DERIVATIVE-FREE GREEKS FOR THE BARNDORFF-NIELSEN AND SHEPHARD STOCHASTIC VOLATILITY MODEL

FRED ESPEN BENTH, MARTIN GROTH, AND OLLI WALLIN

ABSTRACT. We derive derivative-free formulas for the Delta and other Greeks of options written on an asset modeled by a geometric Brownian motion with stochastic volatility of Barndorff-Nielsen and Shephard type. The method applies the Malliavin Calculus in Wiener space which moves differentiation of the payoff function of the option to a random weight function. Our method paves the way for simple Monte Carlo approaches, illustrated by several numerical examples.

1. INTRODUCTION

Option price sensitivities, commonly referred to as *the Greeks*, are essential tools for investors trying to hedge their positions. Being measurements of how a contract respond to shifts in the parameters of the underlying model, the Greeks are used to manage the risk from unfavourable changes. Informally, one can think of the Greeks as derivatives with regards to a parameter θ of the risk-neutral price:

$$\frac{\partial}{\partial \theta} \mathbb{E}[\phi(S(T))]$$

where $\phi(S(T))$ is the payoff function and $S(T)$ the underlying asset, depending on θ . The Greeks are unobservable quantities in the market, and hence, we need to choose a model for the underlying asset to obtain an estimate of them.

Given a model, the option prices can with benefit be calculated using a Monte Carlo method. The flexibility and low implementation threshold often makes them the preferred pricing tool in finance. However, calculating the option sensitivities requires often substantially greater effort than calculating the price of the option. The slow convergence is especially prominent for discontinuous payoffs. To speed up the convergence there are several different methods and variance reduction techniques proposed.

The finite difference method is the simplest and crudest method to approximate the derivative using a Monte Carlo method. Simulating two different paths with a small difference in the parameter and forming a finite difference, gives an approximation of the sensitivity. The method is universally applicable, however, the estimates are known to be biased and prone to large variance. Broadie and Glasserman [8] proposed two different unbiased methods to improve the convergence rate, both assuming we can exchange order of expectation and differentiation. The pathwise method assumes the dynamics of the model depends on the parameter and differentiates the paths of the model. On the contrary, the likelihood ratio method assumes that the probability density of the price depends on the parameter θ and instead differentiate the measure. Both methods are reported to have significantly lower variance than the finite difference method but are not as applicable. The pathwise method is unable to handle discontinuous payoffs, while the likelihood ratio method is restricted by requiring an explicit knowledge of the density of the underlying model.

Date: March 1, 2007.

Key words and phrases. Ornstein-Uhlenbeck process, subordinators, stochastic volatility, Malliavin derivative, Greeks, Monte Carlo methods.

We thank Tommi Sottinen for fruitful and interesting discussions.

Recent development suggests using an approach based on variational stochastic calculus, referred to as Malliavin calculus. Using an integration-by-parts formula, Fournié *et al.* [16] derive expressions for the Greeks involving weight functions such that the payoff function is not differentiated. The method proved to outperform the finite difference method for discontinuous payoffs, while remaining less restricted than the pathwise and likelihood ratio methods. However, for smoother functions, like vanilla options, the Malliavin method is not reported to be significantly better than the finite difference method. The pioneer work done with the Black-Scholes model spun a large research activity to find optimal weighting functions and perform similar analysis for other contracts and models. Chen and Glasserman [11] list some of the important references. Models including a different source of randomness than a Brownian motion provide an additional complexity, because the Malliavin calculus covers only the Wiener space. There exists several papers developing a similar Malliavin theory for Poisson random fields (Benth and Løkka [5], Nualart and Vives [24], Carlen and Pardoux [9] and Bichteler, Gravereaux and Jacod [7]). El-Khatib and Privault [15] derived Malliavin weights for a market driven by Poisson processes using an integration-by-parts formula, but the domain of the differential operator exclude many option types, for example European claims. Jump-diffusion models are considered in several papers; Leon *et al.* [19], Davis and Johansson [13] and Debelley and Privault [14], the two former considering markets where the jump sizes are deterministic. Due to the lack of chain-rule for the jump component the general idea is to take a directional derivative and use the analysis on the Wiener space.

Barndorff-Nielsen and Shephard [2] proposed a stochastic volatility model suitable to capture the characteristics from high-frequency stock price data. Intra-day sampled log-returns are known to experience heavy tails, skewness and volatility clustering. The Barndorff-Nielsen and Shephard (BNS) model features a stock price dynamics driven by a Brownian motion together with a non-Gaussian Ornstein-Uhlenbeck process describing the volatility. The mean-reverting volatility process includes jumps given by a subordinator, a Lévy process with strictly non-negative increments. At the same time as the model is able to generate realistic asset prices it is analytically tractable enough for derivative pricing and portfolio optimisation, see Benth and Groth [3] and Lindberg [21], [20].

For the BNS model the density of the price distribution is not known explicitly. For options with discontinuous payoff function neither the pathwise nor the likelihood ratio method will be directly applicable for simulations of the Greeks. We use the Malliavin calculus on the Wiener space to derive weight functions for the Greeks, assuming the stock price is given by the Barndorff-Nielsen and Shephard model. The weights here resemble the weights in the Black-Scholes market, but now involve a stochastic volatility. We consider both options depending exclusively on the terminal value of the stock and discretely sampled path-dependent options.

The organisation of the paper is as follows. In the next section we introduce the Barndorff-Nielsen and Shephard model and the properties we use in latter sections. Section 3 discuss the Malliavin calculus in the product space we are interested in. The Malliavin weight for the Greeks in the BNS-model are derived in Section 4 while Section 5 gives several numerical examples.

2. THE BARNDORFF-NIELSEN AND SHEPHARD MODEL

In this section, we give a brief review of the Barndorff-Nielsen and Shephard model, with a view towards option pricing.

We consider a financial market where a risk-free asset and a single risky asset (a stock) are traded up to a fixed time $T > 0$. Especially, we assume the asset price dynamics of the stock price $S(t) = x \exp(X(t))$ are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{P}$ denoting the *physical* probability measure and log prices following Black and Scholes type dynamics

$$(2.1) \quad dX(t) = (\mu + \beta\sigma^2(t)) dt + \sigma(t) dW(t) + \rho dZ(\lambda t), \quad X(0) = 0$$

with stochastic volatility given by a non-Gaussian Ornstein-Uhlenbeck (OU) process

$$(2.2) \quad d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(\lambda t), \quad \sigma^2(0) > 0.$$

Here W is Brownian motion, and Z is a subordinator commonly referred to as the background driving Lévy process (BDLP). We denote by $\kappa(\cdot)$ the cumulant generating function $\kappa(z) := \log(\mathbb{E}[\exp(zZ(1))])$, which uniquely specifies the distribution of $Z(t)$ for all $t \in [0, T]$. Moreover, $r > 0$ is the risk-free rate of return and $\lambda > 0$, $\rho \leq 0$ are constants related to the mean-reversion rate of the volatility and the leverage effect, respectively. The Brownian motion W and the subordinator Z are independent, and $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ is assumed to be the augmented natural filtration of (W, Z) . The parameters μ and β are constants. Note that the solution of (2.2) can be written explicitly as

$$(2.3) \quad \sigma^2(t) = \sigma^2(0)e^{-\lambda t} + \int_0^t e^{\lambda(s-t)} dZ(\lambda s), \quad \sigma^2(0) > 0.$$

Clearly the volatility process $\sigma^2 = (\sigma^2(t))_{t \in [0, T]}$ is then bounded from below by the deterministic function $\sigma^2(0)e^{-\lambda t}$ and is, especially, strictly positive on $[0, T]$. Later we shall deal with processes of the form $1/\sigma^n(t)$ for $n = 1, 2$, which are thus bounded from above by a constant. Barndorff-Nielsen and Shephard [2] propose to use a superposition of Ornstein-Uhlenbeck processes as the model of the squared volatility. We restrict ourselves to only one here, but our results can easily be extended to the general case.

Following a standard procedure in mathematical finance literature, we can now choose a concrete model by specifying a distribution for Z through the cumulant κ . Let us start by stating some additional assumptions on Z .

Assumption 1. The subordinator Z has no drift and its Lévy measure has density $w(\cdot)$ so that $\kappa(\cdot)$, when it is well defined, takes the form

$$\kappa(z) = \int_{\mathbb{R}_+} (e^{zy} - 1)w(y) dy.$$

Moreover, $\hat{z} := \sup\{\theta \in \mathbb{R} : \kappa(z) < \infty\}$ satisfies $\hat{z} > \max\{0, 2\lambda^{-1}(1 + \beta + \rho)\}$ and $\lim_{z \rightarrow \hat{z}} \kappa(z) = +\infty$.

It can be seen using the formula for the Laplace transform of $X(t)$, computed in Nicolato and Venardos [22], that the condition $\hat{z} > 2\lambda^{-1}(1 + \beta + \rho)$ is sufficient for square integrability of S . Furthermore, $\hat{z} > 0$ implies that the variance process σ^2 has an invariant distribution which, in particular, is self-decomposable. A deep connection between self-decomposable distributions and OU-processes is that the converse is also true: for every self-decomposable distribution μ on $\mathbb{R}_+$ there is a subordinator Z such that μ is the invariant distribution of σ^2 . The cumulant generating function κ of Z can be easily recovered from the cumulant κ^μ of μ by

$$(2.4) \quad \kappa(z) = z \frac{d\kappa^\mu}{dz}(z).$$

This one-to-one correspondence makes it possible to build stochastic volatility models of OU-type by first stating the invariant distribution for $\sigma^2(t)$. An important example is the case when μ is an inverse Gaussian (IG) distribution, since for $\rho = 0$ the marginal distribution of log-returns are approximately normal inverse Gaussian (NIG). This class of distributions has been shown to have excellent fit with empirical return distributions. The cumulant function of an IG(δ, γ) distribution is

$$\kappa^{IG}(z) = z\gamma - \delta(\gamma - 2z)^{1/2}$$

so it follows from (2.4) that

$$\kappa(z) = z\delta(\gamma - 2z)^{-1/2}$$

is the cumulant function for the corresponding BDLP. For definitions and properties of invariant and self-decomposable distributions, and the connection with OU-processes we refer the reader to the book by Sato [28].

Let us now turn to option pricing theory under the Barndorff-Nielsen and Shephard model. By the first fundamental theorem of option pricing, the arbitrage free price of an option can be expressed as the discounted expectation of the payoff under an equivalent martingale measure (EMM) $\mathbb{Q}$, which is also commonly called the risk neutral measure. For the BNS model, these measures were characterized by Nicolato and Venardos in [22]. In general, under $\mathbb{Q}$ the jump process Z does not remain a Lévy process and W and Z may be dependent. Thus, the log-price process X may no longer be described by a BNS model. In this article, we restrict our main attention to the class of measures $\mathbb{Q}$ that do retain the general form of the model (2.1), (2.2), but with possibly different parameters and Lévy measure for Z . It was shown in [22] that under any such *structure preserving* $\mathbb{Q}$, the risk neutral dynamics of the log-price have the form

$$(2.5) \quad dX(t) = (r - \lambda\kappa(\rho) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW(t) + \rho dZ(\lambda t)$$

$$(2.6) \quad d\sigma^2(t) = -\lambda\sigma^2(t)dt + dZ(\lambda t), \quad \sigma^2(0) > 0$$

where κ is now the cumulant function of Z under the measure $\mathbb{Q}$. In subsequent sections, we shall assume directly that the risk neutral BNS model (2.5), (2.6) has been given and that Assumption 1 holds for κ with respect to the measure $\mathbb{Q}$.

Note that the integrability criteria given in Assumption 1 collapse to $\hat{z} > \lambda^{-1}$ for the no leverage case $\rho = 0$, which is the situation we consider in Section 5. We remark in passing that we could easily have included other measure changes which are not structure preserving. In fact, our theory will be valid for any martingale measure which lets W and Z be independent. One interesting example is the minimal entropy martingale measure for the BNS-model, which turns Z into a Markov process with state-dependent jumps (see [6]). This more general situation require different integrability hypotheses for the jump process Z .

For many specifications of μ or Z , the Laplace transform of $X(t)$ given in [22] has a fairly explicit form. This makes it possible to compute option prices and Greeks using numerical transform methods if the payoff depends on the terminal value only. Although these methods are potentially superior in simple cases, we do not consider them here because of their limited applicability and instead refer the reader to [10], [12], [22], [29] and the references therein.

3. MALLIAVIN CALCULUS WITH RESPECT TO BROWNIAN MOTION

We base our derivation of derivative-free formulas of the sensitivities on the Malliavin Calculus, as presented in Fournié *et al.* [16]. To do this, we work on the product of the canonical spaces for Brownian motion W and the subordinator Z . This allows us to do Malliavin calculus with respect to Brownian motion in the classical setting of [23].

Let $(\Omega^W, \mathcal{F}^W, \mathbb{Q}^W)$ be the canonical Wiener space for Brownian motion (see, for example, [17]) and correspondingly, let $(\Omega^Z, \mathcal{F}^Z, \mathbb{Q}^Z)$ be the canonical space for the Lévy process Z ([1], [28]). Furthermore, let $\mathbb{F}^W$ and $\mathbb{F}^Z$ be the augmented natural filtrations generated by W and Z , respectively. Then, by independence of W and Z , we can model the risk-neutral dynamics of the BNS model (2.5) on the filtered probability space given by the product

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q}) = (\Omega^W \otimes \Omega^Z, \mathcal{F}^W \otimes \mathcal{F}^Z, \mathbb{F}^W \otimes \mathbb{F}^Z, \mathbb{Q}^W \otimes \mathbb{Q}^Z).$$

There exists a regular conditional probability of $\mathbb{Q}$ given the sigma-algebra $\mathcal{G}$ generated by events of the form $\Omega^W \times \{\omega^Z\}$, $\omega^Z \in \Omega^Z$, and it is denoted by $\mathbb{Q}(\cdot|\omega^Z)$. By independence, this measure coincides with the Wiener measure. We denote by $\mathbb{E}_{\mathbb{Q}^W}$, $\mathbb{E}_{\mathbb{Q}^Z}$ the expectations under the measures $\mathbb{Q}^W$ and $\mathbb{Q}^Z$, respectively. Furthermore, we use $\mathbb{E}$ to denote the expectation

under the product measure $\mathbb{Q}$, so that

$$\mathbb{E} = \mathbb{E}_{\mathbb{Q}} = \mathbb{E}_{\mathbb{Q}^Z} \mathbb{E}_{\mathbb{Q}(\cdot|\omega^Z)} = \mathbb{E}_{\mathbb{Q}^Z} \mathbb{E}_{\mathbb{Q}^W} = \mathbb{E}_{\mathbb{Q}^W} \mathbb{E}_{\mathbb{Q}^Z}.$$

Now, let $F = F(\omega^W, \omega^Z)$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{Q})$. From standard measure theory it follows that for every fixed $\omega^Z \in \Omega^Z$, the mapping

$$\omega^W \mapsto F(\omega^W, \omega^Z), \quad \omega^W \in \Omega^W$$

is a random variable on $(\Omega^W, \mathcal{F}^W, \mathbb{Q}^W)$. Assuming further that this random variable is Malliavin differentiable, we can apply the usual Malliavin calculus on the Wiener space. Moreover, it follows by applying this result on each $\mathcal{F}_t, t \in [0, T]$ that if X is an $\mathbb{F}$ -adapted stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, then, for fixed $\omega^Z \in \Omega^Z$, the process $(X(t, \cdot, \omega^Z))_{t \in [0, T]}$ is an $\mathbb{F}^W$ -adapted stochastic process on $(\Omega^W, \mathcal{F}^W, \mathbb{F}^W, \mathbb{Q}^W)$. Finally, suppose a process u is progressively measurable with respect to $\mathbb{F}^Z$. Then, for almost every $\omega^Z \in \Omega^Z$, the mapping $t \mapsto u(t, \omega^Z)$ is measurable and deterministic. Furthermore, if $u \in L^2([0, T] \times \Omega^Z)$, then $t \mapsto u(t, \omega^Z)$ is in $L^2([0, T])$ for almost every $\omega^Z \in \Omega^Z$. We recall here that every adapted process which is measurable has a progressively measurable modification, and henceforth we shall work with this modification.

Let us next recall the Malliavin calculus on Wiener space in view of sensitivity analysis for the Barndorff-Nielsen and Shephard model. The above discussion hints at a natural way to use Malliavin calculus in our setting. We let $\mathcal{S}_{BNS}$ denote the set of *smooth* random variables F of the form

$$F = f\left(\int_0^T h_1(t) dW(t), \dots, \int_0^T h_m(t) dW(t), \omega^Z\right),$$

where $h_1, \dots, h_m \in L^2([0, T] \times \Omega)$ are $\mathbb{F}$ -adapted and $f : \mathbb{R}^m \times \Omega^Z \rightarrow \mathbb{R}$ are such that $f(\cdot, \omega^Z) \in C_2^\infty(\mathbb{R}^m)$ for $\omega^Z \in \Omega^Z$. Note that, for fixed $\omega^Z \in \Omega^Z$, the random variable $F(\cdot, \omega^Z)$ belongs to the set $\mathcal{S}$ of random variables on the Wiener space that are smooth in the classical sense of [23]. Given $F \in \mathcal{S}_{BNS}$, the Malliavin derivative of F with respect to Brownian motion is the process $(D_t F)_{t \in [0, T]}$ in $L^2([0, T] \times \Omega)$ defined by

$$D_t F := \sum_{j=1}^m f\left(\int_0^T h_1(t) dW(t), \dots, \int_0^T h_m(t) dW(t), \omega^Z\right) h_j(t).$$

Again, this is nothing but the classical definition done ω^Z -wise.

On $L^2(\Omega^W, \mathcal{F}^W, \mathbb{Q}^W)$, define the norm

$$\|F\|_{1,2} := \left(\mathbb{E}_{\mathbb{Q}^W}[F^2] + \mathbb{E}_{\mathbb{Q}^W} \left[\int_0^T |D_t F|^2 dt \right] \right)^{1/2}$$

and denote by $\mathbb{D}^{1,2}$ the closure under $\|\cdot\|_{1,2}$ of the set of smooth Wiener random variables $\mathcal{S}$. The normed space $(\mathbb{D}^{1,2}, \|\cdot\|_{1,2})$ is a Banach space, and the Malliavin derivative is a closed linear operator on $\mathbb{D}^{1,2}$ taking values in $L^2([0, T] \times \Omega^W)$. Now, we denote by $\mathbb{D}_{BNS}^{1,2}$ the set of random variables $F \in L^2(\Omega)$ such that $F(\cdot, \omega^Z) \in \mathbb{D}^{1,2}$ for almost every $\omega^Z \in \Omega^Z$. Then we also have the existence of a sequence $\mathcal{F}_n \in \mathcal{S}_{BNS}$ such that

$$\mathbb{E}[F^2] + \mathbb{E} \left[\int_0^T |D_t F|^2 dt \right] = \mathbb{E}_{\mathbb{Q}^Z} [\|F_n - F\|_{1,2}^2] \rightarrow 0.$$

Let us illustrate the calculus with the following

Example 3.1. *Let us consider the random variable*

$$F = \int_0^T \sigma(t) dW(t).$$

Fixing ω^Z , the mapping $t \mapsto \sigma(t, \omega^Z)$ is a deterministic function in $L^2([0, T])$, so $F(\cdot, \omega^Z)$ is a Malliavin differentiable random variable on the Wiener space. We thus have

$$D_t F = \sigma(t), \quad t \in [0, T]$$

almost surely.

It is also clear that since we are doing Malliavin calculus with respect to Brownian motion only, anything that is $\mathcal{F}^Z$ -measurable vanishes on differentiation.

Property 3.1. *If F is $\mathcal{F}^Z$ measurable, then $D_t F = 0$ for $t \in [0, T]$.*

The Malliavin derivative satisfies the chain rule, which we state here in a form suitable for our purposes:

Property 3.2. *Let $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuously differentiable function and let $(F_1, \dots, F_m)$ be a random vector whose components belong to $\mathbb{D}_{BNS}^{1,2}$. Suppose furthermore that*

$$\mathbb{E}[|\phi(F_1, \dots, F_m)|^2] + \mathbb{E}\left[\int_0^T \left|\sum_{j=1}^m \partial_{x_j} \phi(F_1, \dots, F_m) D_t F_j\right|^2 dt\right] < \infty.$$

Then

$$(3.1) \quad D_t \phi(F_1, \dots, F_m) = \sum_{j=1}^m \partial_{x_j} \phi(F_1, \dots, F_m) D_t F_j.$$

In standard references, this result is usually stated only for ϕ with bounded derivatives which would exclude the important case of the exponential function. In the above generality, the proof (for the more general case of $F_j \in \mathbb{D}^{1,1}$) can be found in the Appendix of [25].

The Malliavin derivative has an adjoint operator called the *Skorohod integral* (also known as the *divergence operator*). Let us start by returning once more to the setting of the Wiener space, and then give the corresponding extension.

Property 3.3. *Skorohod integral in the Brownian direction: Let $u \in L^2([0, T] \times \Omega^W)$. Then $u \in \text{Dom}(\delta^W)$ if and only if for all $F \in \mathbb{D}^{1,2}$ we have*

$$\mathbb{E}_{\mathbb{Q}^W} \left(\int_0^T D_t F u(t) dt \right) \leq C(u) \|F\|_{1,2}$$

where $C(u)$ is a constant independent of $F \in \mathbb{D}^{1,2}$. If $u \in \text{Dom}(\delta^W)$, then the Skorohod integral of u is the a.s. unique random variable $\delta(u) \in L^2(\Omega^W)$ satisfying the relation

$$\mathbb{E}_{\mathbb{Q}^W} [F \delta(u)] = \mathbb{E}_{\mathbb{Q}^W} \left[\int_0^T D_t F u(t) dt \right].$$

We define $\text{Dom}(\delta)$ to be the set of processes $u \in L^2([0, T] \times \Omega)$ such that u belongs to $\text{Dom}(\delta^W)$ $\mathbb{Q}^Z$ -almost surely and

$$(3.2) \quad \mathbb{E} \left(\int_0^T D_t F u(t) dt \right) < \infty.$$

If $\delta \in \text{Dom}(\delta)$, we denote by δ the operator $\delta : L^2([0, T] \times \Omega) \rightarrow L^2(\Omega)$ defined by

$$\delta(u)(\omega^W, \omega^Z) = \delta^W(u(\cdot, \omega^W, \omega^Z)).$$

Then it follows by Fubini's theorem that

$$(3.3) \quad \mathbb{E}[F \delta(u)] = \mathbb{E} \left[\int_0^T D_t F u(t) dt \right].$$

The above equality (3.3) is commonly referred to as *the integration-by-parts formula*, and the process u is called Skorohod integrable if $u \in \text{Dom}(\delta)$. One of the main properties of the Skorohod integral δ^W on the Wiener space is that all $\mathbb{F}^W$ -adapted processes in $L^2([0, T] \times \Omega^W)$ are Skorohod integrable and the Skorohod integral of such processes coincides with the usual stochastic integral of Itô. Here we state the corresponding result for adapted integrands in $L^2([0, T] \times \Omega)$.

Property 3.4. *For $\mathbb{F}$ -adapted $u \in L^2([0, T] \times \Omega)$, we have $u \in \text{Dom}(\delta)$ and*

$$\delta(u) = \int_0^T u(s) dW(s).$$

Furthermore,

$$D_t \delta(u) = D_t \int_0^T u(s) dW(s) = u(t).$$

In the above, the claim that (3.2) holds might not seem clear at first. However, we can again use conditioning to estimate

$$\begin{aligned} \left[\mathbb{E} \left(\int_0^T D_t F u(t) dt \right) \right]^2 &= \left[\mathbb{E}_{\mathbb{Q}^Z} \mathbb{E}_{\mathbb{Q}^W} \left(F \delta^W(u) \right) \right]^2 \\ &= \left[\mathbb{E} \left(F \int_0^T u(t) dW(t) \right) \right]^2 \\ &\leq \|F\|_{L^2(\Omega)}^2 \mathbb{E} \left[\left(\int_0^T u(t) dW(t) \right)^2 \right] \\ &= \|F\|_{L^2(\Omega)}^2 \mathbb{E} \left[\int_0^T u^2(t) dt \right] \\ &\leq \|F\|_{L^2(\Omega)}^2 \|u\|_{L^2([0, T] \times \Omega)}^2 < \infty. \end{aligned}$$

where we have used properties of the Skorohod integral on Wiener space, Cauchy-Schwarz inequality and the Itô isometry.

The following lemma facilitates further computation of Skorohod integrals in an important special case where the integrand is no longer adapted.

Property 3.5. *Let $F \in \mathbb{D}_{BNS}^{1,2}$. For all $u \in \text{Dom}(\delta)$ such that $F\delta(u) - \int_0^T D_t F u(t) dt \in L^2(\Omega)$ we have $Fu \in \text{Dom}(\delta)$ and*

$$(3.4) \quad \delta(Fu) = F\delta(u) - \int_0^T D_t F u(t) dt.$$

Finally, it is easily seen that $\theta \mapsto S^\theta$ is pathwise differentiable (with exception of boundary values $x = 0$, $\sigma^2(0) = 0$) for the different parameters $\theta = x, r, \rho, \sigma^2(0)$ and ϵ (to be defined in section 4.2).

Remark. Instead of following the concrete program via pointwise conditioning on ω^Z outlined here, one could also proceed by viewing elements in $L^2(\Omega)$ as $L^2(\Omega^Z)$ -valued random variables on the Wiener space, see [19],[23].

4. MALLIAVIN WEIGHTS FOR THE GREEKS

In this section we apply the previous results to derive formulas for the Greeks as weighted expectations of the payoff. We start by verifying some quite standard but useful lemmas. The first one justifies differentiation under the expectation.

Lemma 4.1. *Let F^θ be a real valued random variable, depending on a parameter $\theta \in \mathbb{R}$. Suppose furthermore that, for almost every $\omega \in \Omega$ the mapping $\theta \mapsto F^\theta(\omega)$ is continuously differentiable in $[a, b]$, and that*

$$\mathbb{E}[\sup_{\theta \in [a, b]} |\partial_\theta F^\theta|] < \infty.$$

Then, the mapping $\theta \mapsto \mathbb{E}[F^\theta]$ is differentiable in (a, b) , and for $\theta \in (a, b)$ we have

$$\partial_\theta \mathbb{E}[F^\theta] = \mathbb{E}[\partial_\theta F^\theta].$$

Proof. First, fix $\bar{\theta} \in (a, b)$ and note by the assumptions we have

$$\frac{1}{h}\{F^{\bar{\theta}+h} - F^{\bar{\theta}}\} \rightarrow \partial_\theta F^{\bar{\theta}}, \quad \text{almost surely,}$$

as $h \rightarrow 0$. Moreover, by the mean value theorem of calculus we see that

$$|\frac{1}{h}\{F^{\bar{\theta}+h} - F^{\bar{\theta}}\}| \leq \sup_{\theta \in [a, b]} |\partial_\theta F^\theta|.$$

Thus, we deduce by dominated convergence theorem that

$$\frac{1}{h}\{\mathbb{E}[F^{\bar{\theta}+h}] - \mathbb{E}[F^{\bar{\theta}}]\} = \mathbb{E}[\frac{1}{h}\{F^{\bar{\theta}+h} - F^{\bar{\theta}}\}] \rightarrow \mathbb{E}[\partial_\theta F^{\bar{\theta}}]$$

as $h \rightarrow 0$, finishing the proof. $\square$

The next lemma allows us to assume infinite smoothness of the payoff function when deriving the formulas. Let us denote by $L^2(S)$ the class of locally integrable functions ϕ such that the set of discontinuities of ϕ has Lebesgue measure zero, and satisfy

$$(4.1) \quad \mathbb{E}[\phi(S(t_1), \dots, S(t_m))^2] < \infty.$$

From now on, we denote $S(\cdot) = S^\theta(\cdot)$ to emphasize the dependence of the model on a parameter θ .

Lemma 4.2. *Suppose that*

$$(4.2) \quad \partial_\theta \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))] = \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))\pi^\theta]$$

holds for $\phi \in C_0^\infty(\mathbb{R}^m)$, $\pi^\theta \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$. Suppose also that the mapping $\theta \mapsto \pi^\theta$ is continuous, almost surely. Then the equality (4.2) holds also for $\phi \in L^2(S)$.

Proof. Let ϕ satisfy (4.1) and let ϕ_k , $k = 1, 2, \dots$ be such that $\phi_k \uparrow \phi$ Lebesgue almost everywhere as $k \rightarrow \infty$. Since X has transition probability that are absolutely continuous with respect to Lebesgue measure (see [22]), and discontinuities of ϕ have measure zero, we have

$$\phi_k(S^\theta(t_1), \dots, S^\theta(t_m)) \uparrow \phi(S(t_1)^\theta, \dots, S^\theta(t_m))$$

almost surely. Furthermore, the family $\phi_k(S(t_1), \dots, S(t_m))^2$ is uniformly integrable so

$$\phi_k(S^\theta(t_1), \dots, S^\theta(t_m)) \rightarrow \phi(S^\theta(t_1), \dots, S^\theta(t_m))$$

in $L^2(\Omega, \mathcal{F}, \mathbb{Q})$ (and thus also in $L^1(\Omega, \mathcal{F}, \mathbb{Q})$) as $k \rightarrow \infty$. Let us now define $u(\theta) := \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))]$, $u_k(\theta) := \mathbb{E}[\phi_k(S^\theta(t_1), \dots, S^\theta(t_m))]$, and note that $u_k(\theta) \rightarrow u(\theta)$ for every $\theta \in [a, b]$. Furthermore, let

$$f(\theta) := \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))\pi^\theta].$$

By the Cauchy-Schwartz inequality,

$$|\partial_\theta u_k(\theta) - f(\theta)| \leq \epsilon_k(\theta)\psi(\theta),$$

where

$$\epsilon_k(\theta) = (\mathbb{E}[(\phi_k(S^\theta(t_1), \dots, S^\theta(t_m)) - \phi(S^\theta(t_1), \dots, S^\theta(t_m)))^2])^{1/2}$$

and $\psi(\theta) = (\mathbb{E}[|\pi^\theta|^2])^{1/2}$. From the assumptions it follows that ψ and ϵ are continuous. Thus, for an arbitrary compact subset $K \subset \mathbb{R}$, we have

$$\sup_{\theta \in K} |\partial_\theta u_k(\theta) - f(\theta)| \leq C_K \sup_{\theta \in K} \epsilon_k(\theta)$$

with $C_K = \sup_{\theta \in K} \psi(\theta)$. Since $\sup_{\theta \in K} \epsilon_k(\theta) \rightarrow 0$ as $k \rightarrow \infty$, it follows that

$$\partial_\theta u_k(\theta) \rightarrow f(\theta)$$

uniformly on compact subsets of $\mathbb{R}$, proving the lemma. $\square$

Note that the class $L^2(S)$ defined before the lemma is not the L^2 -space on $\mathbb{R}^m$. The difference is important as $L^2(\mathbb{R}^m)$ does not contain most of the contracts, including the call option.

We are now prepared to derive formulas for the Greeks. We treat Delta and Gamma first, and then move on to study the other Greeks in a unified manner.

4.1. Delta and Gamma. Delta and Gamma are, respectively, the first and second order derivatives of the option price with respect x , the current price level of the underlying stock price S .

Proposition 4.3. *Let $a \in L^2([0, T])$ be an $\mathbb{F}$ -adapted process such that*

$$\int_0^{t_i} a(t) dt = 1 \quad \text{almost surely}$$

for all $i = 1, 2, \dots, m$. Then:

(i) *The Delta of the option is given by*

$$\partial_x \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m))] = \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m)) \pi^\Delta],$$

where the Malliavin weight π^Δ equals

$$\pi^\Delta = \int_0^T \frac{a(t)}{x\sigma(t)} dW(t).$$

(ii) *The Gamma of the option is given by*

$$(4.3) \quad \partial_x^2 \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m))] = \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m)) \pi^\Gamma],$$

where the Malliavin weight π^Γ equals

$$(4.4) \quad \pi^\Gamma = (\pi^\Delta)^2 - \frac{1}{x} \pi^\Delta - \frac{1}{x^2} \int_0^T \left(\frac{a(t)}{\sigma(t)} \right)^2 dt.$$

Proof. First note that the assumptions of the above Lemma 4.1 and 4.2 hold, and thus we only need to prove the claim for $\phi \in C_0^\infty(\mathbb{R}^m)$.

(i) Applying Lemma 4.1, we compute

$$\begin{aligned} \partial_x \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m))] &= \mathbb{E}[e^{-rT} \partial_x \phi(S^x(t_1), \dots, S^x(t_m))] \\ &= \mathbb{E}[e^{-rT} \sum_{i=1}^m \phi_{x_i}(S^x(t_1), \dots, S^x(t_m)) \partial_x S^x(t_i)] \\ &= \mathbb{E}[e^{-rT} \sum_{i=1}^m \phi_{x_i}(S^x(t_1), \dots, S^x(t_m)) \frac{1}{x} S^x(t_i)]. \end{aligned}$$

Using $D_t S^x(t_i) = \sigma(t) S^x(t_i) 1_{[0, t_i]}(t)$ and $\int_0^{t_i} a(t) dt = 1$, we note that

$$\int_0^T \frac{a(t)}{x\sigma(t)} D_t S^x(t_i) dt = \frac{1}{x} S^x(t_i),$$

so

$$\partial_x \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m))] = \mathbb{E}\left[e^{-rT} \int_0^T \sum_{i=1}^m \phi_{x_i}(S^x(t_1), \dots, S^x(t_m)) \frac{a(t)}{x\sigma(t)} D_t S^x(t_i) dt\right].$$

By the chain rule property it follows that

$$\partial_x \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m))] = e^{-rT} \mathbb{E}\left[\int_0^T D_t \phi(S^x(t_1), \dots, S^x(t_m)) \frac{a(t)}{x\sigma(t)} dt\right].$$

The claim now follows from the integration-by-parts property (3.3) and (3.4).

(ii) Denoting $F^x := \int_0^T \frac{a(t)}{x\sigma(t)} dW(t)$, we note that $\partial_x F^x = -\frac{1}{x} F^x$ so

$$\begin{aligned} \partial_x^2 \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m))] &= \partial_x \mathbb{E}[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m)) F^x] \\ (4.5) \quad &= -\frac{1}{x} \mathbb{E}[e^{-rT} \partial_x \phi(S^x(t_1), \dots, S^x(t_m)) F^x] \\ &\quad + \mathbb{E}\left[e^{-rT} \sum_{i=1}^m \phi_{x_i}(S^x(t_1), \dots, S^x(t_m)) \frac{1}{x} S^x(t_i) F^x\right]. \end{aligned}$$

For the second term on the last line above, we can re-iterate the procedure from (i):

$$\begin{aligned} &\mathbb{E}\left[e^{-rT} \sum_{i=1}^m \phi_{x_i}(S^x(t_1), \dots, S^x(t_m)) \frac{1}{x} S^x(t_i) F^x\right] \\ &= \mathbb{E}\left[e^{-rT} \int_0^T D_t \phi(S^x(t_1), \dots, S^x(t_m)) \frac{a(t)}{x\sigma(t)} F^x dt\right] \\ &= \mathbb{E}\left[e^{-rT} \phi(S^x(t_1), \dots, S^x(t_m)) \delta\left(\frac{a(\cdot)}{x\sigma(\cdot)} F^x\right)\right]. \end{aligned}$$

Finally, applying (3.4) with $D_t F^x = \frac{a(t)}{x\sigma(t)}$, we have

$$\delta\left(\frac{a(\cdot)}{x\sigma(\cdot)} F^x\right) = F^x \int_0^T \frac{a(t)}{x\sigma(t)} dW(t) - \int_0^T \left(\frac{a(t)}{x\sigma(t)}\right)^2 dt = (F^x)^2 - \int_0^T \left(\frac{a(t)}{x\sigma(t)}\right)^2 dt.$$

Combining this with (4.5) finishes the proof. $\square$

4.2. Rho and the three Vegas. Next we investigate sensitivities with respect to other model parameters. We call Rho, as always, the sensitivity with respect to the interest rate level r . It is common practise to call the sensitivity of the option price related to parameters affecting the random fluctuations of the stock price Vega (although this is not a Greek letter), and we have three different measures here. We call sensitivities with respect to starting value $\sigma^2(0)$ of the variance process and the leverage parameter ρ for Vega1 and Vega3, respectively. We name Vega2 the sensitivity of the option price with respect to changes in the whole volatility structure. More precisely, let $\sigma_\epsilon(u) = \sigma(u) + \epsilon \tilde{\sigma}(u)$, where $\tilde{\sigma}$ is a bounded and adapted process such that σ_ϵ is uniformly bounded away from zero. Furthermore, put

$$X_{\tilde{\sigma}}^\epsilon(t) = \int_0^t \left[r - \frac{1}{2} \sigma_\epsilon^2(u)\right] du + \int_0^t \sigma_\epsilon^2(t) dW(u),$$

and

$$S_{\tilde{\sigma}}^\epsilon(t) = x e^{X_{\tilde{\sigma}}^\epsilon(t)}.$$

Then we define

$$\text{Vega2} = \partial_\epsilon \mathbb{E}[e^{-rT} \phi(S_{\tilde{\sigma}}^\epsilon(t_1), \dots, S_{\tilde{\sigma}}^\epsilon(t_m))]|_{\epsilon=0}.$$

In what follows, we let $b : [0, T] \rightarrow \mathbb{R}$ be an $\mathbb{F}$ -adapted process satisfying

$$\int_{t_{j-1}}^{t_j} b(t) dt = 1, \quad \text{almost surely}$$

for all $j = 1, \dots, m$. Furthermore, let

$$(4.6) \quad a(t) = \sum_{j=1}^m b(t)(\partial_\theta X^\theta(t_j) - \partial_\theta X^\theta(t_{j-1}))1_{[t_{j-1}, t_j]}(t)$$

where X is the log-price process. Notice that the process a so defined is not adapted in general. Now, we prove a general formula from which the above Greeks can be derived.

Theorem 4.4. *Let $\phi \in L^2(S)$. Then*

$$(4.7) \quad \partial_\theta \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))] = \mathbb{E} \left[\phi(S^\theta(t_1), \dots, S^\theta(t_m)) \delta \left(\frac{a(\cdot)}{\sigma(\cdot)} \right) \right]$$

Proof. Recall again that by Lemma 4.2 we may assume $\phi \in C_0^\infty(\mathbb{R}^m)$. Here

$$\begin{aligned} \partial_\theta \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))] &= \mathbb{E}[\partial_\theta \phi(S^\theta(t_1), \dots, S^\theta(t_m))] \\ &= \mathbb{E} \left[\sum_{i=1}^m \phi_{x_i}(S^\theta(t_1), \dots, S^\theta(t_m)) \partial_\theta S^\theta(t_i) \right] \\ &= \mathbb{E} \left[\sum_{i=1}^m \phi_{x_i}(S^\theta(t_1), \dots, S^\theta(t_m)) S^\theta(t_i) \partial_\theta X^\theta(t_i) \right]. \end{aligned}$$

We note that

$$\int_0^T \frac{a(t)}{\sigma(t)} D_t S^\theta(t_i) dt = S^\theta(t_i) \partial_\theta X^\theta(t_i),$$

so that

$$\begin{aligned} \partial_\theta \mathbb{E}[\phi(S^\theta(t_1), \dots, S^\theta(t_m))] &= \mathbb{E} \left[\int_0^T \sum_{i=1}^m \phi_{x_i}(S^\theta(t_1), \dots, S^\theta(t_m)) D_t S^\theta(t_i) \frac{a(t)}{\sigma(t)} dt \right] \\ &= \mathbb{E} \left[\int_0^T D_t \phi(S^\theta(t_1), \dots, S^\theta(t_m)) \frac{a(t)}{\sigma(t)} dt \right] \\ &= \mathbb{E} \left[\phi(S^\theta(t_1), \dots, S^\theta(t_m)) \delta \left(\frac{a(t)}{\sigma(t)} \right) \right] \end{aligned}$$

where we have again applied the chain rule and the integration-by-parts properties. $\square$

Next, we study the Malliavin weights π^θ for the Greeks Rho, Vega1, Vega2 and Vega3 in more detail, using the proposition above. That is, we shall find explicit forms of a random variable π^θ such that

$$\partial_\theta \mathbb{E}[e^{-rT} \phi(S^\theta(T))] = \mathbb{E}[e^{-rT} \phi(S^\theta(T)) \pi^\theta].$$

Corollary 4.5. *Let $\phi \in L^2(S)$.*

(Rho) *The Malliavin weight for the sensitivity of the option price with respect to interest rate r is*

$$\pi^{Rho} = T(x\pi^\Delta - 1),$$

that is

$$Rho = Tx \times \text{delta} - T \times \text{price}.$$

(Vega1) *The Malliavin weight for the sensitivity of the option price with respect to initial value $\sigma_0^2 := \sigma^2(0)$ of the variance process is*

$$\pi^{Vega1} = \frac{1}{2} \sum_{j=1}^m \left\{ \left(\int_{t_{j-1}}^{t_j} \frac{e^{-\lambda t}}{\sigma(t)} dW(t) + \frac{1}{\lambda} (e^{-\lambda t_j} - e^{-\lambda t_{j-1}}) \right) \int_{t_{j-1}}^{t_j} \frac{b(t)}{\sigma(t)} dW(t) - \int_{t_{j-1}}^{t_j} \frac{b(t)}{\sigma^2(t)} e^{-\lambda t} dt \right\}.$$

(Vega2) *Sensitivity of the option price with respect to a perturbation $\tilde{\sigma}$ of the volatility process is given by*

$$\partial_{\epsilon} \mathbb{E}[e^{-rT} \phi(S_{\tilde{\sigma}}^{\epsilon}(t_1), \dots, S_{\tilde{\sigma}}^{\epsilon}(t_m))] |_{\epsilon=0} = \mathbb{E}[e^{-rT} \phi(S(t_1), \dots, S(t_m)) \pi_{\tilde{\sigma}}^{Vega2}],$$

with

$$\pi_{\tilde{\sigma}}^{Vega2} = \sum_{j=1}^m F_j \int_{t_{j-1}}^{t_j} \frac{b(t)}{\sigma(t)} dW(t) - \int_0^T b(t) \frac{\tilde{\sigma}(t)}{\sigma(t)} dt,$$

where

$$F_j = \int_{t_{j-1}}^{t_j} \tilde{\sigma}(t) dW(t) - \int_{t_{j-1}}^{t_j} \sigma(t) \tilde{\sigma}(t) dt$$

(Vega3) *The Malliavin weight for sensitivity with respect to the leverage parameter ρ is given by*

$$\pi^{Vega3} = \sum_{j=1}^m (\Delta Z_j - \lambda \kappa'(\rho) \Delta t_j) \int_{t_{j-1}}^{t_j} \frac{b(t)}{\sigma(t)} dW(t)$$

where $\Delta Z_j := Z(\lambda t_j) - Z(\lambda t_{j-1})$ and $\Delta t_j := t_j - t_{j-1}$.

Proof. The results are a straightforward application of the above Theorem 4.4 and properties given in Section 3 to compute the Skorohod integral in a more recognizable form.

Rho: First notice that $\partial_r X(t) = t$. Choosing

$$b(t) = \sum_{k=1}^m \frac{1}{t_j - t_{j-1}} 1_{[t_{j-1}, t_j]}(t),$$

and noticing that $a(\cdot)$ given in (4.6) is now adapted, we have

$$\delta \left(\frac{a(\cdot)}{\sigma(\cdot)} \right) = \int_0^T \frac{a(t)}{\sigma(t)} dW(t) = \int_0^T \frac{1}{\sigma(t)} dW(t) = Tx\pi^{\Delta}.$$

The result now follows from

$$(4.8) \quad \partial_r \mathbb{E}[e^{-rT} \phi(S^r(t_1), \dots, S^r(t_m))] = -T \mathbb{E}[e^{-rT} \phi(S^r(t_1), \dots, S^r(t_m))] + e^{-rT} \partial_r \mathbb{E}[\phi(S^r(t_1), \dots, S^r(t_m))].$$

Vega1: First, $\partial_{\sigma_0^2} \sigma^2(t) = e^{-\lambda t}$ and $\partial_{\sigma_0^2} \sigma(t) = \partial_{\sigma_0^2} (\sigma^2(t))^{1/2} = \frac{e^{-\lambda}}{2\sigma(t)}$, so

$$\partial_{\sigma_0^2} X(t) = \frac{1}{2} \left\{ \int_0^t \frac{e^{-\lambda s}}{\sigma(s)} dW(s) - \int_0^t e^{-\lambda s} ds \right\} = \frac{1}{2} \left\{ \int_0^t \frac{e^{-\lambda s}}{\sigma(s)} dW(s) - \frac{1}{\lambda} (e^{-\lambda t} - 1) \right\}.$$

From this, we have

$$a(t) = \frac{1}{2} \sum_{j=1}^m (C_j + F_j) b(t) 1_{[t_{j-1}, t_j]}(t),$$

where $C_j = \frac{1}{\lambda} (e^{-\lambda t_j} - e^{-\lambda t_{j-1}})$ and

$$F_j = \int_{t_{j-1}}^{t_j} \frac{e^{-\lambda s}}{\sigma(s)} dW(s).$$

Noting finally that $D_t F_j = \frac{e^{-\lambda t}}{\sigma(t)} 1_{[t_{j-1}, t_j]}(t)$, the result follows from Properties 3.5 and 3.4.

Vega2: The proof of Theorem 4.4 does not use the specific form of the process σ^2 , only its integrability properties and that it is adapted to the filtration $\mathbb{F}^Z$. Thus we see that (4.7) holds with σ replaced with σ_{ϵ} and S replaced with S^{ϵ} . Now, applying Theorem 4.4 with

$$\partial_{\epsilon} X^{\epsilon}(t) = - \int_0^t \sigma_{\epsilon}(s) \tilde{\sigma}(s) ds + \int_0^t \tilde{\sigma}(s) dW(s),$$

the result is a straight forward calculation using Properties 3.5 and 3.4.

Vega3: This is a trivial calculation using Properties 3.4 and 3.1. □

Note that Rho can be stated in terms of the price and the Delta, requiring no extra computation. We will now list the Malliavin weights (except Rho which is the same) in the simple case where the option payoff depends on terminal value only. Here, we have taken the simple choice of $a(\cdot) = 1/T$ as the weight function for Delta and Gamma, and similarly $b(\cdot) = 1/T$ for the other Greeks.

$$\begin{aligned}\pi^\Delta &= \frac{1}{Tx} \int_0^T \frac{1}{\sigma(t)} dW(t), \\ \pi^\Gamma &= (\pi^\Delta)^2 - \frac{1}{T^2 x^2} \int_0^T \frac{1}{\sigma^2(t)} dt - \frac{1}{x} \pi^\Delta, \\ \pi^{Vega1} &= \frac{1}{2T} \left[\int_0^T \frac{e^{-\lambda t}}{\sigma(t)} dW(t) + \frac{1}{\lambda} (e^{-\lambda T} - 1) \right] \int_0^T \frac{1}{\sigma(t)} dW(t) - \frac{1}{2T} \int_0^T \frac{e^{-\lambda t}}{\sigma^2(t)} dt, \\ \pi_{\tilde{\sigma}}^{Vega2} &= \frac{1}{T} \int_0^T \frac{1}{\sigma(t)} dW(t) \left[\int_0^T \tilde{\sigma}(t) dW(t) - \int_0^T \sigma(t) \tilde{\sigma}(t) dt \right] - \frac{1}{T} \int_0^T \frac{\tilde{\sigma}(t)}{\sigma(t)} dt, \\ \pi^{Vega3} &= \left(\frac{1}{T} Z(\lambda T) - \lambda \kappa'(\rho) \right) \int_0^T \frac{1}{\sigma(t)} dW(t).\end{aligned}$$

From these representations we also notice that the weights for Delta, Gamma, Vega2 (and Rho) agree with those in the Black and Scholes model if we replace the stochastic volatility by a constant one.

Using the basic principles developed in this chapter, it is also possible to modify numerous results that have already appeared in the diffusion setting to be applicable for the BNS model. We refer to Kohatsu-Higa and Montero [18] for a comprehensive survey and reference list.

5. NUMERICAL EXAMPLES

In the previous sections we derived Malliavin weights for a derivative-free simulation of option sensitivities in the Barndorff-Nielsen and Shephard stochastic volatility model. In this section we provide some examples to show the efficiency of the method and possible pitfalls. We show the superior performance of the Malliavin method compared to the finite difference method in cases where the payoff function is discontinuous, but also that the methods are comparable for smoother functions. The examples will focus on the three Greeks Delta, Gamma and Vega2, but similar results hold for the other Greeks as well.

For the numerical examples we consider the BNS model, without the leverage effect. The invariant distribution of the variance process is assumed to be the inverse Gaussian distribution, which will give marginal log-returns being approximately normal inverse Gaussian distributed. To have relevant parameters for the volatility dynamics we use the estimates found in Benth, Groth and Lindberg [4] for the Volvo B stock traded at the OMX in Stockholm, where $\delta = 0.0116$, $\gamma = 54.2$ and $\lambda = 0.83$. The estimation procedure uses the number of trades as a measurement of volatility in the market, see [4] for an extensive discussion. The spot price is 374.5 SEK and we assume an interest rate of 3%, which is close to the 3-month STIBOR¹ at the time. The contracts tested are a plain vanilla call with strike 400 SEK, a binary call with the same strike and a knock-out option. The knock-out option has an European payoff function with strike at 380 SEK and a knock-out boundary at 400 SEK. The implementations are done in Matlab, using generic (pseudo)-random number generators.

¹Stockholm Interbank Offered Rate

FIGURE 1. Simulation of the Gamma for a Vanilla Call with payoff function $f(x) = (S - K)^+$, $K = 400$.

A variance reduction could be obtained by using low-discrepancy sequences, but would not change the structure of the results and is not applied here.

A variance reduction technique which can be applied with the Malliavin method, introduced by Fournié *et al.* [16], is to localise the Malliavin weights around the strike price K . Let $\phi(S)$ represent the payoff function, being non-smooth at the strike K , and suppose we are interested in the sensitivity with respect to the parameter θ . The Malliavin weight introduce noise but if the otherwise global weight is localised around K the variance is reduced. Assume we can approximate $\phi(S)$ with a smooth function $\phi_\epsilon(S)$ such that $\phi(S) - \phi_\epsilon(S)$ becomes zero outside the interval $[K - \epsilon, K + \epsilon]$. Define $\Psi_\epsilon(S) = \phi(S) - \phi_\epsilon(S)$, then we see that

$$\partial_\theta \mathbb{E}[\phi(S)] = \partial_\theta \mathbb{E}[\phi_\epsilon(S)] + \partial_\theta \mathbb{E}[\Psi_\epsilon(S)] = \mathbb{E}[\phi'_\epsilon(S) \partial_\theta S] + \mathbb{E}[\Psi_\epsilon(S) \pi^\theta]$$

where π^θ is the Malliavin weight for θ . Localising the Malliavin weight reduces the noise at the same time as we avoid taking the derivative of the payoff function close to the strike price. The choice for the function ϕ_ϵ depends on the particular payoff function.

Several things make the implementation of the Malliavin method more complicated in the BNS than in the Black-Scholes model. The foremost complicating factor is that we need to simulate the stochastic volatility process. Also, adding to the complexity is that the weights contain one or several integrals which need to be estimated using a numerical integration algorithm, in this case the extended trapezoidal rule (see Press [26]). A poor numerical integration adds a bias in the simulations and especially the two measures of Vega suffer if we take a coarse discretisation. The execution time scales with the number of time steps used in the simulation and integration, so there will be a trade-off between speed and accuracy.

For the simulation of the variance processes (2.6) we use the series representation proposed in Rośniński [27], see also Barndorff-Nielsen and Shephard [2]. Recall that the variance process (2.6) can be written explicitly as equation (2.3). To simulate paths for this process we need

FIGURE 2. Simulation of the Delta for a binary option with payoff function $f(x) = 1_{\{x \geq K\}}(x)$, $K = 400$.

FIGURE 3. Simulation of the Gamma for a binary option with payoff function $f(x) = 1_{\{x \geq K\}}(x)$, $K = 400$.

FIGURE 4. *Left:* Simulation of Vega1 for a binary option with payoff function $f(x) = 1_{\{x \geq K\}}(x)$, $K = 400$. *Right:* Simulation of Vega2 for a binary option with payoff function $f(x) = 1_{\{x \geq K\}}(x)$, $K = 400$ using the perturbation function $u(t) = 1$.

to simulate integrals of the form

$$(5.1) \quad \exp(-\lambda t) \int_0^{\lambda t} \exp(s) dZ(s).$$

Letting ℓ be the Levy measure of $Z(1)$ we denote by ℓ^{-1} the inverse of the tail mass function ℓ^+ . Then integrals of the form (5.1) can be approximated as

$$\int_0^\lambda f(s) dZ(s) \stackrel{\mathcal{L}}{=} \sum_{i=1}^{\infty} \ell^{-1}(a_i/\lambda) f(\lambda r_i)$$

where a_i and r_i are two independent sequences of random variables with $r_i \sim \text{Unif}[0, 1]$ and $a_1 < a_2 < \dots < a_i < \dots$ being arrival times of a Poisson process with intensity 1. Hence we can simulate (5.1) by

$$(5.2) \quad \exp(-\lambda t) \int_0^{\lambda t} \exp(s) dZ(s) \stackrel{\mathcal{L}}{=} \exp(-\lambda t) \sum_{i=1}^{\infty} \ell^{-1}(a_i/\lambda) \exp(\lambda r_i).$$

For our choice of stationary distribution the explicit form of the inverse of the mass tail function is unknown, so we need to do a numerical inversion of ℓ^+ . It should be noticed that we need to truncate the infinite sum appearing in (5.2) and here it will be another trade-off between speed and accuracy. A numerical inversion using a search method must be done for each part of the sum. Since the sum appears in every time step, this is the time-consuming part of the algorithm. In practice this is too inefficient to be of any use. Instead we make a fine grid, invert it to get a numerical approximation of ℓ^{-1} and use linear interpolation to find the values. Avoiding the numerical inversion in each steps makes way for a remarkable speed-up, to the cost of a slight error in each estimate.

Our first example is a vanilla call option, depending only on the terminal value. The Malliavin method performs, as expected, best for Gamma, where the localised version is slightly better than the finite difference method, see Figure 1. The unlocalised Malliavin method proves to be comparable to or even worse than the finite difference method, something that has been reported previously, see Fournié *et al.* [16]. To really utilise the Malliavin method we need an option known to produce large variance when simulated with the finite difference method. One choice is a binary option with the discontinuous payoff function

$$\phi(x) = 1_{\{x \geq K\}}(x).$$

A similar option, considered in the original paper by Fournié *et al.*[16], is the option with payoff $1_{[a,b]}(x)$. The binary options are discontinuous, leading to a high variance if simulated with the finite difference method. At the same time we can not use the pathwise or the likelihood ratio method because of the discontinuity and the choice of model. Fortunately this is the kind of problem where the Malliavin method is most suited. As we see in Figure 2 and Figure 3 the unlocalised Malliavin simulation outperforms the finite difference method for both Delta and Gamma.

Interesting are the two different measures of the sensitivity with regards to the volatility in the BNS model. The first, Vega1, perturbs the initial value of the variance process while the other, Vega2, adds another stochastic process to the whole volatility process. The two measures are not equal and give different interpretations of the volatility sensitivity. For simplicity, the numerical tests assume that the perturbation function for Vega2 is constant equal to 1, *i.e.* $u(t) = 1$. The two different approaches give different results, but as we see in Figure 4, for the Binary option, the simulation using the Malliavin method is superior to the finite difference method in both cases.

Path dependent options provide some additional problems in the simulations. If we look at the requirements for Delta and Gamma in Proposition 4.3 we notice that the class of functions satisfying the restriction is rather small. One obvious choice is the function

$$a_1(t) = \frac{1_{\{t \in [0, t_1]\}}(t)}{t_1}.$$

We notice that using this function the weight will only depend on the first time period of the paths. Other possible functions are a_1 plus some periodic function with integral equal to zero on each interval (t_i, t_{i+1}) , $i = 1, 2, \dots$. Tests show that including a periodic or alternating function only adds more noise to the simulations, and in the results below we therefore used a_1 . Path dependent options are also more or less suitable for the Malliavin method. We implemented and simulated a few different options, including Asian options and different variants of barrier options, not reported with graphics here. Asian options show similar patterns as vanilla options; the Malliavin methods are comparable or inferior to the finite difference method except for the simulation of Gamma using the localised Malliavin method. For the Asian options the smoothness of the payoff function is similar to the vanilla options which is why the finite difference method performs reasonably well. The Malliavin method performs much better for a discontinuous option like a knock-in or knock-out option. We test a knock-out option, with vanilla call payoff function, having strike at $K = 380$ and a knock-out barrier at 400. In Figures 5 and 6 we see results from simulation of Gamma and Vega2 using the same parameters as above. We see again that the Malliavin method performs much better than the finite difference one.

REFERENCES

- [1] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, UK, 2004.
- [2] O. E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in economics. *J. R. Statist. Soc. B*, 63, Part 2:167–241, 2001.
- [3] F. E. Benth and M. Groth. The minimal entropy martingale measure and numerical option pricing for the Barndorff-Nielsen and Shephard stochastic volatility model. *Submitted*, 2006.
- [4] F. E. Benth, M. Groth, and C. Lindberg. The implied risk aversion from utility indifference option pricing in a stochastic volatility model. *Submitted*, 2007.
- [5] F. E. Benth and A. Løkka. Anticipative calculus for Lévy processes and stochastic differential equations. *Stoch. Stoch. Reports*, 76(3):191–211, 2004.
- [6] F. E. Benth and T. Meyer-Brandis. The density process of the minimal entropy martingale measure in a stochastic volatility model with jumps. *Finance Stoch.*, 9(4):563–575, 2005.
- [7] K. Bichteler, J.-B. Gravereaux, and J. Jacod. *Malliavin calculus for processes with jumps*. Gordon and Breach, New York, 1987.

- [8] M. Broadie and P. Glasserman. Estimating security price derivatives by simulation. *Management Science*, 42:269–285, 1996.
- [9] E. Carlen and E. Pardoux. Differential calculus and integration-by-parts on Poisson space. In *Stochastics, Algebra and Analysis in Classical and Quantum Dynamics*. Kluwer Acad. Publ., Dordrecht, 1990.
- [10] P. Carr and D. Madan. Option valuation using the fast Fourier transform. *J. Comput. Finance*, 2(4):61–73, 1998.
- [11] N. Chen and P. Glasserman. Malliavin Greeks without Malliavin calculus. *Preprint*, 2006.
- [12] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall, London, UK, 2003.
- [13] M. H. A. Davis and M. Johansson. Malliavin Monte Carlo Greeks for jump-diffusions. *Stoch. Proc. Appl.*, 116(1):101–129, 2006.
- [14] V. Debelley and N. Privault. Sensitivity analysis of European options in jump-diffusion models via the Malliavin calculus on the Wiener space. *Preprint*, 2004.
- [15] Y. El-Khatib and N. Privault. Computations of Greeks in a market with jumps via the Malliavin calculus. *Finance and Stochastics*, 8:161–179, 2004.
- [16] E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance and Stochastics*, 3:391–412, 1999.
- [17] I. Karatzas and S.E. Shreve. *Brownian motion and stochastic calculus*. Springer, 1991.
- [18] A. Kohatsu-Higa and M. Montero. Malliavin calculus in finance. In *Handbook of Computational Finance*. Birkhauser, 2004.
- [19] J. A. León, J. L. Solé, F. Utzet, and J. Vives. On Lévy processes, Malliavin calculus and market models with jumps. *Finance and Stochastics*, 6:197–225, 2002.
- [20] C. Lindberg. The estimation of a stochastic volatility model based in the number of trades. *Submitted*, 2006.
- [21] C. Lindberg. Portfolio optimization and a factor model in a stochastic volatility market. *Stochastics*, 78(5):259–279, 2006.
- [22] E. Nicolato and E. Venardos. Option pricing in stochastic volatility models of the ornstein-uhlenbeck type. *Mathematical Finance*, 13, No. 4:445–466, 2003.
- [23] D. Nualart. *The Malliavin Calculus and related topics*. Springer-Verlag, Berlin, 1995.
- [24] D. Nualart and J. Vives. Anticipative calculus for the Poisson process based on the Fock space. In *Séminaire de Probabilités XXIV 1998/99, Lecture Notes in Math. 1426*. Springer-Verlag, Berlin, 1990.
- [25] D. L. Ocone and I. Karatzas. A generalized Clark representation formula, with application to optimal portfolios. *Stochastics and Stochastics Reports*, 34:187–220, 1991.
- [26] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in C*. Cambridge University Press, Cambridge, 1992.
- [27] J. Rosinski. On a class of infinitely divisible processes represented as mixtures of Gaussian processes. In S. Cambanis, G. Samorodnitsky, and M.S. Taqqu, editors, *Stable Processes and Related Topics*, pages 27–41. Birkhäuser, Basel, 1991.
- [28] K. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge Press, Cambridge, 1999.
- [29] W. Schoutens. *Lévy processes in finance*. Wiley, 2003.

(Fred Espen Benth, Martin Groth and Olli Wallin)
 CENTRE OF MATHEMATICS FOR APPLICATIONS
 UNIVERSITY OF OSLO
 P.O. BOX 1053, BLINDERN
 N-0316 OSLO, NORWAY

E-mail address, fredb, martijg, olliw: @math.uio.no

FIGURE 5. Simulation of Gamma for a knock-out option with payoff function $f(x) = (S - K)^+$, $K = 380$ and knock-out boundary at 400.

FIGURE 6. Simulation of Vega2 for a knock-out option with payoff function $f(x) = (S - K)^+$, $K = 380$ and knock-out boundary at 400, using the perturbation function $u(t) = 1$.