

## AN EXTENSION OF THE CLARK-OCONE FORMULA UNDER CHANGE OF MEASURE FOR LÉVY PROCESSES

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ABSTRACT. The classical Clark-Ocone theorem states that any random variable  $F \in \mathbb{D}_{1,2} \subset L^2(\mathcal{F}_T, P)$  can be represented as

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t),$$

where  $\mathbb{E}[\cdot | \mathcal{F}_t]$  denotes the conditional expectation,  $D_t$  denotes the Malliavin derivative and  $W(\cdot)$  is a Brownian motion with canonical filtration  $\{\mathcal{F}_t\}_{t \in [0, T]}$ . Since many applications in financial mathematics require representation of random variables with respect to equivalent martingale measure, a change of measure version of this theorem was stated by Karatzas and Ocone [19]. In this paper we extend these results to be valid for a general Lévy martingale  $F \in L^2(\mathcal{F}_T, P)$  (rather than just  $W(t)$ ) and with general  $F \in L^2(\mathcal{F}_T, P)$ , using Malliavin calculus and white noise analysis. This extension is important for the applications in finance. As an application of our result we calculate explicitly the closest hedge strategy for the digital option whose payoff is  $F = \chi_{[H, K]}(S(T)) \notin \mathbb{D}_{1,2}$ , where the stock price  $S(\cdot)$  is driven by a Lévy process.

### INTRODUCTION

The representation of functionals of Brownian motion by stochastic integrals have been studied for many years. The first contribution for the explicit representation of the integrand in the martingale representation theorem was done by Clark [8]. He proved that the Fréchet differentiable functionals satisfying certain technical regularity conditions have an explicit representation as a stochastic integral in which the integrand consists of the conditional expectations of the Fréchet differential. Subsequently, Haussmann [15] showed the representation formula for the functionals of Itô processes where the proof is based on the results in [8]. Ocone [26] generalized the formula to the more general class of weakly differentiable functionals by using Malliavin calculus. Accordingly, the classical Clark-Ocone representation formula leads to:

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t),$$

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Received by the editors today.

*Key words and phrases.* Clark-Ocone formula, change of measure, Lévy processes, Malliavin calculus, white noise analysis.

The author wishes to express her thanks to Prof. Bernt Øksendal and Prof. David Applebaum for suggesting the problem. The author also acknowledges the many helpful suggestions of Prof. Giulia Di Nunno during the preparation of the paper.

for any Malliavin differentiable  $F \in \mathbb{D}_{1,2}$  where  $\mathbb{D}_{1,2} \subset L^2(\mathcal{F}_T, P)$  is given by

$$\mathbb{D}_{1,2} := \left\{ F = \sum_{n=0}^{\infty} I_n(f_n); \sum_{n=1}^{\infty} n n! \|f_n\|^2 < \infty \right\}$$

and  $t \rightarrow D_t F$  is the Malliavin derivative of  $F$ .

In the following years many achievements were obtained on the Clark-Ocone formula and the theory has been developed further either as analysis on the Wiener space or on white noise theory. The formula can be extended by replacing  $W(\cdot)$  with a general Lévy martingale or generalizing it to be valid for all  $F \in L^2(\mathcal{F}_T, P)$ . Aase *et al.* [1] extend this representation to the space of stochastic distributions using white noise approach. This extension plays an important role in financial mathematics including many interesting financial applications for random variables not belonging to the Stochastic Sobolev space,  $\mathbb{D}_{1,2}$ .

Løkka [21] proved the Clark-Ocone formula for the pure jump Lévy processes which are becoming increasingly important in many applications, and he showed that for any  $F \in \mathbb{D}_{1,2}$ ,

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{t,x} F | \mathcal{F}_t] \tilde{N}(dt, dx),$$

where  $\{\mathcal{F}_t\}_{t \in [0, T]}$  is the filtration generated by pure jump Lévy processes augmented with  $P$ -null sets,  $\tilde{N}(\cdot, \cdot)$  is the compensated Poisson random measure and  $D_{t,x}$  is the Malliavin derivative of a random variable in direction of  $\tilde{N}(dt, dx)$ . The white noise generalization of this formula for Lévy processes is studied by Di Nunno *et al.* [12].

For many applications in financial mathematics, we need to represent functionals with respect to equivalent martingale measure. In order to use the Clark-Ocone formula, it is important to modify it for the equivalent measure. This was established by Karatzas and Ocone [19] for Itô processes to give an explicit representation of the optimal portfolio in a market modeled by Brownian motion. Yolcu Okur [31] generalized the results of Karatzas and Ocone by using Gaussian white noise theory. It is proved in [31] that for any  $F \in L^2(\mathcal{F}_T, P) \supset \mathbb{D}_{1,2}$

$$F = \mathbb{E}_Q[F] + \int_0^T \mathbb{E}_Q[(D_t F - F \int_t^T D_t u(s) dW_Q(t)) | \mathcal{F}_t] dW_Q(t),$$

where  $D_t F$  is the Hida Malliavin derivative and  $W_Q(\cdot)$  is the Brownian motion under equivalent martingale measure  $Q$ . Huehne [17] extended the results of Karatzas and Ocone for pure jump Lévy processes belonging to  $\mathbb{D}_{1,2}$ .

In this paper, we finalize the results for the square integrable Lévy processes. It is important point to note here is that the formula we derived is valid for all random variables with finite second moment. As a main result showed in section 5, for any  $F \in L^2(\mathcal{F}_T, P) \supset \mathbb{D}_{1,2}$ ,

$$\begin{aligned} F &= \mathbb{E}_Q[F] + \int_0^T \mathbb{E}_Q[D_t F - F \int_t^T D_t u(s) dW_Q(s) | \mathcal{F}_t] dW_Q(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_Q[F(\tilde{H} - 1) + \tilde{H} D_{t,x} F | \mathcal{F}_t] \tilde{N}_Q(dt, dx), \end{aligned}$$

where

$$\begin{aligned} \tilde{H} = & \exp \left\{ \int_0^t \int_{\mathbb{R}_0} [D_{t,x}\theta(s,x) + \log(1 - \frac{D_{t,x}\theta(s,x)}{1 - \theta(s,x)})(1 - \theta(s,x))] \nu(dx) ds \right. \\ & \left. + \log(1 - \frac{D_{t,x}\theta(s,x)}{1 - \theta(s,x)}) \tilde{N}_Q(ds, dx) \right\} \end{aligned}$$

The paper is organized as follows. The first three sections constitute sufficient preparation. In section 4, we derive the Clark-Ocone formula under an equivalent probability measure for square integrable pure jump Lévy processes. In section 5, we extend this formula to square integrable general Lévy processes. The last section covers an application of this formula. We calculate explicitly the closest hedge strategy for a digital option by using the formula derived in section 5.

## 1. PRELIMINARIES

Malliavin calculus to processes with discontinuous trajectories was first studied by Bichteler, Gravereaux and Jacod [6]. However, at this study it is basically on the original problem of the smoothness of the densities of the solutions of the stochastic differential equations. Subsequently Kaminsky [18], Nualart and Vives ([24], [25]), Di Nunno *et al.* [11] work on the theory of Malliavin calculus for the stochastic processes with discontinuous trajectories. An extension of this theory to the white noise theory to discontinuous processes was introduced by Albeverio, Konradiev and Streit [2] and later generalized by Konradiev, Da Silva and Streit [9], Konradiev, Da Silva, Streit and Us [10], Di Nunno, Øksendal and Proske [12], Øksendal and Proske [28]. In this section, we will introduce some basic properties of Lévy processes and we will mention about the white noise analysis for Lévy processes. Many versions of these results have already been provided and known by many theorists but we think that it is better to have a unified approach. The general theory on Lévy processes can be found in [3], [5], [29] and [30].

**1.1. Lévy processes.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space equipped with the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ , where  $\mathcal{F}_t$  is the  $\sigma$ -algebra generated by 1-dimensional Lévy process. We assume that  $\mathcal{F}_0$  is augmented of all negligible sets.

**Definition 1.1.1.** A càdlàg, real valued process  $\eta(t), t \geq 0$  is a 1-dimensional Lévy process with  $\eta(0) = 0$   $P$ -a.s. if the following properties hold

- (i)  $\eta(t), t \geq 0$  has independent increments, i.e. for all  $0 \leq s < t$  the increment  $\eta(t) - \eta(s)$  is independent of  $\mathcal{F}_s$ ,
- (ii)  $\eta$  has stationary increments, i.e.  $\eta(t) - \eta(s)$  has the same probability law as  $\eta(s)$  for all  $0 \leq s < t$ ,
- (iii)  $\eta$  is stochastically continuous, i.e. for all  $t \geq 0$  and  $\epsilon > 0$   $\lim_{s \rightarrow t} P(|\eta(t) - \eta(s)| > \epsilon) = 0$ .

The jump at time  $t$  for this process is denoted by

$$\Delta\eta(t) = \eta(t) - \eta(t^-).$$

Define

- $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$
- $\mathbb{B}$  as the family of all Borel subsets  $U \in \mathbb{R}$  s.t. closure of  $U \subset \mathbb{R}_0$ .
- $N(t, U) := \sum_{0 \leq s \leq t} \chi_U(\Delta\eta(s))$  where  $U \in \mathbb{R}_0$ , which is the number of jump size  $\Delta\eta(s) \in U$  for any  $s$  in  $0 \leq s \leq t$ .

**Theorem 1.1.2. (Lévy-Khintchine formula [30])**

Let  $\eta = \eta(t), t \geq 0$  be a Lévy process. Then

$$(1.1.1) \quad \mathbb{E}[e^{iu\eta(t)}] = e^{i\Psi(u)}, \quad u \in \mathbb{R}, i^2 = -1$$

where

$$(1.1.2) \quad \Psi(u) := i\mu u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}_0} (e^{iux} - 1 - iux\chi_{\{|x|<1\}}) \nu(dx),$$

the parameters  $\mu \in \mathbb{R}$  and  $\sigma^2 \geq 0$  are constants and  $\nu(dx), x \in \mathbb{R}_0$ , is a  $\sigma$ -finite measure on  $\mathcal{B}(\mathbb{R}_0)$  satisfying

$$(1.1.3) \quad M := \int_{\mathbb{R}_0} \min(1, x^2) \nu(dx) < \infty.$$

The triplet  $(\mu, \sigma, \nu)$  is called the Lévy or characteristic triplet,  $\Psi$  is called the characteristic exponent and  $\nu = \nu(dx)$  is called the Lévy measure of  $\eta$  and it is defined as  $\nu(U) := \mathbb{E}[N(1, U)], U \in \mathcal{B}(\mathbb{R}_0)$ .

Conversely, given constants  $\mu, \sigma^2$  and a  $\sigma$ -finite measure  $\nu$  on  $\mathcal{B}(\mathbb{R}_0)$  satisfying the condition (1.1.3), there exists a Lévy process  $\eta(t), t \geq 0$  (unique in law) such that (1.1.1) and (1.1.2) hold.

Let us define the compensated jump measure  $\tilde{N}$  as follows:

$$(1.1.4) \quad \tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt.$$

Levy-Itô representation theorem states that there exist constants  $a_1$  and  $\sigma \in \mathbb{R}$  s.t.

$$\eta(t) = a_1 t + \sigma W(t) + \int_0^t \int_{|x|<1} x \tilde{N}(ds, dx) + \int_0^t \int_{|x|\geq 1} x N(ds, dx)$$

where  $B(t) = B(t, \omega), t \geq 0$  is a standard Brownian motion. In the present paper we will deal with square integrable Levy processes, i.e.,

$$\mathbb{E}[\eta^2(t)] < \infty$$

which is equivalent to

$$\int_{|x|\geq 1} |x|^2 \nu(dx) < \infty.$$

Then the representation turns to

$$\eta(t) = at + \sigma W(t) + \int_0^t \int_{\mathbb{R}_0} x \tilde{N}(ds, dx),$$

where

$$a = a_1 + \int_{|x|\geq 1} x \nu(dx).$$

In view of this, it becomes natural to define the stochastic differential equations of the the form

$$(1.1.5) \quad dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t) + \int_{\mathbb{R}_0} \gamma(t, X(t), x) \tilde{N}(dt, dx)$$

for the deterministic functions  $b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\gamma : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_0 \rightarrow \mathbb{R}$  satisfying certain growth conditions.

**Theorem 1.1.3.** (*1-dimensional Itô Formula* [29])

Let  $X = X(t), t \geq 0$  be the Itô-Lévy process given by (1.1.5) and let  $f : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be a function in  $C^{1,2}((0, \infty) \times \mathbb{R})$  and define

$$Y(t) := f(t, X(t)),$$

then the process  $Y(t), t \geq 0$  is also an Itô-Lévy process and

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t, X(t)) + \frac{\partial f}{\partial x}(t, X(t))[\mu(t, X(t))dt + \sigma(t, X(t))dW(t)] \\ &\quad + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X(t)) \sigma^2(t, X(t)) dt \\ &\quad + \int_{\mathbb{R}_0} \{f(t, X(t) + \gamma(t, X(t), x)) - f(t, X(t)) - \frac{\partial f}{\partial x}(t, X(t))\gamma(t, X(t), x)\} \nu(dx) dt \\ &\quad + \int_{\mathbb{R}_0} \{f(t, X(t^-) + \gamma(t, X(t^-), x)) - f(t, X(t^-))\} \tilde{N}(dt, dx). \end{aligned}$$

**1.2. White Noise Analysis for pure jump Lévy Processes.** In this subsection, we will define the white noise probability space and on this settlement we will construct square integrable pure jump Lévy process without drift by means of Bochner-Minlos-Sazanov theorem. For related works in the context of white noise theory for Lévy processes, see also [12], [22] and [28].

In the white noise theory for Lévy processes, the Lévy measure satisfies additional condition, say integrability condition which is stated as follows: for all  $\varepsilon > 0$  there exists  $\lambda > 0$  such that

$$(1.2.1) \quad \int_{|x| \geq \varepsilon} \exp(\lambda |x|) \nu(dx) < \infty.$$

This implies that

$$\int_{\mathbb{R}_0} |x|^i \nu(dx) < \infty, \quad i \geq 2.$$

and the characteristic function of Lévy process is analytic in the neighborhood of zero. Hence, Lévy process satisfying condition (1.2.1) has finite moments of order  $n$  for all  $n \geq 2$  ([23], [27]).

Let  $S(\mathbb{R})$  be the Schwartz space of test functions, i.e. the rapidly decreasing smooth functions  $\phi \in C^\infty(\mathbb{R})$  such that

$$\|\phi\|_{k,n} = \sup_{x \in \mathbb{R}} |\phi^{(j)}(x) x^n| < \infty \quad \text{for all } j \leq k, n \leq N.$$

$S(\mathbb{R})$  equip with the family of seminorm  $\|\cdot\|_{k,N}$  constitutes a Frechet space. Then  $S'(\mathbb{R})$  is the Schwartz space of tempered distributions equip with weak star topology and it is the dual of the space  $S(\mathbb{R})$ . Moreover, the action of  $\omega \in S'(\mathbb{R})$  on  $\phi \in S(\mathbb{R})$  is denoted by

$$\omega(\phi) := \langle \omega, \phi \rangle$$

**Theorem 1.2.1.** (*Bochner-Minlos-Sazanov* [12])

Let

$$F : \phi \rightarrow e^{\int_{\mathbb{R}_0} \Psi(\phi(x)) dx}, \quad \phi \in S(\mathbb{R})$$

be a given function. Then there exists a probability measure  $\mu$  on  $\omega = S'(\mathbb{R})$  s.t.

$$\int_{S'(\mathbb{R})} e^{i \langle \omega, \phi \rangle} d\mu(\omega) = \exp\left(\int_{\mathbb{R}} \Psi(\phi(x)) dx\right)$$

if and only if

- $F$  is positive definite on  $S(\mathbb{R})$ ,
- $F$  is continuous in Frechet topology.

Note that if we take  $\phi(\cdot)$  as the characteristic exponent of the pure jump Lévy process which satisfies the above conditions then there exists a probability measure on  $S'(\mathbb{R})$  by Bochner-Minlos-Sazanov theorem. This probability measure is called *Lévy white noise probability measure*.

**Lemma 1.2.2.** *Let  $\mu$  is the Lévy white noise probability measure. Then for all  $\phi \in S(\mathbb{R})$  we get*

$$\mathbb{E}[\langle \omega, \phi \rangle] := \int_{S'(\mathbb{R})} \langle \omega, \phi \rangle d\mu(\omega) = 0$$

and

$$\text{Var}_\mu[\langle \omega, \phi \rangle] := \mathbb{E}[\langle \omega, \phi \rangle^2] = \left( \int_{\mathbb{R}_0} x^2 \nu(dx) \right) \|\phi\|_{L^2(\mathbb{R})}^2,$$

where  $\|\phi\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \phi^2(x) dx$ .

*Proof.* We will give the sketch of the proof. Since  $\mu$  is the Lévy white noise probability measure then it satisfies the following:

$$\int_{S'(\mathbb{R})} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = \exp\left(\int_{\mathbb{R}} \Psi(\phi(x)) dx\right),$$

where

$$\Psi(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux) \nu(dx), \quad i^2 = -1.$$

Applying this equality to the function  $\phi(x) = tg(x)$  and by using Taylor expansions on both sides we get the properties.  $\square$

Hence, we can extend the definition of  $\langle \omega, \phi \rangle$  from  $\phi \in S(\mathbb{R})$  to  $\phi \in L^2(\mathbb{R})$  by using previous lemma and the fact that  $S(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ , i.e. if  $\phi \in L^2(\mathbb{R})$  choose  $\phi_n \in S(\mathbb{R})$  such that  $\phi_n \rightarrow \phi$  in  $L^2(\mathbb{R})$ . Then

$$\|\langle \omega, \phi_n \rangle - \langle \omega, \phi_m \rangle\|_{L^2(\mu)}^2 = \|\langle \omega, \phi_n - \phi_m \rangle\|_{L^2(\mu)}^2 = M \|\phi_n - \phi_m\|_{L^2(\mathbb{R})}^2 \rightarrow 0$$

So  $\{\langle \omega, \phi_n \rangle\}_{n \geq 1}$  is a Cauchy sequence in  $L^2(\mu)$  and since  $L^2(\mu)$  is complete space this sequence is convergent in  $L^2(\mu)$  and we denote this limit by  $\langle \omega, \phi \rangle$ , where  $\phi \in L^2(\mathbb{R})$ . Therefore, it is natural to define for all  $t \in \mathbb{R}$

$$\tilde{\eta}(t) := \eta(t, \omega) = \langle \omega, \chi_{[0,t]}(\cdot) \rangle$$

where

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq t \\ -1 & \text{if } t \leq s \leq 0 \text{ except } t=s=0 \\ 0 & \text{otherwise} \end{cases}$$

Since  $\tilde{\eta}(t), t \in \mathbb{R}$  has independent and stationary increments with  $\tilde{\eta}(0) = 0$ , then it has càdlàg modification, denoted by  $\eta(t, \omega)$  [3]. When  $t$  restricted to  $t \geq 0$ , it is a pure jump Lévy process with no drift equip with a Lévy measure  $\nu(\cdot)$ . From now on we will work with  $\eta(t, \omega)$  constructed on the white noise probability space. Note that  $\eta(t, \omega), t \in \mathbb{R}_+, \omega \in S'(\mathbb{R})$  can be regarded as a tempered distribution and for all  $\psi \in L^2(\mathbb{R})$ ,

$$\langle \omega, \psi \rangle = \int_0^\infty \psi(t) d\eta(t),$$

since  $\eta$  has the following stochastic integral representation

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} x \tilde{N}(ds, dx),$$

then

$$\langle \omega, \psi \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}_0} \psi(t) x \tilde{N}(dt, dx).$$

To sum up, from now on we will work on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mu)$  where  $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N}$  is the filtration with  $\mathcal{F}_\infty = \mathcal{F}$  and  $\mathcal{G}_t$  the  $\sigma$ -algebra generated by  $\eta$  up to time  $t$  and  $\mathcal{N}$  is the collection of all  $\mu$ -null sets.

## 2. CHAOS EXPANSIONS

In this section we will mention two different types of chaos expansions. For further details see [11], [12] and [28]. Here are some elementary properties of these concepts. First, we will construct the Chaos representation via special functions which are orthonormal basis in  $L^2(\mu)$ . Since we assume that  $\nu(dx)$  satisfies the condition (1.2.1), the polynomials are dense in  $L^2(\rho)$ , where

$$d\rho(x) = x^2 \nu(dx).$$

Let  $\{l_0, l_1, \dots\}$  with  $l_0 = 1$  be the orthogonalization of  $\{1, x, x^2, \dots\}$  with respect to inner product of  $L^2(\rho)$ . Define

$$(2.0.2) \quad p_j(x) = \|l_{j-1}\|_{L^2(\rho)}^{-1} x l_{j-1}(x); \quad j = 1, 2, \dots$$

Then  $\{p_j(x)\}_{j=1}^\infty$  is an orthonormal basis for  $L^2(\nu)$ . In particular, note that

$$\begin{aligned} p_1(x) &= \|l_0\|_{L^2(\rho)}^{-1} x l_0(x) \\ &= \frac{x}{\left(\int_{\mathbb{R}_0} x^2 \nu(dx)\right)^{\frac{1}{2}}}. \end{aligned}$$

Therefore,  $x = p_1(x)M$  where  $M$  is defined in equation (1.1.3).

Let  $\{e_i(t)\}_{i \geq 0}$  be Hermite functions on  $\mathbb{R}$ . These functions have two important properties related with this content. First, for all  $i \geq 0$ ,  $e_i \in S(\mathbb{R})$  and the second property is they constitute an orthonormal basis for  $L^2(\mathbb{R})$ . Define the bijective map  $\kappa : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  as follows:

$$(2.0.3) \quad \kappa : (i, j) \rightarrow j + \frac{(i+j-2)(i+j-1)}{2},$$

where it is so-called *Cantor diagonalization* of the Cartesian product  $\mathbb{N} \times \mathbb{N}$ . Set  $\epsilon^{(k)}$  as a vector with 1 at the  $k$ th place, 0 otherwise. To simplify the notation we call  $\mathcal{I}$  the set of multi-indexes  $\alpha = (\alpha_1, \alpha_2, \dots)$  which have only finitely many non-zero values and for each  $\alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . We denote  $Index(\alpha) = \max\{n : \alpha_n \neq 0\}$  and  $|\alpha| = \sum_n \alpha_n$ , for  $\alpha \in \mathcal{I}$ .

For  $k = \kappa(i, j)$  define

$$\delta_k(t, x) := e_i(t)p_j(x), \quad t \in \mathbb{R}, x \in \mathbb{R}_0,$$

where the tensor product of  $\delta$  for any  $\alpha \in \mathcal{I}$  with  $\text{Index}(\alpha) = n$  and  $|\alpha| = m$  is defined by

$$\begin{aligned} \delta^{\otimes \alpha} &= \delta_1^{\otimes \alpha_1} \otimes \cdots \otimes \delta_n^{\otimes \alpha_n}((t_1, x_1), \dots, (t_m, x_m)) \\ &:= \underbrace{\delta_1(t_1, x_1) \cdots \delta_1(t_{\alpha_1}, t_{x_{\alpha_1}})}_{\alpha_1 \text{ factors}} \cdots \underbrace{\delta_n(t_{m-\alpha_n+1}, x_{m-\alpha_n+1}) \cdots \delta_n(t_m, x_m)}_{\alpha_n \text{ factors}}, \end{aligned}$$

with  $\delta_k^{\otimes 0} = 1$ . Then the symmetrized tensor product of  $\delta_k$ 's is given by

$$\delta^{\hat{\otimes} \alpha} = \delta_1^{\hat{\otimes} \alpha_1} \hat{\otimes} \cdots \hat{\otimes} \delta_n^{\hat{\otimes} \alpha_n}((t_1, x_1), \dots, (t_m, x_m)).$$

For  $\alpha \in \mathcal{I}$  define

$$K_\alpha := I_{|\alpha|}(\delta^{\hat{\otimes} \alpha}),$$

where  $I_0(\delta^{\hat{\otimes} 0}) = 1$ .

**Theorem 2.0.3. (Chaos Expansion I [11])**

The family  $\{K_\alpha\}_{\alpha \in \mathcal{I}}$  constitutes an orthogonal basis for  $L^2(\mu)$  with norm expression

$$(2.0.4) \quad \|K_\alpha\|_{L^2(\mu)}^2 = \alpha! := \alpha_1! \alpha_2! \cdots .$$

Therefore, for every  $F \in L^2(\mu)$  has the unique representation

$$F = \sum_{\alpha \in \mathcal{I}} c_\alpha K_\alpha,$$

where  $c_\alpha \in \mathbb{R}$  for all  $\alpha \in \mathcal{I}$  and we set  $c_0 = \mathbb{E}[F]$ . Moreover, we have the isometry

$$\|F\|_{L^2(\mu)}^2 = \sum_{\alpha \in \mathcal{I}} \alpha^2 c_\alpha^2.$$

**Example 2.0.4.** The chaos expansion of  $F = \eta(t)$  is as follows

$$\eta(t) = \sum_{k=1}^{\infty} M \left( \int_0^t e_k(s) ds \right) K_{e^{\kappa(k,1)}}$$

where  $M$  is defined in (1.1.3).

Let us define

$$S_n := \{(t_1, x_1, t_2, x_2, \dots, t_n, x_n) \mid 0 \leq t_1 \leq \dots \leq t_n \leq T, x_i \in \mathbb{R}_0, i = 0, 1, \dots, n\},$$

and  $L^2(S_n)$  be the set of all real functions  $g$  on  $S_n$  such that

$$\|g\|_{L^2(S_n)}^2 := \int_{S_n} g^2(t_1, x_1, \dots, t_n, x_n) dt_1 \nu(dx_1) dt_2 \nu(dx_2) \cdots dt_n \nu(dx_n) < \infty.$$

**Definition 2.0.5.** For any  $g \in L^2(S_n)$  the  $n$ -fold iterated integral  $J_n(g)$  is the random variable in  $L^2(\mu)$  defined as

$$J_n(g) := \int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} g(t_1, x_1, \dots, t_n, x_n) \tilde{N}(dt_1, dx_1) \cdots \tilde{N}(dt_n, dx_n),$$

where  $J_0(g) = g$  for any  $g \in \mathbb{R}$ .



Note that for any  $g$  in the set of symmetric deterministic real functions, say  $\in \hat{L}^2((\lambda \times \nu)^n)$ , where  $\lambda(dt) = dt$  denotes Lebesgue measure, there exists the following equality;

$$\|g\|_{L^2((\lambda \times \nu)^n)}^2 = n! \|g\|_{L^2(S_n)}^2.$$

It is convenient to define  $n$ -fold iterated integral for  $f \in \hat{L}^2((\lambda \times \nu)^n)$  as follows:

$$(2.0.5) \quad I_n(f) := \int_{([0, T]^n \times \mathbb{R}_0)^n} f(t_1, x_1, \dots, t_n, x_n) \tilde{N}^{\otimes n}(dt, dx) = n! J_n(f),$$

where  $\tilde{N}^{\otimes n}(dt, dx) = \tilde{N}(dt_1, dx_1) \cdots \tilde{N}(dt_n, dx_n)$ .

**Theorem 2.0.6. (Chaos Expansion II [11])**

Let  $F(\omega)$  be an  $\mathcal{F}_T$  measurable random variable in  $L^2(\mu)$ , where

$$L^2(\mu) = \{F : \Omega = S'(\mathbb{R}) \rightarrow \mathbb{R} \mid \|F\|_{L^2(\mu)}^2 := \int_{\Omega} F^2(\omega) d\mu(\omega) < \infty\}.$$

Then there exists a unique sequence  $\{f_n\}_{n=0}^{\infty}$  of functions  $f_n \in \hat{L}^2((\lambda \times \nu)^n)$ , such that

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where  $I_n$  is defined in the equation (2.0.5). Moreover, we have the isometry

$$\|F\|_{L^2(\mu)}^2 = \sum_{n=0}^{\infty} n! \|f_n\|_{L^2((\lambda \times \nu)^n)}^2$$

*Remark 2.0.7.* By Wiener-Itô chaos expansion II, any sequence of functions  $f_n \in \hat{L}^2((\lambda \times \nu)^n)$ , such that  $\sum_{n=0}^{\infty} n! \|f_n\|_{L^2((\lambda \times \nu)^n)}^2 < \infty$  defines a random variable  $F \in L^2(\mu)$ . Then each  $f_n$  is contained in the closure of the linear span of the orthogonal family  $\{\delta^{\otimes \alpha}\}_{|\alpha|=n}$  in  $L^2((\lambda \times \nu)^n)$  for all  $\alpha \in \mathcal{I}$  [12]. Therefore, we obtain

$$f_n = \sum_{|\alpha|=n} c_{\alpha} \delta^{\otimes \alpha}.$$

### 3. STOCHASTIC TEST FUNCTIONS AND DISTRIBUTION SPACES

The space of stochastic test functions,  $\mathcal{G} = \mathcal{G}(\mu)$ , and stochastic distributions  $\mathcal{G}^* = \mathcal{G}^*(\mu)$  have been studied for Lévy processes by [11], [12] and [28]. In this section, we will introduce and prove the relevant properties of these spaces.

**Definition 3.0.8.**

(i) Let  $k \in \mathbb{N}_0$ . The space  $\mathcal{G}_k$  consists of all formal expansions

$$f = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mu)$$

such that

$$(3.0.6) \quad \|f\|_{\mathcal{G}_k}^2 := \sum_{n=0}^{\infty} n! \|f_n\|_{L^2((\lambda \times \nu)^n)}^2 e^{2kn} < \infty.$$

Define

$$\mathcal{G} = \mathcal{G}(\mu) = \bigcap_{k \in \mathbb{N}_0} \mathcal{G}_k(\mu)$$

equip with the projective topology (i.e. as  $n$  goes to infinity,  $f_n \rightarrow f$  in  $\mathcal{G}$  iff  $\|f_n - f\|_{\mathcal{G}_k} \rightarrow 0$ , for all  $k \in \mathbb{N}_0$ ).

(ii) For  $F = \sum_{n=0}^{\infty} I_n(g_n) \in L^2(\mu)$  belongs to the space  $\mathcal{G}_{-q}$ , for  $q \in \mathbb{N}_0$  if

$$\|F\|_{\mathcal{G}_{-q}}^2 := \sum_{n=0}^{\infty} n! \|g_n\|_{L^2((\lambda \times \nu)^n)}^2 e^{-2qn} < \infty.$$

Define

$$\mathcal{G}^* = \mathcal{G}^*(\mu) = \bigcup_{q \in \mathbb{N}_0} \mathcal{G}_{-q}(\mu)$$

equip with inductive topology (i.e. as  $n$  goes to infinity,  $F_n \rightarrow F$  in  $\mathcal{G}^*$  iff there exists  $q$  such that  $\|F_n - F\|_{\mathcal{G}_{-q}} \rightarrow 0$ ).

$\mathcal{G}^*$  is the dual of  $\mathcal{G}$ , and the action of  $F = \sum_{n \geq \infty} I_n(g_n) \in \mathcal{G}^*$  on  $f = \sum_{n \geq \infty} I_n(f_n) \in \mathcal{G}$  is

$$(3.0.7) \quad \langle F, f \rangle_{\mathcal{G}, \mathcal{G}^*} = \sum_{n=0}^{\infty} n! \langle f_n, g_n \rangle_{L^2((\lambda \times \nu)^n)}.$$

We will also introduce Hida test function space and Hida stochastic distribution space. For more information we refer to [16], [27].

**Definition 3.0.9.**

(i) Let  $0 \leq \rho \leq 1$ .  $f = \sum_{\alpha \in \mathcal{I}} c_\alpha K_\alpha \in L^2(\mu)$  belongs to the Hilbert space  $(S)_{\rho, k}$  for  $k \in \mathbb{N}_0$  if

$$\|f\|_{\rho, k}^2 := \sum_{\alpha \in \mathcal{I}} c_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha} < \infty$$

where

$$(3.0.8) \quad (2\mathbb{N})^{k\alpha} = \prod_{i=1}^m (2i)^{k\alpha_i}, \text{ for } \alpha = (\alpha_1, \dots, \alpha_m) \in \mathcal{I}.$$

then define  $(S)_\rho$  as

$$(S)_\rho := \bigcap_{k \in \mathbb{N}} (S)_{\rho, k},$$

with projective topology.

(ii) Similarly, for an expansion  $F = \sum_{\alpha \in \mathcal{I}} a_\alpha K_\alpha \in L^2(\mu)$  define the seminorm

$$\|F\|_{-\rho, -k}^2 := \sum_{\alpha \in \mathcal{I}} a_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-k\alpha}, \quad k \in \mathbb{N}_0,$$

where  $(2\mathbb{N})^\alpha$  defined in (3.0.8). Let  $(S)_{-\rho, -k} := \{F : \|F\|_{-\rho, -k} < \infty\}$  and define  $(S)_{-\rho}$  as

$$(S)_{-\rho} := \bigcup_{k \in \mathbb{N}_0} (S)_{-\rho, -k},$$

with inductive topology.

Note that  $(S)_{-\rho}$  is the dual of  $(S)_\rho$  and one can define the action of  $F = \sum_{\alpha \in \mathcal{I}} b_\alpha H_\alpha \in (S)_{-\rho}$  on  $f = \sum_{\alpha \in \mathcal{I}} a_\alpha H_\alpha \in L^2(P) \in (S)_\rho$  as follows:

$$(3.0.9) \quad \langle F, f \rangle = F(f) = \sum_{\alpha \in \mathcal{I}} a_\alpha b_\alpha \alpha!$$

From the definition of these spaces, we can easily extract the following inclusions for  $0 \leq \rho \leq 1$ .

$$(S)_1 \subset (S)_\rho \subset (S)_0 \subset L^2(\mu) \subset (S)_{-0} \subset (S)_{-\rho} \subset (S)_{-1}.$$

The spaces  $(S) := (S)_0$  and  $(S)^* := (S)_{-\rho}$  are the Lévy version of the *Hida test function space* and the *Hida distribution space* respectively. For arbitrary  $\rho \in [0, 1]$  the spaces are called *Kondratiev spaces*.

**3.1. Malliavin derivative and some properties.** In this section we will introduce the properties of Malliavin derivative and some properties of it which will be used in the proof of Clark-Ocone formula under change of measure for Lévy processes.

**Definition 3.1.1.** For any  $F = \sum_{\alpha \in \mathcal{I}} c_\alpha K_\alpha \in (S)^*$ , the *Hida Malliavin derivative*  $D_{t,x}F$  is defined as

$$(3.1.1) \quad D_{t,x}F := \sum_{\alpha \in \mathcal{I}} c_\alpha \sum_{i,j=1}^{\infty} \alpha_{\kappa(i,j)} e_i(t) p_j(x) K_{\alpha - \epsilon^{\kappa(i,j)}}, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}_0$$

where  $\kappa$  is defined in (2.0.3),  $\{e_i(t)\}_{i \geq 0}$  are the Hermite functions on  $\mathbb{R}$  and  $p(x)$  is the polynomial defined in (2.0.2).

**Theorem 3.1.2. (Chain rule via Wick product [13])** *Let  $f \in (S)^*$  and let  $g(x) = \sum_{n \geq 0} a_n x^n$  be an analytic function. Then*

$$\sum_{n \geq 0} a_n F^{\diamond n} \text{ is convergent in } (S)^*.$$

Moreover, for  $g^\diamond(F) = \sum_{n \geq 0} a_n F^{\diamond n}$ ,

$$D_{t,x}g^\diamond(F) = \left(\frac{\partial g}{\partial x}\right)^\diamond(F) \diamond (D_{t,x}F).$$

Suppose  $F \in L^2(\mu) \subset (S)^*$  is Malliavin differentiable. Let  $\varphi \in C'(\mathbb{R})$ . Then  $\varphi(F)$  is Malliavin differentiable and

$$(3.1.2) \quad D_{t,x}(\varphi(F)) = \varphi(F + D_{t,x}F) - \varphi(F).$$

*Remark 3.1.3.* Note that the product rule for  $F_1, F_2 \in L^2(\mu) \subset (S)^*$  is

$$\begin{aligned} D_{t,x}(F_1 F_2) &= (F_1 + D_{t,x}F_1)(F_2 + D_{t,x}F_2) - F_1 F_2 \\ &= F_1 D_{t,x}F_2 + F_2 D_{t,x}F_1 + D_{t,x}F_1 D_{t,x}F_2 \end{aligned}$$

The following lemma proved in [1] will be used in further proofs.

**Lemma 3.1.4.**

- (i) Suppose  $F \in \mathcal{G}^*$ . Then  $D_{t,x}F \in \mathcal{G}^*$  for a.a.  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}_0$ .
- (ii) Suppose  $F, F_n \in \mathcal{G}^*$  for all  $n \in \mathbb{N}$  and

$$F_n \rightarrow F \text{ in } \mathcal{G}^*.$$

Then there exists a subsequence  $\{F_{n_k}\}_{k \geq 1}$  such that

$$D_{t,x}F_{n_k} \rightarrow D_{t,x}F \text{ in } \mathcal{G}^*,$$

for almost all  $t > 0$ .

The following theorem plays an important role for the proof of the formula that we will derive in the next section. It is one of the important result obtained in this paper.

**Theorem 3.1.5.** *Let  $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mu)$ . Then  $D_{t,x}F \in \mathcal{G}^*$  and it is defined as*

$$(3.1.3) \quad D_{t,x}F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t, x)),$$

where  $f_n(\cdot, t, x)$  is  $(n-1)$ -fold iterated integral defined in (2.0.5) of  $f_n$  regarded as a function of its  $(n-1)$  first pairs of variables  $(t_1, x_1), \dots, (t_{n-1}, x_{n-1})$ , while the last pair  $(t, x)$  kept as a parameter.

*Proof.* Let  $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\mu)$ . Define  $F_m = \sum_{n=0}^m I_n(f_n)$ . Then  $F_m \rightarrow F$  in  $L^2(\mu)$  which implies  $F_m \rightarrow F$  in  $\mathcal{G}^*$ . By Lemma 3.1.4, there exists a subsequence  $F_{m_k}$  such that  $D_{t,x}F_{m_k} \rightarrow D_{t,x}F$  in  $\mathcal{G}^*$ , i.e.,

$$\sum_{n=0}^{m_k} nI_{n-1}(f_n(\cdot, t, x)) \rightarrow D_{t,x}F$$

in  $\mathcal{G}^*$ . We want to prove that

$$\sum_{n=0}^{m_k} nI_{n-1}(f_n(\cdot, t, x)) \rightarrow \sum_{n=0}^{\infty} nI_{n-1}(f_n(\cdot, t, x))$$

in  $\mathcal{G}^*$  as  $m_k$  goes to infinity. Consider

$$\left\| \sum_{n=m_{k+1}}^{\infty} nI_{n-1}(f_n(\cdot, t, x)) \right\|_{\mathcal{G}_{-q-1}}^2 = \sum_{n=m_{k+1}}^{\infty} (n-1)! n^2 \|f_n(\cdot, t, x)\|_{L^2((\lambda \times \nu)^{n-1})}^2 e^{-2q(n-1)}.$$

Then taking the integral of both sides with respect to Lebesgue and Lévy measure, we get the following

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \left\| \sum_{n=m_{k+1}}^{\infty} nI_{n-1}(f_n(\cdot, t, x)) \right\|_{\mathcal{G}_{-q}}^2 \nu(dx) dt &= \sum_{n=m_{k+1}}^{\infty} n e^{-2q(n-1)} n! \|f_n(\cdot, t, x)\|_{L^2((\lambda \times \nu)^n)}^2 \\ &\leq K \sum_{n=m_{k+1}}^{\infty} n! \|f_n\|_{L^2((\lambda \times \nu)^n)}^2 \\ &= K \|F\|_{L^2(\mu)}^2 < \infty \end{aligned}$$

for some constant  $K$ . □

The following lemma plays a crucial rule for proving the fundamental theorem of calculus for any random variable  $F \in L^2(\mu)$ .

**Lemma 3.1.6.**

- (i) *Let  $G \in (S)^*$ . Then  $D_{t,x}G \in (S)^*$  for a.a.  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}_0$ .*
- (ii) *Suppose  $G, G_n \in (S)^*$  for all  $n \in \mathbb{N}$  and*

$$G_n \rightarrow G \text{ in } (S)^*.$$

*Then there exist a subsequence  $\{G_{n_k}\}_{k \geq 1}$  such that*

$$D_{t,x}G_{n_k} \rightarrow D_{t,x}G \text{ in } (S)^*,$$

for almost all  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}_0$ .

*Proof.* Let  $G = \sum_{\alpha \in \mathcal{I}} c_\alpha K_\alpha \in (S)^*$ . Then there exists a  $q < \infty$  such that

$$\begin{aligned} \|G\|_{-\rho, -q}^2 &:= \sum_{\alpha \in \mathcal{I}} c_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} c_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-\alpha q} < \infty. \end{aligned}$$

Note that by Definition 3.1.1 Hida-Malliavin derivative of  $G$  is as follow:

$$\begin{aligned} D_{t,x}G(\omega) &= \sum_{\alpha \in \mathcal{I}} c_\alpha \sum_{i,j=1}^{\infty} \alpha_{\kappa(i,j)} e_i(t) p_j(x) K_{\alpha - \epsilon^\kappa(i,j)} \\ &= \sum_{\beta} \left( \sum_{i,j} c_{\beta^\kappa(i,j)} (\beta_{\kappa(i,j)} + 1) e_i(t) p_j(x) \right) K_\beta(\omega) \\ &= \left( \sum_i c_{\beta+\epsilon(i)} (\beta_i + 1) e_i(t) \right) K_\beta(\omega) \\ &:= \sum_{\beta} g_\beta(t, x) K_\beta(\omega), \end{aligned}$$

where  $g_\beta(\omega)(t, x) = \sum_{i,j} c_{\beta+\epsilon^\kappa(i,j)} (\beta_{\kappa(i,j)} + 1) e_i(t) p_j(x)$ .

We want to prove that for some  $q \in \mathbb{N}_0$

$$\|D_{t,x}G\|_{-\rho, -q-1}^2 := \sum_{n=0}^{\infty} \left( \sum_{|\beta|=n} g_\beta^2(\beta!)^{1-\rho} \right) (2\mathbb{N})^{-\beta(q+1)} < \infty \text{ for a.a. } t, x.$$

Since  $\{p_j(x)\}_{j=1}^{\infty}$  and  $\{e_i(t)\}_{i=1}^{\infty}$  are orthonormal basis for  $L^2(\nu)$  and  $L^2(\mathbb{R})$  respectively,

$$\int_{\mathbb{R}} \int_{\mathbb{R}_0} g_\beta^2(t, x) \nu(dx) dt = \sum_{\beta} c_{\beta+\epsilon^\kappa(i,j)}^2 (\beta_{\kappa(i,j)} + 1)^2.$$

Moreover,

$$\begin{aligned} (2\mathbb{N})^{-\beta q} < (2\mathbb{N})^{-\beta} &= \prod_i (2 \cdot i)^{-\beta_i} \\ &\leq \prod_i e^{-\beta_i (\log 2)} = e^{-|\tilde{\beta}|} \end{aligned}$$

where  $\tilde{\beta}_i = (\log 2) \beta_i$  for all  $i \in \mathcal{I}$ . Hence,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}_0} \|D_{t,x}G\|_{-\rho, -q-1}^2 \nu(dx) dt &= \sum_{\beta} \left( \sum_{i,j} c_{\beta+\epsilon^\kappa(i,j)}^2 (\beta_{\kappa(i,j)} + 1)^2 \right) (\beta!)^{1-\rho} (2\mathbb{N})^{-\beta(q+1)} \\ &= \sum_{\beta, i, j} (\beta_{\kappa(i,j)} + 1) (\beta!)^{-\rho} (2\mathbb{N})^{-\beta(q+1)} \sum_{\alpha, |\alpha|=|\beta|+1} c_\alpha^2 \alpha! \\ &< \sum_n \sum_{|\tilde{\beta}|=n} (n+1) e^{-n} \sum_{|\alpha|=(\log 2)^{-1}n+1} c_\alpha^2 \alpha! (2\mathbb{N})^{-\alpha q} \end{aligned}$$

Using the fact that  $(n+1)e^{-n} \leq 1$  for all  $n$ , we get

$$(3.1.4) \quad \int_{\mathbb{R}} \int_{\mathbb{R}_0} \|D_{t,x}G\|_{-\rho,-q-1}^2 \nu(dx) dt < \sum_n \left( \sum_{|\alpha|=(\log 2)^{-1}n+1} c_\alpha^2 \alpha! \right) (2\mathbb{N})^{-\alpha q} \\ \leq \|G\|_{-\rho,-q} < \infty$$

Therefore,  $D_{t,x}G \in (S)_{-\rho,-q-1} \subset (S)^*$  for a.a.  $t, x$ .

(ii) To prove this part, it suffices to prove that if  $G_n \rightarrow 0$  in  $(S)_{-\rho,-q}$ , then there exist a subsequence  $\{G_{n_k}\}_{k \geq 1}$  such that  $D_{t,x}G_{n_k} \rightarrow 0$  in  $(S)^*$  as  $k$  goes to infinity, for a.a.  $t, x$ . We have proved that

$$(3.1.5) \quad \int_{\mathbb{R}} \int_{\mathbb{R}_0} \|D_{t,x}G_n\|_{-\rho,-q-1}^2 \nu(dx) dt \leq \|G_n\|_{-\rho,-q}^2 \rightarrow 0.$$

Therefore,

$$\|D_{t,x}G_n\|_{-\rho,-q-1} \rightarrow 0 \quad \text{in } L^2(\mathbb{R}).$$

So, there exists a subsequence  $\{\|D_{t,x}G_{n_k}\|_{-\rho,-q-1}\}_{k \geq 1}$  such that  $\|D_{t,x}G_{n_k}\|_{-\rho,-q-1} \rightarrow 0$  for a.a.  $t, x$  as  $k \rightarrow \infty$ .  $\square$

### 3.2. Conditional expectation under $\mathcal{G}^*$ .

**Definition 3.2.1.** ([12]) Let  $F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{G}^*$ . Then the conditional expectation of  $F$  with respect to the filtration  $\mathcal{F}_t$  defined by

$$\mathbb{E}[F | \mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(f_n \cdot \chi_{[0,t]^n}).$$

Note that this coincides with the usual expectation if  $F \in L^2(\mu) \subset \mathcal{G}^*$ . Moreover, if  $F \in \mathcal{G}^*$  and since  $\|\mathbb{E}[F | \mathcal{F}_t]\|_{\mathcal{G}_r} \leq \|F\|_{\mathcal{G}_r}$  for all  $r \in \mathbb{N}$ , then  $\mathbb{E}[F | \mathcal{F}_t] \in \mathcal{G}^*$ .

## 4. CLARK-OCONE FORMULA UNDER CHANGE OF MEASURE FOR PURE JUMP LÉVY PROCESSES

After settlements we are ready to prove the Clark-Ocone formula under change of measure for pure jump Lévy processes. First, we will use the Girsanov theorem introduced in Øksendal and Sulem [29] in order to define the equivalent white noise probability measure. Second, Clark-Ocone formula for pure jump Lévy processes will be used. Then, by using some auxiliary theorems we will show the Clark-Ocone formula under change of measure for square integrable pure jump Lévy processes with no drift.

### Theorem 4.0.2. (Girsanov theorem for pure jump Lévy Processes [29])

For a given process  $\theta(s, x) \leq 1$ , let

(4.0.1)

$$Z(t) = \exp \left\{ \int_0^t \int_{\mathbb{R}_0} \{ \log(1 - \theta(s, x)) + \theta(s, x) \} \nu(dx) ds + \int_0^t \int_{\mathbb{R}_0} \log(1 - \theta(s, x)) \tilde{N}(ds, dx) \right\}$$

exists for all  $0 \leq t \leq T$ . Define a measure  $Q$  on  $\mathcal{F}_T$  by

$$dQ(w) = Z(T)d\mu(w),$$

where  $\mu$  is the pure jump Lévy white noise probability measure defined via Bochner-Minlos-Sazanov theorem. Assume that

$$\mathbb{E}_\mu[Z(T)] = 1.$$

Then  $Q$  is a probability measure on  $\mathcal{F}_T$  and

$$(4.0.2) \quad \tilde{N}_Q(dt, dx) = \theta(t, x)\nu(dx)dt + \tilde{N}(dt, dx)$$

is a  $Q$ -compensated Poisson random measure of  $N(\cdot, \cdot)$ . We can also write the  $Q$ -compensated process as follows;

$$(4.0.3) \quad \tilde{N}_Q(dt, dx) = N(dt, dx) - (1 - \theta(t, x))\nu(dx) dt$$

Moreover,

$$M(t) := \int_0^t \int_{\mathbb{R}_0} \gamma(s, x)\tilde{N}_Q(ds, dx)$$

is a local  $Q$ -martingale, for all predictable processes  $\gamma(s, x)$  s.t.

$$\int_0^T \int_{\mathbb{R}_0} (\gamma(s, x)\theta(s, x))^2 \nu(dx)ds < \infty \quad a.s.$$

*Proof.* Proof and detail information can be found in [29].  $\square$

**Theorem 4.0.3. (Clark-Ocone formula [11])**

Let  $F \in L^2(\mu)$  be  $\mathcal{F}_T$ -measurable. Then

$$\mathbb{E}[D_{t,x}F | \mathcal{F}_t] \in L^2(\lambda \times \nu \times \mu)$$

for all  $t \in [0, T]$ ,  $x \in \mathbb{R}_0$  and

$$(4.0.4) \quad F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{t,x}F | \mathcal{F}_t] \tilde{N}(dt, dx).$$

*Proof.* The proof can be found in Di Nunno *et al.* [11].  $\square$

**Definition 4.0.4.** A function  $\psi(t, x) : \mathbb{R} \times \nu \rightarrow (S)^*$  is  $(S)^*$ -integrable if

$$< \psi(\cdot), \phi > \in L^1(\mathbb{R} \times \nu)$$

for all  $\phi \in (S)$  and the action is defined in (3.0.9). Then the  $(S)^*$ -integral of  $\psi(t, x)$ , denoted by  $\int_{\mathbb{R}} \int_{\mathbb{R}_0} \psi(t, x)\nu(dx)dt$  is the unique element in  $(S)^*$  such that

$$< \int_{\mathbb{R}} \int_{\mathbb{R}_0} \psi(t, x)\nu(dx)dt, \phi > = \int_{\mathbb{R}} \int_{\mathbb{R}_0} < \psi(t, x), \phi > \nu(dx)dt,$$

for all  $\phi \in (S)$ .

**Definition 4.0.5.** Let  $u_n$ ,  $n \in \mathbb{Z}$  be a sequence of Skorohod integrable random fields such that

$$u = \lim_{n \rightarrow \infty} u_n$$

in  $L^2(\mu \times \lambda \times \nu)$ . Then the Skorohod integral of  $F$  is defined as follows

$$\delta(u) := \int_{\mathbb{R}} \int_{\mathbb{R}_0} u(t, x)\tilde{N}(\delta t, dx) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \int_{\mathbb{R}_0} u_n(t, x)\tilde{N}(\delta t, dx),$$

if the limit exists in  $L^2(\mu)$ .

**Theorem 4.0.6. (Fundamental theorem of stochastic calculus)**

Let  $u \in L^2(\mu \times \lambda \times \nu)$  and assume that  $(s, x) \rightarrow D_{t,x}u(s, x)$  is  $(S)^*$ -integrable, for all  $s \in \mathbb{R}$ ,  $x \in \mathbb{R}_0$ ,  $\mathbb{E}[\int_{\mathbb{R}} \int_{\mathbb{R}_0} (\int_{\mathbb{R}} \int_{\mathbb{R}_0} D_{t,x}(u(s, x))\tilde{N}(\delta s, dx))^2 \nu(dx)ds] < \infty$  then  $\delta(u) = \int_{\mathbb{R}} u(s, x)\tilde{N}(\delta s, dx) \in L^2(\mu)$  and

$$(4.0.5) \quad D_{t,x}(\int_{\mathbb{R}} \int_{\mathbb{R}_0} u(s, x)\tilde{N}(\delta s, dx)) = \int_{\mathbb{R}} \int_{\mathbb{R}_0} D_{t,x}(u(s, x))\tilde{N}(\delta s, dx) + u(t, x).$$

*Proof.* Proof is similar to Gaussian settings in [31]. The sketch of proof is as follows. First, we assume that  $u(s, x) = I_n(f_n(\cdot, s, x))$  where  $I_n$  is defined in (2.0.5) and  $f_n$  is the symmetric deterministic function in  $L^2((\lambda \times \nu)^n)$ . After, showing the equality for this case we can generalize it for the infinite summation of symmetric functions by using Lemma 3.1.4 and Lemma 3.1.6 and some properties of Hida stochastic distribution space.  $\square$

**Corollary 4.0.7.** *Let  $G \subseteq [0, T]$  be a Borel set. If  $u(s, x) \in L^2(\mu \times \lambda \times \nu)$  then  $\mathbb{E}[u \mid \mathcal{F}_G] \in L^2(\mu \times \lambda \times \nu)$  and*

$$D_{t,x}(\mathbb{E}[u \mid \mathcal{F}_G]) = \mathbb{E}[D_{t,x} u \mid \mathcal{F}_G] \chi_G(t)$$

*in addition to this if  $u$  is adapted then*

$$D_{t,x} u(s, x) = 0 \text{ for } t > s.$$

*Proof.* Let  $u = I_n(f_n) \in L^2(\mu)$  for some  $f_n \in \hat{L}^2([0, T]^n \times \nu)$ . Then by Definition 3.2.1 and Theorem 3.1.5 we get as follows:

$$\begin{aligned} D_{t,x}(\mathbb{E}[u \mid \mathcal{F}_G]) &= D_{t,x} \left( \mathbb{E} \left[ \sum_{n=0}^{\infty} I_n(f_n \cdot \chi_G^{\otimes n}) \right] \right) \\ &= \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, x) \chi_G^{\otimes(n-1)}) \chi_G(t) \\ &= \mathbb{E} \left[ \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, x)) \mid \mathcal{F}_G \right] \chi_G(t) \\ &= \mathbb{E} [D_{t,x} u \mid \mathcal{F}_G] \cdot \chi_G(t). \end{aligned}$$

$\square$

**Lemma 4.0.8.** *Let  $Z(t)$  is defined as in the equation (4.0.1) and  $F \in L^2(\mu)$  then*

$$(4.0.6) \quad D_{t,x}(Z(T)F) = Z(T)\{F[H - 1] + HD_{t,x}F\}$$

*where  $H$  is defined as follows,*

$$\begin{aligned} H &= (1 - \theta(t, x)) \exp \left\{ \int_0^T \int_{\mathbb{R}_0} [D_{t,x}\theta(s, x) + \log(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)})(1 - \theta(s, x))] \nu(dx) ds \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}_0} \log(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}) \tilde{N}_Q(ds, dx) \right\} \end{aligned}$$

*Proof.* The proof falls naturally into two parts. First, using the chain rule and Theorem 4.0.6 we introduced, we will find the Malliavin derivative of  $Z$ . Second, we will apply the product rule in order to find the Malliavin derivative of  $Z(T)F$ . Let us define

$$Z(T) = \varphi(X)$$

where  $\varphi(X) = \exp(X)$  and  $X$  is defined by as follows:

$$X = \int_0^T \int_{\mathbb{R}_0} \{ \log(1 - \theta(s, x)) + \theta(s, x) \} \nu(dx) ds + \int_0^T \int_{\mathbb{R}_0} \log(1 - \theta(s, x)) \tilde{N}(ds, dx)$$

Then by chain rule,

$$\begin{aligned} D_{t,x}(Z(T)) &= D_{t,x}(\varphi(X)) = \exp(X + D_{t,x}X) - \exp(X) \\ &= Z(T)[\exp(D_{t,x}X) - 1] \end{aligned}$$



and by Theorem 4.0.6 and using the fact that

$$D_{t,x}(\log(1 - \theta(s, x))) = \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right),$$

$$\begin{aligned} D_{t,x}X &= \int_0^T \int_{\mathbb{R}_0} \left\{ \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) + D_{t,x}\theta(s, x) \right\} \nu(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) \tilde{N}(ds, dx) + \log(1 - \theta(t, x)) \\ &= \int_0^T \int_{\mathbb{R}_0} \left\{ D_{t,x}\theta(s, x) + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) (1 - \theta(s, x)) \right\} \nu(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) \tilde{N}_Q(ds, dz) + \log(1 - \theta(t, x)). \end{aligned}$$

Set  $H := \exp(D_{t,z}X)$  then;

$$\begin{aligned} H &= (1 - \theta(t, x)) \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \left[ D_{t,x}\theta(s, x) + \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) (1 - \theta(s, x)) \right] \nu(dx) ds \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) \tilde{N}_Q(ds, dx) \right\}. \end{aligned}$$

Note that by product rule,

$$\begin{aligned} D_{t,x}(Z(T)F) &= FD_{t,x}Z(T) + Z(T)D_{t,x}F + D_{t,x}FD_{t,x}Z(T) \\ (4.0.7) \quad &= Z(T)\{F[\exp(D_{t,x}X) - 1] + D_{t,x}F \exp(D_{t,x}X)\} \end{aligned}$$

Plugging H into the equation (4.0.7), we obtain the result.  $\square$

**Theorem 4.0.9. (Clark-Ocone formula under change of measure for pure jump Lévy processes)**

Suppose  $F \in L^2(\mu)$  is  $\mathcal{F}_T$ -measurable such that that

$$(4.0.8) \quad \mathbb{E}_Q[|F|] < \infty$$

Let  $\tilde{H}$  as follows:

$$\begin{aligned} \tilde{H} &= \exp \left\{ \int_0^T \int_{\mathbb{R}_0} \left[ D_{t,x}\theta(s, x) + \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) (1 - \theta(s, x)) \right] \nu(dx) ds \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}_0} \log\left(1 - \frac{D_{t,x}\theta(s, x)}{1 - \theta(s, x)}\right) \tilde{N}_Q(ds, dx) \right\} \end{aligned}$$

and assume that

$$(4.0.9) \quad \mathbb{E}_Q \left[ \int_0^T \int_{\mathbb{R}_0} |\tilde{H} D_{t,x}F|^2 \nu(dx) dt \right] < \infty$$

$$(4.0.10) \quad \mathbb{E}_Q \left[ \int_0^T \int_{\mathbb{R}_0} |F \tilde{H}|^2 \nu(dx) dt \right] < \infty$$

Then

$$F = \mathbb{E}_Q[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_Q[\tilde{H}D_{t,x}F - F(1 - \tilde{H})|\mathcal{F}_t] \tilde{N}_Q(dt, dx)$$

*Proof.* Let us define

$$\begin{aligned} \Lambda(t) = Z^{-1}(t) &= \exp\left(-\int_0^t \int_{\mathbb{R}_0} \{\log(1 - \theta(s, x)) + \theta(s, x)\} \nu(dx) ds \right. \\ &\quad \left. - \int_0^t \int_{\mathbb{R}_0} \log(1 - \theta(s, x)) \tilde{N}(ds, dx)\right) \end{aligned}$$

Since

$$dZ(t) = -Z(t^-) \int_{\mathbb{R}_0} \theta(t, x) \tilde{N}(dt, dx)$$

Applying Itô formula to  $\Lambda(t)$ , we obtain as follows:

$$d\Lambda(t) = \int_{\mathbb{R}_0} \frac{1}{Z(t)} \left\{ \frac{1}{1 - \theta(t, x)} - 1 - \theta(t, x) \right\} \nu(dx) dt + \int_{\mathbb{R}_0} \frac{1}{Z(t^-)} \left\{ \frac{1}{1 - \theta(t, x)} - 1 \right\} \tilde{N}(dt, dx).$$

By the equation (4.0.2) it follows easily that

$$(4.0.11) \quad d\Lambda(t) = \Lambda(t^-) \int_{\mathbb{R}_0} \frac{\theta(t, x)}{1 - \theta(t, x)} \tilde{N}_Q(dt, dx).$$

Set

$$(4.0.12) \quad Y(t) := \mathbb{E}_Q[F|\mathcal{F}_t]$$

Then using the Bayes' formula (Karatzas and Shreve, Lemma 3.5.3 [20]) and the equation (4.0.11),

$$Y(t) = \Lambda(t) \mathbb{E}[Z(T)F|\mathcal{F}_t].$$

By Theorem 4.0.3 and Corollary 4.0.7,

$$\begin{aligned} \mathbb{E}[Z(T)F|\mathcal{F}_t] &= \mathbb{E}[\mathbb{E}(Z(T)F|\mathcal{F}_t)] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{s,x}(\mathbb{E}(Z(T)F|\mathcal{F}_t))|\mathcal{F}_s] \tilde{N}(ds, dx) \\ &= \mathbb{E}[Z(T)F] + \int_0^t \int_{\mathbb{R}_0} \mathbb{E}[D_{s,x}(Z(T)F)|\mathcal{F}_s] \tilde{N}(ds, dz). \end{aligned}$$

Let us define  $U(t), t \in [0, T]$  such that

$$Y(t) := \Lambda(t)U(t),$$

$$(4.0.13) \quad U(t) = \mathbb{E}[Z(T)F] + \int_0^t \int_{\mathbb{R}_0} \mathbb{E}[D_{s,x}(Z(T)F)|\mathcal{F}_s] \tilde{N}(ds, dx).$$

Applying Itô formula for the Lévy processes and using equation (4.0.11) and (4.0.13) we obtain as follows:

$$\begin{aligned} dY(t) &= \Lambda(t^-)dU(t) + U(t^-)d\Lambda(t) + d[\Lambda, U]_t \\ &= \Lambda(t^-) \int_{\mathbb{R}_0} \frac{\mathbb{E}[D_{t,x}(Z(T)F)|\mathcal{F}_t]}{1 - \theta(t, x)} \tilde{N}_Q(dt, dx) + Y(t^-) \int_{\mathbb{R}_0} \frac{\theta(t, x)}{1 - \theta(t, x)} \tilde{N}_Q(dt, dx). \end{aligned}$$

Using Lemma 4.0.8 and the Bayes' formula one can easily conclude that

$$\begin{aligned} dY(t) &= \int_{\mathbb{R}_0} \frac{\mathbb{E}_Q[F(H-1) + HD_{t,x}F|\mathcal{F}_t]}{1-\theta(t,x)} \tilde{N}_Q(dt, dx) \\ &\quad + Y(t^-) \int_{\mathbb{R}_0} \frac{\theta(t,x)}{1-\theta(t,x)} \tilde{N}_Q(dt, dx) \end{aligned}$$

where  $H$  is defined as in Lemma 4.0.8. Note that by the setting (4.0.12),

$$Y(T) = \mathbb{E}_Q[F|\mathcal{F}_T] = F,$$

$$Y(0) = \mathbb{E}_Q[F|\mathcal{F}_0] = \mathbb{E}_Q[F].$$

Integrating  $dY(t)$  and using the equality (4.0.12), we conclude that

$$\begin{aligned} F - \mathbb{E}_Q[F] &= \int_0^T \int_{\mathbb{R}_0} \frac{\mathbb{E}_Q[FH + HD_{t,x}F|\mathcal{F}_t]}{1-\theta(t,x)} \tilde{N}_Q(dt, dx) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \frac{(\theta(t,x)-1)\mathbb{E}_Q[F|\mathcal{F}_t]}{1-\theta(t,x)} \tilde{N}_Q(dt, dx) \\ &= \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_Q[F(\tilde{H}-1) + \tilde{H}D_{t,x}F|\mathcal{F}_t] \tilde{N}_Q(dt, dx) \end{aligned}$$

where

$$\begin{aligned} \tilde{H} &= \frac{H}{(1-\theta(t,x))} = \exp\left\{ \int_0^T \int_{\mathbb{R}_0} [D_{t,x}\theta(s,x) + \log(1 - \frac{D_{t,x}\theta(s,x)}{1-\theta(s,x)})] \nu(dx) ds \right. \\ &\quad \left. + \int_0^T \int_{\mathbb{R}_0} \log(1 - \frac{D_{t,x}\theta(s,x)}{1-\theta(s,x)}) \tilde{N}_Q(ds, dx) \right\} \end{aligned}$$

□

In particular, if  $\theta(s,x)$  is a deterministic function then for all  $F \in L^2(\mu)$ ,

$$F = \mathbb{E}_Q[F] + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_Q[D_{t,x}F|\mathcal{F}_t] \tilde{N}_Q(dt, dx)$$

## 5. THE CLARK-OCONE FORMULA UNDER A CHANGE OF MEASURE FOR LÉVY PROCESSES

In the previous section, we proved the formula for the square integrable pure jump Lévy processes with no drift. We can extend this formula to the combination of Gaussian and pure jump Lévy processes in the white noise setting by combining the results found in the previous section and the results in Yolcu Okur [31]. Let us denote  $P^W$  the Gaussian white noise probability measure on  $(\Omega, \mathcal{F}_T^W)$  where the sample space is the Schwartz space  $(S'(\mathbb{R}))$  and  $\mathcal{F}_t^W = \sigma\{W(s), s \leq t\} \vee \mathcal{N}_1$ . From the previous sections, we know that  $\mu$  is the pure jump Lévy white noise probability measure on  $(\Omega, \mathcal{F}_T^\eta)$  where  $\Omega = S'(\mathbb{R})$  and  $\mathcal{F}_t^\eta = \sigma\{\eta(s), s \leq t\} \vee \mathcal{N}_2$ . Here  $\mathcal{N}_1$  and  $\mathcal{N}_2$  denote  $P^W$ -null and  $\mu$ -null sets respectively. Let

$$\Omega = S'(\mathbb{R}) \times S'(\mathbb{R}), \quad \mathcal{F}_T = \mathcal{F}_T^W \otimes \mathcal{F}_T^\eta$$

Then, there exists a unique measure on the product  $\sigma$ -algebra such that

$$P = P^W \times \mu$$

and

$$(5.0.14) \quad P(A) = P^W(A')\mu(A''), \quad \text{where } A' \in \mathcal{F}_T^W, A'' \in \mathcal{F}_T^\eta, A = A' \times A''.$$

From now on we will work with this product  $\sigma$ -algebra,  $(\Omega, \mathcal{F}_T, P)$ . Then the orthogonal basis for  $L^2(P)$  is the family of  $\mathbb{K}_\alpha$  with

$$\|\mathbb{K}\|_{L^2(P)} = \alpha! := \alpha^{(1)}!\alpha^{(2)}!$$

and,

$$(5.0.15) \quad \mathbb{K}_\alpha := H_{\alpha^{(1)}}(\omega') \cdot K_{\alpha^{(2)}}(\omega''),$$

where  $(\omega', \omega'') \in \Omega$ ,  $\alpha = (\alpha^{(1)}, \alpha^{(2)})$  and  $\{\alpha^{(i)}\}_{i=1,2} \in \mathcal{I}$  are multi-indexes defined in section 2,  $H_\alpha$  and  $K_\alpha$  are the orthogonal basis for  $L^2(P^W)$  and  $L^2(\mu)$  respectively.

**Theorem 5.0.10.** ([11]) *For all  $F \in L^2(P)$ , there exist unique constants  $c_\alpha \in \mathbb{R}$  such that*

$$F(\omega) = \sum_{\alpha \in \mathcal{I}^2} c_\alpha \mathbb{K}(\omega).$$

Moreover, there exists the following isometry

$$\|F\|_{L^2(P)}^2 = \sum_{\alpha \in \mathcal{I}^2} c_\alpha^2 \alpha!$$

**Definition 5.0.11.** For any random variable  $F \in L^2(P)$ , the Hida Malliavin derivatives,  $D_t$  and  $D_{t,x}$  are the mappings from  $\mathcal{G}^* \rightarrow \mathcal{G}^*$  and defined as

$$D_t F = \sum_{\alpha \in \mathcal{I}^2} \sum_{i \geq 1} c_\alpha \alpha_i^{(1)} \mathbb{K}_{\alpha^{(1)} - \epsilon^i} e_i(t)$$

$$D_{t,x} F = \sum_{\alpha \in \mathcal{I}^2} \sum_{i \geq 1} c_\alpha \alpha_{\kappa(i,j)}^{(2)} e_i(t) p_j(x)$$

**Theorem 5.0.12. (Girsanov theorem for Lévy Processes [29])**

Let  $\theta(s, x) \leq 1$ , and  $u(t)$  be given processes such that the process

$$Z(t) = \exp \left\{ - \int_0^t u(s) dW(s) - \int_0^t u^2(s) ds + \int_0^t \int_{\mathbb{R}_0} \{ \log(1 - \theta(s, x)) + \theta(s, x) \} \nu(dx) ds \right. \\ \left. + \int_0^t \int_{\mathbb{R}_0} \log(1 - \theta(s, x)) \tilde{N}(ds, dx) \right\}$$

exists for all  $0 \leq t \leq T$ . Define a measure  $Q$  on  $\mathcal{F}_T$  by

$$dQ(w) = Z(T) dP(w),$$

where  $P$  is the Lévy white noise probability measure defined above.

$$\mathbb{E}_P[Z(T)] = 1.$$

Then  $Q$  is a probability measure on  $\mathcal{F}_T$  and

$$\tilde{N}_Q(dt, dx) = \theta(t, x) \nu(dx) dt + \tilde{N}(dt, dx)$$

and

$$dW_Q(t) = u(t) dt + dW(t).$$

Then  $\tilde{N}_Q(\cdot, \cdot)$  and  $\tilde{W}(\cdot)$  are compensated Poisson random measure of  $N(\cdot, \cdot)$  and Brownian motion under measure  $Q$  respectively.

**Corollary 5.0.13.** ([11]) *Let  $F \in L^2(P)$  be an  $\mathcal{F}_T$ -measurable random variable. Then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t) + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}[D_{t,x} F | \mathcal{F}_t] \tilde{N}(dt, dx).$$

**Corollary 5.0.14.** *Let  $F \in L^2(P)$  be  $\mathcal{F}_T$ -measurable. Then the representation of  $F$  with respect to equivalent martingale measure is as follows:*

$$\begin{aligned} F &= \mathbb{E}_Q[F] + \int_0^T \mathbb{E}_Q[D_t F - F \int_t^T D_t u(s) dW_Q(s) | \mathcal{F}_t] dW_Q(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \mathbb{E}_Q[F(\tilde{H} - 1) + \tilde{H} D_{t,x} F | \mathcal{F}_t] \tilde{N}_Q(dt, dx), \end{aligned}$$

where

$$\begin{aligned} \tilde{H} &= \exp \left\{ \int_0^t \int_{\mathbb{R}_0} [D_{t,x} \theta(s, x) + \log(1 - \frac{D_{t,x} \theta(s, x)}{1 - \theta(s, x)}) (1 - \theta(s, x))] \nu(dx) ds \right. \\ &\quad \left. + \log(1 - \frac{D_{t,x} \theta(s, x)}{1 - \theta(s, x)}) \tilde{N}_Q(ds, dx) \right\}. \end{aligned}$$

*Proof.* Proof is similar to pure jump Lévy process case. For the proof, one need the auxiliary theorems that is showed in section 4 and the results of Yolcu Okur [31].  $\square$

This result plays a crucial role in financial mathematics and has many applications. In most of previous researches, Malliavin derivative is studied on Wiener space and with this settlement a main requirement arises, which is that random variable  $F$  should be in stochastic Sobolev spaces, i.e.  $F \in \mathbb{D}_{1,2}$ . However, this requirement excludes many interesting applications. In the following section, we will apply Clark-Ocone formula under change of measure for square integrable Lévy processes to obtain explicit formula for the closest hedge for the digital option.

## 6. APPLICATION

Let  $W(\cdot)$  be a 1-dimensional Brownian motion and  $\tilde{N}(\cdot, \cdot)$  be a compensated Poisson random measure on the filtered product space  $(\Omega, \mathcal{F}_T, \{\mathcal{F}_t\}, P)$  -c.f. (5.0.14). Assume that there exist one riskless and one risky asset in the financial market and the behavior of the stock prices are determined by the following stochastic differential equations respectively,

$$dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1$$

and

$$dS_1(t, \omega_1, \omega_2) = S_1(t^-, \omega_1, \omega_2)[\mu(t)dt + \sigma(t)dW(t, \omega_1) + \int_{\mathbb{R}_0} \gamma(t, x)\tilde{N}(dt, dx, \omega_2)],$$

with  $S_1(0) > 0$  for  $\omega = (\omega_1, \omega_2) \in \Omega$  and  $0 \leq t \leq T, x \in \mathbb{R}_0$ . The parameters  $r(t), \mu(t), \sigma(t)$  and  $\gamma(t, x) > -1$  are deterministic functions satisfying

$$\int_0^T \{ |r(t)| + |\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}_0} \gamma^2(t, x)\nu(dx) \} dt < \infty.$$

We will work on normalized financial market in order to simplify the presentation. We use risk free asset,  $S_0$ , as a numéraire. This means that the normalized riskless asset has price 1 and the normalized (discounted) risky asset price is denoted by,

$$X_1(t) := \frac{S_1(t)}{S_0(t)}.$$

Then by Theorem 1.1.3,

$$\begin{aligned} dX_1(t) &= X_1(t)[(\mu(t) - r(t))dt + \sigma(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, x)\tilde{N}(dt, dx)] \\ X_1(0) &> 0. \end{aligned}$$

Assume there exists a digital option (buy-sell) in the financial market whose payoff at the maturity is

$$F = \chi_{[H, K)}(S_1(T)),$$

where  $K > H$  and  $H, K > 0$ .

Note that for Lévy measure of  $\nu(dx) = \chi_{[H, K)}(x) \frac{1}{x^2} dx$ ,

$$F = \chi_{[H, K)}(S_1(T)) \notin \mathbb{D}_{1,2} \subset L^2(P).$$

Let  $\phi(t, \omega) = (\phi_0(t), \phi_1(t))$ ,  $\omega \in \Omega$  and  $\phi_0(t)$  and  $\phi_1(t)$  denote the number of units of risk free and risky assets invested at time  $t$  respectively. They are adapted processes and admissible; that is

$$\mathbb{E}\left[\int_0^T \phi_1^2(t) + \phi_2^2(t) dt\right] < \infty.$$

Assuming the self-financing condition, we get the discounted value of the portfolio  $\tilde{V}^\phi(t) = e^{-\int_0^t r(s)ds} V^\phi(t)$ ,  $\forall t \in [0, T]$  for the dynamic portfolio strategy  $\phi$  as follows:

$$d\tilde{V}^\phi(t) = \phi_1(t)X_1(t)[(\mu(t) - r(t))dt + \sigma(t)dW(t) + \int_{\mathbb{R}_0} \gamma(t, x)\tilde{N}(dt, dx)]$$

Since the financial market is modeled by Lévy processes, it is not complete. Therefore, we can not find the direct formula for the replicating portfolio, but it is possible to find the closest hedging portfolio in sense of minimal variance. The mean-variance hedging problem introduced by Föllmer and Sondermann [14]:

$$(6.0.16) \quad \min_{\phi \in \mathcal{A}} \mathbb{E}[F - V^{x, \phi}(T)]^2,$$

where  $F$  is the payoff a contingent claim,  $V^{x, \phi}(T)$  is the value of the portfolio at the final time and  $\mathcal{A}$  is the set of admissible trading strategies. Extensions to more general cases have been studied by many authors. In this section, we will follow the methodology from Benth *et. al.* [4]. However, there are slight differences between that paper and the present one. In that paper, the results are calculated for the contingent claims belonging to  $\mathbb{D}_{1,2}$  and the market is modeled by Lévy processes with constant coefficients. Therefore, the explicit solution for the closest-hedge we found is the generalization of the results in Benth *et. al.* [4].

The expectation in equation (6.0.16) can be calculated either under an objective measure or under a martingale measure. Whereas the first choice may seem more natural there are practical and theoretical motivations for using the martingale

measures like in [7], [14]. Hence, in this paper, since also the interest rate is deterministic function, equation (6.0.16) turns out to be as follows,

$$(6.0.17) \quad \min_{\phi_1 \in \mathcal{A}} \mathbb{E}_Q \left[ \frac{F}{S_0(T)} - \int_0^T \phi_1 dX(t) \right].$$

Since the market is incomplete, there are many equivalent martingale measures to choose. To exclude arbitrage opportunities, we assume that there exists a probability measure  $Q$  equivalent to  $P$  s.t.  $X(\cdot)$  is a local martingale which leads to the discounted value of the portfolio is a local-martingale. Therefore, the given processes  $u(t)$  and  $\theta(t, x) \leq 1$  in the Theorem 5.0.12 should satisfy the following equality,

$$\mu(t) - r(t) = \sigma(t)u(t) + \int_{\mathbb{R}_0} \gamma(t, x)\theta(t, x)\nu(dx).$$

Then  $Z(t)$  defined in Theorem 5.0.12 is well-defined and there exist an equivalent local martingale measure on  $\mathcal{F}_T$  such that

$$\begin{aligned} dQ(\omega) &= Z(T)dP(\omega), \\ \tilde{N}_Q(dt, dx) &= \theta(t, x)\nu(dx)dt + \tilde{N}(dt, dx) \end{aligned}$$

and

$$dW_Q(t) = u(t)dt + dW(t).$$

where  $\tilde{N}_Q(\cdot, \cdot)$  and  $\tilde{W}(\cdot)$  are compensated Poisson random measure of  $N(\cdot, \cdot)$  and Brownian motion under measure  $Q$  respectively. Then the asset price under  $Q$  evolves the following s.d.e.

$$(6.0.18) \quad dS_1(t) = S_1(t^-)[r(t)dt + \sigma(t)dW_Q(t) + \int_{\mathbb{R}_0} \gamma(t, x)\tilde{N}_Q(dt, dx)]$$

Let us define  $\beta(t) := e^{-\int_0^t r(s)ds}$ , the discounted value process of dynamic hedging strategy in terms of discounted asset price can be written follows:

$$\begin{aligned} \tilde{V}^{x, \phi}(T) &= \tilde{V}^{x, \phi}(0) + \int_0^T \phi_1(t)X_1(t)\sigma(t)dW_Q(t) + \int_0^T \int_{\mathbb{R}_0} \phi_1(t)X_1(t)\gamma(t, x)\tilde{N}_Q(dt, dx) \\ &= V^{x, \phi}(0) + \int_0^T \phi_1(t)\beta(t)S_1(t)\sigma(t)dW_Q(t) + \int_0^T \int_{\mathbb{R}_0} \phi_1(t)\beta(t)S_1(t)\gamma(t, x)\tilde{N}_Q(dt, dx). \end{aligned}$$

Then solving the optimization problem in (6.0.17), we obtain the following main result.

**Theorem 6.0.15.** *Let  $\theta(t, x)$  and  $u(t)$  be deterministic functions in Theorem 5.0.12. The minimal variance hedging strategy for a contingent  $F \in L^2(P)$  the minimal variance portfolio  $\phi_1$  such that*

$$F = \mathbb{E}_Q[F] + \int_0^T \phi_1(s) d\left(\frac{S_1}{S_0}\right)(t)$$

admits the following representation

$$\phi_1(t) = \rho^{-1}\beta(T-t) \left\{ \sigma(t)S_1(t)\mathbb{E}_Q[D_t F | \mathcal{F}_t] + \int_{\mathbb{R}_0} \gamma(t, x)S_1(t)\mathbb{E}_Q[D_{t,x} F | \mathcal{F}_t]\nu(dx) \right\}$$

where  $\rho = [(\sigma(t)S_1(t))^2 + \int_{\mathbb{R}_0} (\gamma(t, x)S_1(t))^2\nu(dx)]$  and  $D_t, D_{t,x}$  are the Hida Malliavin derivatives in direction of Brownian motion and compensated Poisson random measure in white noise settlement respectively.

*Proof.* The proof is quite similar to the proof of Theorem 4.1. in Benth *et al.* [4]. It is basically based on the solution of (6.0.17) and an application of Kunita-Watanabe projection theorem.  $\square$

*Remark 6.0.16.* The explicit formula in Theorem 6.0.15 can be extended for the markets contain  $N$  risky assets. Because it only requires minor changes in the computations. However, for simplicity we will just assume that there is one risky asset in the financial market.

#### REFERENCES

- [1] K. Aase, B. Øksendal, N. Privault, and J. Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. *Finance and Stochastic*, 4:465–496, 2000.
- [2] S. Albeverio, Y. G. Konradiev, and L. Streit. *How to generalize white noise analysis to non Gaussian spaces*. Dynamics of Complex and Irregular Systems. World Scientific, 1993.
- [3] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Cambridge University Press, Cambridge, 2003.
- [4] F. E. Benth, G. Di Nunno, A. Løkka, B. Øksendal, and F. Proske. Explicit representation of the minimal variance portfolio in the markets driven by Lévy processes. *Mathematical Finance*, 13(1):55–72, 2003.
- [5] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge, 1996.
- [6] K. Bichteler, J. Gravereaux, and J. Jacod. *Malliavin Calculus for Processes with Jumps*. Gordon and Breach Science, New York, 1987.
- [7] N. Bouleau and D. Lambertson. Residual risks and hedging strategies in markovian settings. *Stochastic Process. Appl.*, 33:131–150, 1989.
- [8] J. M. C. Clark. The representation of functionals of Brownian motion by stochastic integrals. *The Annals of Math. Statistics*, 41(4):1282–1295, 1970.
- [9] J. L. Da Silva, Y. Konradiev, and L. Streit. Generalized Appell systems. *Methods Funct. Anal. Topology*, 3(3):28–61, 1997.
- [10] J. L. Da Silva, Y. Konradiev, L. Streit, and G. Us. Analysis on Poisson and gamma spaces. *Inf. Dim. Anal. Quant. Prob. Rel. Topics*, 1(1):91–117, 1998.
- [11] G. Di Nunno, B. Øksendal, and F. Proske. Malliavin Calculus for Lévy Processes with Applications to Finance. Forthcoming book to be published by Springer.
- [12] G. Di Nunno, B. Øksendal, and F. Proske. White noise analysis for Lévy processes. *J. Funct. Anal.*, 206:109–148, 2004.
- [13] G. Di Nunno, T. Meyer Brandis, B. Øksendal, and F. Proske. Malliavin calculus and anticipative Itô formulae for lévy processes. University of Oslo, preprint, October 2004.
- [14] H. Föllmer and D. Sondermann. *Hedging of Non-redundant Contingent Claims*, chapter 12, pages 205–223. Amsterdam: North Holland, 1986. Contributions to Mathematical Economics.
- [15] U. G. Haussmann. Functionals of itô processes as stochastic integrals. *Siam J. Control and Optimization*, 16(2):252–269, 1978.
- [16] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic Partial Differential Equations*. Second Edition to appear in 2007/2008. Birkhäuser, 1996.
- [17] F. Huehne. A Clark-Ocone-Haussman Formula for optimal portfolios under Girsanov transformed pure-jump Lévy processes. Working paper, 2005.
- [18] A. D. Kaminsky. A white noise approach to stochastic integration for a Poisson random measure. *Theory of Pro. and Mathematical Statist.*, 57:43–52, 1998.
- [19] I. Karatzas and D. Ocone. A generalized Clark representation formula, with application to optimal portfolios. *Stoch. and Stoch. Rep.*, 34:187–220, 1991.
- [20] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer-Verlag, 1987.
- [21] A. Løkka. Martingale representation of functionals of Lévy processes. *Stochastic Analysis and Applications*, 22(4):867–892, 2004.
- [22] A. Løkka, B. Øksendal, and F. Proske. Stochastic partial differential equations driven by Lévy space-time white noise. *Annals. Appl. probab.*, 14:1506–1528, 2004.
- [23] D. Nualart and W. Schoutens. Chaotic and predictable representations for Lévy processes. *Stoc. Processes and their Appl.*, 90:109–122, 2000.



- [24] D. Nualart and J. Vives. *Anticipative calculus for the Poisson process based on the Fock space*, volume XXIV of *Séminaire de Probabilités*. 1990.
- [25] D. Nualart and J. Vives. *A duality formula on the Poisson process based on the Fock space*, volume 36 of *Séminaire de Probabilités*, pages 205–213. Birkhäuser, Basel, 1995.
- [26] D. Ocone. Malliavin's calculus and stochastic integral representations of functionals of diffusion processes. *Stochastics*, 12:161–185, 1984.
- [27] B. Øksendal. Stochastic partial differential equations driven by multi-parameter white noise of Lévy processes. To appear in *Quarterly of Appl. Math.*
- [28] B. Øksendal and F. Proske. White noise of Poisson random measures. *Potential Analysis*, 21(4):375–403, 2004.
- [29] B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions*. Springer, Berlin Heidelberg New York Hong Kong Paris Tokyo, second edition, 2007.
- [30] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge, 1999.
- [31] Y. Yolcu Okur. White noise generalization of the Clark-Ocone formula under change of measure. To appear in *Banach Center Publications*.

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