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**PREPRINT SERIES
MECHANICS AND
APPLIED MATHEMATICS**

**UNIVERSITY OF OSLO
DEPARTMENT OF MATHEMATICS**

**UNIVERSITETET I OSLO
MATEMATISK INSTITUTT**

Semi-analytical buckling strength analysis of arbitrarily stiffened plates with varying thickness

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Abstract

A computationally efficient method for elastic buckling and buckling strength analysis of biaxially loaded, stiffened plates with varying, stepwise constant thickness, are presented. The stiffeners may be sniped or end-loaded (continuous), and their orientations may be arbitrary. Both global and local plate buckling modes are captured. The method is semi-analytical and makes use of simplified displacement computations that involve the elastic buckling load (eigenvalue), determined using a Rayleigh-Ritz approach, and finally stress computations using large deflection theory in combination with strength assessment using von Mises' yield criterion applied to membrane stresses. The displacements are represented by trigonometric functions, defined over the entire plate. The method is implemented into a Fortran computer code, and numerical results, obtained for a variety of plate and stiffener geometries, are compared to fully nonlinear finite element analysis results.

Key words: Stiffened plates; Stepped plate; Arbitrary stiffener orientations; Buckling strength; Elastic buckling load; Semi-analytical method

1 Introduction

In many branches of engineering, stiffened plates are used as main structural components in order to improve the strength/weight ratio and reduce costs. For analysis of large structures, computationally efficient analysis tools are useful for obtaining results within a reasonable time limit. Also, such tools may be a necessity in the design of structures with complex geometry and stiffener arrangements, for

which explicit strength formulas [1, 2, 3] may not be applicable. Nonlinear finite element method analyses could be used in such cases. However, such analyses are often time consuming to prepare, run and postprocess, and other approaches may be better suited in more practical oriented design contexts.

Computationally efficient analysis tools using semi-analytical methods for buckling and ultimate strength predictions are becoming more common. A rather advanced nonlinear buckling model was developed by Byklum et al. [4, 5] with a basis in previous work by Steen

Revision May 2007: An error in Eq. 10 is corrected. This error was not present in the computer program, and consequently, computed results in the paper are correct.

[6, 7]. These studies deal mainly with unstiffened and regularly stiffened plates. A related model presented in Brubak, Hellesland and Steen [8], deals with buckling strength analysis of constant thickness plates provided with sniped stiffeners with irregular orientations. These models have been adopted by the ship classification and engineering services company Det Norske Veritas (DNV), and implemented into a computerised software code entitled PULS [9]. The semi-analytical models above are based on the Rayleigh-Ritz method. A more detailed review of semi-analytical methods that make use of that method as well as of the Galerkin method (Paik and Lee [10], etc.) is given in Brubak et al. [8].

The studies mentioned above deal with plates of constant thickness. Sometimes it is advantageous to vary the plate thickness locally to increase the buckling strength and obtain more cost efficient structures. Several approaches have been formulated for such plates. For example, Azhari and Shahidi [11] presented a semi-analytical method for analysing the post-buckling behaviour of initially perfect, unstiffened stepped plates (i.e., with varying, stepwise constant thickness). Xiang and Wei [12] developed a method for linear elastic buckling and vibration analysis of such plates. These two papers summarise some earlier works on stepped plates. Most of these works deal mainly with unstiffened plates and not with buckling strength of imperfect plates (onset of yielding, capacity etc.) as such.

The main objective of the present paper is to present and document the applicability of a semi-analytical model for local and global buckling strength analysis of uni-directionally stiffened plates. It represents an extension of an earlier work [8] on constant thickness plates with sniped stiffeners. The present study also includes extensions to end-loaded (continuous) stiffeners and torsional stiffness considerations, but does not include local failure modes of the stiffeners. The plates may have regular or arbi-

Figure 1. Examples of stiffened plates: (a) Stiffened plate enclosed by longitudinal and transverse girders, and (b) girder stiffened by sniped stiffeners.

trarily oriented stiffeners, and may have various restraints at plate edges and in the interior of the plate.

2 Stiffened plate modelling

Two stiffened plate examples are shown in Fig. 1. The stiffeners of the plate enclosed by the strong longitudinal and transverse girders in Fig. 1a, are typically continuous and subject to axial loading at their ends (at the plate boundary). Unlike these, the sniped (“discontinuous”) stiffeners illustrated on the girder web in Fig. 1b, will not be axially loaded at their ends. Normally, this will neither be the case for non-regular stiffeners, such as for instance in the stern and bow of a ship hull.

The plate considered can be defined with reference to Fig. 2. The plate may consist of an arbitrary number of uni-directional plate strips of different thickness (3 shown in the figure). It may be provided with one or several stiffeners with arbitrary orientations. The stiffeners

Figure 2. Simply supported plate with varying, stepwise constant plate thickness, subjected to external in-plane compression or tension $\bar{S}_x$ and S_y in the x- and y-direction, respectively.

may be sniped at the ends or end-loaded (continuous). They may have different cross-section profiles, and may be eccentric, as in Fig. 2b, or symmetric about the middle plane of the plate.

The stiffeners are modelled as simple beams with flexural stiffness only against out-of-plane bending. This implies that possible local buckling of stiffeners, including torsional instability, can not be predicted. This may not represent a serious limitation in practical cases as design rules generally impose constructional design provisions that prevent local buckling of stiffeners. Also, stiffeners are usually proportioned such as to provide sufficient strength to prevent a global plate buckling mode. Without the additional axial stress from global bending, local buckling of stiffeners is even less likely. Thus, the simplified stiffener model seems like a reasonable one.

The torsional stiffness of the stiffeners may be included, but their axial stiffness is neglected. The latter implies that the stiffeners effect on the internal membrane stress distribution in the plate is neglected. This is considered an acceptable simplification.

The usual assumptions that the plate edges remain straight (due to the neighbouring plates) while in-plane movements are allowed,

are adopted. Otherwise, all four plate edges are supported in the out-of-plane direction. An edge, or a part of an edge, may rotationally be simply supported, partly or fully clamped, as illustrated in Fig. 3. Rotational and out-of-plane restraints, modelled by strong translational and rotational springs, can also be added along specified lines with arbitrary orientations in the interior of the plate.

The displacement field is defined over the entire plate. It was initially questioned whether this was adequate for stepped plates. Preliminary eigenvalue and buckling strength results indicated that it is. The alternative of specifying one displacement field for each plate strip and impose continuity requirements along the boundaries between the plate strips, will be considerably more complicated and was not considered.

Hooke's material law for plane stress for an elastic, isotropic material is adopted, and further, Kirchhoff's deformation assumption (a straight line normal to the middle plane prior to loading, remains straight and normal to the plane after deformation). These are the usual thin plate assumptions [13]. The first implies that only in-plane stresses (σ_x , σ_y , τ_{xy}) and in-plane strains (ϵ_x , ϵ_y , γ_{xy}) are present. The

Figure 3. Stepped plate example with partly simple (ss) and partly clamped supports (cc), and irregular stiffener arrangement.

second implies that strains, consisting of membrane strains (constant over the plate thickness) and bending strains, vary linearly across the plate thickness.

The plate, Fig. 2, is subjected to the external, in-plane biaxial compressive or tensile loading defined by the stresses shown in the figure, and defined as positive when compressive. A consequence of the straight edge assumption is that the resultant of the mean external stress $\bar{S}_x$, in the x-direction, will initially be distributed as illustrated by the varying, stepwise constant stress S_x . External shear stresses are not included in the model at present. The effect of out-of-plane displacements, that cause a redistribution of normal stresses and formation of shear stresses, is discussed later.

3 Major Computational steps

The computations are carried out in two major steps:

1. In the first step, labelled the elastic buckling stress limit (ESL) state, the elastic buckling load (first eigenvalue) and corresponding buckling mode of the stiffened plate are calculated.
2. In the second step, labelled the buckling strength limit (BSL) state, a buckling strength (capacity) assessment is made for the plate with a specified out-of-plane imperfection. This step involves displacement computations using an approximate displacement magnifier (based on small deflection theory) that is a function of the applied load and the elastic buckling

Figure 4. Definition of in-plane degrees of freedom for linear analysis of a stepped plate example with three plate strips.

load (ESL), computation of stresses according to large deflection theory and, finally, strength assessment using the von Mises' yield criterion applied to membrane stresses.

The essential parts of the model are described below. Additional details are given in Brubak [14].

4 Initial, reference membrane stresses

In order to determine the initial, reference biaxial membrane stresses in the various plate strips, a linear static analysis is performed for a perfect plate with a chosen reference loading ($\bar{S}_{x0}$ and S_{y0} , positive in compression). For illustration, the stepped plate shown in Fig. 4 is considered. It consists of three plate strips and has four in-plane degrees of freedom d_1 to d_4 . Stiffeners are not included (their axial stiffness is neglected). The resulting axial plate stiffness relationship can be given in matrix form by

$$\mathbf{K}_0 \mathbf{d} = \mathbf{P}_0 \quad (1)$$

where

$$\mathbf{K}_0 = E^* \begin{bmatrix} \sum_{i=1}^2 \frac{t_i L}{b_i} & -\frac{t_2 L}{b_2} & 0 & \nu(t_1 - t_2) \\ -\frac{t_2 L}{b_2} & \sum_{i=2}^3 \frac{t_i L}{b_i} & -\frac{t_3 L}{b_3} & \nu(t_2 - t_3) \\ 0 & -\frac{t_3 L}{b_3} & \frac{t_3 L}{b_3} & \nu t_3 \\ \nu(t_1 - t_2) & \nu(t_2 - t_3) & \nu t_3 & \sum_{i=1}^3 \frac{t_i b_i}{L} \end{bmatrix} \quad (2)$$

$$\mathbf{d} = [d_1, d_2, d_3, d_4]^T \quad (3)$$

$$\mathbf{P}_0 = [0, 0, -P_{y0}, -P_{x0}]^T \quad (4)$$

Here, $E^* = E/(1 - \nu^2)$, E and ν are Young's modulus and Poisson's ratio, respectively, $P_{x0} = \bar{S}_{x0}(b_1 t_1 + b_2 t_2 + b_3 t_3)$ and $P_{y0} = \bar{S}_{y0} L t_3$ are the resultant, reference forces acting on the plate edges. Once Eq. 1 is solved, the initial reference strains can be computed from

$$\epsilon_{x0} = \frac{d_4}{L} \quad \text{and} \quad \epsilon_{y0} = \begin{cases} d_1/b_1 \\ (d_2 - d_1)/b_2 \\ (d_3 - d_2)/b_3 \end{cases} \quad (5)$$

for plate strip 1, 2 and 3, respectively. Inserted into Hooke's law, these strains give initial, reference membrane stresses σ_{x0}^m and σ_{y0}^m in the plate strips. These strains and stresses are defined as positive when tensile (i.e., opposite of the externally applied stress definition). The membrane shear stresses and strains $(\tau_{xy0}^m, \gamma_{x0}^m)$ are zero since external shear loading is not considered.

5 Elastic buckling stress limit (ESL)

In the first step of the analysis, the elastic buckling load (first eigenvalue) of a perfect, stiffened plate is computed using the Rayleigh-Ritz method. For completeness and convenience of the reader, the major elements are summarised below. The assumed out-of-plane displacement field, which satisfies the boundary conditions of a simply supported plate, is given by

$$w(x, y) = \sum_{i=1}^M \sum_{j=1}^N a_{ij} \sin\left(\frac{\pi i x}{L}\right) \sin\left(\frac{\pi j y}{b}\right) \quad (6)$$

where a_{ij} are amplitudes, L the plate length, b the total plate width, $0 \leq x \leq L$ and $0 \leq y \leq b$.

Although each component in a series of sine functions represents a simply supported condition, added together they are, in combination with rotational springs along supports, nearly able also to describe fully or partially restrained

conditions [4, 14]. Therefore, rather than to specify different fields, such as for instance a series of cosine functions for a clamped plate, the sine curve assumption is used for various boundary conditions. To achieve the same accuracy, a higher number of degrees of freedom (number of terms) will normally be required with a sine field than with a field that satisfies the kinematic boundary conditions more appropriately.

Equilibrium of the loaded plate requires that its total potential energy, $\Pi = U + T$, has a stationary value, i.e., $\delta\Pi = \delta(U + T) = 0$. Here, U is the strain energy and T is the potential energy of the external loads. This requirement leads to the eigenvalue problem

$$(\mathbf{K}^M + \Lambda^e \mathbf{K}^G) \mathbf{a}^e = \mathbf{0} \quad (7)$$

where $\mathbf{K}^M$ is the material stiffness matrix (due to U), $\mathbf{K}^G$ the geometrical stiffness matrix (due to T for the initial, reference loading), Λ^e the eigenvalues (load factors at buckling) and $\mathbf{a}^e$ the eigenvectors containing all the displacement amplitudes a_{ij} that together define the buckling mode for a specific eigenvalue. Energy contributions used in the derivation of the eigenvalue problem are given below. Strain energy from membrane stresses is not included as it does not affect computed eigenvalues.

The elastic strain energy from bending of the entire plate, U_{plate}^b , is obtained by adding up the contributions U_s^b from each plate strip s . With N_s number of plate strips, the total bending strain energy becomes

$$U_{\text{plate}}^b = \sum_{s=1}^{N_s} U_s^b \quad (8)$$

where

$$U_s^b = \int_0^L \int_{y_{s1}}^{y_{s2}} \frac{D_s}{2} \left((w_{,xx} + w_{,yy})^2 - 2(1 - \nu)(w_{,xx} w_{,yy} - w_{,xy}^2) \right) dx dy \quad (9)$$

is the contribution from plate strip number s , located between $y = y_{s1}$ and $y = y_{s2}$, with thickness t_s and flexural plate stiffness $D_s = Et_s^3/12(1 - \nu^2)$, and where the conventional “comma” notation $w_{,xy}$ for $\partial^2 w / \partial x \partial y$, etc., is adopted. By substituting the assumed displacement field, an analytical solution of this integral may be derived and written in the form

$$U_s^b = \sum_{i=1}^M \sum_{j=1}^N \sum_{l=1}^N a_{ij} a_{il} \frac{D_s L}{4} \times \left[\left(\left(\frac{\pi i}{L} \right)^2 + \left(\frac{\pi j}{b} \right)^2 \right) \left(\left(\frac{\pi i}{L} \right)^2 + \left(\frac{\pi l}{b} \right)^2 \right) G_{jl} - 2(1 - \nu) \left(\left(\frac{\pi i}{L} \right)^2 \left(\frac{\pi l}{b} \right)^2 G_{jl} - \left(\frac{\pi i}{L} \right)^2 \frac{\pi j}{b} \frac{\pi l}{b} H_{jl} \right) \right] \quad (10)$$

where G_{jl} and H_{jl} are given in Appendix A. Note that b and L are the total plate dimensions. For a constant thickness plate, i.e. a plate with only one plate strip, $G_{jl} = H_{jl} = 0$ if $j \neq l$ and $G_{jj} = H_{jj} = b/2$. Thus, Eq. 10 breaks down into the more well known double sum for that case.

Similarly, the potential energy of the external plate loads can be given by

$$T = \sum_{s=1}^{N_s} T_s \quad (11)$$

where

$$T_s = \Lambda \int_0^L \int_{y_{s1}}^{y_{s2}} \frac{t_s}{2} \left(\sigma_{x0} w_{,x}^2 + \sigma_{y0} w_{,y}^2 \right) dy dx \quad (12)$$

in the case of proportional loading. The analytical solution of this integral can be written

$$T_s = \Lambda \sum_{i=1}^M \sum_{j=1}^N \sum_{l=1}^N a_{ij} a_{il} \left(\frac{\sigma_{x0}^m t_s (i\pi)^2}{4L} G_{jl} + \frac{\sigma_{y0}^m t_s L j l \pi^2}{4b^2} H_{jl} \right) \quad (13)$$

The effect of a constant uniaxial or biaxial pre-stress of the plate can readily be included above

by adding another term that is not increased by the load factor Λ . In the same manner as for the strain energy of the plate, Eq. 13 breaks down into the more well known double sum for constant thickness plates.

The curvature of a stiffener is equal to the curvature of the plate along the stiffener. Then, for an arbitrarily oriented stiffener with length L_s , end coordinates (x_1, y_1) and (x_2, y_2) , and cross-section area A_s , the bending strain energy due to the stiffener can be given by

$$U_{\text{stiff}}^b = \frac{EI_e}{2} \int_{L_s} w_{,ss}^2 dL_s = \frac{EI_e}{2L_s^4} \int_{L_s} \left(L_x^2 w_{,xx} + 2L_x L_y w_{,xy} + L_y^2 w_{,yy} \right)^2 dL_s \quad (14)$$

where I_e is an effective moment of inertia about the axis of bending, $w_{,ss}$ the curvature in the stiffener direction, $L_x = (x_2 - x_1)$ and $L_y = (y_2 - y_1)$. In the case of an eccentric stiffener, the stiffener will “lift” the axis of bending above the middle plane. I_e can be given by

$$I_e = \int_{A_s} (z - z_c)^2 dA_s + b_e t z_c^2 \quad (15)$$

where z_c is the distance from the plate middle plane to the centroidal axis (through the centre of area) of a section consisting of the stiffener and an effective plate area of width b_e . For a stepped plate, t may conservatively be taken as the thickness of the thinnest of the plate strips. Bending of the plate about its own axis (middle plane) is included in the plate strain energy. The strain energy integral above may be solved analytically or by numerical integration. The latter is chosen here.

For a symmetric stiffener, $z_c = 0$ is the correct solution as bending in this case will be about the middle plane. For an eccentric stiffener, Eq. 15 is an approximation, whose accuracy will depend on the assumed value of b_e . It is found that $z_c = 0$ is an acceptable value also for eccentric stiffeners in many practical cases.

In practical design work, a z_c -value calculated with a b_e of about $b_e = 20t$ has been suggested [8, 14].

The torsional stiffness of the stiffeners may be accounted for by including the energy contribution (St. Venant torsion) given by

$$U_{\text{stiff}}^T = \frac{GJ}{2} \int_{L_s} w_{,ns}^2 dL_s = \frac{GJ}{2L_s^4} \int_{L_s} \left(L_x L_y (w_{,yy} - w_{,xx}) + (L_x^2 - L_y^2) w_{,xy} \right)^2 dL_s \quad (16)$$

where J is the torsion constant, $w_{,ns}$ is the partial double derivative of w with respect to the directions normal to and along the stiffener and $G = E/2(1 + \nu)$.

For end loaded (continuous) stiffeners, it is necessary to include the potential energy due to external loads on the stiffener ends. For the arbitrarily oriented stiffener considered above, it can be shown that its effect due to plate shortening can be expressed by

$$T_{\text{stiff}} = -\Lambda \frac{P_{s0}}{2} \int_{L_s} w_{,s}^2 dL_s = -\Lambda \frac{P_{s0}}{2L_s^2} \int_{L_s} \left(L_x w_{,x} + L_y w_{,y} \right)^2 dL_s \quad (17)$$

where P_{s0} is the initial, resultant reference load on the stiffener. It acts in the stiffener direction and is defined positive in compression. The translation of this load due to the rotation of the stiffener end is not included as it does not affect computed eigenvalues.

If plate edges, or portions of edges, are partly or fully clamped (modelled by rotational springs), additional strain energy contributions have to be added. Similarly, contributions have to be added for any rotational or translational restraints in the interior of the plate. For additional details, see Brubak et al. [8, 14].

6 Buckling strength limit (BSL)

6.1 Load incrementation and strength criterion

The present model aims at predicting an approximate buckling strength limit (BSL) of a stiffened plate with an initial displacement imperfection (w_0). Additional displacements (w) at a given load stage are estimated using the approximate displacement magnifier

$$w = \frac{\Lambda}{\Lambda_{cr}^e - \Lambda} w_0 \quad (18)$$

where Λ_{cr}^e is the load factor of the first eigenvalue and Λ the load factor at a given stage of the loading ($\Lambda \bar{S}_{x0}, \Lambda S_{y0}$).

This, and similar magnifiers based on linearised elastic second order theory (small deflection theory), are commonly used for approximate analyses of both columns and plates [13]. The load factor, and the resulting displacements and stresses, redistributed due to the out-of-plane displacements, are increased until the adopted strength criterion (below) is satisfied.

Use of Eq. 18 implies that the buckling strength estimate will never exceed the elastic buckling load limit. In order to capture the load carrying capability of slender (thin) plates beyond this limit, often denoted the post-critical (or reserve) strength, displacements must be computed using large displacement theory. However, in practical design, it may be desirable not to utilise this reserve strength, in order to limit the formation of significant plastic (permanent) deformations of slender plates. It is in such contexts that the present model represents a sound alternative.

A buckling strength criterion, proposed and discussed previously [8], is adopted for the presented model. According to this criterion, the buckling strength is obtained at first yield according to the von Mises' yield criterion [13]

applied to membrane stresses:

$$\sigma_e^m = \sqrt{(\sigma_x^m)^2 + (\sigma_y^m)^2 - \sigma_x^m \sigma_y^m + 3(\tau_{xy}^m)^2} \leq f_Y \quad (19)$$

Here, f_Y is the yield strength. This membrane stress criterion allows in an approximate manner for some additional strength due to redistribution of stresses after yielding in the outer plate fibres.

The critical stress points are typically located in the plate along the edges, along the stiffeners or at the boundary between two neighbouring plate strips. The latter is typical for a plate strip subjected mainly to a stress in the x -direction and that is located between two considerably thicker plate strips preventing the out-of-plane displacements along its boundaries.

6.2 Imperfection shape and amplitude

The imperfection shape can be taken according to any specified shape. Here, w_0 is taken equal to the first buckling mode (eigenmode) from the ESL analysis. Then, w_0 can be given in the same form as Eq. 6, but with the coefficients a_{ij} replaced by b_{ij} . The latter are scaled to give the chosen $w_{0,max}$.

In design, maximum imperfection amplitudes $w_{0,max}$ will normally be taken according to values, $w_{0,spec.}$, specified in relevant design codes [1, 2, etc.]. In order to compensate in part for the conservativeness for slender plates implied by the displacement magnifier (Eq. 18), a slenderness dependent $w_{0,max}$ proposed by Brubak et al. [8] is adopted here. It is shown in Fig. 5 and defined by

$$\frac{w_{0,max}}{w_{0,spec.}} = \begin{cases} (1 - \frac{1}{12}\bar{\lambda}^4) & \text{if } \bar{\lambda} \leq \sqrt{1.56} \\ 3/\bar{\lambda}^4 & \text{if } \bar{\lambda} \geq \sqrt{1.56} \end{cases} \quad (20)$$

where

$$\bar{\lambda} = \sqrt{\frac{\Lambda_Y}{\Lambda_{cr}^e}} \quad (21)$$

Figure 5. Adopted maximum imperfection amplitude versus reduced slenderness.

is the reduced slenderness. It is defined in terms of the load factor Λ_Y at which the von Mises' yield stress is reached ($\sigma_e^m = \Lambda_Y$) and the load factor Λ_{cr}^e of the first eigenvalue of the perfect plate.

6.3 Stress redistribution

Following the classical approach, large deflection theory (large rotations, but small in-plane strains) is used to capture the redistribution of stresses that takes place due to out-of-plane displacements. In this theory, the membrane strains in an imperfect plate can be written [15]

$$\epsilon_x^m = u_{,x} + \frac{1}{2}w_{,x}^2 + w_{0,x}w_{,x} \quad (22)$$

$$\epsilon_y^m = v_{,y} + \frac{1}{2}w_{,y}^2 + w_{0,y}w_{,y} \quad (23)$$

$$\gamma_{xy}^m = u_{,y} + v_{,x} + w_{,x}w_{,y} + w_{0,x}w_{,y} + w_{0,y}w_{,x} \quad (24)$$

where w_0 is the initial imperfection, w is the additional displacements, u and v are the x - and y - displacements at the middle plane of the plate, respectively.

The strain compatibility equation for imperfect plates can now be obtained by differentiation and combination of Eqs. 22-24. By substituting strains from Hooke's law for plane stress into this equation, and introducing Airy's stress function $F(x, y)$, defined by $\sigma_x^m = F_{,yy}$, $\sigma_y^m = F_{,xx}$, $\tau_{xy}^m = -F_{,xy}$, the following nonlin-

ear plate compatibility equation results:

$$\begin{aligned} \nabla^4 F = & E(w_{,xy}^2 - w_{,xx}w_{,yy} \\ & + 2w_{0,xy}w_{,xy} - w_{0,xx}w_{,yy} - w_{0,yy}w_{,xx}) \end{aligned} \quad (25)$$

This equation was given by Marguerre [15], and represents an extension of von Karman's plate theory.

The shape of the imperfection displacement field (w_0) is taken on the same form as Eq. 6, with amplitudes b_{ij} instead of a_{ij} . A solution of Eq. 25 on the form

$$\begin{aligned} F(x, y) = & \Lambda \left(\frac{1}{2} \sigma_{x0}^m y^2 + \frac{1}{2} \sigma_{y0}^m x^2 \right) \\ & + \sum_{i=0}^{2M} \sum_{j=0}^{2N} f_{ij} \cos\left(\frac{i\pi}{L}x\right) \cos\left(\frac{j\pi}{b}y\right) \end{aligned} \quad (26)$$

was given by Levy [16] for perfect plates ($w_0 = 0$). Byklum et al. [4] showed that the same form can be used for imperfect plates and derived the stress amplitudes f_{ij} , given in Appendix A, for such plates. They are found by substituting $F(x, y)$, w and w_0 into Eq. 25.

Airy's stress function identically satisfies in-plane equilibrium within each plate strip. To discuss conditions at interfaces, consider Eq. 26, where the first term is due to the external stresses. The second (summation) term represents the redistribution due to out-of-plane displacements (w_0 and w), and is a function of the load stage factor Λ through the amplitudes a_{ij} and b_{ij} . The second term varies smoothly, also across an interface between two strips (of different thickness), while the first term exhibits stress jumps at interfaces. A consequence of this is that equilibrium will be satisfied only on the average between two strips (of full length), and not along limited strip lengths.

Local inaccuracies in stresses near interfaces increase with increasing dominance of the summation term in Eq. 26, i.e., with increasing plate slenderness since the displacements (w) increase with increasing slenderness. For very slender plates, possible effects on strength predictions are of minor concern as the present

Figure 6. Elastic buckling stress (ESL) in the global and local buckling range of a uniaxially loaded plate ($L = b = 2000$ mm, $t = 20$ mm) with an eccentric stiffener ($t_w = 12$ mm, height h_w , $b_e = 30t$).

model is rather conservative for such plates due to the neglect of the post-critical reserve strength.

7 Sniped vs. end-loaded stiffeners

Fig. 6 shows buckling stresses (ESL) for a uniaxially loaded (S_x), simply supported, quadratic plate having constant thickness and one regular, eccentric flat bar stiffener (t_w, h_w) in the middle (parallel to S_x). The stiffener is either sniped or end-loaded (but with no rotational restraints at the ends). In the latter case, S_x is also applied to the stiffener area. The elastic material properties and modelling details are the same given below (Section 9). With increasing stiffener stiffness, the buckling stress increases as the buckling mode changes from a fully global mode, through a mixed mode to an almost fully local mode at the “threshold value” at which the rather flat plateau begins (at about $h_w/t_w = 12-13$).

The moment of inertia I_e (Eq. 15) is calculated with an effective width $b_e = 30t$. In the local buckling range, the model results would have increased slightly if the I_e -values instead

had been calculated about the middle plane of the plate.

The effect on results of end-loading on a continuous stiffener is also seen to be rather small in the considered case. This is particularly so for local plate buckling cases, to the right of the threshold value, where results for the sniped and the end-loaded stiffener (dashed line) coincide almost exactly. This was to be expected since the stiffener-plate interface will remain nearly straight when local buckling governs. The external loads on a stiffener will then only contribute negligibly to the potential energy, T . As a consequence, it makes little difference for the resulting buckling stresses acting on the plate whether the stiffeners are sniped or end-loaded in local plate buckling cases.

In global buckling cases, the difference is seen to be somewhat greater, but not significantly so. In this case, the smaller global buckling stress in the end-loaded stiffener case is almost completely compensated for by the greater load area. The resultant edge loadings, $P_{\text{cont.}} = S_{x,\text{cont.}}(bt + t_w h_w)$ and $P_{\text{sniped}} = S_{x,\text{sniped}}bt$ are found to be almost the same. For local plate buckling cases, which is the target in most practical design situations, the difference in total loading is simply equal to the additional load carried by the stiffener in the end loaded case.

8 Effect of torsional stiffness

The effect of including the torsional stiffness ($J = h_w t_w^3/3$) of the stiffener can be seen in Fig. 6 by comparing the results obtained by the present model for the sniped stiffener case with and without torsion included (full, thin line versus full, thick line). For global buckling modes, there is no effect since the stiffener does not rotate, due to symmetry. For local plate buckling, there is some, but very small beneficial effects. Local plate buckling results by the present model with torsional effects included,

are almost identical to comparable ANSYS results, which are also shown in the figure (open dots). Such torsional effects are conservatively neglected in the remainder of the paper.

9 Validation

The applicability of the presented model, incorporated into a Fortran computer code, has been assessed for many plate and stiffener dimensions. Elastic buckling stress limits (ESL) are verified against ESL results obtained by the finite element analysis computer program ANSYS [17] using Shell93 elements. Buckling strength limits (BSL) are compared with ultimate strength limits, here labelled USL, obtained from fully nonlinear ANSYS analyses (considering both geometric and material non-linearity).

Ultimate strength limits (USL) are the maximum loads plates can carry without becoming unstable. For thin (slender) plates, they will give information about the reserve strength, beyond the BSL results. The BSL and the USL results should ideally converge as the plate thickness increases

The specified imperfection is taken as $w_{0,\text{spec.}} = 5$ mm for the purpose of making comparisons. In the fully nonlinear USL analyses, this value is used directly as the maximum imperfection amplitude ($w_{0,\text{max}} = w_{0,\text{spec.}} = 5$ mm). In the BSL calculations, on the other hand, the fictitious, slenderness dependent value $w_{0,\text{max}}$ according to Eq. 20 is used. The corresponding imperfection shapes are taken equal to the first eigenmode of the plate, as computed with ANSYS and the present model, respectively.

The elastic material properties used are $E = 208000$ MPa and $\nu = 0.3$, and the yield strength is $f_Y = 235$ MPa. In the fully nonlinear ANSYS analyses, a bilinear stress-strain relationship is used that is defined by the properties above, and additionally by a hardening

modulus $E_T = 1000$ MPa. In the subsequent comparisons, the moment of inertia of stiffeners is approximated by I_e about the middle plain (i.e., Eq. 15 with $z_c = 0$) and the torsional stiffness is neglected.

The displacement field in the present model is defined with 15 terms in each direction (225 degrees of freedom). This generally provides sufficient numerical accuracy. Resulting strength predictions may be up to about 2% greater than those that would have been obtained with two to three times the number of terms in each direction. In many cases, the number of terms could be significantly reduced without reducing the accuracy noticeable. In comparison, in a typical USL analysis by ANSYS, the number of degrees of freedom used was about 20000. Probably, sufficient accuracy could have been obtained with fewer degrees of freedom.

Results are presented in subsequent sections. They are limited to simply supported plates with eccentric or symmetric sniped stiffeners (free and not loaded at the ends). To provide severe test cases for the stepped plates, the relative difference in thickness between the thickest and thinnest strip in some of the plates is chosen rather large. Further, both regular and rather strongly irregular stiffener orientations are considered. Additional comparisons with ANSYS results are given by Brubak [14], where also clamped plates are considered.

10 ESL predictions

Typical elastic buckling stresses (eigenvalues) are shown by the interaction curves in Fig. 7, obtained for an unstiffened plate consisting of two very different plate strips. The present ESL model does not neglect any energy contribution in the case of unstiffened plates. Results will therefore converge towards the exact solution as the number of degrees of freedom increases. The excellent agreement with the ANSYS results

Figure 7. ESL interaction curves for a stepped plate consisting of two plate strips of unequal width and thickness ($L/b_1/t_1 = 1000/1000/20$ mm, and $L/b_2/t_2 = 1000/500/10$ mm).

Table 1

Dimensions [mm] of plates with stepwise constant plate thickness and two inclined, eccentric stiffeners.

	L	b	t_1	t_2	t_3	h_w	t_w	b_f	t_f
Plate 1	1000	2400	25	20	15	200	15	150	25
Plate 2	1200	2400	25	20	15	200	15	150	25
Plate 3	1000	3000	20	18	16	200	16	100	16

shows that the assumed displacement field, defined over the entire plate, is clearly acceptable with the present choice of degrees of freedom.

For stiffened plates, the agreement can not be expected to be equally good due to the neglect of some energy contributions and approximations in initial, reference stress computations (neglect of axial stiffener stiffness). However, as will be seen in ESL results shown below for several stepped plates (Table 1 and 2), the agreement is still considered to be very good for biaxial stress combinations of interest (Fig. 9 and 10).

Figure 9. Interaction curves in the stress space S_x - S_y for plates no. 1, 2 and 3, with stepwise constant thickness and with two eccentric, inclined stiffeners.

11 BSL predictions – Inclined eccentric stiffeners

A number of plates with three strips and two inclined, eccentric, sniped T-section stiffeners (Fig. 2b) have been analysed. Selected results for plates defined in Table 1 and Fig. 9a will be presented and discussed.

The first buckling mode calculated by the present model and by the finite element model

has been found to be quite similar in each case considered. A typical case is shown in Fig. 8a and b. The displacements are largest in the thinnest plate strip. The mode shown can be considered as a local plate buckling mode, as the out-of-plane plate displacements along the stiffeners are small. This is a consequence of the chosen stiffeners, which apparently are sufficiently stiff to prevent a global buckling mode. The same is found to be the case for the

Figure 8. First buckling mode of plate 1 subjected to a uniaxial external stress S_y , calculated by (a) the present model and (b) Ansys.

other plates in Table 1. The main difference between the models is that the finite element model accounts for the sideways deflections of the stiffeners and of their torsional and axial stiffness.

In Fig. 9, biaxial load interaction curves are given for buckling strengths (BSL), elastic buckling stresses (ESL), which have been discussed above, and the “yield limit” according to the von Mises’ yield criterion for the material as such. The latter represents a maximum strength limit. In the context of buckling strength, the most relevant loading situations are those with uniaxial or biaxial compressive stresses (in the first quadrant).

Also shown in the figure are ultimate strength (USL) results. These are seen to be close to the BSL results for the considered plates, which have relatively small to intermediate slenderness values. The slenderness $\bar{\lambda}$ varies along the interaction curves, but it is smaller than 1.05 for all load combinations. The corresponding relative imperfection amplitude $w_{0,\max}/w_{0,\text{spec.}}$ varies between 0.90 and 1. Thus, the reduction of the imperfection am-

Table 2

Dimensions [mm] of plates with stepwise constant plate thickness and two symmetric stiffeners with regular orientation.

	L	b	t_1	t_2	t_3	h_w	t_w	b_f	t_f
Plate 4	3000	3300	30	28	26	300	10	150	20
Plate 5	3000	2100	24	22	20	300	10	150	20
Plate 6	3000	2100	16	14	12	300	8	100	10

plitude in the BSL procedure is small in these cases. More slender plates, for which BSL results will become more conservative relative to USL results, are discussed below.

12 BSL predictions – Regular, symmetric stiffeners

For uni-directionally stepped plates, it is believed that regular stiffeners represent a more common stiffener arrangement than inclined stiffeners. Similar results to those presented above have been obtained for a number of such plates, of which preliminary results were given in [18]. Selected results, for the plates with three strips and two regular, symmetric stiffeners defined in Table 2 and Fig. 10a, are shown in Fig. 10b, c and d. Also, for the plates considered here, closer examinations show that the first buckling modes are local plate buckling modes.

The BSL and USL results are close to each other for plate 4 and 5, which have relatively small to intermediate reduced slenderness values. For instance, for plate 5, the maximum reduced slenderness is approximately $\bar{\lambda} = 1.17$ (for $\bar{S}_x = 1.67S_y$). The corresponding imperfection amplitude is $w_{0,\max} = 0.84w_{0,\text{spec.}}$ according to Eq. 20.

For plate 6, the maximum reduced slenderness is approximately 2.36 (for $\bar{S}_x = 1.43S_y$), which is representative of a very slender case. The corresponding imperfection amplitude is $w_{0,\max} = 0.10w_{0,\text{spec.}}$, which is physically speaking, unreasonably small. Even so, the BSL results calculated by the present method are still

Figure 10. Interaction curves in the stress space S_x - S_y for plates no. 4, 5 and 6, with stepwise constant thickness and with two symmetric, regular stiffeners.

conservative compared to the fully nonlinear USL results. The latter is in this case twice as large as the BSL results.

The reason for the conservativeness in BSL results for slender plates is, as mentioned before, that the present BSL model is not able to capture the post-critical (reserve) strength. Therefore, the present BSL model is most feasible in practical design cases in which loading in the post-buckling range is not accepted.

13 Concluding remarks

An approximate method for global and local buckling strength (BSL) analysis of stepped plates with arbitrary stiffener orientations has been presented. Out-of plane displacements are estimated using an approximate displacement magnifier and plate failure is estimated using von Mises' yield criterion. The assumed displacement field implies that equilibrium is

only satisfied on the average between the plate strips.

The applicability and versatility of the presented model are documented for a various plate geometries and computed BSL results are validated against fully nonlinear USL analyses. The BSL predictions are generally conservative, in particular for slender plates. This is due to the use of the approximate displacement magnifier, which results in buckling strengths that never exceed the elastic buckling loads. Consequently, the postbuckling (reserve) strengths for slender plates are not accounted for. Although conservative, this may be a sound theoretical treatment in design situation in which structural elements are not accepted to buckle elastically.

The method is computationally very efficient. Using a computer code that has not yet been optimised, the computer time for a BSL prediction for a given loading is typically 1-2 seconds on a medium fast computer (1.5 GHz processor, 512 MB RAM). Compared to non-linear ultimate strength (USL) analyses by ANSYS, the presented method is typically more than 1000 times faster for the same problem. Speed is always an advantage, in particular in optimisation, reliability and other situations requiring larger numbers of case studies. The size of the computer code of the method is limited and the number of input data required is minimal. Due to such factors, the method is suitable for incorporation into computerised analysis and design codes. A constant thickness version is already included in a code labelled PULS [9] and can be downloaded from www.dnv.com.

Acknowledgements

The authors would like to thank dr.scient. Eivind Steen and dr.ing. Eirik Byklum, both at DNV, for initially suggesting the topic of the paper, for their interest and discussions, and

additionally for making a computer subroutine for stress calculations available for use in this study.

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A Appendix

A.1 Coefficients in Airy's stress function

The coefficients f_{ij} in Airy's stress function are defined by

$$f_{ij} = \frac{E}{4(i^2 \frac{b}{L} + j^2 \frac{L}{b})^2} \sum_{r=1}^M \sum_{s=1}^N \sum_{p=1}^M \sum_{q=1}^N c_{rspq} (a_{rs} a_{pq} + a_{rs} b_{pq} + a_{pq} b_{rs}) \quad (\text{A.1})$$

where f_{00} is zero, a_{rs} and b_{pq} are the amplitudes of w and w_0 , respectively, and c_{rspq} are integers given by

$$c_{rspq} = rspq + r^2 q^2 \quad (\text{A.2})$$

if $\pm(r - p) = i$ and $s + q = j$, or $r + p = i$ and $\pm(s - q) = j$, or

$$c_{rspq} = rspq - r^2 q^2 \quad (\text{A.3})$$

if $r + p = i$ and $s + q = j$, or $\pm(r - p) = i$ and $\pm(s - q) = j$, or

$$c_{rspq} = 0 \quad (\text{A.4})$$

for other cases. More details of the derivation of the coefficients f_{ij} can be found in the literature [4].

A.2 Definition of G_{jl} and H_{jl}

$$G_{jl} = \int_{y_{s1}}^{y_{s2}} \sin(\frac{\pi j}{b} y) \sin(\frac{\pi l}{b} y) dy \quad (\text{A.5})$$

and

$$H_{jl} = \int_{y_{s1}}^{y_{s1}} \cos(\frac{\pi j}{b} y) \cos(\frac{\pi l}{b} y) dy \quad (\text{A.6})$$

The results of these integrals are

$$G_{jl} = \frac{b}{2\pi(j^2 - l^2)} \left((j+l)\sin\left(\frac{\pi(j-l)}{b}y\right) - (j-l)\sin\left(\frac{\pi(j+l)}{b}y\right) \right) \Bigg|_{y_{s1}}^{y_{s2}} \quad (\text{A.7})$$

if $j \neq l$, and

$$G_{jl} = \left(\frac{y}{2} - \frac{b}{2\pi j} \cos\left(\frac{\pi j}{b}y\right) \sin\left(\frac{\pi j}{b}y\right) \right) \Bigg|_{y_{s1}}^{y_{s2}} \quad (\text{A.8})$$

if $j = l$, and

$$H_{jl} = \frac{b}{2\pi(j^2 - l^2)} \left((j+l)\sin\left(\frac{\pi(j-l)}{b}y\right) + (j-l)\sin\left(\frac{\pi(j+l)}{b}y\right) \right) \Bigg|_{y_{s1}}^{y_{s2}} \quad (\text{A.9})$$

if $j \neq l$, and

$$H_{jl} = \left(\frac{b}{2\pi j} \cos\left(\frac{\pi j}{b}y\right) \sin\left(\frac{\pi j}{b}y\right) + \frac{y}{2} \right) \Bigg|_{y_{s1}}^{y_{s2}} \quad (\text{A.10})$$

if $j = l$.