COMPUTATION OF GREEKS IN MULTI-FACTOR MODELS WITH APPLICATIONS TO POWER AND COMMODITY MARKETS

FRED ESPEN BENTH, GIULIA DI NUNNO, AND ASMA KHEDHER

ABSTRACT. We study the computation of the Greeks of options written on assets modelled by a multi-factor dynamics. For this purpose, we apply the conditional density method in which the knowledge of the density of one factor is enough to derive expressions for the Greeks not involving any differentiation of the payoff function. Several examples are given in applications to power and commodity markets, including numerical examples.

1. INTRODUCTION

Most of the popular spot price dynamics applied in commodity and power markets are so-called multi-factor models. For a market like electricity, it is reasonable to have factors accounting for the spike behavior observed in the spot price series, whereas other factors model the price evolution when the market is in stable conditions. Commodity prices are often said to be mean-reverting, since the law of supply and demand will push prices back if they deviate too much from a mean level. On the other hand, this mean level may be significantly influenced by the resource situation of a commodity (oil say), and thereby also stochastic. Hence, one often encounters two-factor models, essentially trying to capture mean-reverting prices around a randomly fluctuating mean. Typical models are the Schwartz-Smith dynamics applied to commodities or the multi-factor model of Benth, Kallsen, and Meyer-Brandis [5] developed for electricity spot prices.

In this paper we are concerned with the Greeks of options written on such multi-factor dynamics. There exists options in commodity markets which are written on the spot and forward price and to understand the risk involved in option investments one needs to calculate the Greeks. We shall concentrate on the delta and gamma of an option, being the first and second derivative of the price with respect to the current spot price. The traditional methods to find the delta, say, are the so-called *density method*, *finite differencing*, or *differentiation*. For all three methods, the starting point is the discounted risk-neutral expectation of the option payoff. The differentiation method simply computes the derivative of the expectation by exchanging differentiation and integration and thus computing the expectation of the derivative of the payoff. The basic assumption of this technique is the differentiability of the payoff function, which is not always holding. For example, for a plain vanilla call or put option, the payoff has a kink at the strike price. Although you get the right expectation by formally differentiating, the method becomes

Date: March 3, 2010.

Key words and phrases. Options, Greeks, density method, multi-factor models, Levy processes, commodity markets.

numerically very slow when applying the Monte Carlo simulation to evaluate the resulting expectation. For other options, like the digitals, one cannot find the derivative of the payoff function, ruling out this technique. The numerical counterpart to this method is finite differencing. Here one perturbs the option price slightly to calculate the finite difference which is the numerical approximation of the derivative. The computation of the Greek is then carried out via the computation of two similar expectations, which can be efficiently done by Monte Carlo methods if one applies the technique of common random numbers and the payoff function is differentiable. However, for non-differentiable payoffs, the method becomes very inefficient in the sense of slow Monte Carlo convergence. Finally, the density method is based on the knowledge of the probability density of the spot. By moving the dependency of the initial spot price to the density, one may differentiate this rather than the payoff function. The result is an expectation function of the payoff function times the logarithmic derivative of the density evaluated at the spot price at maturity of the option. We refer to Broadie and Glasserman [8] for more on this method.

In this paper we apply the extension called the *conditional density method* as introduced and analyzed in Benth, Di Nunno, and Khedher [3] and further generalized in Benth, Di Nunno, and Khedher [4]. The approach is simple: one applies the conditional expectation with respect to one of the factors and then uses the standard density methods approach. To make this work, we need to have accessible the density of the factor we choose to condition on. As it turns out, the conditional density method is particularly useful for deriving the Greeks in the case of multi-factor models.

As a result, we derive an expression of the delta not involving any differentiation of the payoff. The conclusion of our findings is that as long as there is one component with a density in the spot price dynamics and methods for simulating the spot price exists, one can compute the delta and gamma by simply Monte Carlo simulation of the spot. Furthermore, the delta and gamma are both expressible in terms of the price of an option with payoff equal to the original option's payoff times the density evaluated at the value of the component at exercise.

We illustrate our findings by several examples where we also perform a numerical analysis of efficiency and practical tractability. In particular, we look at a model without any Gaussian component, but with a known stationary distribution. We analyse how one can approximate the delta by calculating the corresponding expectation based on the stationary density instead. Our numerical experiments show that our conditional density method provides expressions which are highly tractable and easily implementable for numerical computation of the Greeks of options on multi-factor models.

There exist other methods, for instance, based on the Malliavin derivative (see Lions et al [10]) or by numerical solution of the partial (integro-) differential equations associated to the option price (see Tankov, Cont, and Voltchkova [15]). We do not discuss these methods any further here, but note that our expressions for the delta and gamma will themselves be solutions of partial (integro-) differential equations. Also, in our set-up, if possible, the Malliavin method will yield the same expressions and therefore not provide any new insight (see Benth, Di Nunno, and Khedher [3] for a discussion). However, when dealing with path-dependent options, the Malliavin approach would be fruitful.

The paper is organized as follows. In the next section, we present a general multi-factor spot model which is a typical model describing commodities and electricity spot prices. In Sect. 3 we discuss the computation of the Greeks delta and gamma using the conditional density method and we give several examples in which the knowledge of the density of just one factor of the multi-factor model is needed. We also present a discussion on options on forwards. In Sect. 4 we consider two numerical examples: in the first one, we derived expressions for the delta using the Gaussian density of the first factor of a two-factor model. In the second example, we use the stationary distribution of the first factor of another two-factor model.

2. MULTI-FACTOR MODELS IN COMMODITY AND POWER MARKETS

Let us consider a probability space $(\Omega, \mathcal{F}, Q)$ equipped with a filtration $\mathcal{F}_t$, $0 \leq t \leq T$, where T is some finite time horizon. Suppose that the spot price dynamics $S(t)$, $0 \leq t \leq T$, is defined by

$$(2.1) \quad S(t) = g(t, X_1(t), \dots, X_n(t)),$$

for n *independent* adapted stochastic processes $X_1(t), \dots, X_n(t)$ representing the factors. The function $g : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous to ensure that $S(t)$ is adapted as well. Note that we work directly under the risk-neutral probability measure Q .

The spot price model in (2.1) is very general and encompasses many interesting cases known in the energy and commodity markets. We list here a few examples and connect them to our model.

The Schwartz model is defined as

$$(2.2) \quad S(t) = S(0) \exp(X(t)),$$

where

$$(2.3) \quad dX(t) = (\theta - \alpha X(t)) dt + \sigma dW(t),$$

for $\theta \in \mathbb{R}$, α, σ positive constants, and W a Brownian motion under Q . By letting $n = 1$, $X_1(t) = X(t)$ and $g(x) = S(0) \exp(x)$ we have identified the Schwartz model to (2.1). The Schwartz model has been applied as a simple model for the oil price dynamics in Schwartz [13]. In this model, the log-price mean-reverts towards a level given by the θ .

A two-factor extension of the model is proposed by Schwartz and Smith [14]. It takes the form

$$(2.4) \quad S(t) = S(0) \exp(X(t) + Y(t)),$$

where $Y(t)$ is the long-term non-stationary drift of the price and it is given by

$$(2.5) \quad dY(t) = \mu dt + \eta d\widetilde{W}(t),$$

with $\widetilde{W}$ a Brownian motion under Q , possibly correlated with W . In the case $\widetilde{W}$ is independent of W , then we obviously choose $n = 2$, $X_1(t) = X(t)$, $X_2(t) = Y(t)$, and the

function g is set equal to $g(x, y) = S(0) \exp(x + y)$. In the correlated case, we represent $Y(t)$ in the following way:

$$Y(t) = Y(0) + \mu t + \eta \rho W(t) + \eta \sqrt{1 - \rho^2} \widetilde{W}_1(t),$$

with ρ being the correlation between $\widetilde{W}$ and W while $\widetilde{W}_1$ is independent of W . Then we can consider an $n = 3$ factor model with $X_1(t) = X(t)$, $X_2(t) = Y(0) + \mu t + \eta \rho W(t)$, and $X_3(t) = \eta \sqrt{1 - \rho^2} \widetilde{W}_1(t)$. The function g is naturally extended: $g(x, y, z) = S(0) \exp(x + y + z)$.

An extension with stochastic volatility of the Schwartz and Smith model is found in Geman [11]. One may choose

$$(2.6) \quad dX(t) = (\theta - \alpha X(t)) dt + \sqrt{Z(t)} dW(t),$$

where $Z(t)$ is some positive adapted stochastic process such that its square-root is Itô integrable. An example, applied by Geman [11], is to assume that Z follows the Heston model. In Benth [2], the Schwartz model is considered with the stochastic volatility Z following a superposition of non-Gaussian Ornstein-Uhlenbeck processes, as proposed by Barndorff-Nielsen and Shephard [1]. In the Schwartz-Smith model with stochastic volatility the elements g , X_1 , and X_2 are identified in a similar way, however, the process $X_1(t)$ becomes more complex.

A natural extension of the models above is to include jumps. In Benth, Saltyte-Benth, and Koekebakker [7] a general class of models based on non-stationary jump processes is discussed. We consider here some examples from this class for illustration. In the power markets, spikes are frequently observed and one natural model for this is

$$(2.7) \quad S(t) = S(0) \exp(X(t) + Y(t)),$$

where

$$(2.8) \quad dY(t) = -\beta Y(t) dt + dL(t).$$

Here, L is a Lévy process, which may possibly be time-inhomogeneous (in this case, also called additive process) in order to model the seasonal jump frequency which is naturally observed in many power markets. The constant β is positive.

In general, realistic models should consider the seasonality. The standard way to include seasonality in the type of model above is, for example, to let

$$S(t) = \Lambda(t) \exp(X(t)) \quad (\Lambda(t) > 0)$$

in the Schwartz model, where the deterministic $\Lambda(t)$ represents the seasonality component. $S(0) = \Lambda(0) \exp(X(0))$ and $g(t, x) = \Lambda(t) \exp(x)$. Modifications of the other examples above to include seasonality are straightforward.

Benth, Kallsen, and Meyer-Brandis [5] proposed an additive model for the electricity spot price defined as a superposition of independent Ornstein-Uhlenbeck processes:

$$(2.9) \quad S(t) = \Lambda(t) \sum_{i=1}^n Y_i(t).$$

with

$$(2.10) \quad dY_i(t) = -\lambda_i Y_i(t) dt + dL_i(t).$$

Here, the constants λ_i are all positive and $L_i(t)$ are subordinator processes (i.e. increasing Lévy processes) possibly being time-inhomogeneous. By using subordinators as jump components, one is assured to have a spot price with positive values. The natural way to apply the model in practice (see *e.g.* Benth, Kiesel, and Nazarova [6]) is to separate the model into base components and one or more spike components. For instance, the two first factors may account for the normal variations in the market, the so-called base signal, and naturally are stationary processes, while a third component may model the spikes, i.e. big jumps followed by a fast mean-reversion. In many markets the jump frequency is seasonally varying, leading to a time-inhomogeneous subordinator for this factor. Thus, the distributional properties are not in general analytically available. We remark that in Meyer-Brandis and Tankov [12] it is proposed to model the base signal using a Brownian motion driven Ornstein-Uhlenbeck process. The identification of the model (2.9) to our general multi-factor dynamics in (2.1) is obvious. Note, however, that this identification is not unique as we see hereafter. In fact, from (2.10) we have

$$Y_i(t) = Y_i(0)e^{-\lambda_i t} + \int_0^t e^{-\lambda_i(t-s)} dL_i(s).$$

Thus, in the multi-factor representation, we can choose

$$X_i(t) = \int_0^t e^{-\lambda_i(t-s)} dL_i(s),$$

and

$$(2.11) \quad g(t, x_1, \dots, x_n) = \Lambda(t) \sum_{i=1}^n \{Y_i(0)e^{-\lambda_i t} + x_i\}.$$

This is the representation we shall use. In the following, when we are going to study Greeks for options, observe that the process $S(t)$, $0 \leq t \leq T$, could also be interpreted as some index, like an index on a stock exchange, or some value of a basket of assets. In this case, the factors $X_i(t)$ would represent the individual assets. This is also covered by our considerations in this present paper, although we particularly focus on models for commodities and power.

3. OPTIONS AND GREEKS

We consider European options written on the spot price $S(t)$, $0 \leq t \leq T$, with exercise time T and payoff function $h : \mathbb{R} \mapsto \mathbb{R}$. The arbitrage-free price is defined as

$$(3.1) \quad C(S(0)) = e^{-rT} \mathbb{E}[h(S(T))],$$

where we have emphasized the dependency on $S(0)$ since we are going to compute the Greeks with respect to this. The parameter r is the risk-free instantaneous interest rate of

a bond used as numéraire. A standing assumption in the sequel is that $h(S(T)) \in L^1(Q)$ to make the price $C(S(0))$ well-defined. To this end, note that we can write

$$h(S(T)) = h(g(T, X_1(T), \dots, X_n(T))).$$

We suppose now that there exist a function f and a differentiable function ζ such that

$$(3.2) \quad h(S(T)) = f(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T)).$$

Hence, we consider prices

$$(3.3) \quad C(S(0)) = e^{-rT} \mathbb{E} [f(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T))].$$

3.1. The delta. Denote by p_1 the density of $X_1(T)$, which we suppose to be known and let $\partial \ln p_1$ be its logarithmic derivative. In the next proposition we derive the delta of C .

Proposition 3.1. *Assume that there exists an integrable function u on $\mathbb{R}$ such that*

$$(3.4) \quad |\mathbb{E} [f(x, X_2(T), \dots, X_n(T))] |p_1(x - \zeta(S(0)))| \leq u(x).$$

Then

$$\frac{\partial C}{\partial S(0)} = -\zeta'(S(0))e^{-rT} \mathbb{E} [h(S(T)) \partial \ln p_1(X_1(T))].$$

Proof. By conditioning on $X_1(T)$ we find

$$\begin{aligned} & \mathbb{E} [f(X_1(T) + \zeta(S(0)), X_2(T), \dots, X_n(T))] \\ &= \int_{\mathbb{R}} \mathbb{E} [f(x + \zeta(S(0)), X_2(T), \dots, X_n(T))] p_1(x) dx \\ &= \int_{\mathbb{R}} \mathbb{E} [f(x, X_2(T), \dots, X_n(T))] p_1(x - \zeta(S(0))) dx. \end{aligned}$$

With our assumption, appealing to Thm 2.27 in Folland [9], we can move the differentiation inside the integration, and find the result after dividing and multiplying by $p_1(x)$. Hence, the proof is complete. $\square$

3.1.1. Remarks.

- (1) In the assumptions above, we have assumed that the density of $X_1(T)$ is defined on the real line, and implicitly that it is strictly positive there. We can easily adapt the result to densities only defined on the positive half-axis, see Example 4.2 in Sect. 4.
- (2) Note that we have assumed the knowledge of the density of $X_1(T)$. In practice, we search for the factor with the most convenient density, among those factors for which the density is known, and use it as the first factor.
- (3) From a computational point of view, the result above is highly advantageous. First of all, we do not need to differentiate explicitly the payoff function h (or equivalently, f), a procedure that is not always possible since the payoff may not be differentiable (e.g. digital options). Furthermore, by applying Monte Carlo methods in conjunction with moving the derivative into the payoff function, we would get a very slow convergence due to very high variability. On the contrary, using

Monte Carlo to compute the expectation in the Prop 3.1 turns out to be much more stable. We shall demonstrate this in the numerical examples in Sect. 4.

The delta is essentially the price of a new option with payoff $h(S(T))\partial \ln p_1(X_1(T))$, namely, the option payoff h modified by the logarithmic derivative of the density of the first factor. The delta takes thus a general form where *only* the payoff changes across options. In most of the examples we will look at, the factor X_1 will be chosen as the Gaussian process appearing in the model considered and therefore very simple to simulate using standard software. In fact, one may simulate the delta in parallel with the option price $C(S(0))$ in a Monte Carlo approach.

We look at some examples.

3.1.2. *Example.* We start with considering the two-factor model of Schwartz and Smith for the case of independence between W and $\widetilde{W}$. We represent the payoff as

$$h(S(T)) = h(\exp(\ln(S(0)) + X_1(T) + X_2(T)))$$

and thus we find $f(x_1 + \zeta(S(0)), x_2) = h(\exp(\ln(S(0)) + x_1 + x_2))$ with $\zeta(s) = \ln s$. Both factors $X_1(T)$ and $X_2(T)$ are Gaussian random variables, with explicit mean and variance, and may be used as factor 1 for the calculation of the delta. If $X_1(t)$ is the mean-reverting process, we have that the mean is $X(0)\exp(-\alpha t) + \theta(1 - \exp(-\alpha t))/\alpha$ and variance $\sigma^2/2\alpha(1 - \exp(-2\alpha t))$. Therefore, it holds that

$$\partial \ln p_1(x) = -\frac{1}{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha T})} \left(x - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1 - e^{-\alpha T}) \right).$$

Hence, the delta is

$$\frac{\partial C}{\partial S(0)} = \frac{e^{-rT}2\alpha}{S(0)\sigma^2(1 - e^{-2\alpha T})} \mathbb{E} \left[h(S(T)) \left(X_1(T) - X(0)e^{-\alpha T} - \frac{\theta}{\alpha}(1 - e^{-\alpha T}) \right) \right].$$

It is simple to modify the above expression for the case of dependent factors. We see that the expression of the delta remains the same if the second factor is a mean-reverting jump process, mimicking spikes, as in (2.8). Hence, we do not see any different delta except of course for the change in the properties of the second factor. Note that when $X_2(t)$ is the Gaussian model, we may apply the density method directly, since $X_1(T) + X_2(T)$ is again a Gaussian random variable. The mean in this case is $e^{-\alpha_1 T} X_1(0) + \theta_1(\frac{1 - e^{-\alpha_1 T}}{\alpha_1}) + e^{-\alpha_2 T} X_2(0) + \theta_2(\frac{1 - e^{-\alpha_2 T}}{\alpha_2})$ and variance (in the independent case) is $\frac{\sigma_1^2}{2\alpha_1}(1 - e^{-2\alpha_1 T}) + \frac{\sigma_2^2}{2\alpha_2}(1 - e^{-2\alpha_2 T})$. Thus, a simple application of the density method would give

$$\begin{aligned} \frac{\partial C}{\partial S(0)} &= \frac{2\alpha_1\alpha_2 e^{-rT}}{S(0)(\sigma_1^2\alpha_2(1 - e^{-2\alpha_1 T}) + \sigma_2^2\alpha_1(1 - e^{-2\alpha_2 T}))} \mathbb{E} \left[h(S(T)) \left(X_1(T) + X_2(T) \right. \right. \\ &\quad \left. \left. - e^{-\alpha_1 T} X_1(0) - \theta_1\left(\frac{1 - e^{-\alpha_1 T}}{\alpha_1}\right) - e^{-\alpha_2 T} X_2(0) - \theta_2\left(\frac{1 - e^{-\alpha_2 T}}{\alpha_2}\right) \right) \right]. \end{aligned}$$

The direct application of the density method to the sums of the factors $X_1(T) + X_2(T)$ is not possible in the case of a jump process in the second component $X_2(T)$, except in

the case when the distribution density of $X_2(t)$ is known. But in that case still assuming independence, the joint distribution of X_1 and X_2 is rather complicated, being the convolution of a Gaussian distribution with the distribution of X_2 . In fact the convolution may give a density which is not analytically tractable. This is a typical situation where the conditional density method provides an easy alternative.

3.1.3. Example. We may extend the Schwartz and Smith model to include a stochastic volatility. In this case it is natural to switch the roles of X_1 and X_2 , and let $X_1(t) = X_1(0) + \mu t + \eta \widetilde{W}(t)$. Thus, $X_1(T)$ is normally distributed with mean $X_1(0) + \mu T$ and variance $\eta^2 T$. The logarithmic derivative of p_1 is in this case

$$\partial \ln p_1(x) = -\frac{1}{\eta^2 T} (x - X(0) - \mu T) .$$

Note that ζ and f remains the same. Hence, again by assuming independence between the two factors for simplicity, we obtain the following expression for the delta

$$\frac{\partial C}{\partial S(0)} = \frac{e^{-rT}}{S(0)\eta^2 T} \mathbb{E} [h(S(T)) (X_1(T) - X(0) - \mu T)] .$$

We observe that this is an alternative expression for the delta in the constant-volatility case as well. In fact, the stochastic volatility only enters in the dynamics of $X_2(t)$ and it is nowhere appearing in the other terms involved in the delta. In this sense we see that the delta is “independent” of the structure of the stochastic volatility process.

3.1.4. Example. Consider the multi-factor model in (2.9). By using the representation in (2.11) with $X_i(t) = \int_0^t e^{-\lambda_i(t-s)} dL_i(s)$, we can write

$$h(S(T)) = h \left(S(0)e^{-\lambda_1 T} \frac{\Lambda(T)}{\Lambda(0)} - \Lambda(T) \sum_{i=2}^n (e^{-\lambda_1 T} - e^{-\lambda_i T}) Y_i(0) + \Lambda(T) \sum_{i=1}^n X_i(T) \right) ,$$

where we recognize that

$$\zeta(S(0)) = \frac{S(0)e^{-\lambda_1 T}}{\Lambda(0)} ,$$

and

$$f(x_1 + \zeta(S(0)), \dots, x_n) = h \left(\Lambda(T)(x_1 + \zeta(S(0))) - \Lambda(T) \sum_{i=2}^n (e^{-\lambda_1 T} - e^{-\lambda_i T}) Y_i(0) + \Lambda(T) \sum_{i=2}^n x_i \right) .$$

So, we find that

$$\zeta'(S(0)) = \frac{e^{-\lambda_1 T}}{\Lambda(0)} .$$

The density p_1 of $X_1(T)$ is not necessarily simple to find explicitly in this model, although its cumulant can be calculated straightforwardly from the cumulant of the subordinator L_1 . However, a reasonable approximation of p_1 , when T is sufficiently large, is to apply its stationary distribution. This is frequently known, since it is specified in the modeling

process and estimated to data. In the numerical examples in Sect. 4, we shall consider this alternative.

One may ask which payoff functions h satisfy the boundedness condition (3.4) on f in Prop. 3.1. If h is bounded, then obviously f will be, and then the boundedness condition will hold since the density p_1 is integrable. This will then include plain vanilla put and digital options, for example. With a view towards call options (and insisting on not applying the call-put parity), it is natural to consider functions h which have at most exponential linear growth (having the exponential models in mind). The boundedness assumption (3.4) will then look like: there exists a function u being integrable such that

$$K(1 + \exp(|x|))p_1(x) \leq u(x).$$

Here, the constant K involves the expectation of the exponential the remaining factors. If p_1 is a Gaussian density, then $p_1(x) \sim \exp(-x^2/c)$, and we again will have integrability. We see that the boundedness condition (3.4) is a balance between the growth of the payoff function versus the properties of the density of the first factor.

3.2. A discussion of options on forwards. Options are frequently written on forwards in the commodity markets. In fact, at markets like NYMEX one trades in options on gas and oil futures, and at the Nordic electricity exchange Nord Pool European options are written on forwards and futures delivering electricity over specific periods. We therefore include a discussion on how our framework above may be incorporated to cover this situation as well.

Let us start our discussion with the two-factor model in (2.7), with $X(t)$ and $Y(t)$ being the base and spike components defined in (2.3) and (2.8), resp. To simplify the exposition, we ignore seasonality here. The forward price $F(t, \tau)$ of a contract at time $t \geq 0$, maturing at time $\tau \geq t$ is defined as

$$(3.5) \quad F(t, \tau) = \mathbb{E}[S(\tau) | \mathcal{F}_t],$$

see Benth, Saltyte-Benth, and Koekebakker [7]. From Prop. 4.6 in the same reference, it follows that the forward price for the spot model in (2.7) becomes

$$(3.6) \quad F(t, \tau) = \Theta(\tau - t) \exp(X(t)e^{-\alpha(\tau-t)} + Y(t)e^{-\beta(\tau-t)}),$$

with

$$(3.7) \quad \ln \Theta(s) = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha s}) + \frac{\theta}{\alpha} (1 - e^{-\alpha s}) + \int_0^s \psi(-ie^{-\beta u}) du.$$

In order to establish the formula for the forward price, the jump process L must satisfy certain exponential integrability conditions, which can be found in Benth, Saltyte-Benth, and Koekebakker [7]. We observe that the forward price is exactly represented in the suitable to embed it directly into our machinery for calculating the delta of a spot.

The price of an option with exercise time $T \leq \tau$ and payoff function h is

$$C(F(0, \tau)) = e^{-rT} \mathbb{E}[h(F(T, \tau))].$$

We want to identify functions f and ζ along with factors $X_1, \dots, X_n$ such that

$$h(F(T, \tau)) = f(X_1(t) + \zeta(F(0, \tau)), X_2(T), \dots, X_n(T)).$$

From the forward price in (3.6) and the explicit solutions of $X(T)$ and $Y(T)$, we find

$$\begin{aligned} F(T, \tau) &= \Theta(\tau - T) \exp \left(X(0)e^{-\alpha\tau} + \frac{\theta}{\alpha}e^{-\alpha\tau}(e^{\alpha T} - 1) + Y(0)e^{-\beta\tau} \right. \\ &\quad \left. + \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s) + \int_0^T e^{-\beta(\tau-s)} dL(s) \right) \\ &= \frac{\Theta(\tau - T)}{\Theta(\tau)} \exp \left(\frac{\theta}{\alpha}e^{-\alpha\tau}(e^{\alpha T} - 1) \right) \\ &\quad \exp \left(\ln F(0, \tau) + \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s) + \int_0^T e^{-\beta(\tau-s)} dL(s) \right). \end{aligned}$$

where we have used the fact that $F(0, \tau) = \Theta(\tau) \exp(X(0)e^{-\alpha\tau} + Y(0)e^{-\beta\tau})$. We then set $\zeta(x) = \ln x$,

$$\begin{aligned} X_1(T) &= \int_0^T \sigma e^{-\alpha(\tau-s)} dW(s), \\ X_2(T) &= \int_0^T e^{-\beta(\tau-s)} dL(s), \end{aligned}$$

and we get

$$f(x_1 + \zeta(F(0, \tau)), \dots, x_n) = h(\Psi(T, \tau) \exp(x_1 + \zeta(F(0, \tau)) + x_2)),$$

where

$$\Psi(T, \tau) = \frac{\Theta(\tau - T)}{\Theta(\tau)} e^{\frac{\theta}{\alpha}e^{-\alpha\tau}(e^{\alpha T} - 1)}.$$

Hence, we are in the framework of the previous subsection and we can apply Prop. 3.1 to obtain an expression for the delta of the option on F with payoff function h . We base the calculation on the density of $X_1(T)$, which is Gaussian with mean zero and variance $\sigma^2 e^{-2\alpha\tau}(e^{2\alpha T} - 1)/2\alpha$.

As already indicated, forward contracts in power markets are not delivering the underlying commodity (that is, electricity) at a fixed time in the future, but rather over a given time period. This is due to the very nature of electricity as commodity. Hence, sometimes one refers to these contracts as *flow forwards*. The forward price $G(t, \tau_1, \tau_2)$ of a flow forward contract at time $t \geq 0$ with delivery in the period $[\tau_1, \tau_2]$, $\tau_1 \geq t$, is defined as

$$(3.8) \quad G(t, \tau_1, \tau_2) = \mathbb{E} \left[\frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} S(u) du \mid \mathcal{F}_t \right],$$

see Benth, Saltyte-Benth, and Koekebakker [7]. Note that the price is defined as the average spot price over delivery and not the aggregated spot. In reality, the aggregated spot is delivered, but, by market convention, the *forward price* is stated per time unit, that

is, in MWh (Mega Watt hours) instead of MW. By commuting integration and expectation, we find

$$(3.9) \quad G(t, \tau_1, \tau_2) = \frac{1}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} F(t, \tau) d\tau.$$

As it turns out, most models do not allow for analytical price formulas for flow forwards. For example, the exponential models discussed above yield in general only a price in terms of the integral in (3.9). However, considering the multi-factor spot model, one may derive an explicit price dynamics for $G(t, \tau_1, \tau_2)$. Hereafter, we investigate this case and relate it to our analysis of the delta.

Consider the multi-factor model for the spot in (2.9) where the factors follow the dynamics in (2.10). To simplify the notation in our exposition, we suppose that the seasonality function is constant and equal to one, that is, $\Lambda(t) = 1$. Then, according to Prop. 4.14 in Benth, Saltyte-Benth, and Koekebakker [7], the flow forward price is

$$(3.10) \quad G(t, \tau_1, \tau_2) = \Theta(t, \tau_1, \tau_2) + \sum_{i=1}^n Y_i(t) \tilde{\lambda}_i(t, \tau_1, \tau_2),$$

for a deterministic function Θ depending on the characteristics of the jump processes $L_i(t)$ (see Prop. 4.14 in [7] for an explicit expression) and

$$(3.11) \quad \tilde{\lambda}_i(t, \tau_1, \tau_2) = \frac{1}{\lambda_i(\tau_2 - \tau_1)} (e^{-\lambda_i(\tau_1 - t)} - e^{-\lambda_i(\tau_2 - t)}).$$

We observe that

$$G(0, \tau_1, \tau_2) = \Theta(0, \tau_1, \tau_2) + \sum_{i=1}^n Y_i(0) \tilde{\lambda}_i(0, \tau_1, \tau_2).$$

Applying the explicit solution of the Ornstein-Uhlenbeck processes $Y_i(t)$, and reshuffling terms, we find

$$\begin{aligned} G(t, \tau_1, \tau_2) &= \tilde{\lambda}_1(t, \tau_1, \tau_2) e^{-\lambda_1 t} \left(\frac{G(0, \tau_1, \tau_2)}{\tilde{\lambda}_1(t, \tau_1, \tau_2)} + \int_0^t e^{\lambda_1 s} dL_1(s) - \frac{\Theta(0, \tau_1, \tau_2)}{\tilde{\lambda}_1(t, \tau_1, \tau_2)} \right) \\ &\quad + \sum_{i=2}^n Y_i(0) \left(\tilde{\lambda}_i(t, \tau_1, \tau_2) e^{-\lambda_i t} - \tilde{\lambda}_1(t, \tau_1, \tau_2) e^{-\lambda_1 t} \frac{\tilde{\lambda}_i(0, \tau_1, \tau_2)}{\tilde{\lambda}_1(0, \tau_1, \tau_2)} \right) \\ &\quad + \sum_{i=2}^n \tilde{\lambda}_i(t, \tau_1, \tau_2) e^{-\lambda_i t} \int_0^t e^{\lambda_i s} dL_i(s) + \Theta(t, \tau_1, \tau_2). \end{aligned}$$

Given a payoff function $h(G(T, \tau_1, \tau_2))$ for an option with exercise time $T \leq \tau_1$, we can read off the factors

$$X_i(T) = \int_0^T e^{\lambda_i s} dL_i(s),$$

for $i = 1, \dots, n$ and $\zeta(x) = x$. The function f is then easily defined, and we have obtained an expression coinciding with the kind we discuss above. The delta of the option $h(G(T, \tau_1, \tau_2))$ can then be calculated by appealing to Prop. 3.1 with appropriate change of

notation. If the exact density function for the factor X_1 is not available, a reasonable way to proceed is given by its stationary distribution, as suggested for the computation of the delta and studied in Sect. 4.

3.3. The gamma. In the next proposition we calculate an expression for the Greek gamma.

Proposition 3.2. *Suppose the hypothesis of Prop. 3.1 holds, and in addition that there exists an integrable function v on $\mathbb{R}$ such that*

$$(3.12) \quad |\mathbb{E}[f(x, X_2(T), \dots, X_n(T))] p_1''(x - \zeta(S(0)))| \leq v(x),$$

and that ζ is twice differentiable in $S(0)$. Then, the gamma is given by

$$\frac{\partial^2 C}{\partial S(0)^2} = e^{-rT} \mathbb{E} \left[h(S(T)) \left\{ (\zeta'(S(0)))^2 \frac{p_1''(X_1(T))}{p_1(X_1(T))} - \zeta''(S(0)) \partial \ln p_1(X_1(T)) \right\} \right]$$

Proof. From the proof in Prop. 3.1, we have that

$$\frac{\partial C}{\partial S(0)} = -\zeta'(S(0)) e^{-rT} \int_{\mathbb{R}} \mathbb{E}[f(x, X_1(T), \dots, X_n(T))] p_1'(x - \zeta(S(0))) dx.$$

Appealing to the boundedness condition (3.12), we obtain the result by commuting the integration and differentiation using Thm. 2.27 in Folland [9]. $\square$

As we saw in the examples following Prop. 3.1, in most cases the function ζ is $\zeta(S(0)) = \ln S(0)$. For such a choice, $\zeta'(S(0)) = 1/S(0)$ and $\zeta''(S(0)) = -1/S^2(0)$, and thus the gamma becomes

$$\frac{\partial^2 C}{\partial S(0)^2} = \frac{1}{S^2(0)} e^{-rT} \mathbb{E} \left[h(S(T)) \left\{ \frac{p_1''(X_1(T))}{p_1(X_1(T))} + \partial \ln p_1(X_1(T)) \right\} \right].$$

Using the known density of $X_1(T)$ as in the examples for computation of the delta, we may give explicit expressions for the terms involving p_1 . In the case of the additive model by Benth, Kallsen, and Meyer-Brandis [5], we have $\zeta(S(0)) = \xi(T)S(0)$ for some known function $\xi(T)$. Then $\zeta''(S(0)) = 0$, thus we have an expression for the gamma given by

$$\frac{\partial^2 C}{\partial S(0)^2} = (\zeta'(S(0)))^2 e^{-rT} \mathbb{E} \left[h(S(T)) \frac{p_1''(X_1(T))}{p_1(X_1(T))} \right].$$

As remarked before in the case of the computation of the delta, we can argue here as well the possible use of the stationary distribution of X_1 exploiting the approximation that the stationary distribution represents to the original one of X_1 .

4. NUMERICAL EXAMPLES

In this section we consider two numerical examples illustrating our conditional density approach. We use the popular finite difference approach for comparison. We look first at an example of a two-factor model where one of the factors has a dynamics based on Brownian noise. This factor will have a normal density suitable for differentiation. In the second example we look at a model for the spot price which is stationary, and we apply the explicit knowledge of the stationary density to approximate the delta by conditioning.

FIGURE 1. Simulation of a delta for a call option

4.1. **Example 1.** In our first numerical example, we considered the two-factor model $S(t)$ given by

$$S(t) = S(0) \exp(X(t) + Y(t)),$$

where $X(t)$ is a mean-reverting process given by equation (2.3). The parameters of $X(t)$ are $\theta = 0$, $\alpha = 0.099$, and $\sigma = 0.032$. This form of a spot price model is rather typical in commodity markets, see for example Meyer-Brandis and Tankov [12]. The process $Y(t)$ is given by equation (2.8), where we choose $\beta = 0.23$ and we set L as a compound Poisson with jump frequency $\lambda = 20/250$ and exponentially distributed jump size, with mean 0.2.

We consider a function h being the payoff of a call option with strike $K = 100$:

$$h(S(T)) = (\exp(S(T)) - 100)_+.$$

In Figure 1, we show the resulting delta for $S(0) = 100$ and exercise time $T = 20$ days. To estimate the expectation operator, we have used Monte Carlo simulation. Along the horizontal axis, we have the number of simulations (in 10^4) used in the estimation of the expectation operator. The solid line shows the derivative using the finite difference method that is

$$\frac{\partial C}{\partial S(0)} \approx \frac{C(S(0) + \delta) - C(S(0))}{\delta},$$

where $\delta = 0.01$. The broken line shows the delta using the conditional density method. Common random numbers are used in the Monte Carlo simulation. In the case of a call option, we clearly see that the conditional density method has higher variance than the finite difference approach, and thus a slower convergence.

FIGURE 2. Simulation of a delta for a digital option

In Figure 2, we consider a digital option with payoff function

$$h(S(T)) = 1_{(100, \infty)}(\exp(S(T))),$$

where $T = 20$ days. The solid line shows the delta using a finite difference method with $\delta = 0.01$ and the broken line shows the delta using the conditional density method. We observe that in this case the conditional density method has much lower variance, and therefore converges faster than the finite difference method. The rather high variation yielding uncertain Monte Carlo estimates that result from the finite difference method, is well-known for payoff functions which are not differentiable. The conditional density method has in this case a much more stable performance. We would get the same conclusions looking at options on forwards. Again the conditional density method would converge faster for singular payoffs. Moreover, this result will carry over to the gamma, the second derivative of the option with respect to the underlying spot price. The computation of the gamma essentially involves the second derivative of the payoff function, and thus the case of the gamma of a call option would show similar features as the case of the delta of a digital option. In this case the conditional density method would outperform the finite difference method.

4.2. Example 2. The second example that we considered is a special case of the additive model of Benth, Kallsen, and Meyer-Brandis [5]. Let the spot price be given as a two-factor model,

$$S(t) = X(t) + Y(t), \quad S(0) > 0.$$

Here, the process $Y(t)$ is given by

$$Y(t) = -\lambda_2 Y(t)dt + dL_2(t), \quad Y(0) = 0,$$

where L_2 is a compound Poisson process with intensity μ and exponentially distributed jumps with parameter ν . The process $X(t)$ is a so called $\Gamma(a, b)$ -OU process. Namely, it is

a Lévy process following the dynamics

$$dX(t) = -\lambda_1 X(t)dt + dL_1(t), \quad X(0) = S(0),$$

where $L_1(t)$ is a subordinator, admitting a stationary distribution which is here $\Gamma(a, b)$ (see Thm 17.5 in Sato [13] and Thm 1 in Barndorff-Nielsen and Shephard [1]).

The problem now is to compute the delta of an option written on the spot. We have not given any explicit density here, so apparently the conditional density method is not working. However, we know that X (in fact also Y) has a stationary distribution, and we can apply this for the conditional density method in order to derive the delta, at least approximately.

To check out the validity of such an approximation, we need to be able to simulate from the processes in the spot model. To simulate a $\Gamma(a, b)$ -OU process, we first remark that $L_1(t)$ is actually a compound Poisson process with intensity parameter a and exponential jump distribution with parameter b (see Example 2 in Section 2 in Barndorff-Nielsen and Shephard [1]). Then from

$$(4.1) \quad X(t) = e^{-\lambda_1 t} X(0) + \int_0^t e^{\lambda_1(s-t)} dL_1(s),$$

we see that in order to simulate $X(t)$, we need to simulate a Poisson process with intensity $\lambda_1 a$ at the discrete times $t_n = n\Delta t$, $n = 0, 1, \dots$. Then, we set

$$x(n\Delta t) = e^{-\lambda_1 \Delta t} x((n-1)\Delta t) + \sum_{N((n-1)\Delta t)+1}^{N(n\Delta t)} z_n e^{-u_n \lambda_1 \Delta t},$$

where z_n are independent $\text{Exp}(b)$ random numbers and u_n are independent uniform random numbers.

Consider the payoff of a call option $h(S(T)) = \max(S(T) - K, 0)$, with strike $K = 1.5$. We apply the conditional density method in the following way. First of all, we observe that the stochastic integral

$$\int_0^t e^{\lambda_1(s-t)} dL_1(s) = X(t) - e^{-\lambda_1 t} X(0)$$

has an asymptotic distribution being $\Gamma(a, b)$ when t goes to infinity, since $e^{-\lambda_1 t} X(0)$ goes to 0 when t goes to infinity. Denoting by Z a random variable which is $\Gamma(a, b)$ -distributed, we consider

$$\tilde{S}(t) = e^{-\lambda_1 t} X(0) + Z + Y(t),$$

which is asymptotically equal in distribution to $S(t)$.

In the notation of Prop 3.1, we have the factors, $X_1(T) = Z$, $X_2(T) = Y(T)$, $\zeta(S(0)) = e^{-\lambda_1 T} S(0)$ and $h(S(T)) = f(X_1(T) + \zeta(S(0)), X_2(T))$. Therefore, for any density $p_1(x)$ defined on the positive half axis, we have in particular that

$$\frac{\partial C}{\partial S(0)} \approx \frac{\partial}{\partial S(0)} e^{-rT} \mathbb{E}[f(X_1(T) + \zeta(S(0)), X_2(T))]$$

$$\begin{aligned}
&= \frac{\partial}{\partial S(0)} e^{-rT} \int_{\zeta(S(0))}^{+\infty} \mathbb{E}[f(x, X_2(T))] p_1(x - \zeta(S(0))) dx \\
&= e^{-rT} \int_{\zeta(S(0))}^{+\infty} \mathbb{E}[f(x, X_2(T))] \frac{\partial p_1}{\partial S(0)}(x - \zeta(S(0))) dx - \mathbb{E}[f(\zeta(S(0)), X_2(T))] p_1(0),
\end{aligned}$$

where in the latter equality, we used the fact that

$$\frac{\partial}{\partial y} \int_y^{+\infty} g(x, y) dx = \int_y^{+\infty} \frac{\partial g}{\partial y}(x, y) dx - g(y, y).$$

Therefore

$$\begin{aligned}
\frac{\partial C}{\partial S(0)} &\approx e^{-rT} \mathbb{E}[f(X(T) + \zeta(S(0)), Y(T))(-\zeta'(S(0))) \frac{\partial}{\partial x} \log p_1(X(T))] \\
&- \mathbb{E}[f(\zeta(S(0)), Y(T))] p_1(0).
\end{aligned}$$

In our study, in the case of a $\Gamma(a, b)$, the density is given by $p_1(x) = x^{a-1} \frac{e^{-x/b}}{\Gamma(a)b^a}$, $x > 0$, $a, b > 0$. Note that when $0 < a < 1$, p_1 is not defined in 0, while when $a = 1$, it is equal to $\frac{1}{\Gamma(1)b}$ and finally for $a > 0$, it is equal to 0. The expression for the delta is then given by

$$\frac{\partial C}{\partial S(0)} \approx e^{-rT} \mathbb{E}\left[h(e^{-\lambda_1 T} X(0) + Z + Y(T)) e^{-\lambda_1 T} (b - \frac{a-1}{Z})\right].$$

This will be our approximation of the delta based on the conditional density method.

To make a numerical example which is relevant for energy markets, we note that Benth, Kiesel, and Nazarova [6] showed empirically that the spot model fitted the Phelix Base electricity price index at the European Power Exchange (EEX) very well. In their paper, they estimated the parameters in the suggested model to be $a = 13.3009$, $b = 8.5689$, $\lambda_1 = 0.2008$, $\lambda_2 = 0.3333$, $\mu = 20/250$, and $\nu = 0.2$. We use these estimates in our example, however, we let the seasonality function be constant equal to one for simplicity. We remark that the inclusion of a seasonal function is straightforward. In our numerical examples we ignore any risk premium.

In Figure 3, we show the resulting derivative for $S(0) = 5$, exercise time $T = 10$ days and interest rate $r = 0$. To estimate the expectation operators in the conditional density and finite difference methods, we use a Monte Carlo simulation technique with common random numbers. Along the horizontal axis, we have the number of simulations (in 10^4) used in the estimation of the expectation operators. The solid line shows the derivative using a finite difference method, that is,

$$(4.2) \quad \frac{\partial C}{\partial S(0)} = \frac{C(S(0) + \delta) - C(S(0))}{\delta},$$

with $\delta = 0.01$. In the expression (4.2), we used the fact that

$$\begin{aligned}
C(S(0) + \delta) &= e^{-rT} \mathbb{E}\left[\max\left(e^{-\lambda_1 T}(X(0) + \delta) + \int_0^T e^{\lambda_1(s-t)} dL_1(s) + Y(T) - K, 0\right)\right] \\
&= e^{-rT} \mathbb{E}\left[\max(e^{-\lambda_1 T} \delta + X(T) + Y(T) - K, 0)\right].
\end{aligned}$$

FIGURE 3. Simulation of a delta for a call option

The broken line shows the delta using the conditional density method. Again we find that the finite difference method converges faster for the delta of a call option, not unexpectedly. But, interestingly, the approximation based on conditional density seems to be reasonably good. Based on 600,000 samples in a Monte Carlo simulation, the true value resulting from the conditional density method is 0.132 with three decimals of accuracy. The finite difference method converges slightly below 0.129, giving an upward bias of approximately 2% for the conditional density approximation relative to the finite difference method.

Motivated by the above, we go further to study a digital option and its delta. Consider a digital option with payoff $h(S(T)) = 1_{(K, \infty)}(S(T))$, where the strike is $K = 2$, $S(0) = 5$, exercise time $T = 10$ days and interest rate $r = 0$. In Figure 4, the solid line shows the delta using a finite difference method with $\delta = 0.01$ and the broken line shows the approximation using the conditional density method. We observe that as in Example 1, the conditional density method converges faster for singular payoffs. Based on 600,000 outcomes, the finite difference method gave the result 0.129 with three decimals of accuracy. The conditional density method is now downward biased, and the error of the conditional density method relative to the finite difference is approximately 7%.

How well the approximation based on the conditional density method works is depending on how far from stationarity the $X(t)$ factor is. We have looked at options with only 10 days left to exercise, and one may argue this is a rather short time for the model to be in stationarity. This taken into account, one may say that the approximation is rather good although the deviation of around 7% relative to the finite difference method. How fast the model goes into stationary is also depending on the speed of mean reversion and the size and frequency of jumps. In conclusion, the approximating method may provide

FIGURE 4. Simulation of a delta for a digital option

an attractive alternative to other methods which like the finite difference method since it converges so much faster.

REFERENCES

- [1] Barndorff-Nielsen, OE and Shephard, N (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. Royal Statist. Soc.. Ser. B (Statistical Methodology)*, **63**, pp. 167–241.
- [2] Benth, F.E. (2008). The stochastic volatility model of Barndorff-Nielsen and Shephard in commodity markets. To appear in *Math. Finance*.
- [3] Benth, F. E., Di Nunno, G., and Khedher, A. (2009). Lévy-models robustness and sensitivity. To appear in Volume XXV of the series QP-PQ, Quantum Probability and White Noise Analysis, H. Ouerdiane and A. Barhoumi (eds.), World Scientific.
- [4] Benth, F. E., Di Nunno, G., and Khedher, A. (2010). Robustness of option prices and their deltas in markets modelled by jump-diffusions. *E-print*, No. 2, January, Department of Mathematics, University of Oslo, Norway.
- [5] Benth, F. E., Kallsen, J., and Meyer-Brandis, T. (2007). A non-Gaussian Ornstein-Uhlenbeck process for electricity spot price modeling and derivatives pricing. *Appl. Math. Finance*, **14**, pp. 153-169.
- [6] Benth, F. E., Kiesel, R., and Nazarova, A. (2009). A critical empirical study of two electricity spot price models. *Manuscript*.
- [7] Benth, F. E., Saltyte-Benth, J., and Koekebakker, S. (2008). *Stochastic Modelling of Electricity and Related Markets*. World Scientific.
- [8] Broadie, M., and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Manag. Science*, **42**, pp. 169–285.

- [9] Folland, G. B. (1984). *Real Analysis – Modern Techniques and their Applications*. John Wiley & Sons.
- [10] Fournié, E., Lasry, J. M., Lbucheux, J., Lions, P. L., and Touzi, N. (1999). Applications of Malliavin calculus to Monte Carlo methods in finance, *Finance Stoch.*, **3**, pp 391-412.
- [11] Geman, H. (2008). *Risk Management in Commodity Markets: from Shipping to Agriculturals and Energy*. Wiley Finance.
- [12] Meyer-Brandis, T. and Tankov, P. (2008). Multi-factor jump-diffusion models of electricity prices. *Intern. J. Theor. Appl. Finance*, **11**, pp 503 - 528.
- [13] Schwartz, E. S. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *J. Finance*, **52**, pp 923-973.
- [14] Schwartz, E.S. and Smith, J. E. (2000). Short-term variations and long-term dynamics in commodity prices. *Manag. Science*, **46**(7), pp. 893–911.
- [15] Tankov, P., Cont, R., and Voltchkova, E. (2007). Hedging with options in models with jumps. In: Benth, Di Nunno, Lindstrm and ksendal (eds.), *Stochastic Analysis and Applications*, Springer Verlag, pp. 197-218.

FRED ESPEN BENTH: CENTER OF MATHEMATICS FOR APPLICATIONS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, N-0316 OSLO, NORWAY, AND, UNIVERSITY OF AGDER, SCHOOL OF MANAGEMENT, SERVICEBOKS 422, N-4604 KRISTIANSAND, NORWAY.

GIULIA DI NUNNO: CENTER OF MATHEMATICS FOR APPLICATIONS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, N-0316 OSLO, NORWAY, AND, NORWEGIAN SCHOOL OF ECONOMICS AND BUSINESS ADMINISTRATION, HELLEVEIEN 30, N-5045 BERGEN, NORWAY.

ASMA KHEDHER: CENTER OF MATHEMATICS FOR APPLICATIONS, UNIVERSITY OF OSLO, PO BOX 1053 BLINDERN, N-0316 OSLO, NORWAY

E-mail address: [fredb,giulian]@math.uio.no, asma.khedher@cma.uio.no

URL: <http://folk.uio.no/fredb,giulian/>