ON THE WELL-POSEDNESS OF
THE DEGASPERIS-PROCESI EQUATION

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ABSTRACT. We investigate well-posedness in classes of discontinuous functions for the nonlinear and third order dispersive Degasperis-Procesi equation

(DP) $\qquad \partial_t u - \partial_{txx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u.$

This equation can be regarded as a model for shallow-water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm equation (one order more accurate than the KdV equation). We prove existence and L^1 stability (uniqueness) results for entropy weak solutions belonging to the class $L^1 \cap BV$, while existence of at least one weak solution, satisfying a restricted set of entropy inequalities, is proved in the class $L^2 \cap L^4$. Finally, we extend our results to a class of generalized Degasperis-Procesi equations.

CONTENTS

1. Introduction

Our aim is to investigate well-posedness in classes of discontinuous functions for the *Degasperis-Procesi equation*

$$(1.1) \qquad \partial_t u - \partial_{txx}^3 u + 4u\partial_x u = 3\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

which is an equation that can be viewed as a shallow water approximation to the Euler equations (there will be more on this later).

We are interested in the Cauchy problem for this equation, so we augment (1.1) with an initial condition u_0:

$$(1.2) \qquad\qquad u(0,x) = u_0(x), \qquad x \in \mathbb{R},$$

where we assume that

$$(1.3) \qquad\qquad u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R}).$$

Degasperis and Procesi [17] studied the following family of third order dispersive nonlinear equations, indexed over six constants $c_0, \gamma, \alpha, c_1, c_2, c_3 \in \mathbb{R}$:

$$\partial_t u + c_0 \partial_x u + \gamma \partial_{xxx}^3 u - \alpha^2 \partial_{txx}^3 u = \partial_x \left(c_1 u^2 + c_2 (\partial_x u)^2 + c_3 u \partial_{xx}^2 u \right).$$

Using the method of asymptotic integrability, they found that only three equations from this family were asymptotically integrable up to third order: the *KdV equation* ($\alpha = c_2 = c_3 = 0$), the *Camassa-Holm equation* ($c_1 = -\frac{3c_3}{2\alpha^2}$, $c_2 = \frac{c_3}{2}$), and one new equation ($c_1 = -\frac{2c_3}{\alpha^2}$, $c_2 = c_3$), which properly scaled reads

$$(1.4) \quad \partial_t u + \partial_x u + 6u\partial_x u + \partial_{xxx}^3 u - \alpha^2 \left(\partial_{txx}^3 u + \frac{9}{2}\partial_x u \partial_{xx}^2 u + \frac{3}{2}u\partial_{xxx}^3 u \right) = 0.$$

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, equation (1.4) can be transformed into the form (1.1), see [15, 16] for details.

The Korteweg-deVries (KdV) equation models weakly nonlinear unidirectional long waves, and arises in various physical contexts. For example, it models surface waves of small amplitude and long wavelength on shallow water. In this context, $u(t,x)$ represents the wave height above a flat bottom, with x being proportional to distance in the propagation direction and t being proportional to the elapsed time. The KdV equation is completely integrable and possesses solitary wave solutions that are solitons. The Cauchy problem for the KdV equation is well studied, see [22] and the references cited therein. For example, if $u_0 \in H^1(\mathbb{R})$ there exists a unique global solution to the KdV equation.

The Camassa-Holm equation entered the arena in the early 1990s [2]. In one interpretation, it models the propagation of unidirectional shallow water waves on a flat bottom, and then $u(t,x)$ represents the fluid velocity at time t in the horizontal direction x [2, 21]. The Camassa-Holm equation is a water wave equation at quadratic order in an asymptotic expansion for unidirectional shallow water waves described by the incompressible Euler equations, while the KdV equation appears at first order in this expansion [2, 21]. In another interpretation, the Camassa-Holm equation was derived by Dai [12] as a model for finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The Camassa-Holm equation possesses many interesting properties, among which we highlight its bi-Hamiltonian structure (an infinite number of conservation laws) [18, 2] and that it is completely integrable [2, 1, 10, 6]. Moreover, it has an infinite number of non-smooth solitary wave solutions called peakons (since their first derivatives at

the wave peak are discontinuous), which interact like solitons. Although the KdV equation admits solitary waves that are solitons, it does not model wave breaking. The Camassa-Holm equation is remarkable in the sense that it admits soliton solutions and at the same time allows for wave breaking. For a discussion of the Camassa-Holm equation as well as other related equations, see the recent paper [20]. From a mathematical point of view the Camassa-Holm equation is rather well studied. Local well-posedness results are proved in [7, 19, 24, 29]. It is also known that there exist global solutions for a certain class of initial data and also solutions that blow up in finite time for a large class of initial data [5, 7, 9]. Existence and uniqueness results for global weak solutions of the Camassa-Holm equation are proved in [3, 8, 11, 32, 33, 13, 14].

Let us now turn to the Degasperis-Procesi equation (1.1). As mentioned before, it was singled out first in [17] by an asymptotic integrability test within a family of third order dispersive equations. Then Degasperis, Holm, and Hone [16] proved the exact integrability of (1.1) by constructing a Lax pair. Moreover, they displayed a relation to a negative flow in the Kaup-Kupershmidt hierarchy by a reciprocal transformation and derived two infinite sequences of conserved quantities along with a bi-Hamiltonian structure. They also showed that the Degasperis-Procesi equation possesses "non-smooth" solutions that are superpositions of multipeakons and described the integrable finite-dimensional peakon dynamics, which were compared with the multipeakon dynamics of the Camassa-Holm equation. An explicit solution was also found in the perfectly anti-symmetric peakon-antipeakon collision case. Lundmark and Szmigielski [26] presented an inverse scattering approach for computing n-peakon solutions to (1.1). Mustafa [28] proved that smooth solutions to (1.1) have infinite speed of propagation, that is, they loose instantly the property of having compact support. Regarding well-posedness (in terms of existence, uniqueness, and stability of solutions) of the Cauchy problem for the Degasperis-Procesi equation (1.1), Yin has studied this within certain functional classes in a series of recent papers [34, 35, 36, 37].

To put the present paper in a proper perspective we shall next comment on the results obtained by Yin. In [34], he studied the Cauchy problem on the unit circle (i.e., the 1-periodic case). He proved the local well-posedness when $u_0 \in H^r(\mathbb{S})$, $r > 3/2$, and provided an estimate of the maximal existence time. If, in addition, the initial function u_0 is odd and $u_0'(0) < 0$, then he proved that the corresponding strong solution blows up in finite time, whereas if the sign of $(1 - \partial_{xx}^2)u_0$ is constant, then the corresponding strong solution is global in time. In [35] he proved similar results for the Cauchy problem on $\mathbb{R}$.

In [36], Yin proved the following strong solution theorem for (1.1), (1.2) (see [37] for the 1-periodic case): Let $u_0 \in H^s(\mathbb{R})$ with $s \geq 3$. Suppose $u_0 \in L^3(\mathbb{R})$ is such that $m_0 := (1 - \partial_{xx}^2)u_0 \in L^1(\mathbb{R})$ is non-negative (non-positive). Then the Cauchy problem (1.1), (1.2) possesses a unique global strong solution $u \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}))$. Furthermore, $I(u) := \int_{\mathbb{R}} u \, dx$ and $E(u) := \int_{\mathbb{R}} u^3 \, dx$ are two conserved quantities. Finally, if $m := (1 - \partial_{xx}^2)u$, then for any $t \in \mathbb{R}_+$ the following properties hold: (i) $m(t, \cdot) \geq 0$ ($m(t, \cdot) \leq 0$), $u(t, \cdot) \geq 0$ ($u(t, \cdot) \leq 0$), and $|\partial_x u(\cdot, t)| \leq -u(t, \cdot)$ ($|\partial_x u(\cdot, t)| \leq u(t, \cdot)$), (ii) $\|u(t, \cdot)\|_{L^1(\mathbb{R})} = \|m(t, \cdot)\|_{L^1(\mathbb{R})} = \|m_0\|_{L^1(\mathbb{R})}$ and $\|\partial_x u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}$, (iii) $\|u(t, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \|u_0\|_{H^1(\mathbb{R})}^2 + t \|u_0\|_{L^3(\mathbb{R})}^3$. The strong solution theorem was then used in conjunction with an approximation procedure to prove existence of a global weak solution to

(1.1), (1.2). But before we discuss Yin's weak solution result, we need to explain what we mean by a weak solution.

Formally, problem (1.1), (1.2) is equivalent to the hyperbolic-elliptic system

$$(1.5) \quad \begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_x P = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\partial_{xx}^2 P + P = \frac{3}{2} u^2, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

For any $\lambda > 0$ the operator $(\lambda^2 - \partial_{xx}^2)^{-1}$ has a convolution structure:

$$(1.6) \quad (\lambda^2 - \partial_{xx}^2)^{-1}(f)(x) = (G_\lambda \star f)(x) = \frac{1}{2\lambda} \int_{\mathbb{R}} e^{-|x-y|/\lambda} f(y)\, dy, \qquad x \in \mathbb{R},$$

where $G_\lambda(x) := \frac{\lambda}{2} e^{-\lambda|x|}$. Hence we have

$$(1.7) \quad P(t, x) = P^u(t, x) := G_1 \star \left(\frac{3}{2} u^2 \right)(t, x),$$

and (1.5) can be written as a conservation law with a nonlocal flux function:

$$(1.8) \quad \begin{cases} \partial_t u + \partial_x \left[\frac{u^2}{2} + G_1 \star \left(\frac{3}{2} u^2 \right) \right] = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

According to [35, 36] a function u is a weak solution of (1.1), (1.2) if it belongs to $L^\infty(0, T; H^1(\mathbb{R}))$ for all $T > 0$ and (1.8) holds in $\mathcal{D}'([0, \infty) \times \mathbb{R})$ (i.e., in the sense of distributions on $[0, \infty) \times \mathbb{R}$). Regarding the existence of a global weak solution to (1.1), (1.2), Yin [36] proved the following result: Suppose u_0 belongs to $H^1(\mathbb{R}) \cap L^3(\mathbb{R})$ and $(1 - \partial_{xx}^2)u_0$ is a nonnegative bounded Radon measure on $\mathbb{R}$, i.e., $(1 - \partial_{xx}^2)u_0 \in \mathcal{M}_+(\mathbb{R})$. Then (1.1), (1.2) possesses a weak solution u belonging to $W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}) \cap L^\infty_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{R}))$. Furthermore, $(1 - \partial_{xx}^2)u(t, \cdot) \in \mathcal{M}_+(\mathbb{R})$ for a.e. $t \in \mathbb{R}_+$ and $I(u), E(u)$ are two conservation laws. Finally, the weak solution is unique. Similar results for the periodic case can be found in [37]. An important tool in Yin's analysis is the quantity $m := u - \partial_{xx}^2 u$, which satisfies

$$(1.9) \quad \partial_t m + 3u\partial_x m + m\partial_x u = 0.$$

The benefit of introducing this quantity becomes evident after noticing that a suitable renormalization turns (1.9) into a divergence-form (linear) transport equation. More precisely, $m^{\frac{1}{3}}$ satisfies (at least formally)

$$\partial_t m^{\frac{1}{3}} + \partial_x \left(u m^{\frac{1}{3}} \right) = 0.$$

With the purpose of motivating the present paper, we stress that a H^1 bound on the weak solution $u(t, \cdot)$ is valid only under restrictive conditions on the initial function u_0. Moreover, the requirement in the weak formulation that $u(t, \cdot)$ should belong to H^1 is much stronger than what is actually needed to make distributional sense to (1.8). For that purpose, it suffices to know that $u \in L^2_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R})$.

Of course, a motivation for insisting on the H^1 space and also for involving the quantity $m(t, \cdot)$ comes from the similitude between the weak formulations of the Degasperis-Procesi and Camassa-Holm equations, where the latter reads

$$(1.10) \quad \begin{cases} \partial_t u + \partial_x \left[\frac{u^2}{2} + G_1 \star \left(\frac{3}{2} u^2 + \frac{1}{2}(\partial_x u)^2 \right) \right] = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

In this equation, due to the additional term $\frac{1}{2}(\partial_x u)^2$, it is natural to impose that u should possess H^1 regularity in the spatial variable.

These considerations lead us to suspect that it should be possible to prove various existence, uniqueness, and stability results for the Degasperis-Procesi equation in functional classes that are significantly lager than the one used in [36], and this is what we set out to do in this paper.

Our starting point is that formally there is an L^2 bound on the solution in terms of the L^2 norm of the initial data u_0. Indeed, if we introduce the quantity $v := G_2 \star u$, then formally the following conservation law can be derived:

$$
(1.11) \quad
\begin{aligned}
&\partial_t \left((\partial_{xx}^2 v)^2 + 5(\partial_x v)^2 + 4v^2 \right) \\
&+ \partial_x \left(\frac{2}{3} u^3 + 4v\, G_1 \star (u^2) + \partial_x v\, \partial_x \left[G_1 \star (u^2) \right] - 4u^2 v \right) = 0.
\end{aligned}
$$

It follows from this that $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$ and thereby also $u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$. The L^2 estimate on u is the key to deriving a series of other (formal) estimates, among which we highlight

$$
(1.12) \quad P \in L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R})),\ \partial_{xx}^2 P \in L^\infty(0,T; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))\ \forall T > 0
$$

and

$$
u \in L^\infty(0,T; L^1(\mathbb{R}) \cap BV(\mathbb{R}))\ \forall T > 0,
$$

where the BV estimate is particularly important as it ensures strong compactness of a sequence of solutions to the Degasperis-Procesi equation.

To prove existence of a global weak solution we construct approximate solutions for which similar bounds can be derived rigorously. To this end, we consider smooth solutions u_ε of the following fourth order viscous approximation of the Degasperis-Procesi equation (1.1):

$$
(1.13) \quad \partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon = 3\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon - \varepsilon \partial_{xxxx}^4 u_\varepsilon.
$$

This equation can be written in the more suggestive form of a viscous conservation law with a non-local flux:

$$
(1.14) \quad \partial_t u_\varepsilon + \partial_x \left[\frac{u_\varepsilon^2}{2} + G_1 \star \left(\frac{3}{2} u_\varepsilon^2 \right) \right] = \varepsilon \partial_{xx}^2 u_\varepsilon.
$$

Assuming that the initial data u_0 satisfy (1.3), we establish a series of ε - uniform estimates that are analogous to the formal ones discussed above. For example, $\{u_\varepsilon\}_{\varepsilon>0} \subset L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$ and

$$
\{u_\varepsilon\}_{\varepsilon>0} \subset L^\infty(0,T; L^1(\mathbb{R}) \cap BV(\mathbb{R}))\ \text{for any } T > 0,
$$

which implies that a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ converges strongly in $L^p_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$, for any $p < \infty$, and also in $L^p(\mathbb{R}_+ \times \mathbb{R})$, for any $p \in [1,2)$, to a limit function u that satisfies (1.11) and (1.12), which we furthermore prove is a weak solution of the Degasperis-Procesi equation. By a *weak solution* we mean a function u that belongs to $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}))$ and satisfies (1.8) in $\mathcal{D}'([0,\infty) \times \mathbb{R})$. In addition to the estimates mentioned above, we also prove that the weak solution u satisfies a one-sided Lipschitz estimate: Fix any $T > 0$. Then $\partial_x u(t,x) \leq \frac{1}{t} + K_T$ for a.e. $(t,x) \in (0,T) \times \mathbb{R}$. Here K_T is a constant that depends on T and the $L^2 \cap BV$ norm of u_0. An implication of this estimate is that if the weak solution u contains discontinuities (shocks) then they must be nonincreasing.

To assert that the weak solution is unique we would need to know somehow that the chain rule holds for our weak solutions. However, since we work in spaces of discontinuous functions, this is not true. Instead we shall borrow ideas from the theory of conservation laws and replace the chain rule with an infinite family of entropy inequalities. Namely, we shall require that an admissible weak solution should satisfy the "entropy" inequality (P^u is defined in (1.7))

$$(1.15) \qquad \partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \leq 0 \quad \text{in } \mathcal{D}'([0,\infty) \times \mathbb{R}),$$

for all convex C^2 entropies $\eta : \mathbb{R} \to \mathbb{R}$ and corresponding entropy fluxes $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = \eta'(u)\, u$. We call a weak solution u that also satisfies (1.15) an *entropy weak solution*. We prove that the above mentioned weak solution, which is obtained as the limit of a sequence of viscous approximations, satisfies the entropy inequality (1.15), and thus is an entropy weak solution of (1.1), (1.2).

At this point we stress that there is a strong analogy with nonlinear conservation laws (Burgers' equation). Indeed, we can view (1.8) as Burgers' equation perturbed by a source term, albeit a nonlocal one. We can take this point of view since $\partial_x P^u$ is bounded, consult (1.12), which formally follows from (1.11). This analogy makes it possible to prove L^1 stability (and thereby uniqueness) of entropy weak solutions to the Degasperis-Procesi equation by a straightforward adaption of Kruzkov's uniqueness proof [23].

Next we prove that there exists at least one weak solution to (1.1), (1.2) under the assumption

$$(1.16) \qquad\qquad u_0 \in L^2(\mathbb{R}) \cap L^4(\mathbb{R}),$$

in which case we are outside the BV/L^∞ framework discussed above. Indeed, in this case we can only bound $\{u_\varepsilon\}_{\varepsilon>0}$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0,T; L^4(\mathbb{R})) \,\forall T > 0$, which is not enough to ensure strong compactness of a sequence of viscous approximations. To obtain the desired strong compactness we use instead Schonbek's L^p version [30] of the compensated compactness method [31]. Another aspect is that we can only prove that the constructed weak solution satisfies the entropy inequality (1.15) for a restricted class of entropies, namely those convex C^2 entropies that have a bounded second order derivative. Unfortunately we are not able to prove L^1 stability/uniqueness based on this restricted class of entropies.

Finally, we mention that existence, uniqueness, and stability results similar to those discussed above for the Degasperis-Procesi equation also hold for more general equations. We refer to these equations as generalized Degasperis-Procesi equations.

The remaining part of this paper is organized as follows: In Section 2 we define the viscous approximations and establish some important a priori estimates. In Section 3, we introduce the notion of entropy weak solution and prove existence, uniqueness, and L^1 stability results for these solutions under assumption (1.3). An existence result under assumption (1.16) is proved In Section 4. Finally, Section 5 is devoted to extending our results to slightly more general equations.

2. Viscous approximations and a priori estimates

We will prove existence of a solution to the Cauchy problem (1.1), (1.2) by analyzing the limiting behavior of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$, where

each function u_ε solves the following viscous problem:

$$(2.1) \quad \begin{cases} \partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + 4u_\varepsilon \partial_x u_\varepsilon \\ = 3\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon - \varepsilon \partial_{xxxx}^4 u_\varepsilon, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

This problem can be stated equivalently as a parabolic-elliptic system:

$$(2.2) \quad \begin{cases} \partial_t u_\varepsilon + \partial_x \left(\frac{u_\varepsilon^2}{2} \right) + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \frac{3}{2} u_\varepsilon^2, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

Observe that we have an explicit expression for P_ε in terms of u_ε:

$$P_\varepsilon(t,x) = P^{u_\varepsilon}(t,x) = G_1 \star \left(\frac{3}{2} u^2 \right)(t,x) = \frac{3}{4} \int_\mathbb{R} e^{-|x-y|} \left(u_\varepsilon(y) \right)^2 \, dy.$$

To begin with, we assume in this section that

$$(2.3) \qquad\qquad u_0 \in L^2(\mathbb{R}),$$

and

$(2.4)\ \ u_{0,\varepsilon} \in H^\ell(\mathbb{R}),\ \ell \geq 2, \qquad \|u_{0,\varepsilon}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \qquad u_{0,\varepsilon} \to u_0 \quad \text{in } L^2(\mathbb{R}).$

We will impose additional conditions on the initial data as we make progress.

We begin by stating a lemma which shows that the viscous problem (1.5) is well-posed for each fixed $\varepsilon > 0$.

Lemma 2.1. *Assume (2.3) and (2.4) hold, and fix any $\varepsilon > 0$. Then there exists a unique global smooth solution $u_\varepsilon = u_\varepsilon(t,x)$ to the Cauchy Problem (2.2) belonging to $C([0,\infty); H^\ell(R))$.*

Proof. We omit the proof since it is similar to the one found in [4, Theorem 2.3]. $\square$

2.1. L^2 estimates and some consequences.

Next we prove a uniform L^2 bound on the approximate solution u_ε, which reinforces the whole analysis in this paper.

Lemma 2.2 (Energy estimate). *Assume (2.3) and (2.4) hold, and fix any $\varepsilon > 0$. Then the following bounds hold for any $t \geq 0$:*

$$(2.5) \qquad \|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \leq 2\sqrt{2}\|u_0\|_{L^2(\mathbb{R})}, \qquad \sqrt{\varepsilon}\|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \leq 2\|u_0\|_{L^2(\mathbb{R})}.$$

For the proof of this lemma we introduce the quantity $v_\varepsilon = v_\varepsilon(t,x)$ defined by

$$v_\varepsilon(t,x) = (G_2 \star u_\varepsilon)(t,x) = \int_\mathbb{R} e^{-2|x-y|} u_\varepsilon(t,y) \, dy, \qquad t \geq 0,\ x \in \mathbb{R}.$$

Since $G_2(x) = e^{-2|x|}$ is the Green's function of the operator $4 - \partial_{xx}^2$, we see that v_ε also satisfies the equation

$$(2.6) \qquad\qquad -\partial_{xx}^2 v_\varepsilon + 4v_\varepsilon = u_\varepsilon \quad \text{in } \mathbb{R}_+ \times \mathbb{R}.$$

The use of the quantity v_ε is motivated by the fact that $\int_\mathbb{R} v \left(u - \partial_{xx}^2 u \right) \, dx$ is a conserved quantity, where $4v - \partial_{xx}^2 v = u$ and u solves (1.1) (see [15]).

To prove Lemma 2.2 we shall need the following estimates on v_ε:

Lemma 2.3. *Assume (2.3) and (2.4) hold, and fix any $\varepsilon > 0$. Then the following identity holds for any $t \geq 0$:*

(2.7)
$$\|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 5\|\partial_x v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 4\|v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2$$
$$+ 2\varepsilon \int_0^t \left(\|\partial_{xxx}^3 v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 + 5\|\partial_{xx}^2 v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 + 4\|\partial_x v_\varepsilon(\tau, \cdot)\|_{L^2(\mathbb{R})}^2 \right) d\tau$$
$$= \|\partial_{xx}^2 v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 + 5\|\partial_x v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2 + 4\|v_\varepsilon(0, \cdot)\|_{L^2(\mathbb{R})}^2.$$

Proof. Multiplying the first equation of (2.2) by $v_\varepsilon - \partial_{xx}^2 v_\varepsilon$ (consult also (2.6)) and integrating over $\mathbb{R}$, we get

(2.8)
$$\int_{\mathbb{R}} \partial_t u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx$$
$$= -\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx.$$

For the left-hand side of this identity, using (2.6), we have

$$\int_{\mathbb{R}} \partial_t u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx$$
$$= \int_{\mathbb{R}} \left(4\partial_t v_\varepsilon - \partial_{txx}^3 v_\varepsilon \right) \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx$$
$$- \varepsilon \int_{\mathbb{R}} \left(4\partial_{xx}^2 v_\varepsilon - \partial_{xxxx}^4 v_\varepsilon \right) \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx$$

(2.9)
$$= \int_{\mathbb{R}} \left(4v_\varepsilon \partial_t v_\varepsilon - 5v_\varepsilon \partial_{txx}^3 v_\varepsilon + \partial_{txx}^3 v_\varepsilon \partial_{xx}^2 v_\varepsilon \right) dx$$
$$- \varepsilon \int_{\mathbb{R}} \left(4v_\varepsilon \partial_{xx}^2 v_\varepsilon - 5v_\varepsilon \partial_{xxxx}^4 v_\varepsilon + \partial_{xxxx}^4 v_\varepsilon \partial_{xx}^2 v_\varepsilon \right) dx$$
$$= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \left(4v_\varepsilon^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2 \right) dx$$
$$+ \varepsilon \int_{\mathbb{R}} \left(4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_x^3 v_\varepsilon)^2 \right) dx.$$

For the right-hand side of (2.8), we calculate

$$- \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx$$

(2.10)
$$= - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx + \int_{\mathbb{R}} \left(P_\varepsilon - \partial_{xx}^2 P_\varepsilon \right) \partial_x v_\varepsilon \, dx$$
$$= - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx - 3 \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon v_\varepsilon \, dx$$
$$= - \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \left(4v_\varepsilon - \partial_{xx}^2 v_\varepsilon \right) dx = - \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \, dx = 0,$$

where have used (2.2), (2.6), and integration-by-parts.

Substituting (2.9) and (2.10) into (2.8) yields

$$\frac{d}{dt} \int_{\mathbb{R}} \left(4v_\varepsilon^2 + 5(\partial_x v_\varepsilon)^2 + (\partial_{xx}^2 v_\varepsilon)^2 \right) dx$$

$$+ 2\varepsilon \int_{\mathbb{R}} \left(4(\partial_x v_\varepsilon)^2 + 5(\partial_{xx}^2 v_\varepsilon)^2 + (\partial_x^3 v_\varepsilon)^2 \right) dx = 0.$$

Integrating this inequality over $[0, t]$ we obtain (2.7). $\qquad\square$

Proof of Lemma 2.2. Observe that, in view of (2.6),

$$(2.11) \qquad \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 2\|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2}^2 + 32\|v_\varepsilon(t, \cdot)\|_{L^2}^2$$
$$\leq 8 \left(\|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2}^2 + 5\|\partial_x v_\varepsilon(t, \cdot)\|_{L^2}^2 + 4\|v_\varepsilon(t, \cdot)\|_{L^2}^2 \right),$$

$$(2.12) \qquad \|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2 \leq 2\|\partial_{xxx}^3 v_\varepsilon\|_{L^2}^2 + 32\|\partial_x v_\varepsilon\|_{L^2}^2$$
$$\leq 8 \left(\|\partial_{xxx}^3 v_\varepsilon\|_{L^2}^2 + 5\|\partial_x^2 v_\varepsilon\|_{L^2}^2 + 4\|\partial_x v_\varepsilon\|_{L^2}^2 \right),$$

$$(2.13) \qquad \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \left(-\partial_{xx}^2 v_\varepsilon + 4v_\varepsilon \right)^2 dx$$
$$= \int_{\mathbb{R}} (\partial_{xx}^2 v_\varepsilon)^2 dx - 8 \int_{\mathbb{R}} v_\varepsilon \partial_{xx}^2 v_\varepsilon \, dx + 16 \int_{\mathbb{R}} v_\varepsilon^2 \, dx$$
$$= \int_{\mathbb{R}} (\partial_{xx}^2 v_\varepsilon)^2 \, dx + 8 \int_{\mathbb{R}} (\partial_x v_\varepsilon)^2 \, dx + 16 \int_{\mathbb{R}} v_\varepsilon^2 \, dx$$
$$\geq \|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2}^2 + 5\|\partial_x v_\varepsilon(t, \cdot)\|_{L^2}^2 + 4\|v_\varepsilon(t, \cdot)\|_{L^2}^2.$$

Then, from (2.4), (2.7), (2.11), and (2.13),

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 8 \left(\|\partial_{xx}^2 v_\varepsilon(t, \cdot)\|_{L^2}^2 + 5\|\partial_x v_\varepsilon(t, \cdot)\|_{L^2}^2 + 4\|v_\varepsilon(t, \cdot)\|_{L^2}^2 \right)$$
$$(2.14) \qquad \leq 8 \left(\|\partial_{xx}^2 v_\varepsilon(0, \cdot)\|_{L^2}^2 + 5\|\partial_x v_\varepsilon(0, \cdot)\|_{L^2}^2 + 4\|v_\varepsilon(0, \cdot)\|_{L^2}^2 \right)$$
$$\leq 8\|u_{0,\varepsilon}\|_{L^2}^2 \leq 8\|u_0\|_{L^2}^2,$$

and, from (2.4), (2.7), (2.12), and (2.13),

$$\varepsilon\|\partial_x u_\varepsilon\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2 \leq 8\varepsilon \left(\|\partial_{xxx}^3 v_\varepsilon\|_{L^2}^2 + 5\|\partial_{xx}^2 v_\varepsilon\|_{L^2}^2 + 4\|\partial_x v_\varepsilon\|_{L^2}^2 \right)$$
$$(2.15) \qquad \leq 4 \left(\|\partial_{xx}^2 v_\varepsilon(0, \cdot)\|_{L^2}^2 + 5\|\partial_x v_\varepsilon(0, \cdot)\|_{L^2}^2 + 4\|v_\varepsilon(0, \cdot)\|_{L^2}^2 \right)$$
$$\leq 4\|u_{0,\varepsilon}\|_{L^2}^2 \leq 4\|u_0\|_{L^2}^2.$$

Clearly, (2.14) and (2.15) imply (2.5). $\qquad\square$

We conclude this subsection with some bounds on the nonlocal term P_ε, which all are consequences of the L^2 bound in Lemma 2.2.

Lemma 2.4. *Assume* (2.3) *and* (2.4) *hold, and fix any* $\varepsilon > 0$. *Then*

$$(2.16) \qquad\qquad\qquad\qquad P_\varepsilon \geq 0,$$

$$(2.17) \qquad \|P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})}, \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12\|u_0\|_{L^2(\mathbb{R})}^2, \qquad t \geq 0,$$

$$(2.18) \qquad \|P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})}, \|\partial_x P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \leq 6\|u_0\|_{L^2(\mathbb{R})}^2,$$

$$(2.19) \qquad\qquad \|\partial_x^2 P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 24\|u_0\|_{L^2(\mathbb{R})}^2, \qquad t \geq 0.$$

Proof. By (2.2),

$$(2.20) \qquad\qquad P_\varepsilon(t, x) = \frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} \left(u_\varepsilon(t, y) \right)^2 dy,$$

$$(2.21) \qquad\qquad \partial_x P_\varepsilon(t, x) = \frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} \mathrm{sign}\,(y - x) \left(u_\varepsilon(t, y) \right)^2 dy.$$

From (2.20), we get (2.16). By (2.5) and the Tonelli theorem,

$$(2.22) \quad \int_{\mathbb{R}} |P_\varepsilon(t,x)| \, dx, \int_{\mathbb{R}} |\partial_x P_\varepsilon(t,x)| \, dx \leq \frac{3}{4} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-|x-y|} \, dx \right) (u_\varepsilon(t,y))^2 \, dy$$

$$\leq \frac{3}{2} \int_{\mathbb{R}} (u_\varepsilon(t,y))^2 \, dy \leq 12 \, \|u_0\|_{L^2(\mathbb{R})}^2 \, ,$$

$$(2.23) \qquad |P_\varepsilon(t,x)| \, , \, |\partial_x P_\varepsilon(t,x)| \leq \frac{3}{4} \int_{\mathbb{R}} (u_\varepsilon(t,y))^2 \, dy \leq 6 \, \|u_0\|_{L^2(\mathbb{R})}^2 \, .$$

Clearly, (2.22) and (2.23) imply (2.17) and (2.18), respectively. Finally, (2.19) is direct consequence of (2.2), (2.5) and (2.17). $\qquad \square$

2.2. L^1 **estimate.** As a consequence of the L^2 bound in Lemma 2.2, we can bound u_ε in L^1, as long as we assume, in addition to (2.3) and (2.4),

$$(2.24) \qquad u_0, u_{0,\varepsilon} \in L^1(\mathbb{R}), \qquad \|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} \, .$$

Lemma 2.5 (L^1-estimate). *Assume* (2.3), (2.4), *and* (2.24) *hold, and fix any* $\varepsilon > 0$. *Then*

$$(2.25) \qquad \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + 12t \|u_0\|_{L^2(\mathbb{R})}^2, \qquad t \geq 0.$$

Proof. Let $\eta \in C^2(\mathbb{R})$ and $q : \mathbb{R} \to \mathbb{R}$ be such that $q'(u) = u \, \eta'(u)$. By multiplying the first equation in (2.2) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$(2.26) \qquad \partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) + \eta'(u_\varepsilon) \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \eta''(u_\varepsilon) \left(\partial_x u_\varepsilon \right)^2 .$$

Choosing $\eta(u) = |u|$ (modulo an approximation argument), and then integrating the resulting equation over $\mathbb{R}$ yield

$$\frac{d}{dt} \int_{\mathbb{R}} |u_\varepsilon| \, dx \leq \int_{\mathbb{R}} \operatorname{sign}(u_\varepsilon) \, \partial_x P_\varepsilon \, dx.$$

By (2.17),

$$\int_{\mathbb{R}} \operatorname{sign}(u_\varepsilon) \, \partial_x P_\varepsilon \, dx \leq \|\partial_x P_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2,$$

and hence

$$(2.27) \qquad \frac{d}{dt} \|u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq 12 \|u_0\|_{L^2(\mathbb{R})}^2.$$

Integrating (2.27) over $[0, t]$ we get (2.25). $\qquad \square$

2.3. BV **and** L^∞ **estimates.** In this subsection we derive supplementary a priori estimates for the viscous approximations, which also are consequences of the L^2 bound in Lemma 2.2. In particular, we prove that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in BV, which yields strong compactness of this sequence. To this end, we need to assume, in addition to (2.3) and (2.4),

$$(2.28) \qquad u_0, u_{0,\varepsilon} \in BV(\mathbb{R}), \qquad |u_{0,\varepsilon}|_{BV(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} \, .$$

Lemma 2.6 (BV estimate in space). *Assume* (2.3), (2.4), *and* (2.28) *hold, and fix any* $\varepsilon > 0$. *Then*

$$(2.29) \qquad \|\partial_x u_\varepsilon(t, \cdot)\|_{L^1(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24t \|u_0\|_{L^2(\mathbb{R})}^2, \qquad t \geq 0.$$

Proof. Set $q_\varepsilon := \partial_x u_\varepsilon$. Then q_ε satisfies the equation

$$(2.30) \qquad \partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + q_\varepsilon^2 + \partial_{xx}^2 P_\varepsilon = \varepsilon \partial_{xx}^2 q_\varepsilon.$$

If $\eta \in C^2(\mathbb{R})$ and $q : \mathbb{R} \to \mathbb{R}$ satisfies $q'(u) = u\,\eta'(u)$, then by by the chain rule

$$(2.31) \qquad \begin{aligned} &\partial_t \eta(q_\varepsilon) + \partial_x \left(u_\varepsilon q(u_\varepsilon)\right) - q_\varepsilon \eta(q_\varepsilon) + \eta'(q_\varepsilon) q_\varepsilon^2 \\ &+ \eta'(u_\varepsilon)\partial_{xx}^2 P_\varepsilon = \varepsilon \partial_{xx}^2 \eta(q_\varepsilon) - \eta''(q_\varepsilon)\left(\partial_x q_\varepsilon\right)^2. \end{aligned}$$

Choosing $\eta(u) = |u|$ (modulo an approximation argument) and then integrating the resulting equation over $\mathbb{R}$ yield

$$\frac{d}{dt} \int_{\mathbb{R}} |u_\varepsilon|\; dx \leq \int_{\mathbb{R}} \operatorname{sign}(q_\varepsilon)\, \partial_{xx}^2 P_\varepsilon\, dx.$$

By (2.19),

$$\int_{\mathbb{R}} \operatorname{sign}(q_\varepsilon)\, \partial_{xx}^2 P_\varepsilon\, dx \leq \|\partial_{xx}^2 P_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq 24\|u_0\|_{L^2(\mathbb{R})}^2,$$

and hence

$$(2.32) \qquad \frac{d}{dt}\|u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq 24\|u_0\|_{L^2(\mathbb{R})}^2.$$

Integrating (2.32) over $[0,t]$ we get (2.29). $\qquad\square$

Lemma 2.7 (L^∞-estimate). *Assume* (2.3), (2.4), *and* (2.28) *hold, and fix any* $\varepsilon > 0$. *Then*

$$(2.33) \qquad \|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24t\|u_0\|_{L^2(\mathbb{R})}^2, \qquad t \geq 0.$$

Proof. Since

$$|u_\varepsilon(t,x)| \leq \int_{\mathbb{R}} |\partial_x u_\varepsilon(t,y)|\; dy = |u_\varepsilon(t,\cdot)|_{BV},$$

the claim is direct consequence of (2.29). $\qquad\square$

Lemma 2.8 (*BV* estimate in time). *Assume* (2.3), (2.4), *and* (2.28) *hold, and fix any* $\varepsilon > 0$. *Then*

$$(2.34) \qquad \|\partial_t u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq C_t, \qquad t \geq 0,$$

where the constant

$$C_t := \left(|u_0|_{BV(\mathbb{R})} + 24t\,\|u_0\|_{L^2(\mathbb{R})}^2\right)^2 + 12\,\|u_0\|_{L^2(\mathbb{R})}^2$$

is independent of ε *but dependent on* t.

Proof. We have, by (2.33), (2.29), and (2.17),

$$\begin{aligned} \|\partial_t u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} &\leq \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon|\; dx + \int_{\mathbb{R}} |\partial_x P_\varepsilon|\; dx \\ &\leq \|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\, |u_\varepsilon(t,\cdot)|_{BV} + \|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq C_t. \end{aligned}$$

$\square$

Lemma 2.9. *Assume* (2.3), (2.4), *and* (2.28) *hold, and fix any* $\varepsilon > 0$. *Then*

$$(2.35) \qquad \|\partial_{xx}^2 P_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq 6\|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2}\left(|u_0|_{BV(\mathbb{R})} + 24t\|u_0\|_{L^2(\mathbb{R})}^2\right)^2,$$

for any $t \geq 0$.

Proof. This is a consequence of the second equation in (2.2) and (2.18), (2.33). $\quad\square$

Remark 2.1. Lemma 2.9 is used later to prove an Oleinik type estimate for u_ε that is independent of ε.

2.4. L^4 **estimate.** Next we prove that the viscous approximations are uniformly bounded in L^4, a fact that we use later to prove the existence of at least one weak solution to (1.1), (1.2) under the mere assumption that (1.16) holds. For this purpose, we need to assume, in addition to (2.3) and (2.4),

$$(2.36) \qquad u_0, u_{0,\varepsilon} \in L^4(\mathbb{R}), \qquad \|u_{0,\varepsilon}\|_{L^4(\mathbb{R})} \le \|u_0\|_{L^4(\mathbb{R})}.$$

Lemma 2.10 (L^4-estimate). *Assume (2.3), (2.4), and (2.36) hold, and fix any $\varepsilon > 0$. Then*

$$(2.37) \quad \|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4 \le e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} \|u_0\|_{L^4(\mathbb{R})}^4 + 8\|u_0\|_{L^2(\mathbb{R})}^2 \left(e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} - 1 \right),$$

for any $t \ge 0$.

Proof. Choosing $\eta(u) = \frac{1}{4}u^4$ in (2.26), writing

$$\varepsilon \partial_{xx}^2 \eta(u_\varepsilon) - \eta''(u_\varepsilon)\left(\partial_x u_\varepsilon\right)^2 = \varepsilon \eta'(u_\varepsilon)\partial_{xx}^2 u_\varepsilon = \varepsilon \partial_{xx}^2 u_\varepsilon u_\varepsilon^3,$$

and integrating the result over $\mathbb{R}$ yield

$$(2.38) \qquad \frac{1}{4}\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4 = -\int_{\mathbb{R}} u_\varepsilon^3 \partial_x P_\varepsilon \, dx + \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_\varepsilon u_\varepsilon^3 \, dx.$$

Observe that by an integration by parts

$$\varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_\varepsilon u_\varepsilon^3 \, dx = -3\varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 u_\varepsilon^2 \, dx \le 0.$$

and, using Hölder's inequality, (2.5), and (2.18),

$$\begin{aligned}
-\int_{\mathbb{R}} u_\varepsilon^3 \partial_x P_\varepsilon \, dx &\le \|\partial_x P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \int_{\mathbb{R}} |u_\varepsilon|^3 dx \\
&\le \|\partial_x P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^2 \\
&\le \frac{1}{2}\|\partial_x P_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times \mathbb{R})} \left(\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4 \right) \\
&\le 24\|u_0\|_{L^2(\mathbb{R})}^4 + 3\|u_0\|_{L^2(\mathbb{R})}^2 \|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4.
\end{aligned}$$

Hence, by (2.38),

$$(2.39) \qquad \frac{d}{dt}\|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4 \le 96\|u_0\|_{L^2(\mathbb{R})}^4 + 12\|u_0\|_{L^2(\mathbb{R})}^2 \|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4.$$

Clearly, (2.37) is a direct consequence of (2.39) and Gronwall's inequality. $\qquad\square$

2.5. **Oleinik type estimate.** In this subsection we show through an estimate of Oleinik type that a solution of the Degasperis-Procesi equation can only contain decreasing discontinuities (shocks), which coincides with what is known for the Burger's equation. However, different from the Burgers equation, the Oleinik type estimate depends on the total variation of the solution and a final time.

Lemma 2.11 (Oleinik type estimate). *Assume (2.3), (2.4), and (2.28) hold, and fix any $\varepsilon > 0$. Then for each $t \in (0, T]$, with $T > 0$ being fixed,*

$$(2.40) \qquad \partial_x u_\varepsilon(t, x) \le \frac{1}{t} + K_T, \qquad x \in \mathbb{R},$$

where

$$K_T := \left[6\|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left(|u_0|_{BV(\mathbb{R})} + 24T\|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 \right]^{1/2}.$$

Proof. Setting $q_\varepsilon := \partial_x u_\varepsilon$, it follows from (2.2) and (2.35) that

$$(2.41) \qquad \partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon + q_\varepsilon^2 - \varepsilon \partial_{xx}^2 q_\varepsilon = -\partial_{xx}^2 P_\varepsilon \leq K_T^2.$$

Comparing q_ε with the solution f of the ordinary differential equation

$$\frac{df}{dt} + f^2 = K_T^2,$$

we find

$$\partial_x u_\varepsilon(t,x) \leq \frac{1}{t} + K_T, \qquad (t,x) \in (0,T] \times \mathbb{R},$$

and hence (2.40) follows. $\qquad\qquad\square$

3. Well-posedness in $L^1 \cap BV$

Relying on the a priori estimates derived in Section 2, we prove in this section existence, uniqueness, and L^1 stability of entropy weak solutions to (1.1), (1.2) under the $L^1 \cap BV$ assumption (1.3).

We begin by introducing a suitable notion of weak solution.

Definition 3.1 (Weak solution). *We call a function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ a weak solution of the Cauchy problem (1.1), (1.2) provided*

     i) $u \in L^\infty \left(\mathbb{R}_+; L^2(\mathbb{R}) \right)$, *and*

     ii) $\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_x P^u = 0$ *in* $\mathcal{D}'([0,\infty) \times \mathbb{R})$, *that is,* $\forall \phi \in C_c^\infty([0,\infty) \times \mathbb{R})$
        there holds the equation

$$(3.1) \qquad \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left(u \partial_t \phi + \frac{u^2}{2} \partial_x \phi - P^u \phi \right) dx \, dt + \int_{\mathbb{R}} u_0(x)\phi(x,0) \, dx = 0,$$

     where

$$P^u(t,x) = G_1 \star \left(\frac{3}{2} u^2 \right)(t,x) = \frac{3}{4} \int_{\mathbb{R}} e^{-|x-y|} (u(t,y))^2 \, dy.$$

Remark 3.1. It follows from part *i)* of Definition 3.1 that $u \in L^1((0,T) \times \mathbb{R})$ for any $T > 0$ and $\partial_x P^u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ (consult the proof of Corollary 2.4). Hence equation (3.1) makes sense.

By extending the definition of a weak solution by requiring some more (BV) regularity and the fulfillment of an entropy condition we arrive at the notion of an entropy weak solution for the Degasperis-Procesi equation.

Definition 3.2 (Entropy weak solution). *We call a function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ an entropy weak solution of the Cauchy problem (1.1), (1.2) provided*

     ii) u *is a weak solution in the sense of Definition 3.1,*

     ii) $u \in L^\infty(0,T; BV(\mathbb{R}))$ *for any $T > 0$, and*

     iii) *for any convex C^2 entropy $\eta : \mathbb{R} \to \mathbb{R}$ with corresponding entropy flux*
        $q : \mathbb{R} \to \mathbb{R}$ *defined by $q'(u) = \eta'(u)\, u$ there holds*

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u) \partial_x P^u \leq 0 \ \text{ in } \mathcal{D}'([0,\infty) \times \mathbb{R}),$$

that is, $\forall \phi \in C_c^\infty([0,\infty) \times \mathbb{R})$, $\phi \geq 0$,

$$(3.2) \quad \int_{\mathbb{R}+} \int_{\mathbb{R}} \left(\eta(u)\partial_t\phi + q(u)\partial_x\phi - \eta'(u)\partial_x P^u \phi \right) dx\, dt + \int_{\mathbb{R}} \eta(u_0(x))\phi(x,0)\, dx \geq 0.$$

Remark 3.2. It takes a standard argument to see that it suffices to verify (3.2) for the Kruzkov entropies/entropy fluxes

$$\eta(u) := |u - c|, \qquad q(u) = \text{sign}\,(u - c) \left(\frac{u^2}{2} - \frac{c^2}{2} \right), \qquad c \in \mathbb{R}.$$

Observe that it follows from part *ii)* of Definition 3.2 that $u \in L^\infty((0,T) \times \mathbb{R})$ for any $T > 0$ (consult the proof of Lemma 2.7). Using the Kruzkov entropies/entropy fluxes it can then be seen that the weak formulation (3.1) is a consequence of the entropy formulation (3.2).

Remark 3.3. It follows from part *ii)* of Definition 3.2 that $u \in C([0,T]; L^1(\mathbb{R}))$ for any fixed $T > 0$ (see the proof of Lemma 2.8). In fact, we have more

$$\|u(t_2, \cdot) - u(t_1, \cdot)\|_{L^1(\mathbb{R})} \leq C_T |t_2 - t_1|, \qquad \forall t_1, t_2 \in [0,T],$$

for some constant C_T. Consequently, it makes sense to interpret the initial condition in the L^1 sense:

$$(3.3) \quad \lim_{t \to 0+} \|u(t, \cdot) - u_0\|_{L^1(\mathbb{R})} = 0,$$

and then restricting the choice of test functions in (3.1) and (3.2) to those that vanish at $t = 0$.

Our main results are collected in the following theorem:

Theorem 3.1 (Well-posedness). *Suppose condition* (1.3) *holds. Then there exists an entropy weak solution to the Cauchy problem* (1.1), (1.2). *Fix any $T > 0$, and let $u, v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be two entropy weak solutions to* (1.1), (1.2) *with initial data $u_0, v_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, respectively. Then for any $t \in (0,T)$*

$$(3.4) \quad \|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})},$$

where

$$(3.5) \quad M_T := \frac{3}{2} \left(\|u\|_{L^\infty((0,T) \times \mathbb{R})} + \|v\|_{L^\infty((0,T) \times \mathbb{R})} \right) < \infty.$$

Consequently, there exists at most one entropy weak solution to (1.1), (1.2).

The entropy weak solution u satisfies the following estimates for any $t \in (0,T)$:

$$(3.6) \quad \|u(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})} + 12t\|u_0\|_{L^2(\mathbb{R})}^2,$$

$$(3.7) \quad |u(t, \cdot)|_{BV(\mathbb{R})}, \|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq |u_0|_{BV(\mathbb{R})} + 24t\|u_0\|_{L^2(\mathbb{R})}^2,$$

$$(3.8) \quad \|u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} \|u_0\|_{L^4(\mathbb{R})}^4 + 8\|u_0\|_{L^2(\mathbb{R})}^2 \left(e^{12\|u_0\|_{L^2(\mathbb{R})}^2 t} - 1 \right).$$

Furthermore,

$$(3.9) \quad \|u(t_2, \cdot) - u(t_1, \cdot)\|_{L^1(\mathbb{R})} \leq C_T |t_2 - t_1|, \qquad \forall t_1, t_2 \in [0,T],$$

where

$$C_T := \left(\|u_0\|_{L^1(\mathbb{R})} + 12T\|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 + 12\|u_0\|_{L^2(\mathbb{R})}^2.$$

Finally, the following Oleinik type estimate holds for a.e. $(t, x) \in (0, T] \times \mathbb{R}$,

$$(3.10) \qquad \partial_x u(t, x) \leq \frac{1}{t} + K_T,$$

where

$$K_T := \left[6\|u_0\|_{L^2(\mathbb{R})}^2 + \frac{3}{2} \left(|u_0|_{BV(\mathbb{R})} + 24T\|u_0\|_{L^2(\mathbb{R})}^2 \right)^2 \right]^{1/2}.$$

This theorem is an immediate consequence of Theorem 3.2, Theorem 3.3, and Corollary 3.1 below.

3.1. **Existence of entropy weak solutions.**

Theorem 3.2 (Existence). *Suppose* (1.3) *holds. Then there exists at least one entropy weak solution to* (1.1), (1.2).

Proof. We assume then that the approximating sequence $\{u_{0,\varepsilon}\}_{\varepsilon>0}$ is chosen such that (2.3), (2.4), (2.24), and (2.28) hold. Then, in view of the a priori estimates obtained in Section 2, it takes a standard argument to see that there exists a sequence of strictly positive numbers $\{\varepsilon_k\}_{k=1}^\infty$ tending to zero such that as $k \to \infty$

$$(3.11) \qquad u_{\varepsilon_k} \to u \text{ a.e. in } \mathbb{R}_+ \times \mathbb{R},$$

and hence

$$(3.12) \qquad u_{\varepsilon_k} \to u \text{ in } L_{\text{loc}}^p(\mathbb{R}_+ \times \mathbb{R}) \text{ for all } p \in [1, \infty).$$

Thanks to (3.11) and estimates (2.25), (2.33) there also holds

$$(3.13) \qquad u_{\varepsilon_k} \to u \text{ in } L^p((0, T) \times \mathbb{R}) \ \forall T > 0, \ \forall p \in [1, \infty).$$

The a priori estimates in Section 2 imply immediately that the limit function u satisfies (3.6) - (3.10).

Let us now prove that as $k \to \infty$

$$(3.14) \qquad P_{\varepsilon_k} \to P^u, \ \partial_x P_{\varepsilon_k} \to \partial_x P^u \text{ in } L^p((0, T) \times \mathbb{R}), \ \forall T > 0, \ \forall p \in [1, \infty),$$

which follows from the following calculation:

$$\|P_{\varepsilon_k} - P^u\|_{L^p((0,T)\times\mathbb{R})}^p, \|\partial_x P_{\varepsilon_k} - \partial_x P^u\|_{L^p((0,T)\times\mathbb{R})}^p$$

$$\leq \left(\frac{3}{4}\right)^p \iint_{\Pi_T} \left(\int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t,y))^2 - (u(t,y))^2 \right| dy \right)^p dx\, dt$$

$$\leq \left(\frac{3}{4}\right)^p \iint_{\Pi_T} \left(\int_{\mathbb{R}} e^{-|x-y|(p-1)/p} e^{-|x-y|/p} \left| (u_{\varepsilon_k}(t,y))^2 - (u(t,y))^2 \right| dy \right)^p dx\, dt$$

$$\leq \left(\frac{3}{4}\right)^p \iint_{\Pi_T} \left(\int_{\mathbb{R}} e^{-|x-y|} dy \right)^{p-1} \left(\int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t,y))^2 - (u(t,y))^2 \right|^p dy \right) dx\, dt$$

$$\leq \left(\frac{3}{4}\right)^p 2^{p-1} \iint_{\Pi_T} \int_{\mathbb{R}} e^{-|x-y|} \left| (u_{\varepsilon_k}(t,y))^2 - (u(t,y))^2 \right|^p dt\, dx\, dy$$

$$= \left(\frac{3}{2}\right)^p \iint_{\Pi_T} \left| (u_{\varepsilon_k}(t,y))^2 - (u(t,y))^2 \right|^p dy\, dt \to 0$$

$$\leq C_T \iint_{\Pi_T} |u_{\varepsilon_k}(t,y) - u(t,y)|^p dy\, dt \to 0 \quad \text{as } k \to \infty \text{ (we use (3.12) here)},$$

where $\Pi_T := (0, T) \times \mathbb{R}$. For the purpose of proving that the limit u satisfies the entropy inequality (3.2), we need to know (3.14) only for the case $p = 1$. Indeed,

equipped with (3.12) and (3.14) (with $p = 1$), this follows by choosing $\varepsilon = \varepsilon_k$ in equation (2.26) (interpreted in $\mathcal{D}'([0, \infty) \times \mathbb{R})$) and then sending $k \to \infty$. $\qquad\square$

3.2. L^1 stability and uniqueness of entropy weak solutions.

Next we prove L^1 stability (and thus uniqueness) of entropy weak solutions. Our method of proof is a straightforward adaption of Kruzkov's device of doubling the variables [23].

Theorem 3.3 (L^1 stability)**.** *Let u and v be two entropy weak solution of* (1.1) *with initial data $u(0, \cdot) = u_0$ and $v(0, \cdot) = v_0$ satisfying* (1.3)*. Fix any $T > 0$. Then*

$$(3.15) \qquad \|u(t, \cdot) - v(t, \cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})}, \qquad t \in (0, T),$$

where M_T is defined in (3.5)*.*

Proof. Set $Q := \mathbb{R}_+ \times \mathbb{R}$, and let $\varphi = \varphi(t, x, s, y)$ be a positive $C^\infty(Q \times Q)$ function with compact support. Since u, v are entropy weak solutions according to Definition 3.2 *iii*), we find by following the standard Kruzkov argument [23] that

$$
\begin{aligned}
(3.16) \qquad \iint_{Q \times Q} \Bigg(&|u(t, x) - v(s, y)| \, \partial_t \varphi \\
&+ \operatorname{sign}(u(t, x) - v(s, y)) \left(\frac{(u(t, x))^2}{2} - \frac{(v(s, y))^2}{2} \right) \partial_x \varphi \\
&- \operatorname{sign}(u(t, x) - v(s, y)) \, \partial_x P^u(t, x) \varphi \Bigg) \, dt \, dx \, ds \, dy \geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
(3.17) \qquad \iint_{Q \times Q} \Bigg(&|v(s, y) - u(t, x)| \, \partial_s \varphi \\
&+ \operatorname{sign}(v(s, y) - u(t, x)) \left(\frac{(v(s, y))^2}{2} - \frac{(u(t, x))^2}{2} \right) \partial_y \varphi \\
&- \operatorname{sign}(v(s, y) - u(t, x)) \, \partial_y P^v(s, y) \varphi \Bigg) \, ds \, dy \, dt \, dx \geq 0.
\end{aligned}
$$

Adding together (3.16) and (3.17) yields

$$
\begin{aligned}
\iint_{Q \times Q} \Bigg(&|u(t, x) - v(s, y)| \, (\partial_t \varphi + \partial_s \varphi) \\
&+ \operatorname{sign}(u(t, x) - v(s, y)) \left(\frac{(u(t, x))^2}{2} - \frac{(v(s, y))^2}{2} \right) (\partial_x \varphi + \partial_y \varphi) \\
&- \operatorname{sign}(u(t, x) - v(s, y)) \, (\partial_x P^u(t, x) - \partial_y P^v(s, y)) \, \varphi \Bigg) \, dx \, dt \, dy \, ds \geq 0,
\end{aligned}
$$

and hence

$$
\iint_{Q \times Q} \left(|u(t,x) - v(s,y)| \, (\partial_t \varphi + \partial_s \varphi) + \text{sign} \, (u(t,x) - v(s,y)) \right.
$$

(3.18)
$$
\left. \times \left(\frac{(u(t,x))^2}{2} - \frac{(v(s,y))^2}{2} \right) (\partial_x \varphi + \partial_y \varphi) \right) dx \, dt \, dy \, ds
$$

$$
\geq - \iint_{Q \times Q} |\partial_x P^u(t,x) - \partial_y P^v(s,y)| \, \varphi \, dx \, dt \, dy \, ds.
$$

Let $\delta \in C^\infty(\mathbb{R})$ be such that

$$
\text{supp}(\delta) \subset [-1,1], \quad 0 \leq \delta(\cdot) \leq 1, \quad \int_{\mathbb{R}} \delta(x) \, dx = 1.
$$

For $h > 0$, define

$$
\delta_h(x) := \frac{1}{h} \delta\left(\frac{x}{h}\right), \qquad \alpha_h(x) := \int_{-\infty}^{x} \delta_h(\xi) \, d\xi, \quad x \in \mathbb{R}.
$$

Consider a $C^\infty(Q)$ function ψ with compact support, and define

$$
\varphi_h(t,x,s,y) = \psi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \delta_h\left(\frac{t-s}{2}\right) \delta_h\left(\frac{x-y}{2}\right).
$$

With $\varphi = \varphi_h$ as the choice of test function and using a standard argument [23], which works since

$$
u, v, \partial_x P^u, \partial_x P^v \in L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}),
$$

sending $h \to 0$ in (3.18) yields

$$
\iint_Q \left(|u(t,x) - v(t,x)| \, \partial_t \psi \right.
$$

(3.19)
$$
\left. + \text{sign} \, (u(t,x) - v(t,x)) \left(\frac{(u(t,x))^2}{2} - \frac{(v(t,x))^2}{2} \right) \partial_x \psi \right) dx \, dt
$$

$$
\geq - \iint_Q |\partial_x P^u(t,x) - \partial_x P^v(t,x)| \, \psi \, dx \, dt.
$$

Taking again the standard route [23] it follows from (3.19) that

(3.20)
$$
\int_{\mathbb{R}} |u(t,x) - v(t,x)| \, dx
$$
$$
\leq \int_{\mathbb{R}} |u_0 - v_0| \, dx + \int_0^t \int_{\mathbb{R}} |\partial_x P^u(\tau,x) - \partial_x P^v(\tau,x)| \, dx \, d\tau,
$$

for any $t \in (0,T)$. Next, observe that for any $\tau \in (0,T)$

$$
\int_{\mathbb{R}} |\partial_x P^u(\tau,x) - \partial_x P^v(\tau,x)| \, dx
$$

$$
\leq \frac{3}{2} \int_{\mathbb{R}} \left| (u(\tau,x))^2 - (v(\tau,x))^2 \right| \, dx
$$

$$
\leq \frac{3}{2} \left(\|u\|_{L^\infty((0,T) \times \mathbb{R})} + \|v\|_{L^\infty((0,T) \times \mathbb{R})} \right) \int_{\mathbb{R}} |u(\tau,x) - v(\tau,x)| \, dx.
$$

Inserting this estimate into (3.20) and applying Gronwall's inequality, we arrive at the desired L^1 stability (3.15). □

Corollary 3.1 (Uniqueness)**.** *Suppose condition* (1.3) *holds. Then the Cauchy problem* (1.1)*,* (1.2) *admits at most one entropy weak solution.*

Proof. This is an immediate consequence of Theorem 3.3. ☐

4. Existence in $L^2 \cap L^4$

In this section we prove that there exists at least one weak solution to (1.1), (1.2) under assumption (1.16), in which case we are outside the BV/L^∞ framework considered in Section 3. Since no L^∞ bound is available we can only prove that this weak solution satisfies the entropy inequality for convex C^2 entropies possessing a bounded second order derivative. Be that as it may, we are not able to prove L^1 stability/uniqueness based on this restricted class of entropies.

Our main existence result is the following theorem:

Theorem 4.1 (Existence)**.** *Suppose* (1.16) *holds. Then there exists a function*

$$u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0, T; L^4(\mathbb{R})) \text{ for any } T > 0,$$

which solves the Cauchy problem (1.1)*,* (1.2) *in* $\mathcal{D}'([0, T) \times \mathbb{R})$.

As before we will construct a weak solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ of viscosity approximations, see (2.1) or (2.2). We make the standing assumption that the approximate initial data $\{u_{0,\varepsilon}\}_{\varepsilon > 0}$ are chosen such that they respect (2.3), (2.4), and (2.36). Having said that, in the present context we do not have have at our disposal a uniform BV estimate. Indeed, the relevant a priori estimates are only those contained in Lemmas 2.2 and 2.10. Instead we use Schonbek's L^p version [30] of the compensated compactness method [31] to obtain strong convergence of a subsequence of viscosity approximations. To avoid strict convexity of the flux function, we will use a refinement of Schonbek's method found in [25], which we recall next.

Lemma 4.1. *Let Ω be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Let $f \in C^2(\mathbb{R})$ satisfy*

$$|f(u)| \leq C |u|^{s+1} \text{ for } u \in \mathbb{R}, \qquad |f'(u)| \leq C |u|^s \text{ for } u \in \mathbb{R},$$

for some $s \geq 0$, and

$$(4.1) \qquad \qquad \text{meas} \{u \in \mathbb{R} : f''(u) = 0\} = 0.$$

Define functions $I_l, f_l, F_l : \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{cases} I_l \in C^2(\mathbb{R}), \ |I_l(u)| \leq |u| \text{ for } u \in \mathbb{R}, \ |I_l'(u)| \leq 2 \text{ for } u \in \mathbb{R}, \\ |I_l(u)| \leq |u| \text{ for } |u| \leq l, \ I_l(u) = 0 \text{ for } |u| \geq 2l, \end{cases}$$

and

$$f_l(u) = \int_0^u I_l'(\zeta) f'(\zeta) \, d\zeta, \qquad F_l(u) = \int_0^u f_l'(\zeta) f'(\zeta) \, d\zeta.$$

Suppose $\{u_n\}_{n=1}^\infty \subset L^{2(s+1)}(\Omega)$ is such that the two sequences

$$(4.2) \qquad \{\partial_t I_l(u_n) + \partial_x f_l(u_n)_x\}_{n=1}^\infty, \qquad \{\partial_t f_l(u_n) + \partial_x F_l(u_n)\}_{n=1}^\infty$$

of distributions belong to a compact subset of $H_{\text{loc}}^{-1}(\Omega)$, for each fixed $l > 0$.

Then there exists a subsequence of $\{u_n\}_{n=1}^\infty$ that converges to a limit function $u \in L^{2(s+1)}(\Omega)$ strongly in $L^r(\Omega)$ for any $1 \leq r < 2(s+1)$.

The following lemma of Murat [27] is useful:

Lemma 4.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. Suppose the sequence $\{\mathcal{L}_n\}_{n=1}^{\infty}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_n^1 + \mathcal{L}_n^2,$$

where $\{\mathcal{L}_n^1\}_{n=1}^{\infty}$ lies in a compact subset of $H_{\mathrm{loc}}^{-1}(\Omega)$ and $\{\mathcal{L}_n^2\}_{n=1}^{\infty}$ lies in a bounded subset of $\mathcal{M}_{\mathrm{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n=1}^{\infty}$ lies in a compact subset of $H_{\mathrm{loc}}^{-1}(\Omega)$.

We now turn to the proof of Theorem 4.1, which will be accomplished through a series of lemmas.

Lemma 4.3. *Suppose (1.16) holds. Then there exists a subsequence $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ of $\{u_\varepsilon\}_{\varepsilon > 0}$ and a limit function*

$$(4.3) \qquad u \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0, T; L^4(\mathbb{R})) \; \forall T > 0$$

such that

$$(4.4) \qquad u_{\varepsilon_k} \to u \; in \; L^p((0, T) \times \mathbb{R}) \; \forall T > 0, \; \forall p \in [2, 4].$$

If, in addition $u_0 \in L^1(\mathbb{R})$, then

$$(4.5) \qquad u_{\varepsilon_k} \to u \; in \; L^p((0, T) \times \mathbb{R}) \; \forall T > 0, \; \forall p \in [1, 4].$$

Proof. Let $\eta : \mathbb{R} \to \mathbb{R}$ be any convex C^2 entropy function that is compactly supported, and let $q : \mathbb{R} \to \mathbb{R}$ be the corresponding entropy flux defined by $q'(u) = \eta'(u) \, u$. We claim that

$$(4.6) \qquad \partial_t \eta(u_{\varepsilon,\alpha}) + \partial_x q(u_{\varepsilon,\alpha}) = \mathcal{L}_{\varepsilon,\alpha}^1 + \mathcal{L}_{\varepsilon,\alpha}^2,$$

for some distributions $\mathcal{L}_{\varepsilon,\alpha}^1$, $\mathcal{L}_{\varepsilon,\alpha}^2$ that satisfy

$$(4.7) \qquad \begin{aligned} &\mathcal{L}_{\varepsilon,\alpha}^1 \to 0 \text{ in } H^{-1}(\mathbb{R}_+ \times \mathbb{R}), \\ &\mathcal{L}_{\varepsilon,\alpha}^2 \text{ is uniformly bounded in } \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}). \end{aligned}$$

Indeed, by (2.26), we have

$$(4.8) \qquad \partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{=:\mathcal{L}_{\varepsilon,\alpha}^1} \underbrace{-\varepsilon \eta''(u_\varepsilon) \left(\partial_x u_\varepsilon\right)^2 + \eta'(u_\varepsilon) \partial_x P_\varepsilon}_{=:\mathcal{L}_{\varepsilon,\alpha}^2},$$

and, using (2.5) and (2.17),

$$(4.9) \qquad \|\varepsilon \partial_x \eta(u_\varepsilon)\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} \leq 2\sqrt{\varepsilon} \, \|\eta'\|_{L^\infty(\mathbb{R})} \, \|u_0\|_{L^2(\mathbb{R})} \to 0,$$

$$(4.10) \qquad \left\| \varepsilon \eta''(u_\varepsilon) \left(\partial_x u_\varepsilon\right)^2 \right\|_{L^1(\mathbb{R}_+ \times \mathbb{R})} \leq 4 \, \|\eta''\|_{L^\infty(\mathbb{R})} \, \|u_0\|_{L^2(\mathbb{R})}^2,$$

$$(4.11) \qquad \|\eta'(u_\varepsilon) \partial_x P_\varepsilon\|_{L^1((0, T) \times \mathbb{R})} \leq 12T \, \|\eta'\|_{L^\infty(\mathbb{R})} \, \|u_0\|_{L^2(\mathbb{R})}^2.$$

Hence, (4.7) follows. Therefore, thanks to Lemma 4.2 and Theorem 4.1, there exists a subsequence $\{u_{\varepsilon_k}\}_{k=1}^{\infty}$ and a limit function u satisfying (4.3) such that as $k \to \infty$

$$(4.12) \qquad \begin{aligned} &u_{\varepsilon_k} \to u \text{ in } L_{\mathrm{loc}}^p(\mathbb{R}_+ \times \mathbb{R}) \text{ for any } p \in [1, 4), \\ &\text{and } u_{\varepsilon_k} \to u \text{ a.e. in } \mathbb{R}_+ \times \mathbb{R}. \end{aligned}$$

Thanks to the L^4 estimate (2.37) we can upgrade (4.12) to (4.4). Similarly, due to the L^1 estimate (2.25), we can improve (4.12) to (4.5). $\qquad \square$

Lemma 4.4. *Suppose (1.16) holds. Then*

$$(4.13) \qquad P_{\varepsilon_k} \to P^u \; in \; L^p(0, T; W^{1,p}(\mathbb{R})) \; \forall T > 0, \; \forall p \in [1, 2).$$

where the sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ and the function u are constructed in Lemma 4.3.

Proof. Observe first that (4.12) implies $u_{\varepsilon_k}^2 \to u^2$ in $L^p((0,T) \times \mathbb{R})$ for all $T > 0$ and for all $p \in [1,2)$. Using this fact and arguing as in the proof of Theorem 3.2 we find that

$$\|P_{\varepsilon_k} - P^u\|_{L^p((0,T)\times\mathbb{R})}^p , \|\partial_x P_{\varepsilon_k} - \partial_x P^u\|_{L^p((0,T)\times\mathbb{R})}^p$$

$$(4.14) \qquad \leq \left(\frac{3}{2}\right)^p \int_0^T \int_{\mathbb{R}} \left| (u_{\varepsilon_k}(t,y))^2 - (u(t,y))^2 \right|^p dy\, dt$$

$$\to 0 \quad \text{as } k \to \infty.$$

This concludes the proof of the lemma. $\qquad\square$

Lemma 4.5. *Suppose* (1.16) *holds. Then the limit u from Lemma 4.3 is a weak solution of* (1.1), (1.2). *Moreover, $u \in L^\infty(0,T; L^4(\mathbb{R}))$ for each $T > 0$. Finally, if u_0 also belongs to $L^1(\mathbb{R})$, then $u \in L^\infty(0,T; L^1(\mathbb{R}))$ for each $T > 0$.*

Proof. This is an immediate consequence of Lemmas 4.3 and 4.4. $\qquad\square$

Lemma 4.6. *Suppose* (1.16) *holds. Then the weak solution u from Lemma 4.5 satisfies the entropy inequality* (3.2) *for any convex C^2 entropy $\eta : \mathbb{R} \to \mathbb{R}$ with η'' bounded and corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = \eta'(u)\, u$.*

Proof. Let (η, q) be as in the lemma. By (2.26),

$$(4.15) \quad \partial_t \eta(u_{\varepsilon_k}) + \partial_x q(u_{\varepsilon_k}) + \eta'(u_{\varepsilon_k})\partial_x P_{\varepsilon_k} \leq \varepsilon_k \partial_{xx}^2 \eta(u_{\varepsilon_k}) \quad \text{in } \mathcal{D}'([0,\infty) \times \mathbb{R}).$$

Observing that

$$|\eta(u)| = O(1 + u^2), \quad |\eta'(u)| = O(1 + u), \quad |q(u)| = O(1 + u^3),$$

we can use (4.4) and (4.13) when sending $k \to \infty$ in (4.15). The result is

$$(4.16) \qquad \partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \leq 0 \quad \text{in } \mathcal{D}'([0,\infty) \times \mathbb{R}),$$

which concludes the proof of the lemma. $\qquad\square$

Proof of Theorem 4.1. This follows from Lemmas 4.5 and 4.6. $\qquad\square$

5. Generalized Degasperis-Procesi equation

The aim of this last section is to show how the previous results can be extended to the equation

$$(5.1) \qquad \partial_t u - \partial_{txx}^3 u + 4\partial_x f(u) = f'''(u)(\partial_x u)^3 + 3f''(u)\partial_x u \partial_{xx}^2 u + f'(u)\partial_{xxx}^3 u,$$

where $f : \mathbb{R} \to \mathbb{R}$ is given. This equation can be properly labeled *generalized Degasperis-Procesi equation* since the choice $f(u) = \frac{u^2}{2}$ reduces (5.1) to (1.1).

The weak and entropy weak formulations of the Cauchy problem (5.1), (1.2) are based on the following hyperbolic-parabolic system:

$$(5.2) \qquad \begin{cases} \partial_t u + \partial_x f(u) + \partial_x P = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\partial_{xx}^2 P + P = 3f(u), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

We modify Definition 3.1 by replacing part *ii)* by $\partial_t u + \partial_x \left(\frac{u^2}{2}\right) + \partial_x P^u = 0$ in $\mathcal{D}'([0,\infty) \times \mathbb{R})$, where

$$P^u(t,x) := \frac{3}{2} \int_{\mathbb{R}} e^{-|x-y|} f(u(t,y))\, dy.$$

Concerning Definition 3.2, we replace part *iii*) by

$$\partial_t \eta(u) + \partial_x q(u) + \eta'(u)\partial_x P^u \le 0 \text{ in } \mathcal{D}'([0,\infty) \times \mathbb{R}),$$

for any convex C^2 entropy $\eta : \mathbb{R} \to \mathbb{R}$ with corresponding entropy flux $q : \mathbb{R} \to \mathbb{R}$ defined by $q'(u) = \eta'(u)f'(u)$.

Regarding the function $f : \mathbb{R} \to \mathbb{R}$ we shall assume that it is a C^3 function satisfying

$$(5.3) \qquad |f'(u)| \le \kappa_0 |u|, \quad |f(u)| \le \kappa_1 |u|^2, \quad u \in \mathbb{R},$$

or

$$(5.4) \qquad |f'(u)| \le \kappa_2, \quad |f(u)| \le \kappa_3 |u|, \quad u \in \mathbb{R},$$

for some constants $\kappa_0, \kappa_1, \kappa_2, \kappa_3 > 0$. When BV estimates are out of reach, which will be the case when (1.16) holds), we shall impose the condition

$$(5.5) \qquad \text{meas}\{u \in \mathbb{R} : f''(u) = 0\} = 0.$$

This condition ensures that f is "genuinely nonlinear".

As with (1.1), we approximate (1.5) with the following parabolic-elliptic system

$$(5.6) \qquad \begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) + \partial_x P_\varepsilon = \varepsilon u_\varepsilon, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = 3f(u_\varepsilon), & t \ge 0, \ x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

which is equivalent to the fourth order equation

$$\partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + 4\partial_x f(u_\varepsilon) = f'''(u_\varepsilon)(\partial_x u_\varepsilon)^3 + 3f''(u_\varepsilon)\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + f'(u_\varepsilon)\partial_{xxx}^3 u_\varepsilon$$
$$+ \varepsilon\partial_{xx}^2 u_\varepsilon - \varepsilon\partial_{xxxx}^4 u_\varepsilon.$$

We assume on the approximated initial conditions $\{u_{\varepsilon,0}\}_{\varepsilon>0}$ that (2.4) holds.

The starting point even in this case is an L^2 bound, see Lemma 2.2 for the Degasperis-Procesi Equation (1.1). Defining v_ε as in (2.6), multiplying (5.6) by $v_\varepsilon - \partial_{xx}^2 v_\varepsilon$ and integrating on $\mathbb{R}$, we get

$$(5.7) \qquad \begin{aligned} &\int_{\mathbb{R}} \partial_t u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx \\ &= -\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx. \end{aligned}$$

For the left hand side of this identity we use (2.9) and for the right hand side, from (5.6) and (2.6), we have

$$\begin{aligned} &-\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx \\ &= -\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon v_\varepsilon \, dx + \int_{\mathbb{R}} \partial_x P_\varepsilon \partial_{xx}^2 v_\varepsilon \, dx \\ (5.8) \qquad &= -\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - \int_{\mathbb{R}} \partial_x P_\varepsilon v_\varepsilon \, dx + \int_{\mathbb{R}} \partial_{xxx}^3 P_\varepsilon v_\varepsilon \, dx \\ &= -\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \left(v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx - 3\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon v_\varepsilon \, dx \\ &= -\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \left(4v_\varepsilon - \partial_{xx}^2 v_\varepsilon\right) dx = -\int_{\mathbb{R}} f'(u_\varepsilon)u_\varepsilon\partial_x u_\varepsilon \, dx = 0. \end{aligned}$$

Then we get back (2.7) and so the bound stated in (2.5) holds also for (5.6).

Given the L^2 estimate, in the quadratic case (5.3) the proofs are essentially the same as the ones for the Degasperis-Procesi Equation (1.1). The unique differences are in the constants in which we now see the presence of the factors κ_0, κ_1. In the Lipschitz case (5.4), the estimates in (2.17) are replaced by

$$\|P_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} , \|\partial_x P_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} \leq \|u_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})} ,$$

which slightly changes the proof of Theorem 3.3. Moreover, the proof of existence of solutions in L^p spaces is simpler because we need only an L^2 estimate (instead of $L^2 \cap L^4$) to use the compensated compactness argument in Section 3.1.

The precise statements are the content of our closing theorems.

Theorem 5.1 (Well-posedness in $L^1 \cap BV$). *Suppose* (1.3) *and* (5.3) *or* (5.4) *hold. Then there exists an entropy weak solution to the Cauchy problem* (5.1), (1.2). *Fix any* $T > 0$, *and let* $u, v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *be entropy weak solutions to* (5.1), (1.2) *with initial data* $u_0, v_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, *respectively. Then for any* $t \in (0, T)$

$$(5.9) \qquad \|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R})} \leq e^{M_T t} \|u_0 - v_0\|_{L^1(\mathbb{R})} ,$$

where

$$M_T := \begin{cases} \frac{3}{4}\kappa_1 \left(\|u\|_{L^\infty((0,T)\times\mathbb{R})} + \|v\|_{L^\infty((0,T)\times\mathbb{R})} \right), & \text{if } (5.3) \text{ holds}, \\ \frac{3}{8}\kappa_3, & \text{if } (5.4) \text{ holds}. \end{cases}$$

Consequently, there exists at most one entropy weak solution to (5.1), (1.2). *What's more, the entropy weak solution* u *belongs to* $L^\infty(0,T; L^1(\mathbb{R}) \cap BV(\mathbb{R}))$ $\forall T > 0$ *and also* $C([0,\infty); L^1(\mathbb{R}))$. *Finally, the Oleinik type estimate* (3.10) *holds for a.e.* $(t,x) \in (0,T] \times \mathbb{R}$.

Theorem 5.2 (Existence in L^p spaces). *Suppose* (1.16), (5.5), *and* (5.3) *or* (5.4) *hold. Then there exists a function*

$$u \in \begin{cases} L^\infty(\mathbb{R}_+; L^2(\mathbb{R})) \cap L^\infty(0,T; L^4(\mathbb{R})) \ \forall T > 0, & \text{if } (5.3) \text{ holds}, \\ L^\infty(\mathbb{R}_+; L^2(\mathbb{R})), & \text{if } (5.4) \text{ holds}, \end{cases}$$

which solves the Cauchy problem (5.1), (1.2) *in* $\mathcal{D}'([0,T] \times \mathbb{R})$.

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