Optimal stopping with delayed information

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Abstract

We study a general optimal stopping problem for a strong Markov process in the case when there is a time lag $\delta > 0$ from the time τ when the decision to stop is taken (a stopping time) to the time $\tau + \delta$ when the system actually stops. Equivalently, we impose the constraint that the admissible times for stopping are stopping (Markov) times with respect to the *delayed* flow of information. It is shown that such a problem can be reduced to a classical optimal stopping problem by a simple transformation. The results are applied

- (i) to find the optimal time to sell an asset
 - (ii) to solve an optimal resource extraction problem,
- in both cases under delayed information flow.

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1 Introduction

Let $Y(t)$ be a strong Markov process in $\mathbb{R}^k$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \{P^y\}_{y \in \mathbb{R}^n})$, where P^y is the probability measure giving the law of $\{Y(t)\}_{t \geq 0}$ when $Y(0) = y \in \mathbb{R}^k$. Let $\delta \geq 0$ be a fixed constant. In this paper we consider optimal stopping problems of the form

$$(1.1) \quad \Phi_\delta(y) := \sup_{\alpha \in \mathcal{T}_\delta} E^y \left[\int_0^\alpha f(Y(t)) dt + g(Y(\alpha)) \right]$$

where E^y denotes expectation with respect to P^y and $f : \mathbb{R}^k \rightarrow \mathbb{R}$, $g : \mathbb{R}^k \rightarrow \mathbb{R}$ are given continuous functions such that

$$(1.2) \quad E^y \left[\int_0^\alpha |f(Y(t))| dt + |g(Y(\alpha))| \right] < \infty \quad \text{for all } \alpha \in \mathcal{T}_\delta$$

where we interpret $g(Y(\alpha))$ as 0 if $\alpha = \infty$. Here $\mathcal{T}_\delta$ is the set of δ -delayed stopping times, defined as follows

Definition 1.1 *A function $\alpha : \Omega \rightarrow [\delta, \infty]$ is called a δ -delayed stopping time if*

$$(1.3) \quad \{\omega; \alpha(\omega) \leq t\} \in \mathcal{F}_{t-\delta} \quad \text{for all } t \geq \delta$$

or, equivalently,

$$(1.4) \quad \{\omega; \alpha(\omega) \leq s + \delta\} \in \mathcal{F}_s \quad \text{for all } s \geq 0$$

The set of all δ -delayed stopping times is denoted by $\mathcal{T}_\delta$.

In other words, if we interpret $\alpha(\omega)$ as the time to stop, then $\alpha \in \mathcal{T}_\delta$ if the decision whether or not to stop at or before time t is based on the information represented by $\mathcal{F}_{t-\delta}$. In particular, if $\delta = 0$ then $\mathcal{T}_\delta = \mathcal{T}_0$ is the family of classical stopping times and (1.1) becomes the classical optimal stopping problem, discussed in many texts (see e.g. [Ø, Ch. 10]).

In the delayed case problem (1.1) models the situation when there is a delay $\delta > 0$ in the flow of information available to the agent searching for the optimal time to stop. An alternative way of stating this, is that there is a delay $\delta > 0$ from the decided stopping time $\tau \in \mathcal{T}_0$ (based on the complete current information available from the system) to the time $\alpha = \tau + \delta \in \mathcal{T}_\delta$ when the system actually stops. This new formulation is based on the following simple observation:

Lemma 1.2 (i) $\tau \in \mathcal{T}_0 \iff \alpha := \tau + \delta \in \mathcal{T}_\delta$
(ii) $\alpha \in \mathcal{T}_\delta \iff \tau := \alpha - \delta \in \mathcal{T}_0$

PROOF. It suffices to prove (i).

First, assume $\tau \in \mathcal{T}_0$. Then, for $t \geq \delta$,

$$\{\omega; \tau(\omega) + \delta \leq t\} = \{\omega; \tau(\omega) \leq t - \delta\} \in \mathcal{F}_{t-\delta},$$

and hence $\alpha := \tau + \delta \in \mathcal{T}_\delta$.

Conversely, if $\alpha := \tau + \delta \in \mathcal{T}_\delta$ then

$$\{\omega; \tau(\omega) \leq t\} = \{\omega; \tau(\omega) + \delta \leq t + \delta\} = \{\omega; \alpha(\omega) \leq t + \delta\} \in \mathcal{F}_{(t+\delta)-\delta} = \mathcal{F}_t,$$

and hence $\tau \in \mathcal{T}_0$. □

Remark 1.3 In view of this result we see that it is possible to give another interpretation of problem (1.1), namely

$$(1.5) \quad \Phi_\delta(y) = \sup_{\tau \in \mathcal{T}_0} E^y \left[\int_0^{\tau+\delta} f(Y(t))dt + g(Y(\tau+\delta)) \right]$$

In this formulation the problem appears as an optimal stopping problem over classical stopping times $\tau \in \mathcal{T}_0$, but with *delayed effect* of the stopping: If the stopping time $\tau \in \mathcal{T}_0$ is chosen, then the system itself is stopped at time $\tau + \delta$, i.e. after a delay $\delta > 0$.

Note that $\mathcal{T}_\delta \subset \mathcal{T}_0$ for $\delta > 0$ and hence

$$\Phi_\delta(y) \leq \Phi_0(y)$$

and we can interpret $\Phi_0(y) - \Phi_\delta(y)$ as the loss of value due to the delay of information.

In this paper we show that the delayed optimal stopping problem (1.1) can be reduced to a classical optimal stopping problem by a simple transformation (Theorem 2.1).

We call $\alpha^* \in \mathcal{T}_\delta$ an *optimal stopping time* for the problem (1.1) if

$$(1.6) \quad \Phi_\delta(y) = E^y \left[\int_0^{\alpha^*} f(Y(t))dt + g(Y(\alpha^*)) \right].$$

This paper may be regarded as a partial extension of [AK2], where the geometric Brownian motion case is studied and solved (see Example 3.1), with a more general (Markovian) delay $\delta(X) \leq 0$. See also [AK1]. For a related type of problem involving impulse control with delivery lags see [BS].

2 Optimal stopping with δ -delayed information

We are now ready to state and prove the main result of this paper:

Theorem 2.1 a) *Consider the two optimal stopping problems:*

$$(2.1) \quad \Phi_\delta(y) := \sup_{\alpha \in \mathcal{T}_\delta} E^y \left[\int_0^\alpha f(Y(t))dt + g(Y(\alpha)) \right]$$

$$(2.2) \quad \tilde{\Phi}(y) := \sup_{\tau \in \mathcal{T}_0} E^y \left[\int_0^\tau f(Y(t))dt + \tilde{g}_\delta(Y(\tau)) \right]$$

where

$$(2.3) \quad \tilde{g}_\delta(y) = E^y \left[\int_0^\delta f(Y(t))dt + g(Y(\delta)) \right].$$

Then we have

$$\Phi_\delta(y) = \tilde{\Phi}(y) \quad \text{for all } y \in \mathbb{R}^k, \delta \geq 0.$$

b) Moreover, $\alpha^* \in \mathcal{T}_\delta$ is an optimal stopping time for the delayed problem (2.1) if and only if

$$(2.4) \quad \alpha^* := \tau^* + \delta$$

where $\tau^* \in \mathcal{T}_0$ is an optimal stopping time for the non-delayed problem (2.2).

PROOF. **a)** Define

$$(2.5) \quad J^{(\alpha)}(y) = E^y \left[\int_0^\alpha f(Y(t))dt + g(Y(\alpha)) \right]; \quad \alpha \in \mathcal{T}_\delta,$$

and

$$(2.6) \quad \tilde{J}^{(\tau)}(y) = E^y \left[\int_0^\tau f(Y(t))dt + \tilde{g}_\delta(Y(\tau)) \right]; \quad \tau \in \mathcal{T}_0.$$

Choose $\alpha \in \mathcal{T}_\delta$ and put

$$\tau = \alpha - \delta \in \mathcal{T}_0$$

Then $\alpha = \tau + \delta$ and hence

$$\begin{aligned} J^{(\alpha)}(y) &= E^y \left[\int_0^\alpha f(Y(t))dt + g(Y(\alpha)) \right] \\ &= E^y \left[\int_0^{\tau+\delta} f(Y(t))dt + g(Y(\tau + \delta)) \right] \\ &= E^y \left[\int_0^\tau f(Y(t))dt + \int_\tau^{\tau+\delta} f(Y(t))dt + g(Y(\tau + \delta)) \right] \\ (2.7) \quad &= E^y \left[\int_0^\tau f(Y(t))dt + E^y \left[\theta_\tau \left\{ \int_0^\delta f(Y(t))dt + g(Y(\delta)) \right\} \right] \right], \end{aligned}$$

where θ_τ is the *shift operator*, defined by

$$\theta_\tau \{h(Y(s))\} = h(Y(\tau + s)) \quad \text{for } s \geq 0, \text{ for all measurable } h : \mathbb{R}^k \longrightarrow \mathbb{R},$$

and we have used that

$$\theta_\tau \left(\int_0^\delta f(Y(t))dt \right) = \int_\tau^{\tau+\delta} f(Y(t))dt.$$

We refer to [BG] for more information about Markov processes. By the strong Markov property we now get from (2.7) that

$$\begin{aligned} J^{(\alpha)}(y) &= E^y \left[\int_0^\tau f(Y(t))dt + E^y \left[\theta_\tau \left\{ \int_0^\delta f(Y(t))dt + g(Y(\delta)) \right\} \middle| \mathcal{F}_\tau \right] \right] \\ &= E^y \left[\int_0^\tau f(Y(t))dt + E^{Y(\tau)} \left[\int_0^\delta f(Y(t))dt + g(Y(\delta)) \right] \right] \\ (2.8) \quad &= E^y \left[\int_0^\tau f(Y(t))dt + \tilde{g}_\delta(Y(\tau)) \right] = \tilde{J}^{(\tau)}(y). \end{aligned}$$

Hence, by Lemma 1.2 (ii),

$$\begin{aligned}\Phi_\delta(y) &= \sup_{\alpha \in \mathcal{T}_\delta} J^{(\alpha)}(y) = \sup_{\alpha - \delta \in \mathcal{T}_0} J^{(\alpha)}(y) \\ &= \sup_{\alpha - \delta \in \mathcal{T}_0} \tilde{J}^{(\alpha - \delta)}(y) = \sup_{\tau \in \mathcal{T}_0} \tilde{J}^{(\tau)}(y) = \tilde{\Phi}(y),\end{aligned}$$

as claimed.

b) Suppose $\tau^* \in \mathcal{T}_0$ is optimal for (2.2). Define

$$\alpha^* := \tau^* + \delta.$$

Then $\alpha^* \in \mathcal{T}_\delta$ by Lemma 1.2 and by (2.8) combined with a) we have

$$J^{(\alpha^*)}(y) = \tilde{J}^{(\tau^*)}(y) = \tilde{\Phi}(y) = \Phi_\delta(y).$$

Hence α^* is optimal for (2.1).

Conversely, if $\alpha^* \in \mathcal{T}_\delta$ is optimal for (2.1) a similar argument gives that $\tau^* := \alpha^* - \delta$ is optimal for (2.2). $\square$

3 Application 1: The optimal time to sell an asset

In this section we illustrate Theorem 2.1 by solving the following problem:

Example 3.1 (The optimal time to sell an asset)

This case (without the jump part) was first solved by [AK1], with a more general (Markovian) delay $\delta(X) \geq 0$.

Suppose the value $X(t)$ of an asset at time t is modelled by a geometric Lévy process of the form

$$(3.1) \quad dX(t) = X(t^-) \left[\mu dt + \sigma dB(t) + \int_{\mathbb{R}} z \tilde{N}(dt, dz) \right], \quad X(0) = x > 0,$$

where μ, σ and x are constants. Here $B(t)$ and $\eta(t) := \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz)$ is a Brownian motion and an independent pure jump Lévy process, respectively, where

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$

is the compensated Poisson random measure of $\eta(\cdot)$, $N(dt, dz)$ is the Poisson random measure of $\eta(\cdot)$ and $\nu(dz)$ is the Lévy measure of $\eta(\cdot)$. We assume that

$$(3.2) \quad 0 \geq z \geq -1 \quad \text{a.s. } \nu.$$

This guarantees that $X(t)$ never jumps down to a negative value. For convenience, we also assume that

$$(3.3) \quad E[\eta^2(t)] < \infty \quad \text{for all } t \geq 0.$$

Then by the Itô formula for Lévy processes (see e.g. [ØS]) the solution of equation (3.1) is

$$(3.4) \quad \begin{aligned} X(t) = x \exp & \left[\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma B(t) \right. \\ & \left. + \int_0^t \int_{\mathbb{R}} \{ \ln(1+z) - z \} \nu(dz) ds + \int_0^t \int_{\mathbb{R}} \ln(1+z) \tilde{N}(ds, dz) \right]; \quad t \geq 0. \end{aligned}$$

We now study the following problem

$$(3.5) \quad \Phi_\delta(s, x) = \sup_{\alpha \in \mathcal{T}_\delta} E^{s,x} [e^{-\rho(s+\alpha)} (X(\alpha) - q)],$$

where $E^{s,x}$ denotes expectation with respect to the probability law $P^{s,x}$ of the time-space process

$$dY(t) = \begin{bmatrix} dt \\ dX(t) \end{bmatrix}; \quad Y(0) = \begin{bmatrix} s \\ x \end{bmatrix}$$

and $\rho > 0$, $q > 0$ are constants. We assume that

$$(3.6) \quad \rho > \mu.$$

One possible interpretation of this problem is that $\Phi_\delta(s, x)$ represents the maximal expected discounted net payment obtained by selling the asset at a δ -delayed stopping time (ρ is the discounting exponent and q is the transaction cost).

It is well-known that in the *no delay case* ($\delta = 0$) the solution of the problem (3.5) is the following (under some additional assumptions on the Lévy measure ν):

$$(3.7) \quad \Phi_0(s, x) = e^{-\rho s} \Psi_0(x)$$

where

$$(3.8) \quad \Psi_0(x) = \begin{cases} x - q & ; \quad x \geq x_0^* \\ C_0 x^\lambda & ; \quad 0 < x < x_0^* \end{cases}$$

Here $\lambda > 1$ is uniquely determined by the equation

$$(3.9) \quad -\rho + \mu\lambda + \frac{1}{2}\sigma^2\lambda(\lambda-1) + \int_{\mathbb{R}} \{ (1+z)^\lambda - 1 - \lambda z \} \nu(dz) = 0,$$

and x_0^* and C_0 are given by

$$(3.10) \quad x_0^* = \frac{\lambda q}{\lambda - 1}$$

$$(3.11) \quad C_0 = \frac{1}{\lambda} (x_0^*)^{1-\lambda}.$$

The corresponding optimal stopping time $\tau^* \in \mathcal{T}_0$ is

$$(3.12) \quad \tau^* = \inf\{t > 0; X(t) \geq x_0^*\}$$

Thus it is optimal to sell at the first time the price $X(t)$ equals or exceeds the value x_0^* . We refer to [OS, Example 2.5] for details.

To find the solution in the *delay* case ($\delta > 0$) we note that we have $f = 0$ and

$$g(y) = g(s, x) = e^{-\rho s}(x - q)$$

Hence, by (2.2),

$$(3.13) \quad \begin{aligned} \tilde{g}_\delta(y) &= E^y[g(Y(\delta))] = E^{s,x}[e^{-\rho(s+\delta)}(X(\delta) - q)] \\ &= e^{-\rho(s+\delta)}(E^x[X(\delta)] - q) = e^{-\rho(s+\delta)}(x e^{\mu\delta} - q) \\ &= e^{-\rho s + \delta(\mu - \rho)}(x - q e^{-\mu\delta}) = K e^{-\rho s}(x - \tilde{q}), \end{aligned}$$

where

$$(3.14) \quad K = e^{\delta(\mu - \rho)} \quad \text{and} \quad \tilde{q} = q e^{-\mu\delta}.$$

Thus $\tilde{g}_\delta$ has the same form as g , so we can apply the results (3.7)–(3.12) to find $\tilde{\Phi}(y)$ and the corresponding optimal τ^* :

$$(3.15) \quad \tilde{\Phi}(y) = \tilde{\Phi}(s, x) = e^{-\rho s}\tilde{\Psi}(x)$$

where

$$(3.16) \quad \tilde{\Psi}(x) = \begin{cases} K(x - \tilde{q}) & ; \quad x \geq \tilde{x}^* \\ \tilde{C} x^\lambda & ; \quad 0 < x < \tilde{x}^*, \end{cases}$$

with λ as in (3.9). Here $\tilde{x}^*$ and $\tilde{C}$ are given by

$$(3.17) \quad \tilde{x}^* = \frac{\lambda \tilde{q}}{\lambda - 1}$$

$$(3.18) \quad \tilde{C} = \frac{1}{\lambda}(\tilde{x}^*)^{1-\lambda}.$$

The corresponding optimal stopping time for problem (2.2) and (2.1), respectively, is

$$(3.19) \quad \tilde{\tau}^* = \inf\{t > 0; X(t) \geq \tilde{x}^*\}$$

$$(3.20) \quad \alpha^* = \tilde{\tau}^* + \delta.$$

Using Theorem 2.1 we conclude the following:

Theorem 3.2 *The value function $\Phi_\delta(y)$ for the delayed optimal stopping problem (3.5) is given by*

$$\Phi_\delta(y) = \tilde{\Phi}(y),$$

where $\tilde{\Phi}$ is as in (3.15)–(3.18). The corresponding optimal stopping time $\alpha^ \in \mathcal{T}_\delta$ is*

$$\alpha^* = \inf\{t > 0; X(t) \geq \tilde{x}^*\} + \delta.$$

Remark 3.3 Assume for example that

$$\mu > 0.$$

Then comparing (3.17) with the non-delayed case (3.10) we see that $\tilde{q} > q$ and hence

$$\tilde{x}^* < x_0^*$$

Thus, in terms of the *delayed effect of the stopping time* formulation (see (1.5)), it is optimal to stop at the first time $t = \tilde{\tau}^*$ when $X(t) \geq \tilde{x}^*$. This is sooner than in the non-delayed case, because of the anticipation that during the delay time interval $[\tau^*, \tau^* + \delta]$ $X(t)$ is likely to increase (since $\mu > 0$). See Figure 1.

Figure 1. The optimal stopping times for Example 3.1 ($\mu > 0$)

4 Application 2: An optimal resource extraction problem

In the no delay case the following example was discussed in $[\emptyset]$ (continuous case) and $[\emptyset S]$ (jump diffusion case). Our example models the situation when there is a time lag $\delta > 0$ between the decided stopping time $\tau \in \mathcal{T}_0$ and the time $\alpha = \tau + \delta \in \mathcal{T}_\delta$ when the result of the stopping decision comes into effect.

Example 4.1 (Optimal time to stop resource extraction) Suppose the price $P(t)$ at time t per unit of a resource (oil, gas, ...) is given by

$$(4.1) \quad dP(t) = P(t^-) \left[\mu dt + \sigma dB(t) + \int_{\mathbb{R}} z \tilde{N}(dt, dz) \right]; \quad P(0) = p > 0$$

where, as in Example 3.1, μ and σ are given constants and we assume that $z \geq 0$ a.s. with respect to ν .

Let $Q(t)$ denote the amount of remaining resources at time t . As long as the extraction field is open, we assume that the extraction rate is proportional to the remaining amount, i.e.

$$(4.2) \quad dQ(t) = -\lambda Q(t) dt; \quad Q(0) = q > 0$$

where $\lambda > 0$ is a known constant.

If we decide to stop the extraction and close the field at a (delayed) stopping time $\alpha \in \mathcal{T}_\delta$, then the expected total discounted net profit $J^\alpha(s, p, q)$ is assumed to have the form

$$(4.3) \quad J^\alpha(s, p, q) = E^{(s, p, q)} \left[\int_0^\alpha e^{-\rho(s+t)} (\lambda P(t) Q(t) - K) dt + \theta e^{-\rho(s+\alpha)} P(\alpha) Q(\alpha) \right]$$

where $K > 0$ is the (constant) running cost rate and $\rho > 0$, $\theta > 0$ are other constants. The expectation $E^{(s, p, q)}$ is taken with respect to the probability law $P^{(s, p, q)}$ of the strong Markov process

$$(4.4) \quad Y(t) := \begin{bmatrix} s+t \\ P(t) \\ Q(t) \end{bmatrix}, \text{ which starts at } y = \begin{bmatrix} s \\ p \\ q \end{bmatrix} \text{ at time } t = 0.$$

The explanation of the quantity $J^\alpha(s, p, q)$ in (4.3) is the following:

As long as the field is open (i.e. as long as $t < \alpha$) the gross income rate from the production is price times production rate, i.e. $P(t)\lambda Q(t)$. Subtracting the running cost rate K we get the net profit rate

$$\lambda P(t) Q(t) - K \quad \text{for } 0 \leq t < \alpha.$$

If the field is closed at time α the net value of the remaining resources is estimated to be $\theta P(\alpha)Q(\alpha)$. Discounting and integrating/adding these quantities and taking expectation we get (4.3).

We want to find the value function $\Phi_\delta(s, p, q)$ and the corresponding optimal delayed stopping time $\alpha^* \in \mathcal{T}_\delta$ such that

$$(4.5) \quad \Phi_\delta(y) = \Phi_\delta(s, p, q) = \sup_{\alpha \in \mathcal{T}_\delta} J^\alpha(s, p, q) = J^{\alpha^*}(s, p, q)$$

In the case of *no delay* ($\delta = 0$) it is shown in [ØS, p. 158–162] that if the following relations between the parameters hold:

$$(4.6) \quad 0 < \theta(\lambda + \rho - \mu) < \lambda$$

then the optimal stopping time $\tau_0^* \in \mathcal{T}_0$ is

$$(4.7) \quad \tau_0^* = \inf\{t > 0; P(t)Q(t) \leq w_0^*\},$$

where

$$(4.8) \quad w_0^* = \frac{(-r_2)K(\lambda + \rho - \mu)}{(1 - r_2)\rho(\lambda - \theta(\lambda + \rho - \mu))},$$

$r_2 < 0$ being the negative solution of the equation

$$(4.9) \quad h(r) := -\rho + (\mu - \lambda)r + \frac{1}{2}\sigma^2 r(r - 1) + \int_{\mathbb{R}} \{(1 + z)^r - 1 - rz\} \nu(dz) = 0$$

In this case we have

$$f(y) = f(s, p, q) = e^{-\rho s}(\lambda pq - K)$$

and

$$g(y) = g(s, p, q) = \theta e^{-\rho s} pq$$

Thus

$$\begin{aligned} \tilde{g}_\delta(y) &= E^y \left[\int_0^\delta e^{-\rho(s+t)} (\lambda P(t)Q(t) - K) dt \right] + E^y [\theta e^{-\rho(s+\delta)} P(\delta)Q(\delta)] \\ &= \int_0^\delta e^{-\rho(s+t)} (\lambda E[P(t)Q(t)] - K) dt + \theta e^{-\rho(s+\delta)} E^y [P(\delta)Q(\delta)] \\ &= \int_0^\delta e^{-\rho(s+t)} (\lambda pq e^{(\mu-\lambda)t} - K) dt + \theta e^{-\rho(s+\delta)} pq e^{(\mu-\lambda)\delta} \end{aligned}$$

$$\begin{aligned}
&= e^{-\rho s} \left[\{(\lambda + \rho - \mu)^{-1} \lambda (1 - e^{-(\lambda + \rho - \mu)\delta}) + \theta e^{-(\lambda + \rho - \mu)\delta}\} pq - \frac{K}{\rho} (1 - e^{-\rho\delta}) \right] \\
(4.10) \quad &= e^{-\rho s} [F_1 pq - F_2], \quad \text{where}
\end{aligned}$$

$$\begin{aligned}
(4.11) \quad &F_1 = (\lambda + \rho - \mu)^{-1} \lambda (1 - e^{-(\lambda + \rho - \mu)\delta}) + \theta e^{-(\lambda + \rho - \mu)\delta} \\
&\text{and}
\end{aligned}$$

$$(4.12) \quad F_2 = \frac{K}{\rho} (1 - e^{-\rho\delta})$$

Therefore, according to Theorem 2.1 we have

$$(4.13) \quad \Phi_\delta(y) = \sup_{\tau \in \mathcal{T}_0} E^y \left[\int_0^\tau e^{-\rho(s+t)} (\lambda P(t)Q(t) - K) dt \right] + E^y [e^{-\rho(s+\tau)} (F_1 P(\tau)Q(\tau) + F_2)]$$

The method used in [ØS] to provide the solution (4.7)–(4.9) in the no delay case can easily be modified to find the optimal stopping time τ^* for the problem (4.13). The result is

$$(4.14) \quad w_\delta^* = \frac{(-r_2)K(\lambda + \rho - \mu)e^{(\lambda - \mu)\delta}}{(1 - r)\rho[\lambda - \theta(\lambda + \rho - \mu)]} = w_0^* e^{(\lambda - \mu)\delta}.$$

We have proved:

Theorem 4.2 *The optimal stopping time $\alpha^* \in \mathcal{T}_\delta$ for the delayed optimal stopping problem is*

$$(4.15) \quad \alpha^* = \tau_\delta^* + \delta,$$

where

$$(4.16) \quad \tau_\delta^* = \inf\{t > 0; P(t)Q(t) \leq w_\delta^*\},$$

with w_δ^* given by (4.14).

Remark 4.3 Note that the threshold w_δ^* for the decision to close down in the case of a time lag in the action only differs from the corresponding threshold w_0^* in the no delay case by the factor $e^{(\lambda - \mu)\delta}$.

Assume, for example, that $\lambda > \mu$. Then we should decide to stop *sooner* in the delay case than in the no delay case, because of the anticipation that $P(t)Q(t)$ will probably decrease during the extra time δ it takes before the closing down actually takes place.

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