5.1 Strong Uniqueness

We fix a filtered probability space $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}_{t \in [0, T]}$ such that W is a $\mathcal{F}_t$ -cylindrical Brownian motion on H . A mild solution of equation (5.1) is a process $X = (X_t)_{t \in [0, T]}$, which is an $\mathcal{F}_t$ -adapted continuous process in H and satisfies

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B(s, X_s) ds + \int_0^t e^{(t-s)A} \sqrt{Q} dW_s.$$

The stochastic integral is well defined since we have assumed Q_t is of trace class.

The following rewriting is essential to our estimates:

Lemma 5.4 *Under the assumptions of Section 5.1.1, let U be the solution given by Theorem 5.3. If $X = (X)_{t \in [0, T]}$ is a mild solution of equation (5.1), then the equation (5.6) is satisfied.*

Proof. Having now Theorem 5.3, the proof is the one given in Section 5.2. The only point is the application of Itô's formula. In order to use elementary versions of it, one can introduce the approximations

$$dX_t^{j,h} = A_j X_t^{j,h} dt + B(t, X_t^{j,h}) dt + P_h \sqrt{Q} dW_t, \quad X_0^{j,h} = x$$

where A_j are the Yosida approximations of A , $P_h x = \sum_{i=1}^h x_i e_i$. The computations of Section 5.2 can be done on these approximations and then one can pass to the limit in the final equation. We omit the details which are classical.

■

Using the previous lemma we proceed to prove pathwise uniqueness for the equation (5.1).

Theorem 5.5 *There exists a $T > 0$ such that pathwise uniqueness holds for (5.1) on $[0, T]$. That is, if X^1 and X^2 are two mild solutions, then we have for $\text{leb} \times P$ almost all $(t, x) \in [0, T] \times \Omega$, $X_t^1(\omega) = X_t^2(\omega)$.*

Proof. Assume X^1 and X^2 are two milds solutions, and define $V_t = X_t^1 - X_t^2$. Then, by Lemma 5.4 we have

$$\begin{aligned} \int_0^T |V_t|^2 dt &\leq 3 \int_0^T |U(t, X_t^1) - U(t, X_t^2)|^2 dt \\ &\quad + 3 \int_0^T \left| \int_0^t A e^{(t-s)A} [U(s, X_s^1) - U(s, X_s^2)] ds \right|^2 dt \\ &\quad + 3 \int_0^T |I_t(X^1) - I_t(X^2)|^2 dt. \end{aligned}$$

From Theorem 5.3 we have

$$|U(t, X_t^1) - U(t, X_t^2)| \leq K_T |X_t^1 - X_t^2|, \quad t \in [0, T].$$

To deal with the second term we use the maximal inequality

$$\left\| \int_0^\cdot A e^{(\cdot-s)A} f(s) ds \right\|_{L^2(0, T; H)}^2 \leq C_T \|f\|_{L^2(0, T; H)}^2$$

where C_T is a constant independent of f . Notice, however, that C_T does not converge to 0 as $T \rightarrow 0$. We then make the following estimate:

$$\int_0^T |V_t|^2 dt \leq (3 + 3C_T) K_T \int_0^T |V_t|^2 dt + 3 \int_0^T |I_t(X^1) - I_t(X^2)|^2 dt.$$

For T small enough we thus have

$$\int_0^T |V_t|^2 dt \leq 6 \int_0^T |I_t(X^1) - I_t(X^2)|^2 dt,$$

and in particular

$$\int_0^T E[|V_t|^2] dt \leq 6 \int_0^T E[|I_t(X^1) - I_t(X^2)|^2] dt.$$

The proof will be complete once we find an estimate on the right-and side of the previous inequality. We have

$$\int_0^T E[|I_t(X^1) - I_t(X^2)|^2]dt = \int_0^t \|e^{(t-s)A}(DU(s, X_s^1) - DU(s, X_s^2))\sqrt{Q}\|_{HS}^2 ds.$$

For the kernel, we write

$$\begin{aligned} & \|e^{(t-s)A}(DU(s, X_s^1) - DU(s, X_s^2))\sqrt{Q}\|_{HS}^2 \\ &= \sum_{n,h \geq 1} \langle e^{(t-s)A}(DU(s, X_s^1) - DU(s, X_s^2))\sqrt{Q}e_h, e_n \rangle^2 \\ &= \sum_{n,h \geq 1} e^{-2(t-s)\alpha_n} \langle (DU_n(s, X_s^1) - DU_n(s, X_s^2)), \sqrt{Q}e_h \rangle^2 \\ &= \sum_{n \geq 1} e^{-2(t-s)\alpha_n} \sum_{h \geq 1} \langle \sqrt{Q}(DU_n(s, X_s^1) - DU_n(s, X_s^2)), e_h \rangle^2 \\ &\leq \|Q\| \sum_{n \geq 1} e^{-2(t-s)\alpha_n} |DU_n(s, X_s^1) - DU_n(s, X_s^2)|^2 \\ &\leq \|Q\| \sum_{n \geq 1} e^{-2(t-s)\alpha_n} \|DU_n\|_\infty^2 |X_s^1 - X_s^2|^2 \end{aligned}$$

From Theorem 5.3 we have

$$\|DU_n\|_\infty \leq C_T \|B_n\|_\alpha,$$

hence

$$E[|I_t(X^1) - I_t(X^2)|^2] \leq C_T^2 \|Q\| \int_0^t \sum_{n \geq 1} e^{-2(t-s)\alpha_n} \|B_n\|_\alpha^2 |V_s|^2 ds.$$

Therefore

$$\begin{aligned} E\left[\int_0^T |I_t(X^1) - I_t(X^2)|^2 dt\right] &\leq C_T^2 \|Q\| \int_0^T \int_0^t \sum_{n \geq 1} e^{-2(t-s)\alpha_n} \|B_n\|_\alpha^2 |V_s|^2 ds \\ &\leq C_T^2 \|Q\| \|B\|_\alpha^2 \int_0^T \left(\int_s^T \sum_{n \geq 1} e^{-2(t-s)\alpha_n} dt \right) |V_s|^2 ds \\ &\leq C_T^2 \|Q\| \|B\|_\alpha^2 \left(\int_0^T \sum_{n \geq 1} e^{-2t\alpha_n} dt \right) \int_0^T |V_s|^2 ds. \end{aligned}$$

By assumption (5.2) we have $\lim_{T \rightarrow 0} \int_0^T \sum_{n \geq 1} e^{-2t\alpha_n} dt = 0$, so that for small enough T we have

$$E\left[\int_0^T |V_s|^2 ds\right] = 0$$

which gives the result. $\blacksquare$

Notice that by the Yamada-Watanabe theorem, the previous theorem coupled with weak existence is enough to guarantee strong existence of equation (5.1). We will not elaborate further on this here.

In this paper we will however use Malliavin calculus to construct the solution. As a by-product of the construction method, we will prove that the solution is Malliavin differentiable.

5.5.2 Malliavin Differentiability

In the remainder of this section we want to use a compactness criterion for L^2 -functionals of W_t based on Malliavin calculus (see Appendix, Theorem 5.14) to construct Malliavin differentiable mild solutions to (5.1).

To this end we need some definitions and auxiliary results.

Denote by $L_2(H)$ the space of Hilbert-Schmidt operators from H into itself with norm $\|\cdot\|_{HS}$. In what follows let $M : D(M) \subset H \rightarrow H$ be a non-negative self-adjoint operator with existing compact inverse M^{-1} . Further consider the space E obtained by completion with respect to the norm $\|\cdot\|_E$ given by

$$\|K\|_E^2 := \sum_{n \geq 1} \|K M e_n\|^2$$

for $K \in L_2(H)$, if defined.

From now on we also assume that $Q^{1/2}M$ has a self-adjoint continuous extension to H such that

$$\left\| e^{(t-u)A} Q^{1/2} \right\|_E^2 \leq C \frac{1}{(t-u)^{1-\delta}} \quad (5.18)$$

for all $t > u \geq 0$ and

$$\begin{aligned} & \left\| e^{(t-u_1)A} Q^{1/2} - e^{(t-u_2)A} Q^{1/2} \right\|_E^2 \\ & \leq C \frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^\mu \end{aligned} \quad (5.19)$$

for all $t > u_1 > u_2 \geq 0$ and some $\mu > 0$. Further we also assume that

$$\int_H \left\| e^{(u_1-u_2)A} y - y \right\|^2 N_{Q_s}(dy) \leq C |u_1 - u_2|^\eta \quad (5.20)$$

for all $u_1 > u_2 \geq 0$, $s \geq 0$ and some $\eta > 0$.

Remark 5.6 Since $\|K\|_{HS} \leq C \|K\|_E$ for all $K \in E$ for a constant C depending on M we also see that

$$\left\| e^{(t-u)A} Q^{1/2} \right\|_{HS}^2 \leq C \frac{1}{(t-u)^{1-\delta}} \quad (5.21)$$

for all $t > u \geq 0$,

$$\begin{aligned} & \left\| e^{(t-u_1)A} Q^{1/2} - e^{(t-u_2)A} Q^{1/2} \right\|_{HS}^2 \\ & \leq C \frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^\vartheta \end{aligned} \quad (5.22)$$

for all $t > u_1 > u_2 \geq 0$.

The next result shows that if B in (5.1) is "nice" then the E -norm of the Malliavin derivative $D_u X_t$ of X_t exists u -a.e., P -a.e.

Lemma 5.7 Suppose that $B \in C([0, T]; C_b^1(H, H))$ in (1). Further assume the conditions (5.18) and (5.19). Then

$$\begin{aligned} & E \left[\int_0^T \|D_u X_t\|_E^2 du \right] \\ & \leq C \left(\frac{1}{\delta} T^\delta + \exp \left(2 \sup_{0 \leq r \leq T} \|e^{rA}\| \sup_{0 \leq s \leq T} \|DB(s, \cdot)\|_0 T \right) \right) \end{aligned}$$

for all $0 \leq t \leq T$.

Proof. Since

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}B(s, X_s)ds + \int_0^t e^{(t-s)A}Q^{1/2}dW_s, \quad 0 \leq t \leq T,$$

we find for all $0 \leq u \leq T$ that

$$D_u X_t = \int_u^t e^{(t-s)A}DB(s, X_s)D_u X_s ds + e^{(t-u)A}Q^{1/2}, \quad u \leq t \leq T.$$

So we obtain by Picard iteration that

$$\begin{aligned} & D_u X_t \\ & = e^{(t-u)A}Q^{1/2} + \sum_{n \geq 2} \int_{u \leq s_1 < \dots < s_{n-1} \leq t} e^{(s_n - s_{n-1})A}DB(s_{n-1}, X_{s_{n-1}}) \dots \\ & \quad e^{(s_2 - s_1)A}DB(s_1, X_{s_1})e^{(s_1 - u)A}Q^{1/2}ds_1 \dots ds_{n-1} \end{aligned} \quad (5.23)$$

in $L^2(\Omega; L_2(H))$ for all $u \leq t \leq T$.

Hence it follows from (5.23) $t > u$ that

$$\begin{aligned} & \|D_u X_t\|_E \\ & \leq \left\| e^{(t-u)A}Q^{1/2} \right\|_E + \sum_{n \geq 2} \int_{u \leq s_1 < \dots < s_{n-1} \leq t} \sup_{0 \leq r \leq T} \|e^{rA}\| \|DB(s_{n-1}, \cdot)\|_0 \dots \\ & \quad \sup_{0 \leq r \leq T} \|e^{rA}\| \|DB(s_1, \cdot)\|_0 \left\| e^{(s_1 - u)A}Q^{1/2} \right\|_E ds_1 \dots ds_{n-1} \\ & \leq \left\| e^{(t-u)A}Q^{1/2} \right\|_E + \sum_{n \geq 2} \sup_{0 \leq r \leq T} \|e^{rA}\|^{n-1} \\ & \quad \cdot \frac{1}{(n-1)!} \int_{[u, T]^{n-1}} \|DB(s_{n-1}, \cdot)\|_0 \cdot \dots \cdot \|DB(s_1, \cdot)\|_0 \left\| e^{(s_1 - u)A}Q^{1/2} \right\|_E ds_1 \dots ds_{n-1} \\ & \leq \left\| e^{(t-u)A}Q^{1/2} \right\|_E + C \sum_{n \geq 2} \frac{1}{(n-1)!} \left(\sup_{0 \leq r \leq T} \|e^{rA}\| \sup_{0 \leq s \leq T} \|DB(s, \cdot)\|_0 \right)^{n-1} (T-u)^{n-2} \\ & \quad \cdot \int_u^T \frac{1}{(s_1 - u)^{1-\delta}} ds_1 \\ & \leq \left\| e^{(t-u)A}Q^{1/2} \right\|_E + C \left(1 + \sum_{n \geq 2} \frac{1}{(n-1)!} \left(\sup_{0 \leq r \leq T} \|e^{rA}\| \sup_{0 \leq s \leq T} \|DB(s, \cdot)\|_0 T \right)^{n-1} \right) \frac{1}{\delta} (T-u)^\delta \frac{1}{T} \\ & \leq \left\| e^{(t-u)A}Q^{1/2} \right\|_E + C_{T, \delta} \exp \left(\sup_{0 \leq r \leq T} \|e^{rA}\| \sup_{0 \leq s \leq T} \|DB(s, \cdot)\|_0 T \right). \end{aligned} \quad (5.25)$$

So

$$\begin{aligned}
& \int_0^T \|D_u X_t\|_E^2 du = \int_0^t \|D_u X_t\|_E^2 du \\
& \leq C \left(\int_0^t \frac{1}{(t-u)^{1-\delta}} du + \exp \left(2 \sup_{0 \leq r \leq T} \|e^{rA}\| \sup_{0 \leq s \leq T} \|DB(s, \cdot)\|_0 T \right) \right) \\
& = C \left(\frac{1}{\delta} t^\delta + \exp \left(2 \sup_{0 \leq r \leq T} \|e^{rA}\| \sup_{0 \leq s \leq T} \|DB(s, \cdot)\|_0 T \right) \right) < \infty.
\end{aligned}$$

■

We shall also use the following Lemma

Lemma 5.8 *Let B in (5.1) be in $C([0, T]; C_b^1(H, H))$. Then $X_t \in D(A^{\gamma/2})$ P -a.e. for all $0 < \gamma < 1$ and*

$$E[\|A^{\gamma/2} X_t\|^2] \leq C \frac{1}{t^\gamma} (1 + \|B\|_0^2)$$

for all $0 \leq t \leq T$.

Proof. Since

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B(s, X_s) ds + \int_0^t e^{(t-s)A} Q^{1/2} dW_s, \quad 0 \leq t \leq T,$$

it is sufficient to prove that

$$E[\|Q_i\|^2] \leq C_i \frac{1}{t^\gamma} (1 + \|B\|_0^2), \quad i = 1, 2, 3,$$

where

$$\begin{aligned}
Q_1 &:= A^{\gamma/2} e^{tA} x, \quad Q_2 := \int_0^t A^{\gamma/2} e^{(t-s)A} B(s, X_s) ds, \\
Q_3 &:= \int_0^t A^{\gamma/2} e^{(t-s)A} dW_s.
\end{aligned}$$

Then using the inequality

$$\|A^{\gamma/2} e^{(t-s)A}\| \leq \frac{C_\gamma}{(t-s)^{\gamma/2}}$$

and Itô's isometry the result follows. ■

The next Lemma will be crucial for the application of the compactness criterion Theorem 5.14 in the Appendix.

Lemma 5.9 *Assume that $B \in C([0, T]; C_b^1(H, H)) \cap C([0, T]; C_b^\alpha(H, H))$. Let X be the mild solution to (5.1) associated with the coefficient B . Then for all $0 \leq \theta \leq T$ there exists a $0 < \beta < \frac{1}{2}$ such that*

$$E \left[\int_0^\theta \|D_u X_\theta\|_E^2 du \right] \leq L_1 (\|B\|_\alpha^2) \quad (5.26)$$

and

$$E \left[\int_0^\theta \int_0^\theta \frac{\|D_{u_1} X_\theta - D_{u_2} X_\theta\|_E^2}{|u_1 - u_2|^{1+2\beta}} du_1 du_2 \right] \leq L_2 (\|B\|_\alpha^2) \quad (5.27)$$

where $L_i, i = 1, 2$ are non-negative continuous functions on $[0, \frac{1}{V_T}]$ with $V_T \rightarrow 0$ for $T \rightarrow 0$.

Proof By applying the chain rule for the Malliavin derivative (see [13]) we know that

$$\begin{aligned}
D_u X_t &= DU(t, X_t) D_u X_t + \int_u^t A e^{(t-s)A} DU(s, X_s) D_u X_s ds \\
&\quad + e^{(t-u)A} Q^{1/2} - e^{(t-u)A} DU(u, X_u) Q^{1/2} \\
&\quad - \int_u^t e^{(t-s)A} D^2 U(s, X_s) Q^{1/2} D_u X_s dW_s \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{5.28}$$

for $u \leq t < T$ P -a.e., where

$$\begin{aligned}
I_1 &:= DU(t, X_t) D_u X_t, \quad I_2 := \int_u^t A e^{(t-s)A} DU(s, X_s) D_u X_s ds, \quad I_3 := e^{(t-u)A} Q^{1/2}, \\
I_4 &:= -e^{(t-u)A} DU(u, X_u) Q^{1/2}, \quad I_5 := - \int_u^t e^{(t-s)A} D^2 U(s, X_s) Q^{1/2} D_u X_s dW_s.
\end{aligned}$$

We want to use Gronwall's Lemma to show (5.26) and (5.27). To this end we need some estimates of $I_1, \dots, I_5$.

1. Estimate for I_1 : By Lemma 5.7 and the estimates of Theorem 3 we find that

$$\begin{aligned}
E[\|I_1\|_E^2] &\leq \|DU\|_0^2 E[\|D_u X_t\|_E^2] \\
&\leq K_T \|B\|_0^2 E[\|D_u X_t\|_E^2] < \infty
\end{aligned}$$

for $t > u$, where $\lim_{T \rightarrow 0} K_T = 0$.

2. Estimate of I_2 : Using the inequalities

$$\|A^\epsilon e^{At}\| \leq \frac{C_\epsilon}{t^\epsilon}$$

and

$$\|A^\epsilon DU(s, X_t)\| \leq C_{\epsilon, T} \|B\|_0$$

for $0 < \epsilon < 1$ (see [8]) we obtain

$$\begin{aligned}
&E[\|I_2\|_E^2] \\
&\leq (t-u) E\left[\int_u^t \left\|A e^{(t-s)A} DU(s, X_s)\right\|^2 \|D_u X_s\|_E^2 ds\right] \\
&\leq (t-u) E\left[\int_u^t \left\|A^{1-\gamma} e^{(t-s)A}\right\|^2 \|A^\gamma DU(s, X_s)\|^2 \|D_u X_s\|_E^2 ds\right] \\
&\leq (t-u) C_{\epsilon, T} \|B\|_0^2 E\left[\int_u^t \frac{C_\gamma}{(t-s)^{2(1-\gamma)}} \|D_u X_s\|_E^2 ds\right] \\
&= (t-u) C_{\epsilon, T} C_\gamma \|B\|_0^2 \int_u^t \frac{1}{(t-s)^{2(1-\gamma)}} E[\|D_u X_s\|_E^2] ds
\end{aligned}$$

for $1 > \gamma > 0$ with $2(1-\gamma) < 1$ (see (5.25)).

3. Estimation of I_3 : We know from (5.18) and (5.19) that

$$\|I_3\|_E^2 \leq C \frac{1}{(t-u)^{1-\delta}}.$$

As for the other two estimates we use the notation of the previous section and let

$$\Phi = \Phi(s) := \langle B(s), D \rangle U(s) - B(s).$$

4. Estimate for I_4 : Because of our assumptions and Theorem 3 we obtain

$$\begin{aligned}
\|I_4\|_E^2 &= \left\| e^{(t-u)A} DU(u, X_u) Q^{1/2} \right\|_E^2 \\
&= \left\| e^{(t-u)A} DU(u, X_u) Q^{1/2} M \right\|_{H.S.}^2 \\
&= \sum_{n,h \geq 1} \langle e^{(t-u)A} DU(u, X_u) Q^{1/2} M e_h, e_n \rangle^2 \\
&= \sum_{n,h \geq 1} e^{-2\alpha_n(t-u)} \langle DU(u, X_u) Q^{1/2} M e_h, e_n \rangle^2 \\
&= \sum_{n,h \geq 1} e^{-2\alpha_n(t-u)} \langle DU_n(u, X_u), Q^{1/2} M e_h \rangle^2 \\
&= \sum_{n \geq 1} e^{-2\alpha_n(t-u)} \sum_{h \geq 1} \langle Q^{1/2} M DU_n(u, X_u), e_h \rangle^2 \\
&\leq \sum_{n \geq 1} e^{-2\alpha_n(t-u)} \left\| Q^{1/2} M \right\|^2 \|DU_n(u, X_u)\|^2 \\
&\leq \left\| Q^{1/2} M \right\|^2 C_T \left(\int_0^T \|\Lambda_s\| ds \right)^2 \sum_{n \geq 1} e^{-2\alpha_n(t-u)} \|B_n\|_0^2 \\
&= \left\| Q^{1/2} M \right\|^2 C_T \left(\int_0^T \|\Lambda_s\| ds \right)^2 \cdot \\
&\quad \cdot \left(\sum_{n \geq 1} (2\alpha_n(t-u))^{(1-\delta)} 2e^{-2\alpha_n(t-u)} (2\alpha_n(t-u))^{-(1-\delta)} \|B_n\|_0^2 \right) \\
&\leq C_\delta \left\| Q^{1/2} M \right\|^2 C_T \left(\int_0^T \|\Lambda_s\| ds \right)^2 \|B\|_0^2 2^{-(1-\delta)} \frac{1}{(t-u)^{(1-\delta)}}
\end{aligned}$$

if T is small enough. Hence

$$E[\|I_4\|_E^2] \leq C_\delta \left\| Q^{1/2} M \right\|^2 C_T \left(\int_0^T \|\Lambda_s\| ds \right)^2 \|B\|_0^2 2^{-(1-\delta)} \frac{1}{(t-u)^{(1-\delta)}}.$$

5. Estimate for I_5 : By our assumptions and the estimates of Theorem 2 we get for fixed

$r \in \mathbb{N}$

$$\begin{aligned}
& \sum_{h \geq 1} \left\| e^{(t-s)A} D^2 U(s, X_s) [Q^{1/2} e_h, D_u X_s M e_r] \right\|^2 \\
&= \sum_{n, h \geq 1} \langle e^{(t-s)A} D^2 U(s, X_s) [Q^{1/2} e_h, D_u X_s M e_r], e_n \rangle^2 \\
&= \sum_{n, h \geq 1} e^{-2\alpha_n(t-s)} \langle D^2 U(s, X_s) [Q^{1/2} e_h, D_u X_s M e_r], e_n \rangle^2 \\
&= \sum_{n, h \geq 1} e^{-2\alpha_n(t-s)} (D^2 U_n(s, X_s) [Q^{1/2} e_h, D_u X_s M e_r])^2 \\
&= \sum_{n \geq 1} e^{-2\alpha_n(t-s)} \sum_{h \geq 1} (D^2 U_n(s, X_s) [Q^{1/2} e_h, D_u X_s M e_r])^2 \\
&\leq \sum_{n \geq 1} e^{-2\alpha_n(t-s)} \|Q\| \|D^2 U_n\|_0^2 \|D_u X_s M e_r\|^2 \\
&\leq C_T \|Q\| \sum_{n \geq 1} e^{-2\alpha_n(t-s)} \|B_n\|_\alpha^2 \|D_u X_s M e_r\|^2 \\
&= C_T \|Q\| \sum_{n \geq 1} (2\alpha_n(t-s))^{(1-\delta)} 2e^{-2\alpha_n(t-s)} \cdot \\
&\quad \cdot (2\alpha_n(t-s))^{-(1-\delta)} \|B_n\|_\alpha^2 \|D_u X_s M e_r\|^2 \\
&\leq C_\delta C_T \|Q\| 2^{-(1-\delta)} \|B\|_\alpha^2 \|D_u X_s M e_r\|^2 \frac{1}{(t-s)^{1-\delta}}
\end{aligned}$$

So it follows from the Itô isometry that

$$\begin{aligned}
& E[\|I_5\|_E^2] \\
&= \sum_{r \geq 1} \sum_{h \geq 1} \int_u^t \left\| e^{(t-s)A} D^2 U(s, X_s) [Q^{1/2} e_h, D_u X_s M e_r] \right\|^2 ds \\
&\leq C_\delta C_T \|Q\| 2^{-(1-\delta)} \|B\|_\alpha^2 \int_u^t \frac{1}{(t-s)^{1-\delta}} \|D_u X_s\|_E^2 ds.
\end{aligned}$$

So using the above estimates we get

$$\begin{aligned}
& E[\|D_u X_t\|_E^2] \\
&\leq C(K_T \|B\|_\alpha^2 E[\|D_u X_t\|_E^2] + (t-u) C_{\varepsilon, T} C_\gamma \|B\|_\alpha^2 \int_u^t \frac{1}{(t-s)^{2(1-\gamma)}} E[\|D_u X_s\|_E^2] ds \\
&\quad + C \frac{1}{(t-u)^{1-\delta}} + C_\delta \left\| Q^{1/2} M \right\|^2 C_T \left(\int_0^T \|\Lambda_s\| ds \right)^2 \|B\|_\alpha^2 2^{-(1-\delta)} \frac{1}{(t-u)^{(1-\delta)}} \\
&\quad + C_\delta C_T \|Q\| 2^{-(1-\delta)} \|B\|_\alpha^2 \int_u^t \frac{1}{(t-s)^{1-\delta}} \|D_u X_s\|_E^2 ds)
\end{aligned}$$

for $u \leq t \leq T$. Thus

$$\begin{aligned}
& E[\|D_u X_t\|_E^2] \\
& \leq \frac{C}{1 - CK_T \|B\|_\alpha^2} \cdot \\
& \quad \cdot (C \frac{1}{(t-u)^{1-\delta}} + C_\delta \|Q^{1/2} M\|^2 C_T (\int_0^T \|\Lambda_s\| ds)^2 \|B\|_\alpha^2 2^{-(1-\delta)} \frac{1}{(t-u)^{(1-\delta)}}) \\
& \quad + \frac{C}{1 - CK_T \|B\|_\alpha^2} \cdot \\
& \quad \cdot ((t-u) C_{\varepsilon, T} C_\gamma \|B\|_\alpha^2 + C_\delta C_T \|Q\| 2^{-(1-\delta)} \|B\|_\alpha^2) \int_u^t \frac{1}{(t-s)^{1-\delta}} \|D_u X_s\|_E^2 ds
\end{aligned}$$

for $u \leq t \leq T$ with T small enough such that $CK_T \|B\|_0 < 1$. Hence by a generalized Lemma of Gronwall for weakly singular kernels (see [1, Theorem 3]) we get

$$E[\|D_u X_t\|_E^2] \leq a(t) + \int_u^t \sum_{n \geq 1} \frac{(g(t)\Gamma(\delta))^n}{\Gamma(n\delta)} (t-s)^{n\delta-1} a(s) ds$$

for $u \leq t \leq T$, where

$$\begin{aligned}
& a(s) \\
& : = \frac{C}{1 - CK_T \|B\|_\alpha^2} \cdot \\
& \quad \cdot (C + C_\delta \|Q^{1/2} M\|^2 C_T (\int_0^T \|\Lambda_r\| dr)^2 \|B\|_\alpha^2 2^{-(1-\delta)} \frac{1}{(s-u)^{1-\delta}}, \\
& \\
& g(t) \\
& : = \frac{C}{1 - CK_T \|B\|_\alpha^2} ((t-u) C_{\varepsilon, T} C_\gamma \|B\|_\alpha^2 + C_\delta C_T \|Q\| 2^{-(1-\delta)} \|B\|_\alpha^2)
\end{aligned}$$

and where Γ is the Gamma function.

Let us now assume that $n_0 \delta < 1$, but $(n_0 + 1)\delta \geq 1$ for $n_0 \in \mathbb{N}$.

Therefore by using the following relation based on the Beta function

$$\int_u^t (t-s)^{n\delta-1} \frac{1}{(s-u)^{1-\delta}} ds = \frac{\Gamma(n\delta)\Gamma(\delta)}{\Gamma(n\delta+\delta)} (t-u)^{(n+1)\delta-1},$$

where Γ is the Gamma function, we obtain

$$\begin{aligned}
& E[\|D_u X_t\|_E^2] \\
& \leq L_1(\|B\|_\alpha) \frac{1}{(t-u)^{1-\delta}} + L_1(\|B\|_\alpha) \sum_{n \geq 1} \frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n}{\Gamma(n\delta)} \frac{\Gamma(n\delta)\Gamma(\delta)}{\Gamma(n\delta+\delta)} (t-u)^{(n+1)\delta-1} \\
& \leq L_1(\|B\|_\alpha) \frac{1}{(t-u)^{1-\delta}} + L_1(\|B\|_\alpha) \sum_{n=1}^{n_0-1} \frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n \Gamma(\delta)}{\Gamma((n+1)\delta)} \frac{1}{(t-u)^{1-(n+1)\delta}} \\
& \quad + L_1(\|B\|_\alpha) \sum_{n \geq n_0} \frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n \Gamma(\delta)}{\Gamma((n+1)\delta)} (t-u)^{(n+1)\delta-1}, \tag{5.29}
\end{aligned}$$

where L_1 and L_2 are non-negative continuous functions on $[0, \frac{1}{2CK_T}]$, where $K_T \rightarrow 0$ for $T \rightarrow 0$.

Altogether we get

$$\begin{aligned}
& E\left[\int_0^T \|D_u X_t\|_E^2 du\right] \\
&= E\left[\int_0^t \|D_u X_t\|_E^2 du\right] \\
&\leq L_1(\|B\|_\alpha) \frac{1}{\delta} t^\delta + L_1(\|B\|_\alpha) \sum_{n=1}^{n_0-1} \frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n \Gamma(\delta)}{\Gamma((n+1)\delta)} \frac{t^{(n+1)\delta}}{(n+1)\delta} \\
&\quad + L_1(\|B\|_\alpha) \sum_{n \geq n_0} \frac{(L_2(\|B\|_\alpha)\Gamma(\delta))^n T^{(n+1)\delta-1} \Gamma(\delta)}{\Gamma((n+1)\delta)} t.
\end{aligned}$$

Let us now show the estimate (5.27). Assume that $t = \theta \geq u_1 > u_2 \geq 0$. Then it follows from (5.28)

$$D_{u_2} X_t - D_{u_1} X_t = \sum_{k=1}^7 J_k,$$

where

$$\begin{aligned}
J_1 &:= DU(t, X_t)(D_{u_2} X_t - D_{u_1} X_t), \\
J_2 &:= \int_{u_2}^{u_1} A e^{(t-s)A} DU(s, X_s) D_{u_2} X_s ds, \\
J_3 &:= \int_{u_1}^t A e^{(t-s)A} DU(s, X_s)(D_{u_2} X_s - D_{u_1} X_s) ds, \\
J_4 &:= e^{(t-u_2)A} Q^{1/2} - e^{(t-u_1)A} Q^{1/2} \\
J_5 &:= -(e^{(t-u_2)A} DU(u_2, X_{u_2}) Q^{1/2} - e^{(t-u_1)A} DU(u_1, X_{u_1}) Q^{1/2}) \\
J_6 &:= - \int_{u_2}^{u_1} e^{(t-s)A} D^2 U(s, X_s) Q^{1/2} D_{u_2} X_s dW_s \\
J_7 &:= - \int_{u_1}^t e^{(t-s)A} D^2 U(s, X_s) Q^{1/2} (D_{u_2} X_s - D_{u_1} X_s) dW_s.
\end{aligned}$$

Let us first estimate the terms J_4, J_5, J_2 and J_6 .

1. Estimation of J_4 : By assumption we have

$$\left\| e^{(t-u_2)A} Q^{1/2} - e^{(t-u_1)A} Q^{1/2} \right\|_E^2 \leq C \frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^\mu$$

for all $0 \leq u_2 < u_1 < t$ and some $0 < \mu < 1$.

2. Estimate for J_5 : We can write J_5 as

$$J_5 = T_1 + T_2 + T_3,$$

where

$$\begin{aligned}
T_1 &:= -(e^{(t-u_2)A} - e^{(t-u_1)A}) DU(u_2, X_{u_2}) Q^{1/2}, \\
T_2 &:= -e^{(t-u_1)A} ((DU(u_2, X_{u_2}) Q^{1/2} - DU(u_2, X_{u_1}) Q^{1/2}) \\
T_3 &:= -e^{(t-u_1)A} (DU(u_2, X_{u_1}) Q^{1/2} - DU(u_1, X_{u_1}) Q^{1/2})
\end{aligned}$$

2.1. T_2 : Because of our assumptions, Theorem 3 and the mean value theorem we obtain

$$\begin{aligned}
\|T_2\|_E^2 &= \left\| e^{(t-u_1)A} (DU(u_2, X_{u_2})Q^{1/2} - DU(u_2, X_{u_1})Q^{1/2}) \right\|_E^2 \\
&= \left\| e^{(t-u_1)A} (DU(u_2, X_{u_2}) - DU(u_2, X_{u_1}))Q^{1/2}M \right\|_{H.S.}^2 \\
&= \sum_{n,h \geq 1} \langle e^{(t-u_1)A} (DU(u_2, X_{u_2}) - DU(u_2, X_{u_1}))Q^{1/2}Me_h, e_n \rangle^2 \\
&= \sum_{n,h \geq 1} e^{-2\alpha_n(t-u_1)} \langle (DU(u_2, X_{u_2}) - DU(u_2, X_{u_1}))Q^{1/2}Me_h, e_n \rangle^2 \\
&= \sum_{n,h \geq 1} e^{-2\alpha_n(t-u_1)} \langle (DU_n(u_2, X_{u_2}) - DU_n(u_2, X_{u_1})), Q^{1/2}Me_h \rangle^2 \\
&= \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \sum_{h \geq 1} \langle Q^{1/2}M(DU_n(u_2, X_{u_2}) - DU_n(u_2, X_{u_1})), e_h \rangle^2 \\
&\leq \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2}M \right\|^2 \|DU_n(u_2, X_{u_2}) - DU_n(u_2, X_{u_1})\|^2 \\
&\leq \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2}M \right\|^2 \int_0^1 \|D^2U_n(u_2, X_{u_2} + s(X_{u_2} - X_{u_1}))(X_{u_2} - X_{u_1})\|^2 ds \\
&\leq \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2}M \right\|^2 \|D^2U_n\|_0^2 \|X_{u_2} - X_{u_1}\|^2 \\
&\leq C_T \left\| Q^{1/2}M \right\|^2 \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \|B_n\|_\alpha^2 \|X_{u_2} - X_{u_1}\|^2 \\
&= C_T \left\| Q^{1/2}M \right\|^2 \sum_{n \geq 1} (2\alpha_n(t-u_1))^{(1-\delta)} e^{-2\alpha_n(t-u_1)} (2\alpha_n(t-u_1))^{-(1-\delta)} \|B_n\|_\alpha^2 \|X_{u_2} - X_{u_1}\|^2 \\
&\leq C_T \left\| Q^{1/2}M \right\|^2 \|B\|_\alpha^2 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}} \|X_{u_2} - X_{u_1}\|^2.
\end{aligned}$$

Further we have that

$$\begin{aligned}
&X_{u_1} - X_{u_2} \\
&= e^{u_1A}x - e^{u_2A}x + \int_0^{u_1} e^{(u_1-s)A}B(s, X_s)ds - \int_0^{u_2} e^{(u_2-s)A}B(s, X_s)ds \\
&\quad + \int_0^{u_1} e^{(u_1-s)A}Q^{1/2}dW_s - \int_0^{u_2} e^{(u_2-s)A}Q^{1/2}dW_s \\
&= e^{u_1A}x - e^{u_2A}x + \int_0^{u_2} (e^{(u_1-s)A} - e^{(u_2-s)A})B(s, X_s)ds + \int_{u_2}^{u_1} e^{(u_1-s)A}B(s, X_s)ds \\
&\quad + \int_0^{u_2} (e^{(u_1-s)A} - e^{(u_2-s)A})Q^{1/2}dW_s + \int_{u_2}^{u_1} e^{(u_1-s)A}Q^{1/2}dW_s.
\end{aligned}$$

On the other hand we find for all $0 < \varepsilon < 1/2$ that

$$\begin{aligned}
& \left\| (e^{(u_1-s)A} - e^{(u_2-s)A})x \right\|^2 \\
&= \sum_{k \geq 1} \langle x, e_k \rangle^2 (1 - e^{-(u_1-u_2)\alpha_k})^2 e^{-2(u_2-s)\alpha_k} \\
&= \sum_{k \geq 1} \langle x, e_k \rangle^2 ((u_1 - u_2)\alpha_k)^{-2\varepsilon} (1 - e^{-(u_1-u_2)\alpha_k})^2 \cdot \\
&\quad \cdot ((u_1 - u_2)\alpha_k)^{2\varepsilon} \frac{(u_2 - s)^{2\varepsilon}}{(u_2 - s)^{2\varepsilon}} e^{-2(u_2-s)\alpha_k} \\
&\leq \|x\|^2 (u_1 - u_2)^{2\varepsilon} \frac{1}{(u_2 - s)^{2\varepsilon}}.
\end{aligned} \tag{5.30}$$

Hence it follows in connection with (5.22) and (5.30) for $T < 1$ that

$$\begin{aligned}
& E[\|X_{u_1} - X_{u_2}\|^2] \\
&\leq C(\|x\|^2 \frac{1}{u_2^{2\varepsilon}} (u_1 - u_2)^{2\varepsilon} + \|B\|_0^2 \frac{1}{1 - 2\varepsilon} u_2^{1-2\varepsilon} (u_1 - u_2)^{2\varepsilon} + \sup_{0 \leq r \leq T} \|e^{rA}\|^2 \|B\|_0^2 (u_1 - u_2) \\
&\quad + \int_0^{u_2} \frac{1}{(u_2 - s)^{1-\delta}} (u_1 - u_2)^\vartheta ds + \int_{u_2}^{u_1} \frac{1}{(u_1 - s)^{1-\delta}} ds) \\
&= C(\|x\|^2 \frac{1}{u_2^{2\varepsilon}} (u_1 - u_2)^{2\varepsilon} + \|B\|_0^2 \frac{1}{1 - 2\varepsilon} u_2^{1-2\varepsilon} (u_1 - u_2)^{2\varepsilon} + \sup_{0 \leq r \leq T} \|e^{rA}\|^2 \|B\|_0^2 (u_1 - u_2) \\
&\quad + \frac{1}{\delta} u_2^\delta (u_1 - u_2)^\vartheta + \frac{1}{\delta} (u_1 - u_2)^\delta) \\
&\leq H(\|B\|_0) \frac{1}{u_2^{2\varepsilon}} (u_1 - u_2)^{2\varepsilon \wedge \delta \wedge 1},
\end{aligned}$$

where H is a non-negative continuous function on $[0, \infty)$.

Therefore we have

$$\begin{aligned}
& E[\|T_2\|^2] \\
&\leq C_T \|Q^{1/2} M\|^2 \|B\|_\alpha^2 H(\|B\|_0) 2^{-(1-\delta)} \frac{1}{(t - u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon}} (u_1 - u_2)^{2\varepsilon \wedge \delta \wedge 1}.
\end{aligned}$$

for all $u_1 \geq u_2$.

2.2. T_3 : We know that

$$\begin{aligned}
& -e^{(t-u_1)A} (DU(u_2, X_{u_1})Q^{1/2} - DU(u_1, X_{u_1})Q^{1/2})h \\
&= \int_0^{u_1} e^{(t-u_1)A} [D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1})Q^{1/2}h]dr - \int_0^{u_2} e^{(t-u_1)A} [D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})Q^{1/2}h]dr \\
&= G_1 + G_2,
\end{aligned}$$

where

$$\begin{aligned}
G_1 &: = \int_{u_2}^{u_1} e^{(t-u_1)A} [D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1})Q^{1/2}h]dr, \\
G_2 &: = \int_0^{u_2} e^{(t-u_1)A} [D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1})Q^{1/2}h]dr - \int_0^{u_2} e^{(t-u_1)A} [D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})Q^{1/2}h]dr.
\end{aligned}$$

We also see that

$$\begin{aligned}
& \left\| e^{(t-u_1)A} \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})) dr Q^{1/2} \right\|_E^2 \\
&= \left\| e^{(t-u_1)A} \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})) dr Q^{1/2} M \right\|_{H.S.}^2 \\
&= \sum_{n,h \geq 1} \langle e^{(t-u_1)A} \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})) dr Q^{1/2} M e_h, e_n \rangle^2 \\
&= \sum_{n,h \geq 1} e^{-2\alpha_n(t-u_1)} \langle \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})) dr Q^{1/2} M e_h, e_n \rangle^2 \\
&= \sum_{n,h \geq 1} e^{-2\alpha_n(t-u_1)} \langle \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi_n)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi_n)(X_{u_1})) dr, Q^{1/2} M e_h \rangle^2 \\
&= \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \sum_{h \geq 1} \langle Q^{1/2} M \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi_n)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi_n)(X_{u_1})) dr, e_h \rangle^2 \\
&\leq \sum_{n \geq 1} e^{-2\alpha_n(t-u_1)} \left\| Q^{1/2} M \right\|^2 \left\| \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi_n)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi_n)(X_{u_1})) dr \right\|^2 \\
&= \left\| Q^{1/2} M \right\|^2 \sum_{n \geq 1} (2\alpha_n(t-u_1))^{1-\delta} e^{-2\alpha_n(t-u_1)} (2\alpha_n(t-u_1))^{-(1-\delta)} \cdot \\
&\quad \cdot \left\| \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi_n)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi_n)(X_{u_1})) dr \right\|^2 \\
&\leq C_\delta \left\| Q^{1/2} M \right\|^2 \left\| \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})) dr \right\|^2 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}}.
\end{aligned}$$

On the other hand it follows from the semigroup property of $\mathcal{R}_t$ that

$$D\mathcal{R}_{u_1-r}(\Phi)(x) = D(\mathcal{R}_{u_2-r}(\mathcal{R}_{u_1-u_2}\Phi))(x).$$

This in connection with Theorem 2 gives

$$\begin{aligned}
& \int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1})) h dr \\
& \int_0^{u_2} \int_H \langle \Lambda_{u_2-r} h, Q_{u_2-r}^{-1/2} y \rangle ((\mathcal{R}_{u_1-u_2}\Phi - \mathcal{R}_0\Phi)(e^{(u_2-r)A} X_{u_1} + y)) N_{Q_{u_2-r}}(dy) dr.
\end{aligned}$$

Further, we find for $a = e^{(u_2-r)A} X_{u_1} + y$ and arbitrarily small $0 < 2\rho < 1$ by using Burkholder's

inequality, Lemma 5.8 and (5.21)

$$\begin{aligned}
& \|(\mathcal{R}_{u_1-u_2}\Phi - \mathcal{R}_0\Phi)(a)\|^2 \\
&= \|E[\Phi(r, Z_{u_1-u_2}^a) - \Phi(r, a)]\|^2 \\
&\leq \|\Phi\|_\alpha^2 E[\|Z_{u_1-u_2}^a - Z_0^a\|^{2\alpha}] \\
&= \|\Phi\|_\alpha^2 E\left[\left\|e^{(u_1-u_2)A}a - a + \int_0^{u_1-u_2} e^{(u_1-u_2-s)A} Q^{1/2} dW_s\right\|^{2\alpha}\right] \\
&\leq C \|\Phi\|_\alpha^2 \left(\left\|e^{(u_1-r)A}X_{u_1}(\omega_2) - e^{(u_2-r)A}X_{u_1}(\omega_2)\right\|^{2\alpha}\right. \\
&\quad \left.+ \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} + E\left[\left\|\int_0^{u_1-u_2} e^{(u_1-u_2-s)A} Q^{1/2} dW_s\right\|^{2\alpha}\right]\right) \\
&\leq C_\alpha \|\Phi\|_\alpha^2 \left(\left\|e^{(u_1-r)A}A^{-\rho}(A^\rho X_{u_1}(\omega_2)) - e^{(u_2-r)A}A^{-\rho}(A^\rho X_{u_1}(\omega_2))\right\|^{2\alpha}\right. \\
&\quad \left.+ \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} + \int_0^{u_1-u_2} \left\|e^{(u_1-u_2-s)A} Q^{1/2}\right\|_{H.S.}^{2\alpha} ds\right) \\
&\leq C_\alpha \|\Phi\|_\alpha^2 (|u_1 - u_2|^{2\rho\alpha} \|A^\rho X_{u_1}(\omega_2)\|^{2\alpha} + \|e^{(u_1-u_2)A}y - y\|^{2\alpha} \\
&\quad + \frac{1}{1 - (1 - \delta)\alpha} (u_1 - u_2)^{1-(1-\delta)\alpha}).
\end{aligned}$$

Hence by (5.20) in connection with Hölder's inequality we get

$$\begin{aligned}
& \left\|\int_0^{u_2} (D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) - D\mathcal{R}_{u_2-r}(\Phi)(X_{u_1}))dr\right\|^2 \\
&\leq \left(\int_0^{u_2} \|\Lambda_{u_2-r}\| \left(\int_H \left\|((\mathcal{R}_{u_1-u_2}\Phi - \mathcal{R}_0\Phi)(e^{(u_2-r)A}X_{u_1} + y))\right\|^2 N_{Q_{u_2-r}}(dy)\right)^{1/2} dr\right)^2 \\
&\leq \left(\int_0^{u_2} \|\Lambda_{u_2-r}\| (C_\alpha \|\Phi\|_\alpha^2 (|u_1 - u_2|^{2\rho\alpha} \|A^\rho X_{u_1}(\omega_2)\|^{2\alpha}\right. \\
&\quad \left.+ \int_H \left\|e^{(u_1-u_2)A}y - y\right\|^{2\alpha} N_{Q_{u_2-r}}(dy)dr + \frac{1}{1 - (1 - \delta)\alpha} (u_1 - u_2)^{1-(1-\delta)\alpha})^{1/2} dr\right)^2 \\
&\leq C_\alpha \left(\int_0^{u_2} \|\Lambda_{u_2-r}\| dr\right)^2 (C_\alpha \|\Phi\|_\alpha^2 (|u_1 - u_2|^{2\rho\alpha} \|A^\rho X_{u_1}(\omega_2)\|^{2\alpha} \\
&\quad + |u_1 - u_2|^{\eta(\alpha/(1-\alpha))} + \frac{1}{1 - (1 - \delta)\alpha} (u_1 - u_2)^{1-(1-\delta)\alpha})
\end{aligned}$$

So for $T < 1$ we have

$$\begin{aligned}
& E[\|G_2\|_E^2] \\
&\leq C_{\delta,\alpha,\eta,\rho,T} \left\|Q^{1/2}M\right\|^2 \left(\int_0^T \|\Lambda_{u_2-r}\| dr\right)^2 (\|\Phi\|_\alpha^2 E[\|A^\rho X_{u_1}\|^{2\alpha}] + 1)^2 \\
&\quad \cdot \frac{1}{(t - u_1)^{1-\delta}} |u_1 - u_2|^{(2\rho\alpha) \wedge \eta(\alpha/(1-\alpha)) \wedge (1-(1-\delta)\alpha)} \\
&\leq C_{\delta,\alpha,\eta,\rho,T} \left\|Q^{1/2}M\right\|^2 \left(\int_0^T \|\Lambda_{u_2-r}\| dr\right)^2 ((C_1 \|B\|_\alpha^3 + C_2 \|B\|_\alpha^2) E[\|A^\rho X_{u_1}\|^{2\alpha}] + 1)^2 \\
&\quad \cdot \frac{1}{(t - u_1)^{1-\delta}} |u_1 - u_2|^{(2\rho\alpha) \wedge \eta(\alpha/(1-\alpha)) \wedge (1-(1-\delta)\alpha)}
\end{aligned}$$

On the other hand we know by Lemma 5.8 in connection with Hölder's inequality that

$$E[\|A^\rho X_{u_1}\|^{2\alpha}] \leq C(1 + \|B\|_\alpha^2)^{\alpha/(1-\alpha)} \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}}$$

for arbitrarily small $\rho > 0$.

$$\begin{aligned} & E[\|G_2\|_E^2] \\ & \leq C_{\delta,\alpha,\eta,\rho,T} \left\| Q^{1/2} M \right\|^2 \left(\int_0^T \|\Lambda_{u_2-r}\| dr \right)^2 ((C_1 \|B\|_\alpha^3 + C_2 \|B\|_\alpha^2)(1 + \|B\|_\alpha^2)^{\alpha/(1-\alpha)} + 1)^2 \\ & \quad \cdot \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^{(2\rho\alpha) \wedge \eta(\alpha/(1-\alpha)) \wedge (1-(1-\delta)\alpha)} \end{aligned}$$

for arbitrarily small $\rho > 0$.

As for the term G_1 we argue just as above and get

$$\begin{aligned} & \left\| e^{(t-u_1)A} \int_{u_2}^{u_1} D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) dr Q^{1/2} \right\|_E^2 \\ & \leq C_\delta \left\| Q^{1/2} M \right\|^2 \left\| \int_{u_2}^{u_1} D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) dr \right\|^2 2^{-(1-\delta)} \frac{1}{(t-u_1)^{1-\delta}}. \end{aligned}$$

But

$$\begin{aligned} & \left\| \int_{u_2}^{u_1} D\mathcal{R}_{u_1-r}(\Phi)(X_{u_1}) dr \right\|^2 \\ & \leq C \left(\int_{u_2}^{u_1} \|\Lambda_{u_1-r}\| dr \right)^2 \|\Phi\|_0^2 \\ & \leq C \left(\int_0^{u_1-u_2} \|\Lambda_r\| dr \right)^2 C_T (K_1 \|B\|_\alpha^2 + K_2 \|B\|_\alpha)^2 \\ & \leq C \left(\int_0^T \|\Lambda_{u_1-r}\|^{1+\theta} dr \right)^{2/(1+\theta)} C_T (K_1 \|B\|_\alpha^2 + K_2 \|B\|_\alpha)^2 (u_1 - u_2)^{\frac{\theta}{1+\theta}} \end{aligned}$$

Hence

$$\begin{aligned} & E[\|G_1\|_E^2] \\ & C_\delta \left\| Q^{1/2} M \right\|^2 2^{-(1-\delta)} C \left(\int_0^T \|\Lambda_{u_1-r}\|^{1+\theta} dr \right)^{2/(1+\theta)} C_T (K_1 \|B\|_\alpha^2 + K_2 \|B\|_\alpha)^2 (u_1 - u_2)^{\frac{\theta}{1+\theta}} \frac{1}{(t-u_1)^{1-\delta}}. \end{aligned}$$

So we obtain

$$\begin{aligned} & E[\|T_3\|_E^2] \\ & \leq C_{\delta,\alpha,\eta,\rho,T} \left\| Q^{1/2} M \right\|^2 \left(\int_0^T \|\Lambda_{u_2-r}\| dr \right)^2 ((C_1 \|B\|_\alpha^3 + C_2 \|B\|_\alpha^2)(1 + \|B\|_\alpha^2)^{\alpha/(1-\alpha)} + 1)^2 \\ & \quad \cdot \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^{(2\rho\alpha) \wedge \eta(\alpha/(1-\alpha)) \wedge (1-(1-\delta)\alpha)} \\ & \quad + 2C_\delta \left\| Q^{1/2} M \right\|^2 2^{-(1-\delta)} (C_1 \|B\|_\alpha^3 + C_2 \|B\|_\alpha^2)^2 \sup_{0 \leq s \leq T} \left\| e^{sA} Q^{1/2} \right\|_{HS}^{2\alpha} (u_1 - u_2)^{2\alpha} \frac{1}{(t-u_1)^{1-\delta}}. \end{aligned}$$

2.3. T_1 : We find that

$$\begin{aligned}
& \left\| (e^{(t-u_2)A} - e^{(t-u_1)A}) DU(u_2, X_{u_2}) Q^{1/2} \right\|_E^2 \\
&= \sum_{h \geq 1} \left\| (e^{(t-u_2)A} - e^{(t-u_1)A}) DU(u_2, X_{u_2}) Q^{1/2} M e_h \right\|^2 \\
&= \sum_{n, h \geq 1} \langle (e^{(t-u_2)A} - e^{(t-u_1)A}) DU(u_2, X_{u_2}) Q^{1/2} M e_h, e_n \rangle^2 \\
&= \sum_{n, h \geq 1} (e^{-(t-u_2)\alpha_n} - e^{-(t-u_1)\alpha_n})^2 \langle DU_n(u_2, X_{u_2}), Q^{1/2} M e_h \rangle^2 \\
&= \sum_{n, h \geq 1} (e^{-(t-u_2)\alpha_n} - e^{-(t-u_1)\alpha_n})^2 \langle Q^{1/2} M DU_n(u_2, X_{u_2}), e_h \rangle^2 \\
&\leq \left\| Q^{1/2} M \right\|^2 \|DU\|_0^2 \sum_{n \geq 1} (e^{-(t-u_2)\alpha_n} - e^{-(t-u_1)\alpha_n})^2 \\
&= \left\| Q^{1/2} M \right\|^2 \|DU\|_0^2 \sum_{n \geq 1} e^{-2(t-u_1)\alpha_n} (1 - e^{-(u_1-u_2)\alpha_n})^2 \\
&= \left\| Q^{1/2} M \right\|^2 \|DU\|_0^2 \cdot \\
&\quad \cdot \sum_{n \geq 1} (2(t-u_1)\alpha_n)^{1-\delta} e^{-2(t-u_1)\alpha_n} (2(t-u_1)\alpha_n)^{\delta-1} (1 - e^{-(u_1-u_2)\alpha_n})^2 \\
&= \left\| Q^{1/2} M \right\|^2 \|DU\|_0^2 \cdot \\
&\quad \cdot \sum_{n \geq 1} (2(t-u_1)\alpha_n)^{1-\delta} e^{-2(t-u_1)\alpha_n} 2^{\delta-1} (t-u_1)^{\delta-1} (u_1-u_2)^{1-\delta} ((u_1-u_2)\alpha_n)^{\delta-1} (1 - e^{-(u_1-u_2)\alpha_n})^2 \\
&\leq C_\delta \left\| Q^{1/2} M \right\|^2 \|DU\|_0^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{1-\delta} \\
&= C_T C_\delta \left\| Q^{1/2} M \right\|^2 \|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{1-\delta}.
\end{aligned}$$

Hence

$$\begin{aligned}
& E[\|T_1\|_E^2] \\
&\leq C_T C_\delta \left\| Q^{1/2} M \right\|^2 \|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{1-\delta}.
\end{aligned}$$

Altogether we see for $T < 1$ that

$$\begin{aligned}
& E[\|J_5\|_E^2] \\
&\leq C(T, \delta, \alpha, M, Q, \theta, \mu, \vartheta, \varepsilon, A) G(\|B\|_\alpha) \\
&\quad \cdot \left(\frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{1-\delta} + \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon}} (u_1-u_2)^{2\varepsilon \wedge \delta \wedge 1} \right. \\
&\quad \left. + \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_1)^{1-\delta}} |u_1-u_2|^{(2\rho\alpha) \wedge \eta(\alpha/(1-\alpha)) \wedge (1-(1-\delta)\alpha \wedge 2\alpha)} \right),
\end{aligned}$$

where G is a non-decreasing continuous function on $[0, \infty)$.

3. Estimate for J_2 : The calculation in **2.** for the estimate I_2 shows that

$$\begin{aligned} & E[\|J_2\|_E^2] \\ & \leq (u_1 - u_2) C_{\varepsilon, T} C_\gamma \|B\|_0^2 \int_{u_2}^{u_1} \frac{1}{(u_1 - s)^{2(1-\gamma)}} E[\|D_{u_2} X_s\|_E^2] ds \end{aligned}$$

Then employing the estimate (5.29) we have

$$\begin{aligned} & E[\|J_2\|_E^2] \\ & \leq (u_1 - u_2) C C_{\varepsilon, T} C_\gamma \|B\|_0^2 \\ & \quad \cdot (G_1(\|B\|_\alpha) \int_{u_2}^{u_1} \frac{1}{(u_1 - s)^{2(1-\gamma)}} \frac{1}{(s - u_2)^{1-\delta}} ds \\ & \quad + G_2(\|B\|_\alpha) \int_{u_2}^{u_1} \frac{1}{(u_1 - s)^{2(1-\gamma)}} (s - u_2)^\delta ds) \\ & = (u_1 - u_2) C C_{\varepsilon, T} C_\gamma \|B\|_0^2 \\ & \quad \cdot (G_1(\|B\|_\alpha) (u_1 - u_2)^{1-2(1-\gamma)-(1-\delta)} \frac{\Gamma(1-2(1-\gamma))\Gamma(1-\delta)}{\Gamma(1-2(1-\gamma)+1-\delta)} \\ & \quad + G_2(\|B\|_\alpha) (u_1 - u_2)^{1-2(1-\gamma)+\delta} \frac{\Gamma(1-2(1-\gamma))\Gamma(1+\delta)}{\Gamma(1-2(1-\gamma)+1+\delta)}) \\ & = C(\varepsilon, T, \gamma, \delta) (G_1(\|B\|_\alpha) (u_1 - u_2)^{1-2(1-\gamma)+\delta} \\ & \quad + G_2(\|B\|_\alpha) (u_1 - u_2)^{1-2(1-\gamma)+\delta}), \end{aligned}$$

where we used the Beta function and where $0 < 2(1-\gamma) < 1$ and $G_i, i = 1, 2$ are non-negative functions on an interval $[0, V(T)]$ with $V(T) \rightarrow \infty$ for $T \rightarrow 0$.

4. Estimate for J_6 : We argue just as in **5.** for the estimate I_5 and get in connection with the inequality (5.29) for $T < 1$

$$\begin{aligned}
& \sum_{r \geq 1} \sum_{h \geq 1} \int_{u_2}^{u_1} \left\| e^{(t-s)A} D^2 U(s, X_s) [Q^{1/2} e_h, D_{u_2} X_s M e_r] \right\|^2 ds \\
&= \sum_{r \geq 1} \sum_{n, h \geq 1} \int_{u_2}^{u_1} \langle e^{(t-s)A} D^2 U(s, X_s) [Q^{1/2} e_h, D_{u_2} X_s M e_r], e_n \rangle^2 ds \\
&= \sum_{r \geq 1} \sum_{n, h \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \langle D^2 U(s, X_s) [Q^{1/2} e_h, D_{u_2} X_s M e_r], e_n \rangle^2 ds \\
&= \sum_{r \geq 1} \sum_{n, h \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} (D^2 U_n(s, X_s) [Q^{1/2} e_h, D_{u_2} X_s M e_r])^2 ds \\
&= \sum_{r \geq 1} \sum_{n \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \sum_{h \geq 1} (D^2 U_n(s, X_s) [Q^{1/2} e_h, D_{u_2} X_s M e_r])^2 ds \\
&\leq \sum_{n \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \|Q\| \|D^2 U_n\|_0^2 \|D_{u_2} X_s\|_E^2 ds \\
&\leq C_T \sum_{n \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \|Q\| \|B\|_\alpha^2 \|D_{u_2} X_s\|_E^2 ds \\
&\leq C_T F(\|B\|_\alpha) \|B\|_\alpha^2 \sum_{n \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \frac{1}{(s - u_2)^{1-\delta}} ds,
\end{aligned}$$

where F is a non-decreasing continuous function on $[0, \infty)$.

On the other hand we see by integration by parts and Hölder's inequality that

$$\begin{aligned}
& \sum_{n \geq 1} \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \frac{1}{(s-u_2)^{1-\delta}} ds \\
&= \sum_{n \geq 1} \left(\frac{1}{\delta} e^{-2\alpha_n(t-u_1)} (u_1-u_2)^\delta - 2\alpha_n \int_{u_2}^{u_1} e^{-2\alpha_n(t-s)} \frac{1}{\delta} (s-u_2)^\delta ds \right) \\
&\leq \sum_{n \geq 1} \left(\frac{1}{\delta} e^{-2\alpha_n(t-u_1)} (u_1-u_2)^\delta + \frac{1}{\delta} 2\alpha_n \left(\int_{u_2}^{u_1} e^{-2\alpha_n(1+\tau)(t-s)} ds \right)^{1/(1+\tau)} \left(\int_{u_2}^{u_1} (s-u_2)^{\delta(1+\tau^{-1})} ds \right)^{1/(1+\tau^{-1})} \right) \\
&= \sum_{n \geq 1} \left(\frac{1}{\delta} e^{-2\alpha_n(t-u_1)} (u_1-u_2)^\delta + \frac{1}{\delta} 2\alpha_n \left(\frac{1}{2(1+\tau)\alpha_n} (e^{-2\alpha_n(1+\tau)(t-u_1)} - e^{-2\alpha_n(1+\tau)(t-u_2)}) \right)^{1/(1+\tau)} \right. \\
&\quad \cdot \left. \left(\frac{1}{(1+\tau^{-1})\delta+1} \right)^{1/(1+\tau)} (u_1-u_2)^{\delta+\frac{1}{(1+\tau^{-1})}} \right) \\
&= \sum_{n \geq 1} \left(\frac{1}{\delta} (2\alpha_n(t-u_1))^{1-\delta} e^{-2\alpha_n(t-u_1)} \frac{1}{(2\alpha_n(t-u_1))^{1-\delta}} (u_1-u_2)^\delta \right. \\
&\quad + \frac{1}{\delta} \left(\frac{1}{2(1+\tau)} \right)^{1/(1+\tau)} \left(\frac{1}{(1+\tau^{-1})\delta+1} \right)^{1/(1+\tau)} \\
&\quad \cdot \sum_{n \geq 1} \frac{1}{\alpha_n^{\frac{1}{1+\tau}-\delta}} (2\alpha_n(t-u_1))^{1-\delta} e^{-2\alpha_n(t-u_1)} \frac{1}{(t-u_1)^{1-\delta}} (1 - e^{-2\alpha_n(1+\tau)(u_1-u_2)})^{1/(1+\tau)} (u_1-u_2)^{\delta+\frac{1}{(1+\tau^{-1})}} \Big) \\
&\leq C \left(\frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^\delta + \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^{\delta+\frac{1}{(1+\tau^{-1})}} \right) \\
&\leq C \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^\delta
\end{aligned}$$

for $\tau > 0$ small enough.

So we get

$$\begin{aligned}
& E[\|J_6\|_E^2] \\
&\leq C_T F(\|B\|_\alpha) \|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1-u_2)^\delta
\end{aligned}$$

Let us now consider the terms J_1, J_3 and J_7 . But this case just corresponds to the calculations for the estimates of I_1, I_2 and I_5 and we obtain

$$\begin{aligned}
& E[\|J_1\|_E^2] + E[\|J_3\|_E^2] + E[\|J_7\|_E^2] \\
&\leq K_T \|B\|_0^2 E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
&\quad + (t-u_1)C_{\varepsilon,T} \|B\|_0^2 E\left[\int_{u_1}^t \frac{C_\gamma}{(t-s)^{2(1-\gamma)}} \|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds\right] \\
&\quad + C_\delta C_T \|Q\|^2 2^{-(1-\delta)} \|B\|_\alpha^2 \int_{u_1}^t \frac{1}{(t-s)^{1-\delta}} \|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds
\end{aligned}$$

Altogether it follows from the above estimates that

$$\begin{aligned}
& E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq & C\left(\sum_{i=1}^7 E[\|J_i\|_E^2]\right) \\
\leq & C(K_T \|B\|_\alpha^2 E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
& + C(\varepsilon, T, \gamma, \delta)(G_1(\|B\|_\alpha)(u_1 - u_2)^{1-2(1-\gamma)+\delta} + G_2(\|B\|_\alpha)(u_1 - u_2)^{1-2(1-\gamma)+\delta}) \\
& + C\frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^\mu \\
& + C(T, \delta, \alpha, M, Q, \theta, \mu, \vartheta, \varepsilon, A)G(\|B\|_\alpha) \\
& \cdot \left(\frac{1}{(t-u_1)^{1-\delta}}(u_1 - u_2)^{1-\delta} + \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon}}(u_1 - u_2)^{2\varepsilon\wedge\delta\wedge 1}\right. \\
& + \frac{1}{u_1^{2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_1)^{1-\delta}} |u_1 - u_2|^{(2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha\wedge 2\alpha)} \\
& + C_T F(\|B\|_\alpha) \|B\|_\alpha^2 \frac{1}{(t-u_1)^{1-\delta}} (u_1 - u_2)^\delta \\
& + (t-u_1)C_{\varepsilon,T} \|B\|_0^2 E\left[\int_{u_1}^t \frac{C_\gamma}{(t-s)^{2(1-\gamma)}} \|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds\right] \\
& + C_\delta C_T \|Q\| 2^{-(1-\delta)} \|B\|_\alpha^2 \int_{u_1}^t \frac{1}{(t-s)^{1-\delta}} \|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds).
\end{aligned}$$

Since $K_T \rightarrow 0$ for $T \rightarrow 0$ and $1 > T > u_1 > u_2$ we get

$$\begin{aligned}
& E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq & V(\|B\|_\alpha) \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} (u_1 - u_2)^\lambda \\
& + V(\|B\|_\alpha) \int_{u_1}^t \frac{1}{(t-s)^{1-\delta}} \|D_{u_1}X_s - D_{u_2}X_s\|_E^2 ds,
\end{aligned}$$

where $\lambda = (1-2(1-\gamma)+\delta)\wedge\mu\wedge(1-\delta)\wedge(2\varepsilon\wedge\delta\wedge 1)\wedge((2\rho\alpha)\wedge\eta(\alpha/(1-\alpha))\wedge(1-(1-\delta)\alpha\wedge 2\alpha))\wedge\delta > 0$ and where V is a non-negative continuous function on $[0, \frac{1}{A_T}]$ for $A_T \rightarrow 0$ for $T \rightarrow 0$.

So by a Lemma of Gronwall for weakly singular kernels (see [1, Theorem 3]) we get

$$E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \leq a(t) + \int_{u_1}^t \sum_{n \geq 1} \frac{(g(t)\Gamma(\delta))^n}{\Gamma(n\delta)} (t-s)^{n\delta-1} a(s) ds$$

for $u_1 \leq t \leq T$, where

$$a(t) := V(\|B\|_\alpha) \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} (u_1 - u_2)^\lambda$$

and

$$g(t) \equiv V(\|B\|_\alpha).$$

Therefore we have by means of the Beta function

$$\begin{aligned}
& E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq & V(\|B\|_\alpha) \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\sqrt{2\rho(\alpha/(1-\alpha))}}} (u_1 - u_2)^\lambda \\
& + \sum_{n=1}^{n_0-1} \frac{(V(\|B\|_\alpha)\Gamma(\delta))^n \Gamma(\delta)}{\Gamma((n+1)\delta)} \frac{1}{(t-u_1)^{1-(n+1)\delta}} \frac{1}{u_2^{2\varepsilon\sqrt{2\rho(\alpha/(1-\alpha))}}} (u_1 - u_2)^\lambda \\
& + \sum_{n \geq n_0} \frac{(V(\|B\|_\alpha)\Gamma(\delta))^n \Gamma(\delta)}{\Gamma((n+1)\delta)} (t-u_1)^{(n+1)\delta-1} \frac{1}{u_2^{2\varepsilon\sqrt{2\rho(\alpha/(1-\alpha))}}} (u_1 - u_2)^\lambda \\
\leq & E(\|B\|_\alpha) \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\sqrt{2\rho(\alpha/(1-\alpha))}}} (u_1 - u_2)^\lambda,
\end{aligned}$$

where V is a non-negative continuous function on $[0, \frac{1}{A_T}]$ for $A_T \rightarrow 0$ for $T \rightarrow 0$ and where $n_0\delta < 1$, but $(n_0 + 1)\delta \geq 1$ for $n_0 \in \mathbb{N}$.

Thus it follows for all $0 \leq u_1, u_2 < t$ that

$$\begin{aligned}
& E[\|D_{u_1}X_t - D_{u_2}X_t\|_E^2] \\
\leq & E(\|B\|_\alpha) \left(\frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\sqrt{2\rho(\alpha/(1-\alpha))}}} |u_1 - u_2|^\lambda \right. \\
& \left. + \frac{1}{(t-u_2)^{1-\delta}} \frac{1}{u_1^{2\varepsilon\sqrt{2\rho(\alpha/(1-\alpha))}}} |u_1 - u_2|^\lambda \right)
\end{aligned}$$

Now choose in (5.27) $\beta > 0$ such that $\kappa := 1 + 2\beta - \lambda < 1$. Then

$$\begin{aligned}
& \int_0^t \int_0^t \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1-u_2|^\kappa} du_1 du_2 \\
&= \int_0^t \int_{u_2}^t \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1-u_2|^\kappa} du_1 du_2 \\
&\quad + \int_0^t \int_0^{u_2} \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1-u_2|^\kappa} du_1 du_2 \\
&\leq \int_0^t \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} \int_{u_2}^t \frac{1}{(t-u_1)^{1-\delta}} \frac{1}{|u_1-u_2|^\kappa} du_1 du_2 \\
&\quad + \int_0^t \frac{1}{(t-u_2)^{1-\delta}} \int_0^{u_2} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{|u_1-u_2|^\kappa} du_1 du_2 \\
&= \frac{\Gamma(\delta)\Gamma(1-\kappa)}{\Gamma(\delta+1-\kappa)} \int_0^t \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))}} \frac{1}{(t-u_2)^{\kappa-\delta}} du_2 \\
&\quad + \frac{\Gamma((1-2\varepsilon)\vee 2\rho(\alpha/(1-\alpha)))\Gamma(1-\kappa+\delta)}{\Gamma(1-2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+1-\kappa+\delta)} \\
&\quad \cdot \int_0^t \frac{1}{(t-u_2)^{1-\delta}} \frac{1}{u_2^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+2\beta-\lambda-\delta}} du_2 \\
&= \frac{\Gamma(\delta)\Gamma(1-\kappa)}{\Gamma(\delta+1-\kappa)} \frac{\Gamma(1-2\varepsilon\vee 2\rho(\alpha/(1-\alpha)))\Gamma(1-\kappa+\delta)}{\Gamma(1-2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+1-\kappa+\delta)} \\
&\quad \cdot \frac{1}{t^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+2\beta-\lambda-\delta}} \\
&\quad + \frac{\Gamma(1-2\varepsilon\vee 2\rho(\alpha/(1-\alpha)))\Gamma(1-\kappa+\delta)}{\Gamma(1-2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+1-\kappa+\delta)} \\
&\quad \cdot \frac{\Gamma(\delta)\Gamma(1-(2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+2\beta-\lambda-\delta))}{\Gamma(\delta+1-(2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+2\beta-\lambda-\delta))} \frac{1}{t^{2\varepsilon\vee 2\rho(\alpha/(1-\alpha))+2\beta-\lambda-2\delta}} \\
&< \infty,
\end{aligned}$$

since $2\beta - \lambda < 0$ and since $\varepsilon > 0$ and $\rho > 0$ can be chosen arbitrarily small.

So we get the estimate (5.27), which completes the proof. $\blacksquare$

Let now $B^n \in C([0, T]; C_b^1(H, H)) \cap C([0, T]; C_b^\alpha(H, H))$, $n \geq 1$ be a sequence of functions and $B \in C([0, T]; C_b^\alpha(H, H))$ such that

$$B^n(t, x) \longrightarrow B(t, x) \quad (5.31)$$

for $n \longrightarrow \infty$ in H for all x and such that

$$\|B^n\|_\alpha \leq K \quad (5.32)$$

for a constant K independent of n . See e.g. [11].

We also need the following Lemma:

Lemma 5.10 *Suppose that $X_t^n, 0 \leq t \leq T, n \geq 1$ are the unique mild solutions to (5.1) with respect to the coefficients B^n in (5.31) and (5.32). Let $X_t^{n,i} = \langle e_i, X_t^n \rangle$. Then there exists for all i a subsequence $(n_k^i)_{k \geq 1}$ which only depends on (a sufficiently small) T and i such that for all $0 \leq t \leq T$ $X_t^{n_k^i}$ converges in $L^2(\Omega)$ for $k \longrightarrow \infty$.*

Proof. We know that

$$X_t^n = e^{tA}x + \int_0^t e^{(t-s)A}B^n(s, X_s)ds + \int_0^t e^{(t-s)A}Q^{1/2}dW_s, 0 \leq t \leq T.$$

So

$$X_t^{n,i} = \langle e_i, e^{tA}x \rangle + \int_0^t \langle e_i, e^{(t-s)A}B^n(s, X_s) \rangle ds + \int_0^t \langle e_i, e^{(t-s)A}Q^{1/2}dW_s \rangle, 0 \leq t \leq T.$$

Hence

$$\begin{aligned} & X_{t_1}^{n,i} - X_{t_2}^{n,i} \\ &= \langle e_i, e^{t_1A}x - e^{t_2A}x \rangle + \int_0^{t_2} \langle e_i, (e^{(t_1-s)A} - e^{(t_2-s)A})B^n(s, X_s^n) \rangle ds \\ & \quad + \int_{t_2}^{t_1} \langle e_i, e^{(t_1-s)A}B^n(s, X_s^n) \rangle ds + \int_0^{t_2} \langle e_i, e^{(t_1-s)A} - e^{(t_2-s)A}Q^{1/2}dW_s \rangle \\ & \quad + \int_{t_2}^{t_1} \langle e_i, e^{(t_1-s)A}Q^{1/2}dW_s \rangle \end{aligned} \quad (5.33)$$

for all $0 \leq t_2 \leq t_1 \leq T$.

Now let f be an element of the Hida test function space $(\mathcal{S}) \subset L^2(\Omega)$. Denote by $(\mathcal{S})^*$ its topological dual (Hida distribution space). See [10] for further information on these spaces. Then $\langle (X_{t_1}^{n,i} - X_{t_2}^{n,i}), f \rangle_{(\mathcal{S})^*, (\mathcal{S})} = E[(X_{t_1}^{n,i} - X_{t_2}^{n,i})f]$, where $\langle \cdot, \cdot \rangle_{(\mathcal{S})^*, (\mathcal{S})}$ is the dual pairing.

So using (5.33) we get

$$\begin{aligned} & E[(X_{t_1}^{n,i} - X_{t_2}^{n,i})f] \\ &= \langle e_i, e^{t_1A}x - e^{t_2A}x \rangle E[f] + \int_0^{t_2} E[\langle e_i, (e^{(t_1-s)A} - e^{(t_2-s)A})B^n(s, X_s^n) \rangle, f] ds \\ & \quad + \int_{t_2}^{t_1} E[\langle e_i, e^{(t_1-s)A}B^n(s, X_s^n) \rangle, f] ds + E[\int_0^{t_2} \langle e_i, e^{(t_1-s)A} - e^{(t_2-s)A}Q^{1/2}dW_s, f \rangle] \\ & \quad + E[\int_{t_2}^{t_1} \langle e_i, e^{(t_1-s)A}Q^{1/2}dW_s, f \rangle]. \end{aligned}$$

Thus it follows from (5.21) and (5.22)

$$\begin{aligned} & \left| E[(X_{t_1}^{n,i} - X_{t_2}^{n,i})f] \right| \\ & \leq |\langle e_i, x \rangle| |e^{-t_1\alpha_i} - e^{-t_2\alpha_i}| |E[f]| + \int_0^{t_2} E[|\langle e_i, B^n(s, X_s^n) \rangle f|] |e^{-(t_1-s)\alpha_i} - e^{(t_2-s)\alpha_i}| ds \\ & \quad + \int_{t_2}^{t_1} E[|\langle e_i, B^n(s, X_s^n) \rangle f|] e^{-(t_1-s)\alpha_i} ds + (\int_0^{t_2} \|e^{(t_1-s)A} - e^{(t_2-s)A}Q^{1/2}\|_{H.S.}^2 ds)^{1/2} (E[f^2])^{1/2} \\ & \quad + (\int_{t_2}^{t_1} \|e^{(t_1-s)A}Q^{1/2}\|_{H.S.}^2 ds)^{1/2} (E[f^2])^{1/2} \\ & \leq C_{i,T} |t_1 - t_2| \|B^n\| E[|f|] + C(\int_0^{t_2} \frac{1}{(t_2-s)^{1-\delta}} |t_1 - t_2|^\vartheta ds)^{1/2} (E[f^2])^{1/2} \\ & \quad + C(\int_{t_2}^{t_1} \frac{1}{(t_1-s)^{1-\delta}} ds)^{1/2} (E[f^2])^{1/2} \\ & \leq C(i, T, \vartheta, \delta, K, f) |t_1 - t_2|^{\vartheta \wedge (\delta/2)}. \end{aligned}$$

So

$$\sup_{n \geq 1} m^T(\langle (X^{n,i}, f) \rangle_{(\mathcal{S})^*, (\mathcal{S})}, \delta) \longrightarrow 0 \text{ for } \delta \searrow 0,$$

where m^T is the modulus of continuity given by

$$m^T(g, \delta) := \max_{\substack{|t-s| \leq \delta \\ 0 \leq t, s \leq T}} |g(t) - g(s)|.$$

So $\langle (X^{n,i}, f) \rangle_{(\mathcal{S})^*, (\mathcal{S})}$ is relatively compact in $C([0, T])$ for all $f \in (\mathcal{S})$. Since $(\mathcal{S})^*$ is the dual of a countably Hilbertian nuclear space $(\mathcal{S})$, we can apply a result of I. Mitoma [14] and find that there exists for all i a subsequence $(n_k^i)_{k \geq 1}$ which only depends on (a sufficiently small) T and i such that $X^{n_k^i, i}$ converges in $C([0, T]; (\mathcal{S})^*)$.

On the other hand it follows from Lemma 5.9 that there exists (for fixed t) a $C < \infty$ and a $0 < \beta < \frac{1}{2}$ such that

$$E \left[\int_0^t \|D_u X_t^n\|_E^2 du \right] \leq L_1 (\|B^n\|_\alpha^2) \leq C < \infty$$

and

$$\begin{aligned} & E \left[\int_0^t \int_0^t \frac{\|D_{u_1} X_t^n - D_{u_2} X_t^n\|_E^2}{|u_1 - u_2|^{1+2\beta}} du_1 du_2 \right] \\ & \leq L_2 (\|B^n\|_\alpha^2) \leq C < \infty \end{aligned}$$

for all $n \geq 1$, provided that T is sufficiently small.

Then, if we apply Theorem 5.14 in connection with Remark 5.15 in the Appendix to the sequence $X_t^{n_k^i, i}$ we see that for all t and i there exists a subsequence $m_l = m_l^{t,i}, l \geq 1$ of $n_k^i, k \geq 1$ and a $\tilde{X}_t^i \in L^2(\Omega)$ such that

$$X_t^{n_{m_l}^i, i} \longrightarrow \tilde{X}_t^i \text{ for } l \longrightarrow \infty \quad (5.34)$$

in $L^2(\Omega)$.

We claim that

$$X_t^{n_k^i, i} \longrightarrow \tilde{X}_t^i \text{ for } k \longrightarrow \infty \text{ in } L^2(\Omega)$$

for all t, i . To see this assume that there exists for some t, i a $\varepsilon > 0$ and a subsequence $\varphi_l, l \geq 1$ such that

$$\left\| X_t^{n_{\varphi_l}^i, i} - \tilde{X}_t^i \right\|_{L^2(\Omega)} \geq \varepsilon.$$

On the other hand we know by Theorem 5.14 that there exists a subsequence $\phi_r, r \geq 1$ of such that

$$X_t^{n_{\phi_r}^i, i} \longrightarrow \tilde{Y}_t^i \text{ for } r \longrightarrow \infty \text{ in } L^2(\Omega).$$

But since

$$X_t^{n_k^i, i} \longrightarrow \tilde{X}_t^i \text{ for } k \longrightarrow \infty \text{ in } (\mathcal{S})^*$$

because of (5.34), we see that

$$\tilde{Y}_t^i = \tilde{X}_t^i.$$

But this leads to the contradiction

$$\left\| X_t^{n_{\varphi_{\phi_r}^i, i}} - \tilde{X}_t^i \right\|_{L^2(\Omega)} \geq \varepsilon.$$

This completes the proof. ■

We are coming to the main result of this section

Theorem 5.11 *Assume that the functions $B : [0, T] \times H \rightarrow H$ and $B^n : [0, T] \times H \rightarrow H, n \geq 1$ satisfy the conditions (5.31) and (5.32). Then there exists a Malliavin differentiable unique mild solution $X_t, 0 \leq t \leq T$ to the stochastic differential equation*

$$dX_t = AX_t dt + B(t, X_t) dt + Q^{1/2} dW_t, X_0 = x. \quad (5.35)$$

Proof. Let $X_t^n, n \geq 1$ be the mild solutions associated with the coefficients B^n and denote by $X_t^{n,i}$ the i -th component of X_t^n . Then it follows from Lemma 5.10 that there exists for all i a subsequence $(n_k^i)_{k \geq 1}$ which only depends on (a sufficiently small) T and i such that for all $0 \leq t \leq T$

$$X_t^{n_k^i, i} \rightarrow X_t^i \text{ in } L^2(\Omega) \text{ for } k \rightarrow \infty$$

for some $X_t^i \in L^2(\Omega), 0 \leq t \leq T, i \geq 1$.

Now let us denote by $(\varphi_n)_{n \geq 1}$ the diagonal sequence of the sequences $(n_k^1)_{k \geq 1}, (n_k^2)_{k \geq 1}, (n_k^3)_{k \geq 1}, \dots$
So

$$X_t^{\varphi_n, i} \rightarrow X_t^i \text{ for } n \rightarrow \infty$$

in $L^2(\Omega)$ for all t, i .

We now want to show that

$$X_t^{\varphi_n} \rightarrow X_t \text{ for } n \rightarrow \infty \quad (5.36)$$

in $L^2(\Omega; H)$ for all t , where $X_t = \sum_{k \geq 1} X_t^k e_k$. For this purpose choose a $\epsilon > 0$. By a weak compactness argument we also see from Lemma 5.8 that

$$E[(\sum_{k \geq 1} |X_t^k|^2 \alpha_k^{1-\delta})] < \infty.$$

This implies

$$\begin{aligned} & \sup_{n \geq 1} \sum_{k \geq m} E[|X_t^{\varphi_n, k} - X_t^k|^2] \\ & \leq \sup_{n \geq 1} E[\sum_{k \geq m} |X_t^{\varphi_n, k} - X_t^k| \frac{1}{\alpha_k^{(1-\delta)/2}} |X_t^{\varphi_n, k} - X_t^k| \alpha_k^{(1-\delta)/2}] \\ & \leq \sup_{n \geq 1} E[(\sum_{k \geq m} |X_t^{\varphi_n, k} - X_t^k|^2 \frac{1}{\alpha_k^{1-\delta}})^{1/2} (\sum_{k \geq m} |X_t^{\varphi_n, k} - X_t^k|^2 \alpha_k^{1-\delta})^{1/2}] \\ & \leq \sup_{n \geq 1} E[(\sum_{k \geq m} |X_t^{\varphi_n, k} - X_t^k|^2 \frac{1}{\alpha_k^{1-\delta}})]^{1/2} (E[(\sum_{k \geq m} |X_t^{\varphi_n, k} - X_t^k|^2 \alpha_k^{1-\delta})])^{1/2} \\ & \leq C(\sum_{k \geq m} \frac{1}{\alpha_k^{1-\delta}})^{1/2} \\ & < \epsilon \end{aligned}$$

for $m \geq m_0$. Choosing a n_0 such that for all $n \geq n_0$

$$\sum_{k \geq 1}^{m_0-1} E[|X_t^{\varphi_n, k} - X_t^k|^2] < \epsilon$$

we find that

$$E[\|X_t^{\varphi_n, k} - X_t\|^2] < 2\epsilon$$

for all $n \geq n_0$. So (5.36) holds.

Finally it follows from dominated convergence that (measurability/continuous modification of $(\omega, t) \mapsto X_t(\omega)$ can be shown)

$$\begin{aligned} & E\left[\left\|\int_0^t e^{(t-s)A}(B^n(s, X_s^{\varphi_n}) - B(s, X_s))ds\right\|^2\right] \\ & \leq 2 \sup_{0 \leq s \leq T} \|e^{sA}\|^2 (E[\int_0^t \|B^n(s, X_s^{\varphi_n}) - B^n(s, X_s)\|^2 ds] \\ & \quad + E[\int_0^t \|B^n(s, X_s) - B(s, X_s)\|^2 ds]) \\ & \leq 2 \sup_{0 \leq s \leq T} \|e^{sA}\|^2 K^2 (\int_0^t (E[\|X_s^n - X_s\|^2])^\alpha ds \\ & \quad + E[\int_0^t \|B^n(s, X_s) - B(s, X_s)\|^2 ds]) \\ & \longrightarrow 0 \text{ for } n \longrightarrow \infty. \end{aligned}$$

From this we see that X_t is a mild solution to (5.35). Uniqueness was shown in Section 5.1.

■

Remark 5.12 Another approach based on the so called S -transform to verify X_t as a unique solution to SDE's is discussed in [16], [15].

Example 5.13 Consider the equation

$$dX(t, \xi) = (\Delta X(t, \xi) + B(t, X(t, \cdot))(\xi))dt + \sigma(-\Delta)^{-\gamma/2} dW(t, \xi)$$

for $t \geq 0$ and $\xi \in [0, 2\pi]$, with periodic boundary conditions. In this case we have $H = L^2(0, 2\pi)$ and $A = \Delta$. We let $Q = (-\Delta)^{-\gamma}$ with $0 < \gamma < \frac{1}{3}$, $\theta = \frac{1}{2}$ in Section 5.1.1 and $M = (-\Delta)^\lambda$ for a sufficiently small $\lambda > 0$, then the conditions of Section 5.1.1 and the conditions of Theorem 5.11.

Proof. Let us show that (5.19) holds. We have

$$\begin{aligned} & \left\| e^{(t-u_1)A} Q^{1/2} - e^{(t-u_2)A} Q^{1/2} \right\|_E^2 \\ & = \sum_{n=1}^{\infty} \left(e^{-(t-u_1)\alpha_n} \alpha_n^{-\gamma/2} - e^{-(t-u_2)\alpha_n} \alpha_n^{-\gamma/2} \right)^2 \alpha_n^{2\lambda} \\ & = \sum_{n=1}^{\infty} \left(1 - e^{-(u_1-u_2)\alpha_n} \right)^2 e^{-2(t-u_1)\alpha_n} \alpha_n^{2\lambda-\gamma}. \end{aligned}$$

Now, for $\epsilon, \nu \in (0, 1)$ we can find positive constants C_ϵ and C_ν such that

$$(1 - e^{-a}) \leq C_\epsilon a^\epsilon \quad \text{and} \quad e^{-2a} \leq C_\nu a^{-\nu}$$

for every $a \geq 0$. Thus, the above is bounded by

$$\begin{aligned} & C_\epsilon |u_1 - u_2|^{2\epsilon} \sum_{n=1}^{\infty} \alpha_n^{2\epsilon+2\lambda-\gamma} e^{-2(t-u_1)\alpha_n} \\ & \leq C_\epsilon C_\nu |u_1 - u_2|^{2\epsilon} |t - u_1|^{-\nu} \sum_{n=1}^{\infty} \alpha_n^{2\epsilon+2\lambda-\gamma+\nu}. \end{aligned}$$

Rewriting this condition in the Fourier basis on the interval $[0, 2\pi]$, we get the condition

$$\sum_{k \in \mathbb{Z}} k^{4\epsilon+4\lambda-2\gamma+2\nu} < \infty.$$

Let $\nu = 1 - \delta$ and choose ϵ and λ small to get (5.19). Inequality (5.18) is proved similarly.

To see (5.20) we write

$$\int_H \|e^{(u_1-u_2)A} y - y\|^2 N_{Q_s}(dy) = E[\|e^{(u_1-u_2)A} \sqrt{Q_s} W_1 - \sqrt{Q_s} W_1\|^2]$$

where $W = (W_t)_{t \geq 0}$ is a cylindrical Brownian motion, $W_t = \sum_{n=1}^{\infty} W_t^n e_n$. We get

$$(e^{(u_1-u_2)A} \sqrt{Q_s} - \sqrt{Q_s}) W_1 = \sum_{n=1}^{\infty} (e^{-(u_1-u_2)\alpha_n} - 1) \left(\int_0^s q_n e^{-2u\alpha_n} du \right)^{1/2} W_1^n,$$

so that

$$\begin{aligned} E[\|e^{(u_1-u_2)A} \sqrt{Q_s} W_1 - \sqrt{Q_s} W_1\|^2] &= \sum_{n=1}^{\infty} (e^{-(u_1-u_2)\alpha_n} - 1)^2 \int_0^s q_n e^{-2u\alpha_n} du \\ &\leq C_\epsilon \sum_{n=1}^{\infty} ((u_1 - u_2)\alpha)^{2\epsilon} \frac{q_n (1 - e^{-2s\alpha_n})}{2\alpha_n} \\ &\leq C_\epsilon |u_1 - u_2|^{2\epsilon} \sum_{n=1}^{\infty} \alpha_n^{2\epsilon-1-\gamma}. \end{aligned}$$

As before we rewrite this in the Fourier basis we get the condition

$$\sum_{k \in \mathbb{Z}} k^{4\epsilon-2-2\gamma} < \infty$$

which is satisfied for small ϵ .

Finally we show that we have $\int_0^T \|\Lambda_t\|^{1+\theta} dt < \infty$ when $\theta = \frac{1}{2}$. We have

$$\begin{aligned} \|\Lambda_t x\|^2 &= \|A^{(1+\gamma)/2} (I - e^{2tA})^{-1/2} e^{tA} x\|^2 \\ &= t^{-(1+\gamma)} \sum_{n=1}^{\infty} (t\alpha_n)^{1+\gamma} (1 - e^{-2t\alpha_n})^{-1} e^{-2t\alpha_n}. \end{aligned}$$

The mapping $s \mapsto \frac{s^{1+\gamma}}{e^{2s}-1}$ is bounded on $(0, \infty)$, so that we get

$$\|\Lambda_t\| \leq C t^{-(1+\gamma)/2}$$

and thus we get that $t \mapsto \|\Lambda_t\|^{3/2}$ is integrable on any interval $[0, T]$. ■

5.6 Appendix

The following result which is based on Malliavin calculus and which is essentially due to [4] provides a compactness criterion for subsets of $L^2(\Omega)$ of square integrable functionals of a cylindrical Wiener process $W_t, 0 \leq t \leq 1$ on the Hilbert space H . See e.g. [17], [13] or [7] for more information about Malliavin calculus.

Theorem 5.14 *Assume that L is a self-adjoint compact operator on H^* with dense image. Denote by $DX \in L^2(\Omega; L^2([0, 1]) \otimes H^*)$ the Malliavin derivative of a square integrable X in the domain of D and by $\mathbf{D}_{1,2}$ the space of such functionals. Then for $0 < \beta < 1/2$ and $c > 0$ the set*

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \left(\int_0^1 \|L^{-1}D_u G\|_{L^2(\Omega)}^2 du \right)^{1/2} + \left(\int_0^1 \int_0^1 \frac{\|L^{-1}(D_{u_1}G - D_{u_2}G)\|_{L^2(\Omega)}^2}{|u_1 - u_2|^{1+2\beta}} du_1 du_2 \right)^{1/2} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

Remark 5.15 *Denote by $J : H^* \longrightarrow H$ the standard isometric isomorphism for Hilbert spaces H . Then an example of L which satisfies the conditions of Theorem 5.14 is given by*

$$La := \langle MJ(a), \cdot \rangle, a \in H^*,$$

where $M = A^\tau$ for some sufficiently small $\tau > 0$, where A is the densely defined operator in Section 1.

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